February 20, 1992

The problem is to relate Connes's unnormalized b, B approach to cyclic cohomology with the normalized approach you use. Let A be a unital algebra.

Proposition. 1) There is an exact sequence of mixed complexes

$$\begin{tikzcd} 0 \arrow[r] & \Sigma C(A) \arrow[r, "i"] & \Omega\tilde{A} \arrow[r, "j"] & C(A) \arrow[r] & 0 \\ & (-b', 0) & (\hat{b}, \hat{B}) & (b, B) & \end{tikzcd}$$

where $i = \begin{pmatrix} -(1-\lambda)s \\ 1 \end{pmatrix}$ $\qquad j = \begin{pmatrix} 1 & (1-\lambda)s \end{pmatrix}$

$$\hat{b} = \begin{pmatrix} b & 1-\lambda \\ 0 & -b' \end{pmatrix} \qquad \hat{B} = \begin{pmatrix} 0 & 0 \\ N_\lambda & 0 \end{pmatrix}$$

(these are the standard operators on $\Omega\tilde{a}$ for a nonunital)

and
$$B = (1-\lambda)sN$$

where s is a contracting homotopy wrt b' ($s(a_1,..,a_n) = (1, a_1,..,a_n)$, but as Kassel points out one can use any s such that $b's + sb' = 1$, and such an s exists $\Longleftrightarrow$ the algebra is h-unital.)

2) There is a splitting of the corresponding exact sequence of $\mathbb{Z}/2$-graded complexes given by

$$r = \begin{pmatrix} -sN_\lambda & 1 \end{pmatrix} \qquad l = \begin{pmatrix} 1-(1-\lambda)s^2N_\lambda \\ sN_\lambda \end{pmatrix}$$

Proof by computation:

$$ri = \begin{pmatrix} -sN_\lambda & 1 \end{pmatrix} \begin{pmatrix} -(1-\lambda)s \\ 1 \end{pmatrix} = 1$$

$$jl = \begin{pmatrix} 1 & (1-\lambda)s \end{pmatrix} \begin{pmatrix} 1-(1-\lambda)s^2N_\lambda \\ sN_\lambda \end{pmatrix} = 1$$

$$ir + lj = \begin{pmatrix} -(1-\lambda)s \\ 1 \end{pmatrix} \begin{pmatrix} -sN_\lambda & 1 \end{pmatrix} + \begin{pmatrix} 1-(1-\lambda)s^2N_\lambda \\ sN_\lambda \end{pmatrix} \begin{pmatrix} 1 & (1-\lambda \end{pmatrix}$$

$$= \begin{pmatrix} (1-\lambda)s^2N_\lambda & -(1-\lambda)s \\ -sN_\lambda & 1 \end{pmatrix} + \begin{pmatrix} 1-(1-\lambda)s^2N_\lambda & (1-\lambda)s \\ sN_\lambda & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$j\hat{b} = \begin{pmatrix} 1 & (1-\lambda)s \end{pmatrix} \begin{pmatrix} b & 1-\lambda \\ & -b' \end{pmatrix} = \begin{pmatrix} b & (1-\lambda)-(1-\lambda)sb' \end{pmatrix}$$

$$= \begin{pmatrix} b & (1-\lambda)b's \end{pmatrix} = \begin{pmatrix} b & b(1-\lambda)s \end{pmatrix} = b\,j$$

$$j\hat{B} = \begin{pmatrix} 1 & (1-\lambda)s \end{pmatrix} \begin{pmatrix} 0 & 0 \\ N_\lambda & 0 \end{pmatrix} = \begin{pmatrix} (1-\lambda)sN_\lambda & 0 \end{pmatrix}$$

$$= (1-\lambda)sN_\lambda \begin{pmatrix} 1 & (1-\lambda)s \end{pmatrix} = Bj$$

$$\hat{b}i = \begin{pmatrix} b & 1-\lambda \\ & -b' \end{pmatrix} \begin{pmatrix} -(1-\lambda)s \\ 1 \end{pmatrix} = \begin{pmatrix} 1-\lambda - b(1-\lambda)s \\ -b') \end{pmatrix} = \begin{pmatrix} 1-\lambda-(1-\lambda)b' \\ -b' \end{pmatrix}$$

$$= \begin{pmatrix} (1-\lambda)sb' \\ -b' \end{pmatrix} = i(-b')$$

$$\hat{B}i = \begin{pmatrix} 0 & 0 \\ N_\lambda & 0 \end{pmatrix}\begin{pmatrix} -(1-\lambda)s \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} = i \cdot 0$$

$$r(\hat{b}+\hat{B}) = \begin{pmatrix} -sN_\lambda & 1 \end{pmatrix}\begin{pmatrix} b & 1-\lambda \\ N_\lambda & -b' \end{pmatrix} = \begin{pmatrix} -s\underbrace{N_\lambda b}_{b'N_\lambda} + N_\lambda & -b' \end{pmatrix}$$

$$= \begin{pmatrix} b'sN_\lambda & -b' \end{pmatrix} = (-b')\begin{pmatrix} -sN_\lambda & 1 \end{pmatrix} = (-b')r$$

$$(\hat{b}+\hat{B})\ell = \begin{pmatrix} b & 1-\lambda \\ N_\lambda & -b' \end{pmatrix}\begin{pmatrix} 1-(1-\lambda)s^2N_\lambda \\ sN_\lambda \end{pmatrix} = \begin{pmatrix} b - \overbrace{b(1-\lambda)}^{(1-\lambda)b'}s^2N_\lambda + \boxed{(1-\lambda)sN_\lambda} \\ N_\lambda - b'sN_\lambda \end{pmatrix}$$

$$= \begin{pmatrix} b + (1-\lambda)sb'sN_\lambda \\ sb'N_\lambda \end{pmatrix}$$

$$\ell(b+B) = \begin{pmatrix} 1-(1-\lambda)s^2N_\lambda \\ sN_\lambda \end{pmatrix}\left(b + (1-\lambda)sN_\lambda\right)$$

$$= \begin{pmatrix} b + (1-\lambda)sN_\lambda - (1-\lambda)s^2\underbrace{N_\lambda b}_{b'N_\lambda} \\ sN_\lambda b \end{pmatrix}$$

$$= \begin{pmatrix} b + (1-\lambda)sb'sN_\lambda \\ sb'N_\lambda \end{pmatrix}$$

Note in this last proof (i.e. $(\hat{b}+\hat{B})\ell = \ell(b+B)$) one has [illegible] established

$$b(1-\lambda)s^2N_\lambda = (1-\lambda)s^2N_\lambda b$$

which follows from

$$b(1-\lambda)s^2N_\lambda = (1-\lambda)b's^2N_\lambda = (1-\lambda)s^2b'N_\lambda = (1-\lambda)s^2N_\lambda$$

and $[b', s^2] = [b', s]s - s[b', s] = s - s = 0$.

Another minor point is that one can make the contracting homotopy s special by replacing s by $sb's$

i.e. $s^2 = 0$

In effect $[b', sb's] = [b',s]b's - s[b',b']s + sb'[b',s]$

$= b's + sb' = 1.$

and $sb's \cdot sb's = s s^2 b'b's = 0.$

I claim we have a 3×3 diagram of exact sequences

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & & \\
0 \arrow[r] & K \arrow[r] \arrow[d] & P^{\perp}\bar{\Omega}\tilde{A} \arrow[r] \arrow[d] & \Sigma C \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & C \arrow[r, "l"] \arrow[d] & \bar{\Omega}\tilde{A} \arrow[r, "r"] \arrow[d, "p"] & \Sigma C \arrow[r] & 0 \\
 & P\bar{\Omega}\tilde{A} \arrow[r, phantom, "="] \arrow[d] & P\bar{\Omega}\tilde{A} \arrow[d] & & \\
 & 0 & 0 & &
\end{tikzcd}$$

of $\mathbb{Z}/2$-graded complexes with differential $b+B$ (or $\hat{b}+\hat{B}$, $-b'$ as the case may be.) It is equivalent to say Pl maps C onto $P\bar{\Omega}\tilde{A}$, or that r maps $P^{\perp}\bar{\Omega}A$ onto ΣC.

Lemma: $rP^{\perp} : \bar{\Omega}\tilde{A} \longrightarrow \Sigma C$ is surjective.

$$P^{\perp} = Gdb + bGd = \begin{pmatrix} 0 & 0 \\ G_\lambda b & P_\lambda^{\perp} \end{pmatrix} + \begin{pmatrix} P_\lambda^{\perp} & 0 \\ -b'G_\lambda & 0 \end{pmatrix}$$

$$rGdb = \begin{pmatrix} -sN_\lambda & 1 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ G_\lambda b & P_\lambda^{\perp} \end{pmatrix} = \begin{pmatrix} G_\lambda b & P_\lambda^{\perp} \end{pmatrix}$$

$$rbGd = \begin{pmatrix} -sN_\lambda & 1 \end{pmatrix}\begin{pmatrix} P_\lambda^{\perp} & 0 \\ -b'G_\lambda & 0 \end{pmatrix} = \begin{pmatrix} -b'G_\lambda & 0 \end{pmatrix}$$

Thus $\boxed{rP^{\perp}\begin{pmatrix} x \\ y \end{pmatrix} = (G_\lambda b - b'G_\lambda)x + P_\lambda^{\perp} y}$

To show $(G_\lambda b - b'G_\lambda)x + P_\lambda^\perp y$ exhausts all chains, it suffices to show $P_\lambda\{(G_\lambda b - b'G_\lambda)x + P_\lambda^\perp y\} = -P_\lambda b' G_\lambda x$ exhausts all λinvariant chains. Now

$$P_\lambda b G_\lambda = P_\lambda b(1-\lambda) G_\lambda^2 = P_\lambda (1-\lambda) b' G_\lambda^2 = 0$$

so

$$-P_\lambda b' G_\lambda = P_\lambda c G_\lambda \qquad c = b - b'$$

Next we calculate

$$P_\lambda c(1-\lambda^{-1}) s(a_1,\dots,a_n) = P_\lambda c(1-\lambda^{-1})(1,a_1,\dots,a_n)$$

$$= P_\lambda c\{(1,a_1,\dots,a_n) - (-1)^n(a_1,\dots,a_n,1)\}$$

$$= P_\lambda\{(-1)^n(a_n,a_1,\dots,a_{n-1}) - (a_1,\dots,a_n)\} = -2P_\lambda(a_1,\dots,a_n)$$

$$\therefore \circledast \quad \boxed{P_\lambda c\left(\frac{1-\lambda^{-1}}{-2}\right)s = P_\lambda}$$

$$\boxed{P_\lambda c G_\lambda\left(\frac{(1-\lambda)(1-\lambda^{-1})}{-2}s\right) = P_\lambda}$$

In principale we have now a lifting of ΣC into $P^\perp\Omega\tilde{A}$. Given $z \in \Omega C$ take $x = hz$ where $h = \frac{(1-\lambda)(1-\lambda^{-1})}{-2}s$, then take

$$y = -(G_\lambda b - b'G_\lambda)hz + z$$

Now

$$-G_\lambda b\frac{(1-\lambda)(1-\lambda^{-1})}{-2} s z = -P_\lambda^\perp b' \frac{(1-\lambda^{-1})}{-2} s z$$

$$b'G_\lambda \frac{(1-\lambda)(1-\lambda^{-1})}{2} s z = b' \frac{(1-\lambda^{-1})}{2} s z$$

$$\text{so}\quad \left(-G_\lambda b + b' G_\lambda\right) h z = P_\lambda b' \left(\frac{1-\lambda^{-1}}{-2}\right) s z$$
$$= -P_\lambda c \left(\frac{1-\lambda^{-1}}{-2}\right) s z = -P_\lambda z \qquad \text{by } ⊛$$

Thus $y = -P_\lambda z + z = P_\lambda^\perp z$. So we have a lifting

$$z \longmapsto \begin{pmatrix} h z \\ P_\lambda^\perp z \end{pmatrix} \qquad \Sigma C \longrightarrow P^\perp \bar{\Omega} \tilde{A}$$

This lifting is apparently not compatible with $b+B$, but because of s, which contracts ΣC, we can modify the lifting so as to be compatible with $b+B$. Then we should get a lifting of $P \bar{\Omega} \tilde{A}$ into C compatible with differential $b+B$.

February 23, 1992

Tate-Farrell cohomology:

The reason for looking at this is the following: I would like to understand Goodwillie's comment that cyclic homology is the homotopy inductive limit over the cyclic category while negative cyclic homology is the homotopy inverse limit. My idea originally is that if X is a cyclic module, then $L\varinjlim_{Cyc} X$ gives the cyclic homology and $R\varprojlim_{Cyc} X$ gives the negative cyclic homology. (However the latter is wrong it seems, because $R\varprojlim_{Cyc} X$ should be a cochain complex, so $R^i\varprojlim_{Cyc} X$ should be zero for $i<0$.)

Another point is that as for Tate cohomology for finite groups it might be possible to fit the homology & cohomology together, more precisely to fit $R\varprojlim X$ and $L\varinjlim X$ together to get the periodic cyclic homology. This leads us to look at Tate-Farrell cohomology for certain groups Γ, those having property VFT in the sense of Serre. (Even though $R\varprojlim X$ does not give the correct negative cyclic homology, there may be a twisted version using a "Steinberg" module.)

Recall that Γ has property FT when there is a finite resolution of $\mathbb{Z}$ by finitely gen. projective $\mathbb{Z}[\Gamma]$-modules. Denote this resolution $P_\cdot$. We have

$$H^i(\Gamma, M) = H^i\{\operatorname{Hom}_{\mathbb{Z}[\Gamma]}(P_\cdot, M)\} =$$

$$H^i\left(\check{P}\otimes_{\mathbb{Z}[\Gamma]} M\right) = H^i\left((P.\otimes M)_\Gamma\right)$$

~~Let us assume to simplify that~~

[illegible]

where $\check{P} = \mathrm{Hom}_{\mathbb{Z}[\Gamma]}(P., \mathbb{Z}[\Gamma])$

Here I use the isom.

$$\mathrm{Hom}_{\mathbb{Z}[\Gamma]}(E, M) = \mathrm{Hom}_{\mathbb{Z}[\Gamma]}(E, \mathbb{Z}[\Gamma]) \otimes_{\mathbb{Z}[\Gamma]} M$$

for any fin. gen. projective $\mathbb{Z}[\Gamma]$-module E.

Now suppose to simplify that

$$H^i(\check{P}) = H^i(\Gamma, \mathbb{Z}[\Gamma]) = \begin{cases} 0 & i \neq n \\ St & i = n \end{cases}$$

where $St = H^n(\Gamma, \mathbb{Z}[\Gamma])$ is called the Steinberg module. Then we have an isomorphism

$$\check{P} \simeq St[-n] \quad \left(\begin{array}{c} n \text{ fold} \\ \text{negative suspension} \end{array}\right)$$

in the derived category. It's likely one can assume P is of length n, because $\check{P}$ is an acyclic complex in degrees $> n$ consisting of f.g. projectives and its bounded above. So $\check{P}$ is actually of proj resolution of $St[-n]$ and we have

$$H^i(\Gamma, M) = H^i\left((\check{P}\otimes M)_\Gamma\right) = H_{n-i}(\Gamma, St\otimes M)$$

a basic duality theorem.

Next suppose Γ has property VFL, which means it has a subgroup of finite index having property FL. One has

$$H^i(\Gamma, \mathbb{Z}[\Gamma]) = H^i(\Gamma, \mathrm{Ind}_{\Gamma' \to \Gamma} \mathbb{Z}[\Gamma'])$$

$$= H^i(\Gamma', \mathbb{Z}[\Gamma'])$$

ind and coind same in this finite index case

Let P be a $\mathbb{Z}[\Gamma]$-module resolution of $\mathbb{Z}$ by fin. gen. projective modules. (This should exist by the VFT property. In effect if Q is a f.g. proj $\mathbb{Z}[\Gamma']$-module resolution of $\mathbb{Z}$ of finite length, then $\mathbb{Z}[\Gamma] \otimes_{\mathbb{Z}[\Gamma']} Q$ is a f.g. proj f.l. resolution of $\mathbb{Z}[\Gamma/\Gamma']$. Then we combine this with the standard $\mathbb{Z}[\Gamma/\Gamma']$-resolution of $\mathbb{Z}$.)

Now again we have

$$H^i(\Gamma, M) = H^i((\check{P} \otimes M)_\Gamma)$$

but because $\check{P}$ is unbounded above, we only have quasi-isomorphisms

$$\begin{tikzcd} & \check{P} \arrow[d] \\ St[-n] \arrow[r, hook] & \check{P}[\geq n] \end{tikzcd}$$

where $\check{P}[\geq n] :\ \to 0 \to \check{P}^n / B^n(\check{P}) \longrightarrow \check{P}^{n+1} \longrightarrow \cdots$

has the same cohomology groups in degrees $\geq n$. Thus if $E \longrightarrow St[-n]$ is a projective resolution we have quasi-isos.

$$\begin{tikzcd} E \arrow[r] \arrow[d] & \check{P} \\ St[-n] & \end{tikzcd}$$

This then gives a map

$$H_{n-i}(\Gamma, M) = H^i((E \otimes M)_\Gamma) \longrightarrow H^i((\check{P} \otimes M)_\Gamma)$$

Thus we get a map

$$H_{n-i}(\Gamma, M) \longrightarrow H^i(\Gamma, M)$$

which can be completed to a triangle involving the Farrell-Tate cohomology.

$$\check{H}^i(\Gamma, M) = H^i\left(\left(\mathrm{Cone}(E \to \check{P}) \otimes M\right)_\Gamma\right)$$

The ct Cone$(E \to \check{P})$ is then an analogue of a complete resolution for computing Tate cohomology.

Remarks. Because $E \to \check{P}$ is a quis, the complex $\mathrm{Cone}(E \to \check{P})$ has trivial homology.

<u>Example</u>: $\Gamma = \mathbb{Z}/2 \ltimes \mathbb{Z}$ generators ε, g, relations $\varepsilon^2 = 1$ $\varepsilon g \varepsilon^{-1} = g^{-1}$. $\Lambda = \mathbb{Z}[\Gamma]$. $\Lambda/\Lambda(g-1) = \mathbb{Z}[\varepsilon]$.

We have ideals

$$I^{\pm} = \mathrm{Ker}\{\Lambda \longrightarrow \mathbb{Z}\} \qquad \begin{matrix} g \longmapsto 1 \\ \varepsilon \longmapsto \pm 1 \end{matrix}$$

Let's construct a projective $\mathbb{Z}[\Gamma]$ resolution of $\mathbb{Z}$. First step:

$$0 \longrightarrow I^+ \longrightarrow \Lambda \longrightarrow \mathbb{Z} \longrightarrow 0$$

2nd step.

$$\begin{tikzcd}
0 \arrow[r] & K \arrow[r] & \Lambda^2 \arrow[r, "(\varepsilon-1 \quad g-1)"] \arrow[d, "pr_1"] & I^+ \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & K \arrow[r] & \Lambda \arrow[r, "\varepsilon-1"] & I^+/\Lambda(g-1) \arrow[r] & 0
\end{tikzcd}$$

But we have

$$\begin{tikzcd}
 & \Lambda \arrow[r, two heads, "\varepsilon-1"] & I^+/\Lambda(g-1) \arrow[d, phantom, "\cap"] & & \\
0 \arrow[r] & \mathbb{Z} \arrow[r, "\varepsilon-1"] & \Lambda/\Lambda(g-1) \arrow[r, "\varepsilon \mapsto 1"] & \mathbb{Z} \arrow[r] & 0
\end{tikzcd}$$

Thus, since $\varepsilon(\varepsilon-1) = -(\varepsilon-1)$, we have that $\varepsilon = -1$ on $I^+/\Lambda(g-1)$. $\therefore$ $K = I^-$ and we have an exact sequence

$$0 \longrightarrow I^- \xrightarrow{\begin{pmatrix}1\\ -(g-1)^{-1}(\varepsilon-1)\end{pmatrix}} \Lambda^2 \xrightarrow{(\varepsilon-1 \quad g-1)} I^+ \longrightarrow 0$$

Similarly we have

$$0 \longrightarrow I^+ \xrightarrow{\begin{pmatrix}1\\ -(g-1)^{-1}(\varepsilon+1)\end{pmatrix}} \Lambda^2 \xrightarrow{(\varepsilon+1 \quad g-1)} I^- \longrightarrow 0$$

As a check

$$\begin{pmatrix}1\\ -(g-1)^{-1}(\varepsilon+1)\end{pmatrix}(\varepsilon-1 \quad g-1) = \begin{pmatrix}\varepsilon-1 & g-1\\ 0 & -(g-1)^{-1}(\varepsilon+1)(g-1)\end{pmatrix}$$

Now $-(g-1)^{-1}(\varepsilon+1)(g-1) = -(g-1)^{-1}((g^{-1}-1)\varepsilon + g-1)$

$$= \boxed{-1+g^{-1}\varepsilon}$$

Similarly

$$\begin{pmatrix}1\\ -(g-1)^{-1}(\varepsilon-1)\end{pmatrix}(\varepsilon+1 \quad g-1) = \begin{pmatrix}\varepsilon+1 & g-1\\ 0 & -(g-1)^{-1}(\varepsilon-1)(g-1)\end{pmatrix}$$

$$-(g-1)^{-1}(\varepsilon-1)(g-1) = -(g-1)^{-1}((g^{-1}-1)\varepsilon - (g-1)))$$

$$= \boxed{1+g^{-1}\varepsilon}$$

So we should have the following situation

$$\begin{tikzcd}
P\otimes St: \arrow[r] & \Lambda^2 \arrow[r] & \Lambda^2 \arrow[r] \arrow[dr, dashed] & \Lambda \arrow[dr, dashed] & & \\
\check{P}: & & & \Lambda \arrow[r] & \Lambda^2 \arrow[r] & \Lambda^2 \arrow[r] & {}
\end{tikzcd}$$

↙ $\mathbb{Z}$ homology here (probably $\varepsilon = -1$)

← we can construct this map induced an isom on homology

↑ $\mathbb{Z}$ homology here

The Cone $(P\otimes St[-1] \longrightarrow \check{P})$ I guess should be the acyclic complex

$$\longrightarrow \Lambda^2 \longrightarrow \Lambda^2 \longrightarrow \Lambda^2 \longrightarrow \Lambda^2 \longrightarrow \cdots$$

which is periodic of period 2.

February 24, 1992

The goal is to understand cyclic [redacted] modules and their cyclic 'theory'. Let us begin with simplicial modules, where simplicial means without degeneracies.

Let Δ be the category of non-empty finite totally ordered sets and ~~inclusions~~ injections. A simplicial module is a contravariant functor F from Δ to modules. We want to see that $H_n(\Delta, F)$ is calculated in the usual way by the complex consisting of $F_n = F([n])$ in degree n, where the differential is the alternating sum of the faces.

The idea will be to use the natural increasing filtration $F_n\Delta$ = full subcategory of $[p]$ [redacted] for $p \leq n$. This gives rise to an increasing filtration $\gamma_n F$ defined by

$$(\gamma_n F)[p] = \begin{cases} F([p]) & p \leq n \\ 0 & p > n \end{cases}$$

Another description: Let $j_n : F_n\Delta \longrightarrow \Delta$ be the inclusion. If $G \in (F_n\Delta)^\vee$, (this means $G : (F_n\Delta)^{op} \to ab$), then

$$\left((j_n)_! G\right)[p] = \varinjlim_{\substack{[q] \in F_n\Delta \\ j_n[q] \xleftarrow{u} [p]}} G([q])$$

$$= \varinjlim_{\substack{[q] \text{ for } q \leq n \\ \text{with } [p] \xrightarrow{u} [q]}} G([q]) = \begin{cases} 0 & \text{if } p > n \\ G([p]) & \text{if } p \leq n \end{cases}$$

Thus $\gamma_n F = j_{n!} j_n^* F$. Claim

$$H_i(\Delta, \gamma_n F / \gamma_{n-1} F) = \begin{cases} F_n & i = n \\ 0 & i \neq n \end{cases}$$

Now $j_n!$ is exact, so from the spectral sequence

$$H_p(\Delta, L_q j_n! G) \Rightarrow H_{p+q}(F_n\Delta, G)$$

we obtain $\boxed{H_p(\Delta, j_n! G) = H_p(F_n\Delta, G).}$

The point is that $\gamma_n F/\gamma_{n-1} F$ is unchanged if we replace F by the constant functor A with value $A = F_n$. But then $H_i(\Delta, \gamma_n A) = H_i(F_n \Delta, A)$ is the homology of the category $F_n\Delta$ with values in the abelian grp A, and similarly $H_i(\Delta, \gamma_n A/\gamma_{n-1}A) = H_i(F_n\Delta, F_{n-1}\Delta; A)$.

Let us admit the fact that this relative homology can be calculated as if there is a cocartesian square

$$\begin{tikzcd}
\{[p] \to [n] \mid p < n\} \arrow[r] \arrow[d] & {[n]} \arrow[d] \\
F_{n-1}\Delta \arrow[r, hook] & F_n\Delta
\end{tikzcd}$$

It follows then that $F_n\Delta/F_{n-1}\Delta \sim \Delta(n)/\dot{\Delta}(n) = S^n$.

Thus

$$H_i(F_n\Delta, F_{n-1}\Delta; A) = \begin{cases} 0 & i \neq n \\ A & i = n. \end{cases}$$

so in the spectral sequence assoc. to the filtration $\gamma_n F$:

$$E^1_{pq} = H_{p+q}(\Delta, \gamma_p F/\gamma_{p-1} F) \Rightarrow H_{p+q}(\Delta, F)$$

we have

$$E^1_{pq} = \begin{cases} 0 & q \neq 0 \\ F_p & q = 0 \end{cases}$$

thus

$$H_n(\Delta, F) = H_n\{ F_0 \xleftarrow{d^1} F_1 \xleftarrow{d^1} F_2 \leftarrow \cdots \}$$

Let's admit that d' is given by the alternating sum of the faces.

Remarks about the above: When dealing with contravariant functors a subcategory $\mathcal{U}\subset\mathcal{X}$ like $F_n\Delta$ of Δ having the property: given $X\to X'$ and $X'\in\mathcal{U}\Rightarrow X\in\mathcal{U}$. is like an open set. Let's discuss this more carefully using covariant functors. Suppose we have

$$\mathcal{U} \xhookrightarrow{j} \mathcal{X} \xhookleftarrow{i} \mathcal{Y}$$

inclusions of full subcategories which are complementary. such that given $X\to X'$, then $X\in\mathcal{U}\Rightarrow X'\in\mathcal{U}$
equiv. $X'\in\mathcal{Y}\Rightarrow X\in\mathcal{Y}$

Then for covariant functors we have

$$(j_!G)(X) = \varinjlim_{(U,\, jU\to X)} G(U) = \begin{cases} 0 & X\notin\mathcal{U} \\ G(X) & X\in\mathcal{U} \end{cases}$$

$$(i_*G')(X) = \varprojlim_{(Y,\, X\to iY)} G'(Y) = \begin{cases} 0 & X\notin\mathcal{Y} \\ G'(X) & X\in\mathcal{Y} \end{cases}$$

Thus $j_!$ and i_* extend by zero in the familiar way.

Next consider Δ' the category of finite totally ordered sets including the empty set. Define $F_n\Delta'$ to consist of sets of card $\leq n$. In this case

$$H_i(F_n\Delta', F_{n-1}\Delta') = 0$$

because the poset $\{[p]'\to[n]' \mid p<n\}$ has an initial element. Thus $H_i(\Delta', F) = \begin{cases} F(\phi) & i=0 \\ 0 & i\neq 0 \end{cases}$

which is clear also because $\varinjlim_{\Delta'} F = F(\phi)$

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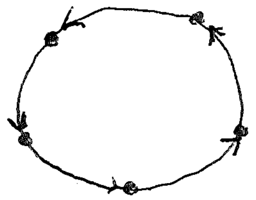
since ϕ is an initial object of Δ' and we are contravariant functors.

The reason for looking at Δ and Δ' is that they occur naturally in the cyclic context. Let Cyc be the category of finite non-empty cyclically ordered sets. Such a set S can be viewed as a circular oriented graph

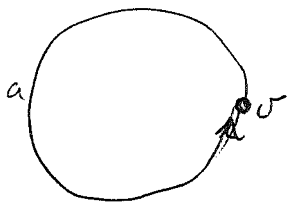


or a set with a cyclic permutation automorphism.

The goal is to show that Cyc $\sim BS^1$; one way to prove this is to exhibit an S^1 bundle over Cyc whose total space is contractible. It seems we can do this as follows.

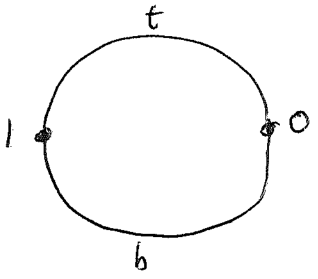
Given $S \in$ Cyc, let $T(S)$ be the category whose objects are the elements of S and the arrows. Thus if S has $n+1$ elements, $T(S)$ has $2(n+1)$ objects. The morphisms are the identity maps and for each edge or arrow one has a ~~map~~ morphism from the arrow to the source and a morphism from the arrow to the target. For $n > 0$, $T(S)$ is a poset - the opposite poset to the poset of simplices in the triangulation of the circle, but for $n = 0$ we have a category

$n = 0$



$$\begin{tikzcd} a \arrow[r, shift left] \arrow[r, shift right] & v \end{tikzcd}$$

$n = 1$



$$\begin{tikzcd} & t \arrow[dl] \arrow[dr] & \\ \bullet & & \bullet \\ & b \arrow[ul] \arrow[ur] & \end{tikzcd}$$

Clearly $T(S)$ has the homotopy type of S^1!

Next $T(S)$ is a contravariant functor of S, namely given $S' \hookrightarrow S$ an injection of cyclic ordered sets, then to each vertex or arrow — say stratum — belonging to S, one associates the stratum of S' containing it.

Next we form the fibred category $\mathcal{T}$ over Cyc associated to this functor $S \mapsto T(S)$. It is useful at this point to think of an object of $T(S)$ as a generic point of a stratum in the circular oriented graph. The morphisms in $T(S)$ allow generic points to specialize, more precisely, the generic point of an arc can specialize either forward or backward to a vertex. Now given $(S, \xi) \in \mathcal{T}_S$ and $(S', \xi') \in \mathcal{T}_{S'}$, then a morphism $(S, \xi) \longrightarrow (S', \xi')$ is an injection $S \hookrightarrow S'$ of cyclic ordered sets together with a specialization $\xi \longrightarrow \xi'$. This means that if ξ is a vertex, so is ξ' and $\xi = \xi'$; if ξ is an arrow then ξ' can be a arrow or vertex contained in ξ, or ξ' could be a vertex at either end of ξ. (End here includes the direction from the generic point.)

This gives a nice category which clearly up to homotopy is a circle bundle over Cyc. Notice that in so far as contravariant functors are concerned, the full subcategory of (S, ξ) with ξ an arc (or edge) is "open" and the complementary set where ξ is a vertex is "closed". Call these $\mathcal{U}$ and $\mathcal{Y}$ respectively. It is clear that

$\mathcal{U}$ equivalent to Δ, $\mathcal{Y}$ is equivalent to Δ'

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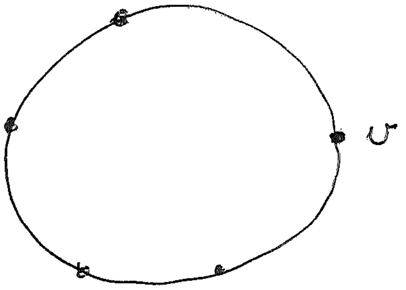
Somehow from this I would like to deduce the contractibility of $\mathcal{T}$.

Let us next consider the analogue of our filtration $F_n \Delta$. We an obvious increasing filtration by "open" subcategories, which first uses n, the cardinality of S minus one, and then the fact that open arcs specialize to vertices.

Take the object $([n], a)$, meaning $n+1$ vertices and an open arc. ~~that the fiber~~ Since the pairs (S, ξ) with ξ an arc form an "open" subcat equivalent to Δ, it follows that the relative homology for attaching $([n], a)$ to $F_n \mathcal{T}$ (= pairs lying over $F_n C_{cyc}$). is the homology of $\Delta(n)/\dot{\Delta}(n) = S^n$. Now consider what happens when we attach $([n], v)$ to $F_n\mathcal{T} \cup ([n], a)$. Now we must consider the category of pairs consisting of and (S, ξ) and a map $(S,\xi) \longrightarrow ([n], v)$, which is not an isomorphism. Draw $([n], v)$:



Then S is a subset of the vertices, which is nonempty, and ~~[illegible]~~ there are four cases: 1) If $v \notin S$, then ξ is the arc of S containing v. 2) $v \in S$ and $\xi = v$. 3) $v \in S$ and ξ is the arc with specialization to v from the top. 4) $v \in S$ and ξ is the arc with specialization to v from the bottom. The way to understand the category of $(S, \xi) \longrightarrow ([n], v)$ is to "double" v i.e. replace it by $\overset{v^+}{\underset{v^-}{\vdots}}$ then look at nonempty subsets S' of the resulting cyclic set with one more vertex. The four cases correspond to 1) S' not containing either of v^+, v^-; 2) $v^+, v^- \in S'$, 3) $v^+ \in S', v^- \notin S'$

and 4) $v^+ \notin S', v^- \in S'$.

So we conclude that the relative homology on attaching $([n];v)$ is $\Delta(n+1)/\dot{\Delta}(n+1) \simeq S^{n+1}$.

At this point things look very close to being able to prove that $L \varinjlim_{Cyc} F$ gives the cyclic homology of the cyclic module F.

February 27, 1992

Problem. Can we use Kadison's method to establish the Morita invariance for entire cyclic cohomology? Specifically, given a Banach algebra A with a separable subalgebra S, can we refine the quasi-isomorphism

$$(1) \qquad \Omega A \longrightarrow \Omega_S A \otimes_S$$

to an explicit strong deformation retraction, such that the corresponding homotopy h is suitable for the convergence of the method of homological perturbation theory. (This means that the geometric series for $\frac{1}{1-hB}$ and $\frac{p}{1-Bh}$ on entire cochains should converge.) According to Ezra one needs to know h is uniformly bounded.

A sub-problem is to show that the canonical

quis

$$(2) \qquad \bar{\Omega}\tilde{A} \longrightarrow \Omega A$$

leads to [redacted] an SDR on the level of entire cochains.

Let's begin by trying to refine the quis (1) to an SDR. We will proceed on the level of bimodule resolutions, then obtain (1) by passing to commutator quotient spaces.

Recall that the standard normalized A-bimodule resolution $(\Omega A \otimes A, b')$ of A is constructed from the basic exact sequence

$$(3)_0 \qquad 0 \longrightarrow \Omega^1 A \longrightarrow A \otimes A \longrightarrow A \longrightarrow 0$$

by iteration as follows. Tensoring with $\Omega^n A$ on the left gives a sequence of short exact sequences

$$(3)_1 \qquad 0 \longrightarrow \Omega^2 A \longrightarrow \Omega^1 A \otimes A \longrightarrow \Omega^1 A \longrightarrow 0$$

$$(3)_2 \qquad 0 \longrightarrow \Omega^3 A \longrightarrow \Omega^2 A \otimes A \longrightarrow \Omega^2 A \longrightarrow 0$$

which one splices to obtain the standard resolution. The relative resolution is constructed similarly from

$$(4)_0 \qquad 0 \longrightarrow \Omega^1_S A \longrightarrow A \otimes_S A \longrightarrow A \longrightarrow 0$$

by iteration. The point is that when S is separable (more generally $A\otimes_S A$ is a projective bimodule), then one has a splitting:

$$\begin{tikzcd}
0 \arrow[r] & \Omega^1 A \arrow[r] \arrow[d, equal] & A \otimes A \arrow[r] \arrow[d, equal] & A \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & \Omega^1_S A \arrow[r] & A \otimes_S A \arrow[r] & A \arrow[r] & 0 \\
 & \oplus & \oplus & & \\
 & AdSA \arrow[r, "\sim"] & AdSA & &
\end{tikzcd}$$

In other words $(4)_0$ is a direct summand of $(3)_0$; hence the iterated sequence $(4)_n$ is a direct summand of $(3)_n$ as we have seen. Our problem is now to construct a contracting homotopy on the [illegible] complementary complex.

Let's simplify the notation. Write

$$(5) \qquad \begin{aligned} E &= A \otimes A = (A \otimes_S A) \oplus (AdSA) = E' \oplus K \\ M &= \Omega^1 A = (\Omega^1_S A) \oplus (AdSA) = M' \oplus K \end{aligned}$$

The basic sequence and its iterates are

$$(6)_n \qquad \begin{array}{ll} 0 \longrightarrow M \longrightarrow E \longrightarrow A \longrightarrow 0 & \text{0th} \\ 0 \longrightarrow M^2 \longrightarrow ME \longrightarrow M \longrightarrow 0 & \text{1st} \\ 0 \longrightarrow M^3 \longrightarrow M^2E \longrightarrow M^2 \longrightarrow 0 & \text{2nd} \end{array}$$

where ME stands for $M \otimes_A E$ etc.

The basic sequence splits into the sum of

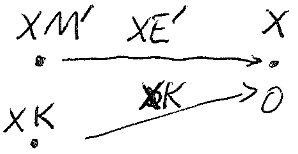
$$(7)_0 \qquad \begin{matrix} 0 \longrightarrow M' \longrightarrow E' \longrightarrow A \longrightarrow 0 \\ 0 \longrightarrow K \xrightarrow{=} K \longrightarrow 0 \longrightarrow 0 \end{matrix}$$

Also $M^n = (M' \oplus K)^n$ splits into 2^n pieces, which are written as words in M' and K of length n. Thus the nth iterated exact sequence is a dried sum of 2^{n+1} short exact sequences, which appear as either

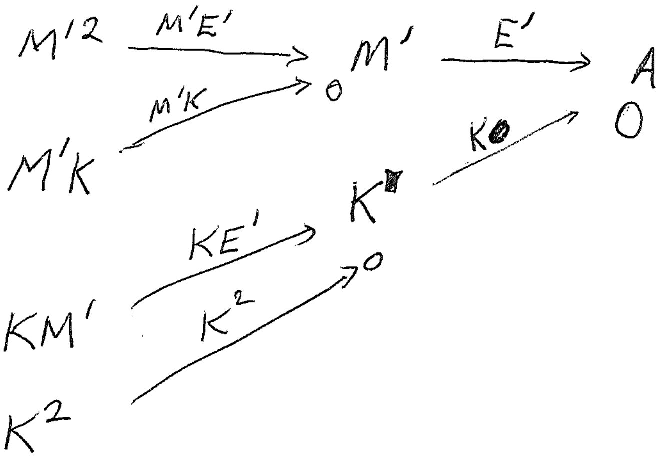
$$(7)_n \qquad \begin{matrix} 0 \longrightarrow XM' \longrightarrow XE' \longrightarrow X \longrightarrow 0 \\ 0 \longrightarrow XK \xrightarrow{=} XK \longrightarrow 0 \longrightarrow 0 \end{matrix}$$

where X is a word in M' and K of length n.

I feel that a graph picture is appropriate. One can think of one of the above exact sequences in $(7)_n$ as an edge joining the words at the ends of the exact sequence:



The graph then looks as follows



The next step is break this into subcomplexes. More precisely, a path in this graph is a subcomplex of the standard resolution. I think at this point I have to treat the edges

$$XK \xrightarrow{XK} 0$$

differently. We have to focus on vertices, i.e. words in M', K which end with K. Let's describe these vertices as pieces of

$$T(M)K = T(M' \oplus K)K$$

Each word XK gives rise to the subcomplex belonging to the graph

$$\xrightarrow{XKM'^{2}E'} XKM'^{2} \xrightarrow{XKM'E'} XKM' \xrightarrow{XKE'} XK \xrightarrow{XK} 0$$

which corresponds to the exact sequence

$$\longrightarrow XKM'^{2}E' \longrightarrow XKM'E' \longrightarrow XKE' \longrightarrow XK \longrightarrow 0.$$

I want to show this exact sequence is canonically ontractible. Notice that this results by tensoring with X on the left on the sequence

$$(8) \quad \longrightarrow KM'^{2}E' \longrightarrow KM'E' \longrightarrow KE' \longrightarrow K \longrightarrow 0,$$

so it should be enough to describe the homotopy in this case.

The basic bookkeeping is as follows. We divide our vertices according to the last appearance of K in the words. Thus our vertices are divided

$$T(M) = T(M' \oplus K) = T(M)KT(M') \oplus T(M')$$

The edges are then divided

$$T(M)E = T(M)(K \oplus KT(M')E') \oplus T(M')E'$$

March 5, 1992

The problem is to make an explicit SDR for Kadison's theorem. Recall that we have the standard resolution

$$\Omega A \otimes A \;=\; \Omega A \otimes_A \underset{\displaystyle T(M)E}{\underset{\shortparallel}{(A\otimes A)}}$$

and the compatible splittings

$$\begin{array}{ccccc} A\otimes A & = & AdSA & \oplus & A\otimes_s A \\ E & = & K & \oplus & M \end{array}$$

and

$$\begin{array}{ccccc} \Omega^1 A & = & AdSA & \oplus & \Omega^1_s A \\ E' & = & K & \oplus & M' \end{array}$$

Then the standard resolution splits as follows. We have

$$T(M) \;=\; T(M)KT(M') \oplus T(M')$$

so

$$T(M)E = T(M)K \oplus \big(T(M)KT(M') \oplus T(M')\big)E'$$

$$= \underbrace{\{T(M)(K \oplus \text{[redacted]}\, KT(M')E')\}}_{\text{complex to be 'contracted'}} \oplus \underbrace{T(M')E'}_{\substack{\text{rel st. resoln} \\ \Omega_s A\otimes_s A}}$$

Note

$$K \oplus KT(M')E' \;=\; AdSA \;\oplus\; AdSA \otimes_A \Omega_s A \otimes_s A$$

The point [redacted] is that this complex, which is the relative standard resolution of [redacted] the bimodule AdSA together with the augmentation, can be constracted using a right connection (relative) on AdSA.

Let N be a right module. We have an exact sequence

$$\begin{tikzcd} 0 \arrow[r] & N \otimes_A \Omega^1_S A \arrow[r, "j"] & N \otimes_S A \arrow[r, "m"] \arrow[l, bend left] & N \arrow[r] \arrow[l, bend left] & 0 \end{tikzcd}$$

Consider splittings given by p and l[redacted]. Then l[redacted] is compatible with right multiplication iff p is. In this case $p(\xi \otimes a) = (\nabla \xi) a$ where $\nabla \xi = p(\xi \otimes 1)$ and we have

$$\xi da = p[redacted] j(\xi da) = p(\xi a \otimes 1 - \xi \otimes a) = \nabla(\xi a) - (\nabla \xi) a$$

Thus ∇ is a relative connection in the right module N, by which I mean $\nabla : N \to N \otimes_A \Omega^1_S A$ satisfies Leibniz wrt right A-multiplication. Conversely given such a ∇ then $\nabla(\xi s) = \nabla \xi s$, so $p(\xi \otimes a) = (\nabla \xi) a$ well-defined and one checks $pj = 1$.

Now suppose given $l : N \to N \otimes_S A$ a right module map such that $ml = 1$, and let $\nabla : N \to N \otimes_A \Omega^1_S A$ be the corresponding connection:

$$\xi \otimes 1 = l(\xi) + j(\nabla \xi)$$

Then look at the relative standard resolution

$$N \otimes_A \Omega_S A \otimes A \qquad \text{diffl} \quad b'(\alpha da \otimes 1) = (-1)^{|\alpha|}(\alpha a \otimes 1 - \alpha \otimes a).$$

Claim $\nabla \otimes 1$ together with l is a contracting homotopy. Calculation: In degrees > 0.

$$(\nabla \otimes 1) b'(\xi da \otimes 1) = (-1)^{|\xi|}(\nabla \otimes 1)(\xi a \otimes 1 - \xi \otimes a)$$
$$= (-1)^{|\xi|}\left((\nabla \xi) a \otimes 1 + (-1)^{|\xi|} \xi da \otimes 1 - \nabla \xi \otimes a\right)$$

$$b'(\nabla \otimes 1)(\xi da \otimes 1) = b'(\nabla \xi da \otimes 1 \text{ [redacted]})$$
$$= (-1)^{|\xi|+1}\left((\nabla \xi) a \otimes 1 - \nabla \xi \otimes a\right) \qquad \therefore [\nabla \otimes 1, b'] = 1.$$

We apply this to contract

$$K \oplus KT(M')E' = AdSA \oplus AdSA \otimes_A \ell_S A \otimes_S A$$

Observe that [redacted] when S is separable then $AdSA = A \otimes_S \Omega^1 S \otimes_S A$ has a canonical right relative connection compatible with right mult, namely the one corresponding to the lifting

$$\begin{tikzcd} A \otimes_S \Omega^1 S \otimes_S A \arrow[r] & A \otimes_S \Omega^1 S \otimes_S A \otimes_S A \\ 1 \otimes \omega \otimes 1 \arrow[r, mapsto] & 1 \otimes \omega \otimes 1 \otimes 1 \end{tikzcd}$$

It appears therefore that we have a really nice contraction for $K \oplus KT(M')E'$ and hence also for $T(M)(K \oplus KT(M')E')$. Observe that because the above right connection is $1 \otimes d_S$ on $(A \otimes_S \Omega^1 S) \otimes_S A$ the curvature should be zero, and this should imply the homotopy h satisfies $h^2 = 0$. Thus we should have a SDR.

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March 6, 1992

In general let $\mathcal{C}$ be a small category, and suppose we compute homology using covariant functors M, and cohomology using contravariant functors F. Then

$$H^i(\mathcal{C}, F) = \mathrm{Ext}^i_{\mathrm{Fun}(\mathcal{C}^o, \mathcal{A}b)}(\mathbb{Z}, F)$$

$$= H^i\left(\mathrm{Hom}_{\mathrm{Fun}(\mathcal{C}^o, \mathcal{A}b)}(P_\cdot, F)\right)$$

where $P_\cdot$ is a projective resolution of the constant functor $\mathbb{Z}$ in $\mathrm{Fun}(\mathcal{C}^o, \mathcal{A}b)$. I claim that

$$H_i(\mathcal{C}, M) = H_i(P_\cdot \otimes_{\mathcal{C}} M)$$

for a suitable tensoring operation:

$$F \otimes_{\mathcal{C}} M = \mathrm{Coker}\left\{ \bigoplus_{X \to Y} F(X) \otimes M(Y) \rightrightarrows \bigoplus_{X} F(X) \otimes M(X) \right\}$$

~~Example $F = \mathbb{Z}[h_X]$, $h_X(Y) = \mathrm{Hom}_{\mathcal{C}}(X, Y)$.~~

~~Then $\mathbb{Z}[h_X] \otimes_{\mathcal{C}} M = h_X \times_{\mathcal{C}} M$~~

~~where $F \times_{\mathcal{C}} M = \mathrm{Coker}\left\{ \coprod_{X \to Y} F(X) \times M(Y) \rightrightarrows \coprod_{X} F(X) \times M(X) \right\}$~~

~~$\mathbb{Z}[h_X] \otimes_{\mathcal{C}} M = \mathrm{Coker}\{ \bigoplus$~~

Example: $F = \mathbb{Z}[h_S]$ $\quad h_S(X) = \mathrm{Hom}_{\mathcal{C}}(X, S)$

$$\mathbb{Z}[h_S] \otimes_{\mathcal{C}} M = \mathrm{Coker}\left\{ \bigoplus_{Y \to X} \mathbb{Z}[\mathrm{Hom}_{\mathcal{C}}(X, S)] \otimes M(Y) \rightrightarrows \bigoplus_{X} \mathbb{Z}[\mathrm{Hom}_{\mathcal{C}}(X, S)] \otimes M(X) \right\}$$

$$= \mathrm{Coker}\left\{ \bigoplus_{\substack{Y \to X \\ X \to S}} M(Y) \rightrightarrows \bigoplus_{X \to S} M(X) \right\} =$$

$$\varinjlim_{\mathcal{C}/S} M = M(S).$$

Now I want to apply this in the case of $(\mathrm{Cyc})^{op}$, so F, P are covariant functors over Cyc and M is contravariant, i.e. M is a (pre)cyclic module.

Connes writes down the bicomplex of projective covariant functors which gives rise to the cyclic bicomplex of M upon tensoring or the dual cyclic bicomplex for F upon taking Hom. This is the bicomplex $P_{\bullet\bullet}$ as follows:

$$\begin{tikzcd}
\, \arrow[d] & \, \arrow[d] & \\
\mathbb{Z}[h^1] \arrow[d, "b"] & \mathbb{Z}[h^1] \arrow[l, "1-\lambda"'] \arrow[d, "-b'"] & \, \arrow[l, "N_\lambda"'] \\
\mathbb{Z}[h^0] & \mathbb{Z}[h^0] \arrow[l, "1-\lambda"'] & \, \arrow[l, "N_\lambda"']
\end{tikzcd}$$

Check the variances: $[p] \longmapsto \left(h^p([n]) = \operatorname{Hom}_{\mathrm{Cyc}}([p],[n])\right)$ is contravariant in $[p]$, so $[p] \longmapsto \mathbb{Z}[h^p]$ is a ~~[illegible]~~ contravariant functor with values in covariant functors. Next for each p, $\mathbb{Z}[h^p]$ is a proj covariant functor since

$$\operatorname{Hom}_{\mathrm{Fun}(\mathrm{Cyc},\,\mathrm{ab})}(\mathbb{Z}[h^p], F) = F([p])$$

is exact in F. So the only point to check is that the above bicomplex of covariant functors resolves the constant functor $\mathbb{Z}$. This means applying all these functors to a fixed object $[n]$ and checking acyclicity. Let us look at the q-th row. We have $\mathbb{Z}/q+1$ acting on $\mathbb{Z}[h^q([n])] = \mathbb{Z}[\mathrm{Hom}_{Cyc}([q],[n])]$. But $\mathbb{Z}/q+1$ acts freely on $\mathrm{Hom}_{Cyc}([q],[n])$ and there is a slice for the action consisting of embeddings $[q] \hookrightarrow [n]$ preserving the total ordering. Taking the horizontal homology leaves us with the complex $[q] \longmapsto \mathbb{Z}\left(\mathrm{Hom}_{\Delta}([q],[n])\right)$ with b diffl. This is the complex for computing the homology of the n simplex, so it resolves $\mathbb{Z}$.

The above is Connes's proof. It is closely related to acyclic models. The complex $P.$ associated to this bicomplex is exact for the models which are the cyclic ordered sets $[n] \in Cyc$.

Let us continue now with the attempt to explain negative cyclic + periodic cyclic homology via analogy with Tate-Farrell. Recall that for a group Γ with suitable finiteness properties (VFT) we have

$$H^i(\Gamma, F) = H^i\left(\mathrm{Hom}_{\mathbb{Z}[\Gamma]}(P, F)\right)$$

$$= H^i\left(\mathrm{Hom}_{\mathbb{Z}[\Gamma]}(P, \mathbb{Z}[\Gamma]) \otimes_{\mathbb{Z}[\Gamma]} F\right)$$

(This uses $P.$ is a complex of finite type projective modules.) On the other hand one might have a quis

$$St \longrightarrow \mathrm{Hom}_{\mathbb{Z}[\Gamma]}(P, \mathbb{Z}[\Gamma])$$

where St is bounded below for the lower indexing. Then $St \overset{L}{\otimes} F$ gives something like negative cyclic cohomology, while the cofibre gives the Tate Farrell cohomology which is periodic cyclic cohomology.

Now we want to work out the dual complex in the cyclic setting ~~starting~~ from Connes's bicomplex $(P_{..})$.

One has $P_{pq} = \mathbb{Z}[h^q]$; it is a covariant functor on Cyc. We have $\underset{Fun(Cyc,\,Ab)}{\mathrm{Hom}}(P_{pq}, F) = F([q])$. Recall

$h_q \underset{Cyc}{\otimes} F = F[q]$. Thus $\check{P}$, the dual complex, should appear as the total complex associated to a bicomplex of contravariant functors. Use the lower grading and it appears in the 3rd quadrant

$$\begin{tikzcd}
\mathbb{Z}[h_0] \arrow[d] & \mathbb{Z}[h_0] \arrow[l] \arrow[d] & \mathbb{Z}[h_0] \arrow[l, "1-\lambda"'] \arrow[d] \\
{} & \mathbb{Z}[h_1] \arrow[l] \arrow[d] & \mathbb{Z}[h_1] \arrow[l, "1-\lambda"'] \arrow[d] \\
{} & {} & \mathbb{Z}[h_2] \arrow[l] \arrow[d] \\
{} & {} & {}
\end{tikzcd}$$

Let us rewrite this ~~appar~~ indexing which is appropriate to cyclic cohomology; this is closer to the group setting. Then we have the bicomplex of projective cyclic modules (i.e. contravariant functors on Cyc):

$$\begin{tikzcd}
\mathbb{Z}[h_2] \arrow[r, "1-\lambda"] & {} & {} \\
\mathbb{Z}[h_1] \arrow[u, "b"] \arrow[r, "1-\lambda"] & \mathbb{Z}[h_1] \arrow[u, "-b'"] \arrow[r] & {} \\
\mathbb{Z}[h_0] \arrow[u, "b"] \arrow[r, "1-\lambda"] & \mathbb{Z}[h_0] \arrow[u, "-b'"] \arrow[r] & {}
\end{tikzcd} \qquad (\ast)$$

Upon tensoring over Cyc with a covariant functor F we get the cyclic bicomplex computing $H^i(\mathrm{Cyc}, F)$.

Look at the q-th row. It is the [redacted] complex calculating the cohomology of $\mathbb{Z}/q+1$ acting on $[n] \longmapsto \mathbb{Z}[h_q]([n]) = \mathbb{Z}\{\mathrm{Hom}_{\mathrm{Cyc}}([n],[q])\}$. [redacted] Now $\mathbb{Z}/q+1$ acts freely on the set $\mathrm{Hom}_{\mathrm{Cyc}}([n],[q])$, and we have a cross-section (slice) for the action consisting of maps preserving 0.

At this point I have described $\check{P}$ as a double cochain complex of cyclic modules $\mathbb{Z}[h_q]$ and I would like the homology $\mathcal{H}^n(\check{P})$ which should be certain cyclic modules. I can go through the following reductions.

First because

$$h_q([n]) = \mathrm{Hom}_{\mathrm{Cyc}}([n],[q])$$

is a free $\mathbb{Z}/q+1$ set, the rows in ⊛ are acyclic [redacted] except in degree $p=0$, where we get

$$\mathbb{Z}[h_q]^{(\mathbb{Z}/q+1)}$$

Let's use norm isom.

$$\mathbb{Z}[h_q]_{(\mathbb{Z}/q+1)} \xrightarrow{\sim} \mathbb{Z}[h_q]^{(\mathbb{Z}/q+1)}$$

to reduce to computing the homology of the cochain complex

$$\mathbb{Z}[h_0/(\mathbb{Z}/1)] \longrightarrow \mathbb{Z}[h_1/(\mathbb{Z}/2)] \longrightarrow$$

where the differential is b'. Now use

$$h_q([n])/\mathbb{Z}/(q+1) \ [redacted] = \mathrm{Hom}_{\mathrm{Cyc}}([n],[q])_{\mathbb{Z}/q+1} = [redacted] \left\{ \begin{array}{l} f:[n]\to[q] \\ \text{order-preserving} \\ \ni f(n)=q \end{array} \right\}$$

What I feel ought to be the case is that we have now a complex which depends only on category of finite totally ordered sets including the empty set. It also should be the corresponding $\check{P}$ for Δ. It seems likely that the answer is trivial. Let's leave aside then the *conjecture*:

$\check{P}$ *is a resolution of the constant* ~~covariant~~ *contravariant functor* $\mathbb{Z}$, *and this should also be true for the* Δ *analogue*.

Note that $\check{P}([0])$ has homology [illegible] 0? Maybe the conjecture has to be modified.

March 6, 1992 (cont.)

Idea: Look for analogues of the fibrings $B\mathbb{Z} \to B\mathbb{R} \to BS^1$, $B(\mathbb{Z}/m) \to BS^1 \to BS^1$.

Inspired by previous experience with buildings let us consider universal coverings of a cyclic graph. Then we get infinite cyclic graphs equipped with a positive direction translation. Again we make a category out of these, where the morphisms are injections compatible with cyclic ordering and the given translations. Up to isomorphism there is one object for each $m \geq 1$, namely, the infinite cyclic graph given by $\mathbb{Z}$ where the translation is $n \mapsto n+m$.

We have an obvious functor from this category to Cyc, which is given by taking the quotient by the translation action. Observe that the fibre over the cyclic graph $\mathbb{Z}/m$ is the group of autom. of $\mathbb{Z}$ generated by the translation $n \mapsto n+m$. It seems clear that this functor is fibred. Let us denote by $\widetilde{\text{Cyc}}$ this category of cyclic graphs with translation and embeddings. Given

$$\begin{tikzcd}
 & (\mathbb{Z}, m) \arrow[d, mapsto] \\
\mathbb{Z}/m' \arrow[r, hook, "i"] & \mathbb{Z}/m
\end{tikzcd}$$

there is a pull-back graph which is a union of $m\mathbb{Z}$-cosets in $\mathbb{Z}$. Each map from $(\mathbb{Z}, m')$ into this pull back over $\mathbb{Z}/m'$ corresponds in 1-1 fashion to a ~~cartesian~~ map $(\mathbb{Z}, m') \hookrightarrow (\mathbb{Z}, m)$ lying over i. (I guess the thing to prove, since the fibres are groupoids is that every map in $\widetilde{\text{Cyc}}$ is cartesian.)

So we have a fibred category $\widetilde{\text{Cyc}}$ over Cyc whose fibres are the group $\mathbb{Z}$.

So one wants to show $\widetilde{\text{Cyc}}$ is contractible. The method will to introduce the fibred category over $\widetilde{Cyc}$ consisting of infinite cyclic graphs equipped with either a vertex or an edge, such that the edge can specialize to either vertex. We then have a commutative square

$$\begin{tikzcd}
\widetilde{Cyc}' \arrow[r, "f'"] \arrow[d, "\tilde{g}"] & Cyc' \arrow[d, "g"] \\
\widetilde{Cyc} \arrow[r, "f"'] & Cyc
\end{tikzcd}$$

The fibre of $\tilde{g}$ over $(\mathbb{Z}, m)$ is essentially a triangulation of $\mathbb{R}$ with vertices $\mathbb{Z}$. So $\tilde{g}$ should be a homotopy equivalence. f' is the functor which takes $(\mathbb{Z}, m, \sigma)$ where σ is either vertex or arc and takes the quotient by the translation. I think it's clear that f' is fibred, since if we have $(\mathbb{Z}, m, \sigma) \in \widetilde{Cyc}'$ and a map $(\mathbb{Z}/m', \bar{\sigma}') \longrightarrow (\mathbb{Z}/m, \bar{\sigma})$ then there really should be a pullback of $\bar{\sigma}'$ specializing to σ. What is the fibre of f'? It consists over $(\mathbb{Z}/m, \bar{\sigma})$ of liftings of $\bar{\sigma}$ to σ in $(\mathbb{Z}, m)$, and all these liftings are related by unique isomorphisms. Thus f' appears to be an equivalence of categories. We can check this by checking that there are up to isom. only two objects in $\widetilde{Cyc}'$ of order m, and this is clear.

The question still remains as to why Cyc' or $\widetilde{Cyc}'$ is contractible. This I feel should follow from dividing $\widetilde{Cyc}'$ into "arcs" and "vertices", the former being equivalent to Δ and the latter to $\Delta \cup \phi$.

March 8, 1992

Simplicial object $\overset{\text{defn.}}{=}$ $\{X_n, n \geq 0\}$. [redacted]
equipped with $d_i : X_n \longrightarrow X_{n-1}$, $0 \leq i \leq n > 0$
and $s_i : X_n \longrightarrow X_{n+1}$, $0 \leq i \leq n \geq 0$ satisfying

$$d_i s_j = \begin{cases} s_{j-1} d_i & i<j \\ 1 & i=j, j+1 \\ s_j d_{i-1} & i>j+1 \end{cases}$$

$$s_i s_j = s_{j+1} s_i \qquad i \leq j$$

$$d_i d_j = d_j d_{i+1} \qquad i \geq j$$

Normal form for a simplicial operator is

$$s_{j_p} \cdots s_{j_1} d_{i_1} \cdots d_{i_q} \qquad \begin{array}{l} j_1 < j_2 < \cdots < j_p \\ i_1 < \cdots < i_q \end{array}$$

In terms of the corresponding map $[m] \xrightarrow{f} [n]$, the j_k are the elements j of $[m]$ such that $f(j) = f(j+1)$ and the i_k are the elements of $[n]$ not in the image of f.

Simplicial normalization theorem. Let $d = \sum_{i=0}^{n} (-1)^i d_i$ on X_n. Then

$$ds_0 = (d_2 - d_3 + \cdots + (-1)^n d_n) s_0$$
$$= s_0 (d_1 - d_2 + \cdots + (-1)^{n-1} d_{n-1})$$
$$= -s_0 d + s_0 d_0$$

so $\boxed{ds_0 + s_0 d = s_0 d_0}$. But $s_0 d_0$ is a projector, so we have a splitting of complexes

$$X = s_0 d_0 X \oplus \underbrace{(1 - s_0 d_0 X)}_{\mathrm{Ker}(d_0 \text{ on } X)}$$

Next if $j \geq 1$ $d_0 s_j = s_{j-1} d_0$ [redacted] $\Longrightarrow$ $s_j(\mathrm{Ker}\, d_0) \subset \mathrm{Ker}(d_0)$
and $d_0 d_j = d_{j-1} d_0 \Longrightarrow d_j (\mathrm{Ker}\, d_0) \subset \mathrm{Ker}\, d_0$ [redacted].

so we have a splitting

$$X = \underbrace{s_0 d_0 X}_{s_0 \text{ htpy}} \oplus \underbrace{s_1 d_1 (1 - s_0 d_0 X)}_{s_1 \text{ htpy}} \oplus \underbrace{s_2 d_2 (1 - s_1 d_1)(1 - s_0 d_0 X)}_{s_2 \text{ htpy}} \oplus \cdots$$

which yields the normalization theorem.

Formulas for <u>Homological Perturbation Theory.</u>

$$\begin{tikzcd} E \arrow[d, shift right, "p"'] \\ \bar{E} \arrow[u, shift right, "s"'] \end{tikzcd}$$

Given $(E, \bar{E}, p, s, h, \theta)$, where $[d,p] = [d,s] = 0$ and $sp - 1 = [d,h]$, $(d+\theta)^2 = 0$. Set

$$p' = p \frac{1}{1-\theta h}, \quad s' = \frac{1}{1-h\theta} s, \quad h' = h \frac{1}{1-\theta h} = \frac{1}{1-h\theta} h$$

$$\theta' = p\,\theta \left(\frac{1}{1-h\theta}\right) s = p\left(\frac{1}{1-\theta h}\theta\right) s$$

$$= p\,\theta\, s' = p'\theta s$$

Claim

$$[d+\theta, h'] = s'p' - 1$$

$$p'(d+\theta) = (d+\theta')\,p'$$

$$(d+\theta)\,s' = s'(d+\theta')$$

Moreover $ps = 1$, $ph = hs = h^2 = 0 \Big\} \Longrightarrow$ same for p', s', h'.
(this means we have a Special Deformation Retraction)

Special case of interest: Suppose we have an SDR as above such that θ descends via p: $p\theta = \theta p$. Then $p' = p$ and $\theta' = \theta$.

Homology and cohomology of small categories. Let C be a small category. By left C-module (resp. right C-module) we mean covariant functor $M : C \longrightarrow Ab$ (resp. contravariant functor). (Bimodules?)

It is natural to consider together homology

$H_i(C, M) = L_i \varinjlim_{C} M$ for C-modules, and cohomology

$H^i(C^{op}, F) = R^i \varprojlim_{C^{op}} F$ for right C-modules.

If $P.$ is a projective right C-module resolution of the constant functor $\mathbb{Z}$, then

$$H_i(C, M) = H_i(P. \otimes_C M)$$

$$H^i(C^{op}, F) = H^i(Hom_{C^{op}}(P., F))$$

Let's suppose P_n is a finite sum of functors of the form $\mathbb{Z}[h_X]$, $h_X(Y) = Hom_C(Y, X)$. Since

$$Hom_{C^{op}}(\mathbb{Z}[h_X], F) = F(X) = \mathbb{Z}[h^X] \otimes_{C^{op}} F$$

we have

$$Hom_{C^{op}}(P., F) = (P.)^{\vee} \otimes_{C^{op}} F$$

where $(P.)^{\vee}$ is a cochain complex of projective C-modules, each being a finite direct sum of functors of the form $\mathbb{Z}[h^X]$.

In the case of a group, which is a self dual category pursuing this line of inquiry leads to Tate-Farrell cohomology. I need more examples, especially Connes cyclic graph category with degeneracies, which is self-dual.

Leray spectral sequence. Given $f : \mathcal{X} \to \mathcal{Y}$ recall we have adjoint functors

$$\begin{tikzcd} Hom(\mathcal{X}, ab) \arrow[r, shift left=2, "f_!"] \arrow[r, shift right=2, "f_*"'] & Hom(\mathcal{Y}, ab) \arrow[l, "f^*"'] \end{tikzcd}$$

where
$$(f_! F)(Y) = \varinjlim_{\substack{(X, fX \to Y) \\ \in \mathcal{X}/Y}} F(X)$$

$$(f_* F)(Y) = \varprojlim_{\substack{(X, Y \to fX) \\ \in Y \backslash \mathcal{X}}} F(X)$$

Leray spectral sequence in homology is

$$E^2_{pq} = H_p(\mathcal{Y}, Y \longmapsto H_q(\mathcal{X}/Y, F)) \Rightarrow H_{p+q}(\mathcal{X}, F)$$

It results from the composite functor

$$\varinjlim_{\mathcal{Y}} f_! = \varinjlim_{\mathcal{X}}$$

together with the result that for the canonical functor $j : \mathcal{X}/Y \longrightarrow \mathcal{X}$ one has j^* preserving projectives because j_* is exact:

$$(j_* G)(X) = \prod_{w \in Hom_{\mathcal{Y}}(fX, Y)} G(X, fX \xrightarrow{w} Y)$$

In case f is cofibred one can use the inclusion $i : \mathcal{X}_Y \longrightarrow \mathcal{X}$ of the fibre, and

$$(i_* G)(X) = \prod_{w : fX \to Y} G(w_* X)$$

Let's go on to (pre) simplicial and cyclic objects. Consider first Δ the category of non-empty finite totally-ordered set and order-preserving injections. We can identify Δ with the full subcategory of sets $\Delta_n = \{0, 1, \dots, n\}$ for $n \geq 0$ with usual order. A presimplicial object is a contravariant functor in Δ, i.e. a sequence $X_n = X(\Delta_n)$ together with $f^* : X_m \longrightarrow X_n$ for each $f : \Delta_n \longrightarrow \Delta_m$ satisfying compatibility with composition an identities.

Question: Supposedly it suffices to given operators

$$d_i : X_n \longrightarrow X_{n-1} \qquad 0 \leq i \leq n > 0$$

satisfying $\quad d_i d_j = d_{j+1} d_i \quad$ if $i \geq j$. Why?

Next let's describe how to calculate homology of simplicial module, i.e. Δ^{op}-module M. Use Connes's method of writing down a projective resolution P of the constant functor $\mathbb{Z}$ of Δ-modules. This is

$$\longrightarrow \mathbb{Z}[h^2] \xrightarrow{d_0 - d_1 + d_2} \mathbb{Z}[h^1] \xrightarrow{d_0 - d_1} \mathbb{Z}[h^0]$$

where $\quad h^p(\Delta_n) = \mathrm{Hom}_\Delta(\Delta_p, \Delta_n)$.

The cosimplicial modules $\mathbb{Z}[h^p]$ are obviously projective. Whis is this a resolution? Because if we apply it to Δ_n for some n, then we have the complex of chains on the ordered n-simplex with vertices $\{0, \dots, n\}$.

Question: Consider $P_{\cdot}^{\vee}$ which is the ~~cosimplicial~~

cochain complex

$$\mathbb{Z}[h_0] \xrightarrow{\partial_0-\partial_1} \mathbb{Z}[h_1] \xrightarrow{\partial_0-\partial_1+\partial_2} \mathbb{Z}[h_2] \longrightarrow$$

of simplicial modules [redacted]

$\mathbb{Z}[h_n](\Delta_p) = \mathbb{Z}[\mathrm{Hom}_{\Delta}(\Delta_p, \Delta_n)]$. What is the cohomology for Δ_p fixed?

Ultimately the study of $\check{P}$ in various cases should be highly interesting. Probably you should first look at the case of posets to get some feeling for the situation, especially the links with duality theory.

Question: Suppose we consider homology associated to covariant functors on Δ.

If degeneracies are included in Δ, then [redacted] cosimplicial abelian groups should be the same as cochain complexes, and it should be easy to calculate $\varinjlim_{\Delta}$. In fact I guess we get

$$\varinjlim_{\Delta} F = F(\Delta_0).$$

Next let's consider cyclic objects.

Here are some ideas worth recording.

Analogue of fibring $B\mathbb{Z} \longrightarrow B\mathbb{R} \longrightarrow BS^1$. [redacted]

Category $\tilde{\Lambda}$ with objects $(\mathbb{Z}, m)$ for each $m \geq 1$, $\mathbb{Z}$ considered with its usual order and translation action $n \longmapsto n+m$. Maps $(\mathbb{Z},m) \to (\mathbb{Z},m')$ are strictly increasing $f: \mathbb{Z} \longrightarrow \mathbb{Z}$ such that $f(n+m) = f(n) + m'$, $\forall n$.

Category Λ of $\mathbb{Z}/m\mathbb{Z}$ $m \geq 1$ with usual cyclic order where maps $\mathbb{Z}/m\mathbb{Z} \longrightarrow \mathbb{Z}/m'\mathbb{Z}$ are injections compatible with cyclic order.

Can form fibred category $\tilde{\Lambda}'$ over $\tilde{\Lambda}$ consisting of vertices and arcs in the triangulation of $\mathbb{R}$ with vertices $\mathbb{Z}$, where an arc is allowed to specialize to its endpoints. Description of $\tilde{\Lambda}'$ as a union [redacted] Δ and Δ^{op}.

Another idea is the filtration by m, but this seems not to yield very good results.

General points:

1) Problem of explaining periodic + negative cyclic homology. Reason for looking at Tate-Farrell

2) $N^{cy}(G)$ and the whole Madsen etc. picture, especially p-adic aspects. Idea: $N^{cy}(G)$ fits into the [redacted] free G-biset resolution of G.

3) Pattern $-\log\det(1-K) = \sum_{m\geq 1} \frac{1}{m} \operatorname{tr}(K^m)$

How is this related to S^1, especially the equivariant cohomology picture which should link to $b+B$

March 9, 1992

Let $E \xrightarrow{\pi} A$ be a surjection of A-bimodules. Consider the tensor algebra $T_A(E) = A \oplus E \oplus E \otimes_A E \oplus \cdots$

We can define a differential on $T_A(E)$ compatible with product of degree -1 by using interior product wrt π:

$$d(\xi_1, \dots, \xi_n) = \sum_{i=1}^{n} (-1)^{i-1} (\xi_1, \dots, \xi_{i-1}\pi(\xi_i), \dots, \xi_n)$$

More precisely we have

$$(\xi_1, \dots, \xi_{i-1}\pi(\xi_i), \xi_{i+1}, \dots, \xi_n) = (\xi_1, \dots, \xi_{i-1}, \pi(\xi_i)\xi_{i+1}, \dots, \xi_n)$$

and we use the latter when $i = 1$.

I claim that $(T_A(E), d)$ is **acyclic**, i.e. we have a bimodule resolution

$$\xrightarrow{d} E\otimes_A E \otimes_A E \xrightarrow{d} E \otimes_A E \xrightarrow{d} E \xrightarrow{d} A \longrightarrow 0$$

Fix $z \in E$ with $\pi(z) = 1$. Then

$$d(z\alpha) = (dz)\alpha - z\,d\alpha$$

or $\boxed{\alpha = z.d\alpha + d(z.\alpha)}$ so that $\alpha \longmapsto z.\alpha$ is a contracting homotopy. This homotopy is compatible with right A-multiplication, and

$$d(\alpha z) = (d\alpha)z + (-1)^{|\alpha|}\alpha$$

$$d((-1)^{|\alpha|}\alpha z) + (-1)^{|d\alpha|}(d\alpha)z = \alpha$$

so $\alpha \longmapsto (-1)^{|\alpha|}\alpha z$ is a contracting homotopy compatible with left multiplication by A.

Example: $E = A \otimes A$, $z = 1 \otimes 1$. This is in some sense the universal example.

We use the identification

$$(A\otimes A)\otimes_A \cdot \overset{n}{\cdot} \cdot \otimes_A (A\otimes A) = A^{\otimes n+1}$$

so we have the standard resolution

$$\cdots \xrightarrow{b'} A^{\otimes 3} \xrightarrow{b'} A\otimes A \xrightarrow{b'} A \longrightarrow 0$$

Check the differential in degree 2.

$$\begin{tikzcd}
(x\otimes y)\otimes(1\otimes z) & & & \\
(A\otimes A)\otimes_A (A\otimes A) \arrow[d, "\cong"] \arrow[drr, "\pi\otimes 1 - 1\otimes\pi"] & & & xy(1\otimes z) - (x\otimes y)z \arrow[d, phantom, "\parallel"] \\
A\otimes A\otimes A \arrow[rr, "b'"] & & A\otimes A & xy\otimes z - x\otimes yz \\
x\otimes y\otimes z \arrow[uuu, mapsto, bend left=60] & & &
\end{tikzcd}$$

Of course $\alpha \longmapsto z\alpha$ is $(a_0,\dots,a_n) \longmapsto (1,a_0,\dots,a_n)$, the familiar contracting homotopy s for b'.

The next point is to consider the (pre)cyclic module (vector space)

$$n \longmapsto [E\otimes_A]^{(n+1)}$$

The claim is that the corresponding cyclic homology is the cyclic homology of A, assuming E is a flat bimodule.

<u>Proof</u>: Choose $z\in E$ such that $\pi z = 1$. Then we get a map of cyclic objects

$$[(A\otimes A)\otimes_A]^{(n+1)} \longrightarrow [E\otimes_A]^{(n+1)}$$

The former gives the cyclic homology of A, so the only thing required is to see that we have a qis wrt b. But the b complex is obtained from the bimodule resolution $T_A(E)$ by applying the commutator quotient space functor over A. So it suffices to show this resolution is by flat bimodules, and this follows because E is assumed to be a flat bimodule.

This result tells us that the cyclic homology is given by the complex

$$\longrightarrow [E\otimes_A]^{(3)}_\lambda \longrightarrow [E\otimes_A]^{(2)}_\lambda \longrightarrow E\otimes_A$$

So what we have done is to embed A inside a quasi-free DG algebra T relative to A which is acyclic, then the quotient $T_\natural/A_\natural$. One might be able to check this is independent of the choice of T by some direct homotopy argument.

We have proved that the (pre) simplicial module $[n] \longmapsto [E\otimes_A]^{(n+1)}$ calculates the Hochschild homology. It is natural to ask if there is a B-operator on this b-complex. Let us consider the cyclic bicomplex associated to this simplicial module:

$$\begin{tikzcd}
{} \arrow[d, "d_0\ d_1\ d_2"'] & {} \arrow[d, "d_0\ d_1"'] & \\
{[E\otimes_A]^{(2)}} \arrow[d, "d_0\ d_1"'] & {[E\otimes_A]^{(2)}} \arrow[l, "1-\lambda"'] \arrow[d, "d_0"'] & {} \arrow[l, "N_\lambda"'] \arrow[d] \\
E\otimes_A & E\otimes_A \arrow[l, "1-\lambda=0"'] & {} \arrow[l, "N_\lambda=1"']
\end{tikzcd}$$

(I need to check the identities at some point.)

One knows from Kassel's paper that all we need is a contracting homotopy for the b' complex. If s satisfies $b's+sb'=1$, then $B=(1-\lambda)sN_\lambda$ is the desired differential anti-commuting with b.

The reason this is true is that the b'-cx. results by applying $?\otimes_A$ to the complex

$$\circledast \quad \Rrightarrow E\otimes_A E\otimes_A E \xrightarrow[1\otimes\pi\otimes 1]{\pi\otimes 1\otimes 1} E\otimes_A E \xrightarrow{\pi\otimes 1} E \longrightarrow 0$$

which is just $(T_A(E), d)\otimes_A E$. This tensor with E on the right preserves exactness, so $\circledast$ is an exact complex of A-bimodules, which are flat when E is a flat A-bimodule. Thus the sequence remains exact after applying $?\otimes_A$.

March 10, 1992

Recall that we have can. maps $K_iA \longrightarrow HP_iA$, $i=0,1$, and that in the case $i=1$ the map can be refined to a map from K_1A into some negative cyclic homology group. The details are as follow. Recall

$$H^{c-}_n M \overset{def}{=} H_{n+2\mathbb{Z}}(F^{n-1}\hat{M})$$

$$\longrightarrow HC^-_{n+2} \longrightarrow HP_{n+2\mathbb{Z}} \longrightarrow HC_n \longrightarrow HC^-_{n+1} \longrightarrow$$

Take $n=-1$. Then we get a surjection $HC^-_1A \longrightarrow HP_1A$ where

$$HC^-_1A = H_{od}(F^0\hat{\Omega}A).$$

Use $\hat{X}(RA,IA)$; then we have

$$\begin{tikzcd} \hat{X}: & \widehat{RA} \arrow[r, shift right, "\delta"'] & \Omega^1(RA)^\wedge_\natural \arrow[l, shift right, "\beta"'] \end{tikzcd}$$

$$F^0\hat{X}: \quad \begin{array}{cc} [A,A] & \\ \times & \Omega^1A \\ \Omega^2A & \times \\ \times & \Omega^3A \\ \Omega^4A & \times \\ \times & \end{array}$$

On the other hand we have constructed

$$K_1A \longrightarrow \mathrm{Ker}\{\Omega^1(RA)^\wedge_\natural \longrightarrow \widehat{RA}\}/\delta\hat{I}A$$

Claim the latter is just HC^-_1A. The point is that $\delta\hat{I}A = \delta\{[RA,RA]+\hat{I}A\} = \delta\{[A,A]\times\Omega^2A\times\cdots\}$.

March 11, 1992

Let's try to understand duality - Poincaré and Alexander duality - on the simplicial complex level, and then see if we can find a version that includes Poincaré duality groups. Let's start with an argument used by Verdier which employs cohomology with compact support. To fix the ideas suppose we have a nice space, say a manifold M. Take a resolution of the constant sheaf, say we take smooth forms

$$0 \longrightarrow \mathbb{C} \longrightarrow \underline{\Omega}^0 \longrightarrow \underline{\Omega}^1 \longrightarrow \underline{\Omega}^2 \longrightarrow \cdots$$

This can be used to compute $H^i(U, \mathbb{C})$ for any open set U. Now take sections with compact support: $U \longmapsto \Gamma_c(U, \underline{\Omega}^\cdot)$. The point is that this is a cosheaf. Actually we will use that

$$U \longmapsto \Gamma_c(U, \underline{\Omega}^\cdot)^*$$

is a sheaf. The reason is because of partitions of unity. The next point is to determine the cohomology sheaves of this complex of sheaves. Now when dealing with a locally contractible space, say a manifold or simplicial complex, the stalk of

$$U \longmapsto H^i\{\Gamma_c(U, \underline{\Omega}^\cdot)^*\}$$

at x should be $H^i_c(U, \mathbb{C})^* = H^i(M, M-x)^*$.

For a manifold we know this is zero for $i \neq n$ and $\mathbb{C}$ for $i = n$. If we have some argument that the sheaf $\Gamma_c(U, \underline{\Omega}^\cdot)^*$ is cohomologically trivial, then we get

$$H^i_c(M, \mathbb{C})^* = H^{-i}(\Gamma_c(M, \underline{\Omega}^\cdot)^*) = H^{-i}(M, \omega[+n])$$
$$= H^{n-i}(M, \omega)$$

where ω is the orientation sheaf.

Now I want to understand this in a simplicial complex context. Perhaps I should take S^1 or a surface to see what really happens.

Idea: Associate to a finite poset the incidence algebra say [illegible] over $\mathbb{C}$. Then covariant and contravariant functors with vector space values are left + right modules over the incidence alg. Any f.t. module has a composition series in terms of simple modules. I think one ought to be able to analyze the structure quite easily for homology + cohomology on a simplicial complex using module theory as a guide.

March 13, 1992

Let us review Weil's proof of the de Rham theorem and the related proof (maybe also due to Weil) of Poincaré duality.

Let M be a compact smooth manifold, $\mathcal{U}$ an open covering, finite, by "convex" open sets. The de Rham theorem à la Weil shows the de Rham cohomology calculated by $\Gamma(M, \Omega^\cdot)$ is the same as the cohomology of the nerve of $\mathcal{U}$ calculated by $C^\cdot(\mathcal{U}, \mathbb{C})$. The proof uses the double complex $C^\cdot(\mathcal{U}, \Omega^\cdot)$. One has quasi-isos.

$$C^\cdot(\mathcal{U}, \mathbb{C}) \longrightarrow C^\cdot(\mathcal{U}, \Omega^\cdot) \longleftarrow \Gamma(M, \Omega^\cdot)$$

The first results from the fact that $\Gamma(U_\sigma, \Omega^\cdot)$ is quis to $\mathbb{C}[0]$ - this is Poincaré's lemma and the fact that U_σ is convex. The second results from the fact that $C^\cdot(\mathcal{U}, \Omega^p)$ is quis $\Omega^p[0]$ - the uses a smooth partition of unity subordinate to $\mathcal{U}$. (The concrete steps are explained in Weil's Kählerian varieties.)

For Poincaré duality one considers sections with compact support, which gives a cosheaf in a certain sense. Consider the complex $C_\cdot(\mathcal{U}, F)$:

$$\longrightarrow \bigoplus_{d(\sigma)=1} \Gamma_c(U_\sigma, F) \longrightarrow \bigoplus_{i_0} \Gamma_c(U_{i_0}, F)$$

and the double complex $C_\cdot(\mathcal{U}, \Omega^\cdot)$. We have a quis $C_\cdot(\mathcal{U}, \Omega^p) \longrightarrow \Gamma_c(M, \Omega^p)$ by a partition of unity; (one ought to be able to do this for M noncompact). On the other hand one has a quis $\Gamma_c(U_\sigma, \Omega^\cdot) \longrightarrow \mathbb{C}[n]$ depending on a choice of orientation.

Picture of the double complex

$$\begin{tikzcd}
\arrow[r] & C_1(\mathcal{U}, \Omega^n) \arrow[r] & C_0(\mathcal{U}, \Omega^n) \\
\arrow[r] & C_1(\mathcal{U}, \Omega^{n-1}) \arrow[r] \arrow[u] & C_0(\mathcal{U}, \Omega^{n-1}) \arrow[u] \\
& \arrow[u] & \arrow[u]
\end{tikzcd}$$

Thus we have quasi-isom.

$$C_\cdot(\mathcal{U}, \mathcal{O})[n] \longrightarrow C_\cdot(\mathcal{U}, \Omega^\cdot) \longleftarrow \Gamma_c(M, \Omega^\cdot)$$

where $\mathcal{O}$ is $\mathbb{C} \otimes$ orientation. This gives then

$$H_c^i(M, \mathbb{C}) \simeq H_{n-i}(\mathcal{U}, \mathcal{O})$$

and by dualizing we get

$$H_c^i(M, \mathbb{C})^* \simeq H^{n-i}(M, \mathcal{O})$$

Now I would like to understand this proof entirely in a simplicial complex context.

March 14, 1992

The problem is to understand Poincaré duality on a combinatorial level. We consider a manifold M, say oriented. Poincaré duality then gives a canonical isomorphism

⊛ $$H^i(K) \xrightarrow{\sim} H_{n-i}(M, M-K)$$

for any compact subset K.

Let us supposed M triangulated. Then we can hope to prove ⊛ by using a version of the Eilenberg-Steenrod uniqueness theorem for cohomology on simplicial complexes. This means that we consider the cohomology functor

$$F^i(K) = H_{n-i}(M, M-K)$$

and its relative version

$$F^i(K, K') = H_{n-i}(M-K', M-K)$$

for $K' \subset K$. Here K is to run over subcomplexes for the triangulation.

Consider the skeletal filtration

$$\phi \subset K_0 \subset K_1 \subset \cdots$$

We then have an exact sequence

⊛⊛ $$\begin{tikzcd} 0 \arrow[r] & F^0(K_1, K_0) \arrow[r] & F^0(K_1) \arrow[r] & F^0(K_0) \arrow[dll, "\delta"] \\ & F^1(K_1, K_0) \arrow[r] & F^1(K_1) \arrow[r] & F^1(K_0) \arrow[dll] \\ & {} & & \end{tikzcd}$$

Suppose, as should be, we have

$$F^i(K_p, K_{p-1}) = \begin{cases} 0 & i \neq p \\ \prod_{\dim(\sigma)=p} \mathcal{O}_\sigma & i = p \end{cases}$$

where $\mathcal{O}_\sigma$ denotes the orientation sheaf ($H^n(M, M-x)$ for $x \in \mathrm{Int}(\sigma)$).

From $\circledast$ we then get

$$0 \to F^0(K_1^{\bullet}) \longrightarrow \prod_{\dim(\sigma)=0} \mathcal{O}_\sigma \xrightarrow{\delta} \prod_{\dim(\sigma)=1} \mathcal{O}_\sigma \longrightarrow F^1(K_1) \to 0$$

$$F^j(K_1) = 0 \qquad j \geq 2.$$

and then from

$$\begin{tikzcd}
0 \arrow[r] & F^0(\overset{\circ}{\cancel{K_2}}, K_1) \arrow[r] & F^0(K_2) \arrow[r] & F^0(K_1) \arrow[dll] \\
& F^1(\cancel{K_2}, \overset{\circ}{K_1}) \arrow[r] & F^1(K_2) \arrow[r] & F^1(K_1) \arrow[dll] \\
& F^2(K_2, K_1) \arrow[r] & F^2(K_2) \arrow[r] & 0
\end{tikzcd}$$

we see $\quad F^0(K_2) = F^0(K_1) = H^0(K, \mathcal{O})$

[redacted]

$$F^1(K_2) = \operatorname{Ker}\left(C^1(K,\mathcal{O}) / \delta C^0(K,\mathcal{O}) \longrightarrow C^2(K,\mathcal{O}) \right)$$

$$= H^1(K, \mathcal{O})$$

$$F^2(K_2) = C^2(K,\mathcal{O}) / \delta C^1(K,\mathcal{O})$$

$$F^j(K_2) = 0 \qquad j \geq 3$$

More generally we have an exact couple

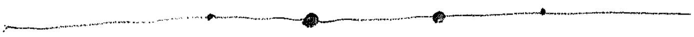
$$\begin{tikzcd}
\bigoplus F^{p+q}\text{[redacted]}(K_{p-1}) \arrow[rr] & & \bigoplus F^{p+q}(K_p)\text{[redacted]} \arrow[dl] \\
& \bigoplus F^{p+q}(K_p, K_{p-1}) \arrow[ul, dotted] &
\end{tikzcd}$$

and a spectral sequence which degenerates.

The next step is to understand combinatorially the homology groups

$$H_j(M, M-K)$$

The point is that $M-K$ is open in M; it is not a simplicial complex. However, it ought to deform to a subcomplex. Except we have to subdivide. For example if K is the set of two vertices joined by a 1-simplex



in a triangulation of $M=\mathbb{R}$, then $M-K$ will not deform to a subcomplex.

Let us therefore consider the barycentric subdivision of our triangulation of M. Recall that vertices for the barycentric subdivision are simplices in the original triangulation, and that more generally simplices for the barycentric subdivision are chains of simplices in the original triangulation. A subcomplex K of the original triang. is a subset of the poset P of simplices which is closed under specialization. Look at the complement $P-K$ which is closed under generalizing. The claim is that $M-K$ has as strong deformation retract the subcomplex L of the barycentric subdivision whose simplices are chains in $P-K$.

To see this consider a point of $M-K$ and look at the (open) simplex containing it. The vertices will be a chain in P not contained in K. The simplex is either contained in L or is the join of a simplex in K and one in L. Then one has a standard deformation of $\sigma * \tau - \sigma$ to τ.

At this point everything becomes combinatorial. We have the poset P of simplices in the original triangulation of M. A subcomplex K is just a subset of P closed under specialization.

What is the homology of M relative to $M-K$? It is the homology of $M = |P|$ ~~modulo the~~ relative to the complementary subcomplex $|P-K|$. So

$$H_i(M, M-K) = H_i\left\{ \coprod_{\substack{\sigma_0<\cdots<\sigma_p \\ \text{in } P}} \mathbb{Z} \Big/ \coprod_{\substack{\sigma_0<\cdots<\sigma_p \\ \text{in } P-K}} \mathbb{Z} \right\}$$

and this can be interpreted as the homology of a covariant functor on P

$$\mathbb{Z}_K(\sigma) = \begin{cases} \mathbb{Z} & \sigma \in K \\ 0 & \sigma \notin K \end{cases}$$

In effect

$$C_p(P, \mathbb{Z}_K) = \coprod_{\sigma_0<\cdots<\sigma_p} \mathbb{Z}_K(\sigma_0)$$

Let's check that $K \longmapsto \mathbb{Z}_K$ is contravariant. But since we are dealing with covariant functors we know that we have

$$0 \longrightarrow \mathbb{Z}_{P-K} \longrightarrow \mathbb{Z} \longrightarrow \mathbb{Z}_K \longrightarrow 0$$

and clearly $K' \subset K \implies P-K' \supset P-K$

$\implies \mathbb{Z}_{P-K'} \supset \mathbb{Z}_{P-K} \implies \mathbb{Z}_{K'} \twoheadleftarrow \mathbb{Z}_K$

The next point to check is ~~what happens~~ whether this is the cohomology of K. Let us set

$$F^i(K) = H_{n-i}(M, M-K) = H_{n-i}(P, \mathbb{Z}_K^{\text{cov}})$$

There will be no problem with exact sequences for a pair, so we have to check $F^i(K_p, K_{p-1})$. This involves the covariant functors $\mathbb{Z}_x$ for $\dim(x) = p$. So we have to look at $H_{n-i}(P, \mathbb{Z}_x)$. Notice that

$$\mathbb{Z}_x = \mathbb{Z}_{\{\geq x\}}/\mathbb{Z}_{\{>x\}}$$

and it's clear that

$$H_{n-i}(P,\mathbb{Z}_x) = H_{n-i}(\{\geq x\},\{>x\};\mathbb{Z})$$

$$\cong \tilde{H}_{n-i}(S^{n-p},\mathbb{Z}) = \begin{cases} \mathbb{Z} & i=p \\ 0 & i\neq p. \end{cases}$$

Thus it appears that

$$F^i(K_p,K_{p-1}) = \begin{cases} \bigoplus\limits_{\dim(x)=p} \mathbb{Z} & i=p \\ 0 & i\neq p \end{cases}$$

But the basic contravariance gives cohomology rather than cohomology.

Next let us forget the simplicial complexes and instead just consider a finite poset P with all maximal chains of length n. Let us suppose that

$$\{>x\} \sim S^{n-1-p} \qquad p=\dim(x)$$

$$\{<x\} \sim S^{p-1} \qquad \text{''}.$$

Then we should have a kind of global P.D. How can we do this? I want to compute $H^i(P,\mathbb{Z})$ and $H_{n-i}(P,\mathbb{Z})$ and show they agree. For the latter we use covariant functors and we use the [redacted] system

$$\mathbb{Z}_P \twoheadrightarrow \underset{\dim\leq n-1}{\mathbb{Z}_{[\text{redacted}]}} \twoheadrightarrow \underset{\dim\leq n-2}{\mathbb{Z}} \twoheadrightarrow \cdots \twoheadrightarrow \mathbb{Z}_{\dim=0} \to 0$$

In both cases we want to use covariant functors. The constant functor $\mathbb{Z}$ is taken apart into the functors $\mathbb{Z}_x$ for $x\in P$.

If we look at cohomology, then

$$H^i(P,F) = H^i\Big(\prod_{x_0} F(x_0) \longrightarrow \prod_{x_0<x_1} F(x_1) \longrightarrow \prod_{x_0\leq x_1<x_2} F(x_2) \longrightarrow\Big)$$

so

$$H^i(P,\mathbb{Z}_x) = H^i\Big(\mathbb{Z} \longrightarrow \prod_{x_0<x} \mathbb{Z} \longrightarrow \prod_{x_0<x_1<x} \mathbb{Z} \longrightarrow \cdots\Big)$$

$$= H^i(\{\leq x\}, \{<x\}; \mathbb{Z})$$

If we consider homology, then

$$H_i(P,F) = H_i\Big(\longrightarrow \bigoplus_{x_0<x_1} F(x_0) \longrightarrow \bigoplus_{x_0} F(x_0)\Big)$$

so

$$H_i(P,\mathbb{Z}_x) = H_i\Big(\to \bigoplus_{x<x_1<x_2} \mathbb{Z} \longrightarrow \bigoplus_{x<x_1} \mathbb{Z} \longrightarrow \mathbb{Z}\Big)$$

$$= H_i(\{\geq x\}, \{>x\}; \mathbb{Z})$$

Also observe we have for covariant functors the exact sequences

$$0 \longrightarrow \mathbb{Z}_x \longrightarrow \mathbb{Z}_{\{\leq x\}} \longrightarrow \mathbb{Z}_{\{<x\}} \longrightarrow 0$$

$$0 \longrightarrow \mathbb{Z}_{\{>x\}} \longrightarrow \mathbb{Z}_{\{\geq x\}} \longrightarrow \mathbb{Z}_x \longrightarrow 0$$

At this point you are missing the punch line which should tell you that the cohomology $H^i(P,\mathbb{Z})$ is canonically isomorphic to the homology $H_{n-i}(P,\mathbb{Z})$. Can this be true for an arbitrary covariant F?

March 15, 1992

Summary formulas. [redacted] Let F be a covariant functor on a poset P.

$$H_i(P,F) = H_i\left\{ \to \bigoplus_{x_0<x_1<x_2} F(x_0) \to \bigoplus_{x_0<x_1} F(x_0) \to \bigoplus_{x_0} F(x_0) \right\}$$

$$H^i(P,F) = H^i\left\{ \prod_{x_0} F(x_0) \longrightarrow \prod_{x_0<x_1} F(x_1) \longrightarrow \prod_{x_0<x_1<x_2} F(x_2) \to \right\}$$

$$H_i(P,\mathbb{Z}_x) = H_i\left(\{\geq x\}, \{>x\}; \mathbb{Z}\right)$$

$$H^i(P,\mathbb{Z}_x) = H^i\left(\{\leq x\}, \{<x\}; \mathbb{Z}\right)$$

Introduce decreasing filtration on F:

$$(\gamma^p F)(x) = \begin{cases} F(x) & \dim(x) \geq p \\ 0 & \dim(x) < p \end{cases}$$

Then $F = \gamma^0 F \supset \gamma^1 F \supset \cdots\cdots$ and

$$\gamma^p F/\gamma^{p+1} F = \bigoplus_{\dim(x)=p} F(x)_x$$

where $(A_x)(y) = A$ if $y = x$ and 0 otherwise.

Thus

$$H_i(P, \gamma^p F/\gamma^{p+1}F) = \bigoplus_{\dim(x)=p} H_i(\{\geq x\}, \{>x\}; F(x))$$

We have a complex

$$⊛ \quad H_n(F/\gamma^1 F) \xrightarrow{\text{[redacted]}} H_{n-1}(\gamma^1 F/\gamma^2 F) \longrightarrow H_{n-2}(\gamma^2 F/\gamma^3 F) \longrightarrow$$

In fact we have a spectral sequence

$$E^1_{pq} = H_{p+q}(\gamma^{-p}F/\gamma^{-p+1}F) \Longrightarrow H_{p+q}(F)$$

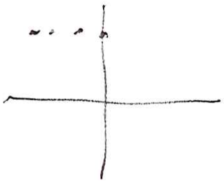
and if we suppose $H_i(\{\geq x\}, \{>x\}) = 0 \quad i \neq n-\dim(x)$

then $E^1_{pq} \neq 0 \Longrightarrow p+q = n+p \Longrightarrow n = q$, so ⊛

gives the homology $H_i(F)$. Now if $H_i(\{\geq x\}, \{>x\}) = \begin{cases} \mathbb{Z} & i = n-\dim(x) \\ 0 & \text{otherwise} \end{cases}$

then ⊛ has the form

① $$\underset{\dim(x)=0}{\overset{(\deg n)}{\bigoplus}} F(x) \longrightarrow \underset{\dim(x)=1}{\overset{(\deg n-1)}{\bigoplus}} F(x) \longrightarrow \cdots$$



On the cohomology side we have

$$E_1^{pq} = H^{p+q}(\gamma^p F/\gamma^{p+1}F) \Longrightarrow H^{p+q}(F)$$
$$\Vert$$
$$\prod_{\dim(x)=p} H^{p+q}(\{\leq x\},\{<x\};F(x))$$

so if we assume $H^i(\{\leq x\}, \{<x\}) = 0$ for $i \neq \dim(x)$, then $E_1^{pq} \neq 0 \Rightarrow q = 0$, so the cohomology is given by the complex E_1^{p0}:

$$H^0(F/\gamma^1 F) \longrightarrow H^1(\gamma^1 F/\gamma^2 F) \longrightarrow \cdots$$

If in addition $H^{\dim(x)}(\{\leq x\}, \{<x\}) = \mathbb{Z}$, then this complex has the form

② $$\prod_{\dim(x)=0} F(x) \longrightarrow \prod_{\dim(x)=1} F(x) \longrightarrow \cdots$$

Supposedly for P finite the complexes ① and ② are the same giving the isomorphism $H_{n-i}(P,F) = H^i(P,F)$. This requires more work.

Discussion. There are a lot of angles to work out. [redacted] Let's mention some. In the duality theory of Grothendieck homology is not mentioned, involved directly. The above version fits nicely with my previous experience with Tate-Farrell homology. A puzzle in this connection is what happened to the contravariant functors. Also there is I believe a classical picture of Poincaré-Alexander duality for a triangulated manifold which uses the dual cell complex.

Another idea which ought to be important at some point is the diagonal $X \xrightarrow{\Delta} X \times X$, the diagonal approximation, the proof of Poincare duality for a manifold using the cohomology class of the diagonal. My feeling is that all of these viewpoints can be nicely understood in the context of simplicial complexes.

Let us next work out in the poset case various aspects of Grothendieck's theory. Let's try to understand supports, filtration by the dimension of the support, Cousin resolution.

We consider a poset X which we suppose to be finite and such that all maximal chains have length n. We consider contravariant functors on X. We know these are the same as sheaves on X considered as a topological space where the open sets are closed under generalization.

Let us now try to understand $\mathrm{Ext}^i(F,G)$ in the category of sheaves. Assume $F(x)$ projective over $\mathbb{Z}$ for all x we can calculate this by the cochain complex

$$(1) \qquad \prod_{x_0} \mathrm{Hom}(F(x_0), G(x_0)) \longrightarrow \prod_{x_0 < x_1} \mathrm{Hom}(F(x_0), G(x_1)) \longrightarrow$$

To see this we construct a projective resolution of F. Claim this is

$$(2) \qquad \longrightarrow \bigoplus_{x_0 < x_1} F(x_0) \otimes \mathbb{Z}[h^{x_1}] \longrightarrow \bigoplus_{x_0} F(x_0) \otimes \mathbb{Z}[h^{x_0}] \longrightarrow F \longrightarrow 0$$

If the latter is evaluated at x we get the complex

$$\begin{tikzcd}
{} \arrow[r] & \bigoplus\limits_{x_0<x_1} F(x_0)\otimes\mathbb{Z}[\mathrm{Hom}(x_1,x)] \arrow[r] \arrow[d, equal] & \bigoplus\limits_{x_0} F(x_0)\otimes\mathbb{Z}[\mathrm{Hom}(x_0,x)] \arrow[r] \arrow[d, equal] & F(x) \arrow[r] & 0 \\
{} \arrow[r] & \bigoplus\limits_{x_0<x_1\leq x} F(x_0) \arrow[r] & \bigoplus\limits_{x_0\leq x} F(x_0) & &
\end{tikzcd}$$

which calculates $L_i \varinjlim_{X/x} F$, and this is $F(x)[0]$, since X/x has a final object. Thus we have a resolution of F by complexes $F(x_0)\otimes\mathbb{Z}[h^{x_1}]$ which are projective since

$$(3) \qquad \mathrm{Hom}\left(F(x_0)\otimes\mathbb{Z}[h^{x_1}],\ G\right) = \mathrm{Hom}_{\mathbb{Z}}\left(F(x_0),\ G(x_1)\right)$$

is exact in G, as $F(x_0)$ is projective. From (3) it's clear that using the resolution (2) of F to compute $\mathrm{Ext}^i(F,G)$ we get the cohomology of the complex (1).

An interesting point is that the homology $H_i(X,F) = L_i \varinjlim_{X} F$ can be recovered from these Ext groups: If A is an injective abelian group, then we should have

$$\boxed{\mathrm{Ext}^i(F,A) = \mathrm{Hom}(H_i(X,F),A)}$$

This should be clear from the dual resolution of G:

$$0 \longrightarrow G \longrightarrow \prod_{x_0} \underbrace{G(x_0)_{\{\leq x_0\}}}_{i_{x_0*}(G(x_0))} \longrightarrow \prod_{x_0<x_1} G(x_1)_{\{\leq x_1\}} \longrightarrow \cdots$$

So now what happens is that we obtain cohomology with support $H^i_Z(X,F)$ as $\mathrm{Ext}^i(\mathbb{Z}_Z, F)$.

Moreover we have local Ext's.
Recall $\mathcal{E}xt^i(F, G)$ is the sheaf associated to the presheaf $U \longmapsto \operatorname{Ext}^i(U; F, G) = \operatorname{Ext}^i(F_U, G)$.
Thus

$$\mathcal{E}xt^i(F,G)(x) = \operatorname{Ext}^i(F_{\{\geq x\}}, G)$$

$$= H^i\left\{ \prod_{x \leq x_0} \operatorname{Hom}(F(x_0), G(x_0)) \longrightarrow \prod_{x\leq x_0 < x_1} \operatorname{Hom}(F(x_0), G(x_1)) \to \cdots \right\}$$

I am interested in

$$\mathcal{E}xt^i(\mathbb{Z}_y, G)(x) = H^i\left\{ \prod_{x \leq x_0} \operatorname{Hom}(\mathbb{Z}_y(x_0), G(x_0)) \longrightarrow \cdots \right\}$$

Now $\mathbb{Z}_y(x_0) = \begin{cases} \mathbb{Z} & \text{if } x_0 = y \\ 0 & \text{otherwise.} \end{cases}$

so we find

$$\mathcal{E}xt^i(\mathbb{Z}_y, G)(x) = \begin{cases} 0 & \text{if } x \not\leq y \\ H^i\left\{ G(y) \to \prod_{y<x_1} G(x_1) \to \cdots \right\} & \text{if } x \leq y \end{cases}$$

$$\mathcal{E}xt^i(\mathbb{Z}_y, G)(x) = \begin{cases} H^i(\{\geq y\}, \{>y\}; G) & \text{if } x \leq y \\ 0 & \text{otherwise.} \end{cases}$$

i.e. $$\boxed{\mathcal{E}xt^i(\mathbb{Z}_y, G) = H^i(\{\geq y\}, \{>y\}; G)_{\{\leq y\}}}$$

At this point I should have a Cousin resolution of G assuming G satisfies the Cohen-Macaulay type conditions.

Let us try to understand the duality theorem.

March 16, 1992

Summarize. The problem is to prove the duality theorem

$$H^i(X,F) = H_{n-i}(X,F)$$

for F a covariant functor on a suitable n-diml finite poset X. Suitable means the "links" $\{<x\}$, $\{>x\}$ are homology spheres of the appropriate dimension. Here

$$H_i(F) = H_i(X,F) = L_i \varinjlim_X (F)$$

$$= H_i\Big(\to \bigoplus_{x_0<x_1} F(x_0) \longrightarrow \bigoplus_{x_0} F(x_0) \Big)$$

$$H^i(F) = H^i(X,F) = R^i \varprojlim_X F$$

$$= H^i\Big(\prod_{x_0} F(x_0) \longrightarrow \prod_{x_0<x_1} F(x_1) \longrightarrow \cdots \Big)$$

$$H_i(\mathbb{Z}_x) = H_i(\{\geq x\}, \{>x\}; \mathbb{Z}) = \mathbb{Z}[n-\dim x]$$

$$H^i(\mathbb{Z}_x) = H^i(\{\leq x\}, \{<x\}; \mathbb{Z}) = \mathbb{Z}[\dim x]$$

Filtration decreasing $\quad F_{\geq p}(x) = \begin{cases} F(x) & \dim(x) \geq p \\ 0 & \text{otherwise} \end{cases}$

$$\begin{tikzcd}
① & H_n(F_{\geq 0}/F_{\geq 1}) \arrow[r, "\partial"] \arrow[d, equal] & H_{n-1}(F_{\geq 1}/F_{\geq 2}) \arrow[r, "\delta"] \arrow[d, equal] & \cdots \\
& \bigoplus_{\dim(x)=0} \mathbb{Z} \arrow[d, equal] & \bigoplus_{\dim(x)=1} \mathbb{Z} \arrow[d, equal] & \\
② & H^0(F_{\geq 0}/F_{\geq 1}) \arrow[r, "\delta"] & H^1(F_{\geq 1}/F_{\geq 2}) \arrow[r, "\partial"] & \cdots
\end{tikzcd}$$

We know ① computes $H_{n-i}(F)$, ② computes $H^i(F)$. We are missing a proof that ∂ and δ correspond. It [redacted] should be possible I think to fill this in computationally, but I feel that there should be a better [redacted] method.

Basically one needs a map. For [redacted] example a map

③ $$H^i(K) \longrightarrow H_{n-i}(M, M-K)$$

given by cap product with a fundamental class μ whose existence isn't trivial. Observe that the problem of relating ① and ② is not a formal triviality because of the business with orientation.

(Aside: Observe that if Z is closed in X, i.e. closed under specialization, then

$$H^i(\mathbb{Z}_Z) = H^i\left\{ \prod_{x_0 \in Z} \mathbb{Z} \longrightarrow \prod_{x_0 < x_1 \in Z} \mathbb{Z} \longrightarrow \cdots \right\}$$

$$= H^i(Z)$$

$$H_i(\mathbb{Z}_Z) = H_i\left\{ \rightarrow \bigoplus_{\substack{x_0<x_1<x_2 \\ x_0 \in Z}} \mathbb{Z} \longrightarrow \bigoplus_{\substack{x_0<x_1 \\ x_0 \in Z}} \mathbb{Z} \longrightarrow \bigoplus_{x_0 \in Z} \mathbb{Z} \right\}$$

$$= H_i(C_\cdot(X)/C_\cdot(X-Z)) = H_i(X, X-Z)$$

$$= H_i(|X|, [redacted] |X| - |Z|)$$ (by a deformation argument)

Thus the cap product map ③ becomes a special case of a map

$$H^i(F) \longrightarrow H_{n-i}(F)$$

we might hope to define.)

In the Grothendieck version one uses the Yoneda pairing

$$H^i(F) \times \mathrm{Ext}^{n-i}(F, \mathbb{Z}) \longrightarrow H^n(\mathbb{Z})$$

together with an distinguished map $\mu : H^n(\mathbb{Z}) \to \mathbb{Z}$.

(One should be working with $\mathbb{Z}$ replaced by a field here.)

The sheaf theory gives strong control over the phenomaena in the presence of "Cohen-Macaulay" (maybe even "Gorenstein") hypotheses. Thus one method to prove the duality theorem would be to show that the coh. functor $H_{n-i}(F)$ satisfies the axioms characterizing $H^i(F)$. I think this is one of Grothendieck's approaches, but I feel that there ought to be a more basic derived category viewpoint. Let's carefully review possible ideas.

Let's first return to some earlier ideas which arose from thinking about Tate-Farrell cohomology. Let's consider a small category (say our poset X) and covariant functors. Start with cohomology

$$H^i(F) = \mathrm{Ext}^i(\mathbb{C}, F) = H^i\Big(\mathrm{Hom}_X(P_\cdot, F)\Big)$$

where $P_\cdot$ is a projective resolution of the constant functor $\mathbb{C}$. (We work with values in vector spaces $/\mathbb{C}$.).

Example of P:

$$(4) \qquad \longrightarrow \bigoplus_{x_0 \to x_1} \mathbb{C}[h^{x_1}] \longrightarrow \bigoplus_{x_0} \mathbb{C}[h^{x_0}]$$

assuming $P_\cdot$ finitely generated in each degree, we have

$$\mathrm{Hom}_X(P_\cdot, F) = \check{P}_\cdot \otimes_X F$$

where $\check{P}_\cdot$ is determined by

$$\mathbb{C}[h^x]^\vee = \mathbb{C}[h_x]$$

Thus we have

$$\mathrm{Hom}_X(\mathbb{C}[h^x], F) = F(x) = \mathbb{C}[h_x] \otimes_X F$$

The problem is to understand the complex $\check{P}$; this is a cochain complex of contravariant functors. In the case of the resolution (4) we have

$$P_n = \bigoplus_{x_0 \to \cdots \to x_n} \mathbb{C}[h^{x_n}]$$

$$\check{P}_n = \prod_{x_0 \to \cdots \to x_n} \mathbb{C}[h_{x_n}]$$

More precisely

$$\check{P}_\cdot : \quad \prod_{x_0} \mathbb{C}[h_{x_0}] \longrightarrow \prod_{x_0 \to x_1} \mathbb{C}[h_{x_1}] \longrightarrow$$

Thus $\check{P}_\bullet = C^\bullet(X, \underset{\text{covariant in } x}{x \mapsto \mathbb{C}[h_x]})$.

with values in contravariant functors

What are the homology sheaves of the complex $\check{P}_\bullet$? Use

$$\check{P}_\bullet(y) = \check{P}_\cdot \otimes_X \mathbb{C}[h^y] = \mathrm{Hom}_X(P_\cdot, \mathbb{C}[h^y])$$

Thus (5) $\boxed{\mathcal{H}^i(\check{P})(y) = H^i(X, \mathbb{C}[h^y]) \quad \text{contravariant in } y}$

For duality we would like to have

$$H^i(X, \mathbb{C}[h^y]) = \begin{cases} 0 & i \neq n \\ \underset{(\text{orientation?})}{\mathbb{C}} & i = n \end{cases}$$

In fact if we just assume the "Gorenstein" type condition $H^\cdot(X, \mathbb{C}[h^y]) = \mathrm{St}(y)[n]$, then we have the duality theorem

(6) $$\boxed{H^i(X, F) = \mathrm{Tor}_{n-i}(\mathrm{St}, F)}$$

Definitely we need more details. We have (at least in the case of our nice poset X)

that $\check{P}$ is a [redacted] length n complex of projectives. They are projective contravariant functors. The Gorenstein condition tells us that $\check{P}.$ is a resolution

$$0 \longrightarrow \check{P}_0 \longrightarrow \check{P}_1 \longrightarrow \cdots\cdots \longrightarrow \check{P}_n \longrightarrow St \longrightarrow 0$$

of the Steinberg contravariant functor

$$St(y) = H^n(X, \mathbb{C}[h^y])$$

and it should be clear that the duality thm. (6) follows easily. It should be [redacted] true that [redacted] the Tor groups $\mathrm{Tor}_i^X(G,F)$ can be calculated using the gadget

$$\bigoplus_{x_2 \leftarrow x_1 \leftarrow x_0} G(x_2)\otimes F(x_0) \longrightarrow \bigoplus_{x_1 \leftarrow x_0} G(x_1)\otimes F(x_0) \longrightarrow \bigoplus_{x_0} G(x_0)\otimes F(x_0)$$

and so if G is locally constant, then we have just $H_{n-i}(X, St \otimes F)$.

Comments on the significance of the above. If you are start with the right [redacted]-derived functor $H^i(F)$ and make suitable hypotheses it begins to look like a [redacted] left-derived functor for the backwards indexing. For it to be a left [redacted]-derived functor like $H_{n-i}(F)$, you need satisfied the vanishing on projectives, i.e. the condition

$$H^i(\mathbb{C}[h^y]) = 0 \quad \text{for} \quad i \neq n.$$

and the rest should be formal.

Suppose we proceed dually starting with

the homology

$$H_i(F) \simeq H_i(Q_\cdot \otimes_X F) = H^{-i}(\mathrm{Hom}_X(\check{Q}_\cdot, F))$$

where $Q_\cdot$ is a projective right X-module resolution of $\mathbb{k}$:

$$Q : \quad \longrightarrow \bigoplus_{x_0 \to x_1} \mathbb{k}[h_{x_0}] \longrightarrow \bigoplus_{x_0} \mathbb{k}[h_{x_0}]$$

$$\check{Q} : \quad \longleftarrow \prod_{x_0 \to x_1} \mathbb{k}[h^{x_0}] \longleftarrow \prod_{x_0} \mathbb{k}[h^{x_0}]$$

We would like the homology functors (or sheaves) of $\check{Q}$. One can do this most easily dually, but taking F to be $(\iota_x)_* V$, $\iota_x : pt \longrightarrow X$ being the functor with image x. We have

$$((\iota_x)_* V)(x_0) = \begin{cases} V & \text{if} \quad x_0 \leq x \\ 0 & \text{otherwise} \end{cases}$$

Thus $(\iota_x)_* V = V_{\{\leq x\}}$. Here V is a vector space, say $\mathbb{C}$. Now we have

$$\mathrm{Hom}_X(\check{Q}, (\iota_y)_* V) = \mathrm{Hom}_{\mathbb{C}}(\check{Q}(y), V)$$

so passing to cohomology gives

$$\mathrm{Hom}_{\mathbb{C}}(H_i(\check{Q})(y), V) = H_i(X, \underbrace{(\iota_y)_* V}_{V_{\{\leq y\}}})$$

The real point is that $(\iota_y)_* V$ is injective, so if we want $H_i(X, F) = H^{n-i}(X, F)$ we need

$$H_i(X, V_{\{\leq y\}}) = 0 \qquad i \neq n$$

and the rest should be formal.

March 17, 1992

Discussion: The problem remains to obtain a good understanding of Poincaré duality on a combinatorial level. There seem to be two formulations

① $H^i(K) = H_{n-i}(M, M-K)$

② $H^i_c(U) = H_{n-i}(U)$

The former is nicely related to Grothendieck's sheaf theory machinery. The latter fits well with sections with compact support, and differential forms. (Recall the proof. Let $U = \bigcup U_i$ be a nice covering and consider [redacted] $C_\cdot(\mathcal{U}, \Omega^\cdot)$

$C_p($[redacted]$\mathcal{U}, \Omega^q) = \bigoplus_{|\sigma|=p} \Gamma_c(U_\sigma, \Omega^q)$

Then we have quis's

$$\Gamma_c(U, \Omega^\cdot) \longleftarrow C_\cdot(\mathcal{U}, \Omega^\cdot) \longrightarrow C_\cdot(\mathcal{U}, \theta[n])$$

which yields ②.

Now ① is nicely related to [redacted] sheaves [redacted] on a poset, because (see 5p. earlier)

$$H^i(X, \mathbb{C}_Z) = H^i(Z, \mathbb{C}) = H^i(|Z|)$$

$$H_i(X, \mathbb{C}_Z) = H_i(X, X-Z; \mathbb{C})$$ $(0 \to \mathbb{C}_{X-Z} \to \mathbb{C}_X \to \mathbb{C}_Z^{\sim})$

$$= H_i(|X|, |X|-|Z|)$$ (explicit deformation)

Note that homology can be obtained in the sheaf theory by duality:

$$\mathrm{Hom}(\underset{n-}{H_i}(X, F), V) = \mathrm{Ext}^{n-i}_X(F, V)$$ V vector space

The natural formulation of ① is

$$① ' \qquad H^i(X,F) = H_{n-i}(X,F)$$

for any sheaf F. This translates to a perfect pairing

$$H^i(X,F) \times \operatorname{Ext}_X^{n-i}(F,\mathbb{C}) \longrightarrow \mathbb{C}$$

which is the formulation Grothendieck used for Serre duality.

We have studied ①,①' closely by filtering F by the decreasing filtration $F_{\geq p}$. Another possibility is to use the same filtration on $\mathbb{C}$ in

$$\operatorname{Ext}^i(\mathbb{C},F) = H^i(F)$$

This amounts to considering cohomology with supports à la Grothendieck. We have

$$\mathcal{E}xt^i(\mathbb{C}_y,F) = H^i(\{\geq y\},\{>y\},F)_{\leq y}$$

and I think this leads in the case $F=\mathbb{C}$ to a Cousin resolution.

In more detail, the Cousin relation of G results from the tower

$$\begin{tikzcd}
R\mathcal{H}om(\mathbb{C},G) \arrow[r] & R\mathcal{H}om(\mathbb{C}_{\geq 1},G) \arrow[r] & \cdots \arrow[r] & R\mathcal{H}om(\mathbb{C}_{\leq n},G) \\
R\mathcal{H}om(\mathbb{C}_{=0},G) \arrow[u] \arrow[d, equal] & R\mathcal{H}om(\mathbb{C}_{=1},G) \arrow[u] \arrow[d, equal] & & R\mathcal{H}om(C_{=n},G) \arrow[u, phantom, "\cong" rotate=90] \\
\prod\limits_{\dim(x)=0} H^n(\{\geq x\},\{>x\};G)_{\leq x} & \prod\limits_{\dim(x)=1} H^{n-1}(\{\geq x\},\{>x\};G)_{\leq x} & &
\end{tikzcd}$$

If G is "Gorenstein"

giving the injective resolution

$$0 \longrightarrow G \longrightarrow \prod_{\dim(x)=n} H^0(\{\geq x\},\{>x\};G)_{\leq x} \longrightarrow \prod_{\dim(x)=n-1} H^1(\{\geq x\},\{>x\},G)_{\leq x} \longrightarrow \cdots$$

Apply $R\mathrm{Hom}(F, ?)$ and we find $\mathrm{Ext}^i(F, G)$ is calculated by

$$\prod_{\dim(x)=n} \mathrm{Hom}_{\mathbb{C}}\left(F(x), H^0(\{\geq x\}, \{> x\}, G)\right)$$

$$\longrightarrow \prod_{\dim(x)=n-1} \mathrm{Hom}_{\mathbb{C}}\left(F(x), H^1(\{\geq x\}, \{> x\}; G)\right) \longrightarrow \cdots$$

Take $G = \mathbb{C}$ and we see $\mathrm{Ext}^i(F, \mathbb{C}) = H_i(F)^*$ is dual to the homology calculated by

$$\bigoplus_{\dim(x)=0} F(x) \otimes H_n(\{\geq x\}, \{> x\}) \longrightarrow \bigoplus_{\dim(x)=1} F(x) \otimes H_{n-1}(\{\geq x\}, \{> x\}) \rightarrow$$

So I conclude with reasonable certainty that using the Cousin resolution for $\mathbb{C}$ in $\mathrm{Ext}^i(F, \mathbb{C}) = H_i(F)^*$ is equivalent to the standard filtration on F and its effect on $H_i(F)$.

At this point I feel that the sheaf approach with its restriction to covariant functors alone is missing something important. A computational proof of the duality theorem is probably possible by computing the differentials in the two complexes of the form

$$\bigoplus_{\dim(x)=0} F(x) \longrightarrow \bigoplus_{\dim(x)=1} F(x) \longrightarrow \cdots$$

Other ideas: cosheaves, Lefschetz fixpt. formula.

The key results are

$$H_i(X, \mathbb{C}_{\leq y}) = 0 \qquad i \neq n$$

$$H^i(X, \mathbb{C}_{\geq y}) = 0 \qquad i \neq n$$

March 19, 1992

Recall that for a manifold there are two formulations of duality

① $H^i(K) = H_{n-i}(M, M-K)$

② $H^i_c(U) \simeq H_{n-i}(U)$

Combinatorially both are special cases of

③ $H^i(F) = H_{n-i}(F)$,

the former being the case $F = \mathbb{C}_Z$ with Z closed, and the latter being the case $F = \mathbb{C}_U$ with U open. To prove the duality result ③ by homological algebra methods we need the effaceability results

④ $H_i(\mathbb{C}_{\leq y}) = 0 \qquad i \neq n$

⑤ $H^i(\mathbb{C}_{\geq y}) = 0 \qquad i \neq n$

together with a little more when $i = n$. In the case of the poset of simplices in a simplicial complex we have

$$\boxed{H_i(\mathbb{C}_{\leq y}) = H_i(M, M - |y|)}$$

and this appears to be hard to analyze. (Come back to this latter.)

Let's examine $H^i(\mathbb{C}_{\geq y})$, more generally $H^i(\mathbb{C}_U)$, where U is open. Let $\bar{U} = \{x \mid \exists x' \in U, x \leq x'\}$ be the closure of U in the poset X, and $\partial U = \bar{U} - U$. One has an exact sequence

$$0 \longrightarrow \mathbb{C}_U \longrightarrow \mathbb{C}_{\bar{U}} \longrightarrow \mathbb{C}_{\partial U} \longrightarrow 0$$

therefore $H^i(\mathbb{C}_U) \simeq H^i(\bar{U}, \partial U)$. Recall that for a closed subset Z one has

$$H^i(X, \mathbb{C}_Z) = H^i(Z, \mathbb{C}) = H^i(|Z|).$$

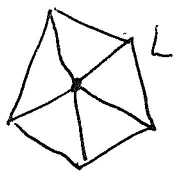
Thus

$$H^i(\mathbb{C}_\mathcal{U}) = H^i(|\bar{\mathcal{U}}|, |\partial\mathcal{U}|)$$

is the cohomology with compact supports of $|\bar{\mathcal{U}}| - |\partial\mathcal{U}|$.

Let us now [redacted] consider the case where X is the poset of simplices in a simplicial complex. Take $\mathcal{U} = \mathcal{U}_y = \{\geq y\}$. Then $\bar{\mathcal{U}}$ $= \{x \mid \exists x' \ni x' \geq x \text{ and } x' \geq y\} = \{x \mid x \cup y \text{ is a simplex}\}$ is (the poset of simplices in) the closed star of y. [redacted] $\partial\mathcal{U}$ is the poset of simplices x such that $x \cup y$ is a simplex not containing y; thus x is a [redacted] simplex in the closed star not containing the generic point of y. Thus $\partial\mathcal{U}$ is the [redacted] poset of simplices in the closed star minus the open star of y. So $H^i(\mathbb{C}_\mathcal{U}) = H^i(|\bar{\mathcal{U}}|, |\partial\mathcal{U}|)$ is the cohomology with compact supports of the open star of y.

[redacted] Geometrically it should be clear that the [redacted] closed star with boundary collapsed to a point is the $(p+1)$-th suspension of the link L of y. For example, if $p = 0$, then the closed star is the cone on the link



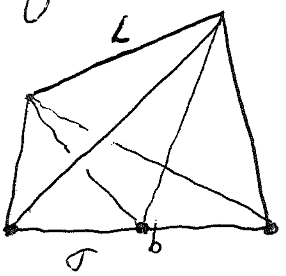
and collapsing L gives the suspension of L. Strictly speaking one means the suspension of $L_+ = L \cup pt$. In general the closed star is the join

$$L * [redacted] |\leq y| = L * \Delta_p$$

and the complement of the open star is

$$L * |<y| = L * \dot{\Delta}_p$$

Observe that the closed star of a simplex σ is a cone with vertex the barycenter of σ.



Put another way, the closed star of σ is $L * \sigma = L * (\dot{\sigma} * b) = (L * \dot{\sigma}) * b$, where L is the link and b is the barycenter of σ. Thus the link of b is the boundary $L * \dot{\sigma}$ of the [redacted] closed star. Hence the closed star $L * \sigma$ [redacted] with boundary $L * \dot{\sigma}$ collapsed is the suspension of $L * \dot{\sigma}$, more precisely $L * \dot{\sigma} * S^0$. But $\dot{\sigma} * S^0$ is a sphere of dimension $= \dim \sigma$. [redacted]

I think the real conclusion might be as follows. The closed star of σ is $L * \sigma = (L * \dot{\sigma}) * b$, [redacted] b being the barycenter of σ. Thus the closed star is the cone on $L * \dot{\sigma}$ with vertex b. The next point is that joining with a sphere $\dot{\sigma}$ of dimension $\dim(\sigma) - 1$ is the same as taking the $\dim(\sigma)$-th suspension. Thus the link of a generic point of σ is the $\dim(\sigma)$-th suspension of the link of σ.

Let's summarize what we have learned. Let X be the poset of simplices in a finite simplicial complex. Then [redacted]

$$\boxed{H^i(X, \mathbb{C}_{\geq \sigma}) = H^{i-p}(\{\geq \sigma\}, \{> \sigma\})}$$

$p = \dim(\sigma)$

March 20, 1992

Problem: What is the combinatorial analogue of the cohomology class of the diagonal?

First look at the Thom class for an oriented vector bundle $E \xrightarrow{f} M$. We consider $f_! \Omega_E^\bullet$ which is the sheaf $\Gamma(U, f_! \Omega_E) = \Omega_{pr}^\bullet(E_U)$, where pr means with proper supports over M. Then we take a nice Weil covering $\mathfrak{U}$ of M and consider the double complex $C^p(\mathfrak{U}, f_! \Omega_E^q)$. Trivializing E over U_σ we have by the contractibility of U_σ a quis

$$\Gamma(U_\sigma, f_! \Omega_E^\bullet) \longrightarrow \Omega_c^\bullet(\text{fibre}) \overset{\text{quis}}{\sim} \mathbb{C}[n]$$

Actually integrating over the fibre gives a quis

$$\Gamma(U_\sigma, f_! \Omega_E^\bullet) \longrightarrow \Gamma(U_\sigma, \Omega_M^\bullet)[n].$$

So we have

$$\begin{tikzcd}
 & \vdots & \vdots & \\
{} & {} \arrow[u] & {} \arrow[u] & \\
\Omega^1_{pr}(E) \arrow[u] \arrow[r] & C^0(f_!\Omega^1_E) \arrow[u] \arrow[r] & C^1(f_!\Omega^1_E) \arrow[u] \arrow[r] & {} \\
\Omega^0_{pr}(E) \arrow[u] \arrow[r] & C^0(f_!\Omega^0_E) \arrow[u] \arrow[r] & C^1(f_!\Omega^0_E) \arrow[u] \arrow[r] & {}
\end{tikzcd}$$

The rows are exact by a partition of unity subordinate to $\mathfrak{U}$. Thus we have quis

$$\begin{tikzcd}
\Omega^\bullet_{pr}(E) \arrow[r, hook] \arrow[rrr, bend right, "f_*"'] & C^\bullet(\mathfrak{U}, f_!\Omega^\bullet_E) \arrow[r, "f_*"] & C^\bullet(\mathfrak{U}, \Omega^\bullet_M) & \Omega^\bullet(M) \arrow[l, hook']
\end{tikzcd}$$

I guess the interesting point here is the fact that one gets a construction of a Thom form, i.e.

a closed $U \in \Omega^n_{pr}(E)$ with $f_* U = 1$.

To do this one chooses $\alpha^0 \in C^0(f_! \Omega^n_E)$ such that $f_* \alpha^0 \equiv 1 \in C^0(\Omega^0_M)$. Then because the columns are exact in degrees $\neq n$, one can complete α^0 to a cocycle in the double complex $\alpha^0, \dots, \alpha^n$ where $\alpha^i \in C^i(f_! \Omega^{n-i}_E)$. But then using the exactness of the rows (this explicitly can be done using a partition of unity on M subordinate to $\mathfrak{U}$) one can successively modify and arrange $\alpha^n = 0$, $\alpha^{n-1} = 0, \dots, \alpha^1 = 0$. Then $\delta \alpha^0 \equiv 0$, so $\alpha^0 \in \Omega^n_{pr}(E)$ is the desired Thom form.

I'd like to understand the case of the diagonal cohomology class in $H^n(M \times M)$ assuming M is closed. This class is described in terms of the duality between $H^i(M)$ and $H^{n-i}(M)$ for all i.

Review Poincaré duality. Let $\mathfrak{U}$ be a Weil covering, let $C_p(\mathfrak{U}, \Omega^q) = \prod_{|\sigma| = p} \Gamma_c(U_\sigma, \Omega^q)$ be the double cx chains on the nerve with values in the forms with compact support. Then one has a quis $C_\cdot(\mathfrak{U}, \Omega^q) \longrightarrow \Omega^q_c(M)$ by a partition of unity (dually $\Omega^q_c(M)'$ is the space of q-currents with arbitrary support, and these currents form a sheaf which is a sheaf of modules over the sheaf of smooth functions, hence fine.) Also $\Omega^\cdot_c(U_\sigma)$ quis $\mathbb{C}[n]$ for each σ. Thus we have quis

$$C_\cdot(\mathfrak{U}, \mathbb{C})[n] \longleftarrow C_\cdot(\mathfrak{U}, \Omega^\cdot) \longrightarrow \Omega^\cdot_c(M)$$

which yields $H^i_c(M) \simeq H_{n-i}(\mathfrak{U})$. This is dual to $H^{n-i}(\mathfrak{U})$ which via $C^\cdot(\mathfrak{U}, \Omega^\cdot)$ is the cohomology of $\Omega^\cdot(M)$.

$$\begin{tikzcd} & C^\cdot(\mathfrak{U}, \Omega^\cdot) & \\ C^\cdot(\mathfrak{U}, \mathbb{C}) \arrow[ur, "\text{quis}"] & & \Omega^\cdot(M) \arrow[ul, "\text{quis}"'] \end{tikzcd}$$

(By doing things this way we can avoid the problem of whether dualizing $\Omega^\cdot_c$ commutes with cohomology, although this is perhaps not critical as our quis are probably actually homotopy equivalences.)

The preceding argument gives a duality between $H^i_c(M)$ and $H^{n-i}(M)$. I have in mind the case where we have a [redacted] finite Weil covering, so that this sort of cohomology is finite dimensional.

Now because of this duality we have a canonical element in

$$\bigoplus_{i=0}^{n} H^{n-i}(M) \otimes H^i_c(M) = \bigoplus_{i=0}^{n} H^{n-i}(\Omega^\cdot(M)) \otimes H^i(\Omega^\cdot_c(M))$$

[redacted] $$= H^n(\Omega^\cdot(M) \underset{alg}{\otimes} \Omega^\cdot_c(M))$$

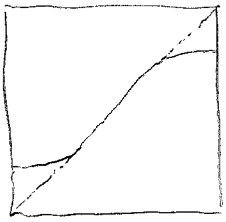
and this maps to $H^n(\Omega_{pr}(M \times M))$, where pr means proper supports with respect to $pr_1 : M \times M \to M$.

Now the [redacted] idea is that we ought to be able to use our quis

$$\begin{tikzcd} C_\cdot(\mathcal{U})[n] & C_\cdot(\mathcal{U}, \Omega^\cdot) \arrow[l] \arrow[r] & \Omega^\cdot_c(M) \\ C^\cdot(\mathcal{U}) \arrow[r] & C^\cdot(\mathcal{U}, \Omega^\cdot) & \Omega^\cdot(M) \arrow[l] \end{tikzcd}$$

to [redacted] construct [redacted] a closed form in $\Omega_{pr}(M \times M)$ realizing the Poincaré duality.

Observe that for M open this [redacted] form one constructs is not supported in a nbd. of ΔM but is apt to be flat as one goes towards the ends of M.



March 25, 1992

Program: Consider a finite simplicial complex M, let X be the poset of its simplices. There is a 1-1 correspondence between subcomplexes of M and subsets Z of X closed under specializing. Write this $Z \longmapsto |Z|$. We then get a 1-1 correspondence between subsets U of X closed under generalization and certain open sets in M given by

$$U \longmapsto M - |X-U| = |X| - |X-U|$$

The open sets of M obtained in this way are the complements of subcomplexes. They should be the open sets in the lattice generated by the open star of vertices, equivalently the finite unions of open stars of simplices.

Given an open set $V \subset M$ complementary to a subcomplex, we have its cohomology $H^i(V)$ (with $\mathbb{C}$ coefficients) and its cohomology with compact supports $H^i_c(V)$. In general in a compact space M the cohomology of an open set V with compact supports is

$$H^i_c(V) = H^i(M, M-V).$$

Reason: $$H^i_c(V) = \varinjlim_{\substack{K \text{ compact} \\ \text{in } V}} H^i_K(V) = \varinjlim_{\substack{K \text{ closed in } M \\ K \subset V}} H^i_K(V)$$

$$= \varinjlim_{\substack{K \text{ closed in } M \\ K \subset V}} H^i_K(M) = \varinjlim_{\substack{K \text{ closed in } M \\ K \subset V}} H^i(M, M-K)$$

$$= \varinjlim_{\substack{U \text{ open in } M \\ U \supset M-V}} H^i(M, U) = H^i(M, M-V)$$

by the general fact that for a closed subset A of M we have

$$H^i(A) = \varinjlim_{\substack{U \supset A \\ U \text{ open}}} H^i(U)$$

So to each $U \subset X$ closed under generalization we have two kinds of cohomology

$$H^i(U) \overset{\text{defn}}{=} H^i(|X| - |X-U|)$$

$$H^i_c(U) = H^i(|X|, |X-U|).$$

If I write V_U for $|X| - |X-U|$, then

$$H^i(U) = H^i(V_U)$$

$$H^i_c(U) = H^i_c(V_U) = H^i(M, M - V_U)$$

$$= H^i(\bar{V}_U, \bar{V}_U - V_U)$$

The former is a cohomology theory in U; contravariant and satisfying Mayer-Vietoris. The latter is a homology theory in U defined similarly. This roughly means that ~~these~~ functors $H^i(U)$, $H^i_c(U)$ on the "open" subsets of X are determined essentially by their values on the open sets $U_x = \{\geq x\}$ for $x \in X$. By "essentially determined" we mean more precisely that there should exist complexes $\Omega^\cdot$ of $\varprojlim$ acyclic covariant functors on X and $\Omega^\cdot_c$ of $\varinjlim$ acyclic contravariant functors on X such that

$$H^i(U) = H^i\left(\varprojlim_{x \in U} \Omega^\cdot(x)\right)$$

$$H^i_c(U) = H^i\left(\varinjlim_{x \in U} \Omega^\cdot_c(x)\right)$$

March 26, 1992

The program: In order to organize the different aspects of duality, I propose to start with the Weil proof of Poincare duality on a manifold with nice covering by geodesically convex open subsets. This should provide a model for a combinatorial proof.

In the Weil proof one considers $\Omega^\cdot(U)$ and $\Omega^\cdot_c(U)$ for U an open set which we can suppose is in the finite distributive lattice generated by the nice covering, i.e. a [redacted] union of the finite intersections of the members of the covering. Then $U \mapsto \Omega^\cdot(U)$ is a contravariant functor which is acyclic with respect to glueing. Similarly $U \mapsto \Omega^\cdot_c(U)$ is covariant and acyclic wrt glueing.

We look for an analogue of this situation for a finite simplicial complex M. [redacted] Let X be the poset of simplices. Regard X as topological space where subsets closed under generalization are open, and let $f: M \longrightarrow X$ be the map such that $f^{-1}(x)$ is the open simplex corresponding to x (the points whose support is x). Then $Z \longmapsto f^{-1}(Z)$ is a 1-1 correspondence between closed subsets of X and subcomplexes of M. In particular f is continuous.

For U open in X we consider $H^i(f^{-1}U)$ and $H^i_c(f^{-1}U)$; these are defined using the topology on the simplicial complexes. We claim that

$$H^i(f^{-1}U) = \mathrm{Ext}^i_X(\mathbb{C}_U, \mathbb{C}) = H^i(U, \mathbb{C})$$

ⓧ

$$H^i_c(f^{-1}U) = \mathrm{Ext}^i_X(\mathbb{C}, \mathbb{C}_U) = H^i(X, \mathbb{C}_U)$$

To prove the latter is easy: We have

$$H^i_c(f^{-1}U) = H^i(M, M - f^{-1}U) = H^i(f^{-1}X, f^{-1}(X-U))$$

For a closed set Z in X we know that

$$H^i(f^{-1}Z) = \mathrm{Ext}^i_X(\mathbb{C}_{\text{[redacted]}}, \mathbb{C}_Z) = H^i(X, \mathbb{C}_Z)$$
$$= H^i(Z, \mathbb{C}).$$

This is the fact that the cohomology of a simplicial complex is the cohomology of the poset of simplices. Thus

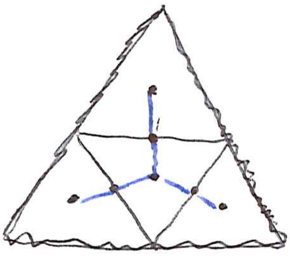
$$H^i(f^{-1}X, f^{-1}(X-U)) = H^i(X, X-U) = H^i(X, \mathbb{C}_U)$$

in view of the exact sequence

$$0 \longrightarrow \mathbb{C}_{\text{[redacted]}U} \longrightarrow \mathbb{C}_X \longrightarrow \mathbb{C}_{X-U} \longrightarrow 0$$

Here are two ways to prove $H^i(f^{-1}U) = \mathrm{Ext}^i_X(\mathbb{C}_U, \mathbb{C}) = H^i(U, \mathbb{C})$. [redacted] The second equality results because an injective resolution of $\mathbb{C}$ over X restricts to an injective resolution [redacted] over U, because j^* has the left adjoint $j_!$ which is exact ($j: U \hookrightarrow X$ the inclusion), and $j_!\mathbb{C} = \mathbb{C}_U$.

To see that $H^i(f^{-1}U) = H^i(U, \mathbb{C})$ we can argue geometrically that $f^{-1}U$ deforms to the full subcomplex of the barycentric subdivision having U as set of vertices:



Take $f^{-1}U$ = interior of the big triangle. It deforms to the subcomplex of the barycentric subdivision whose vertices are the barycenters of the simplices corresponding to the elements of U.

We can also consider the covering of $f^{-1}U$ by [redacted] $f^{-1}(\geq x)$ for $x \in U$. $f^{-1}(\geq x)$ is the open star of x which is contractible. Algebraically we have a projective resolution

$$\longrightarrow \bigoplus_{\substack{x_0 < x_1 \\ x_0 \in U}} \mathbb{C}_{\geq x_1} \longrightarrow \bigoplus_{x_0 \in U} \mathbb{C}_{\geq x_0} \longrightarrow \mathbb{C}_U \longrightarrow 0$$

This is a special case of

$$\to \bigoplus_{x_0<x_1} F(x_0)\otimes \mathbb{C}[h^{x_1}] \to \bigoplus_{x_0} F(x_0)\otimes \mathbb{C}[h^{x_0}] \to \mathbb{C}_U \to 0$$

Apply f^* to get the resolution

$$\to \bigoplus_{\substack{x_0<x_1\\ x_0\in U}} \mathbb{C}_{f^{-1}(\geq x_1)} \to \bigoplus_{x_0\in U} \mathbb{C}_{f^{-1}(x_0)} \to \mathbb{C}_{f^{-1}(U)} \to 0$$

But $f^{-1}(\geq x)$ is the open star of x and it is contractible. So

$$E_1^{pq} = \operatorname{Ext}^p\Big(\bigoplus_{\substack{x_0<\cdots<x_q\\ x_0\in U}} \mathbb{C}_{f^{-1}(\geq x_q)}, \mathbb{C}\Big) \Longrightarrow \operatorname{Ext}^n(\mathbb{C}_{f^{-1}(U)}, \mathbb{C})$$

$$\|$$

$$\prod_{\substack{x_0<\cdots<x_q\\ x_0\in U}} \underbrace{H^p(f^{-1}(\geq x_q))}_{\begin{cases}\mathbb{C} & p=0\\ 0 & p\neq 0\end{cases}}$$

$$\therefore \quad H^n(f^{-1}(U)) = H^n\Big(\to \prod_{\substack{x_0<\cdots<x_q\\ x_0\in U^{pq}}} \mathbb{C} \to \cdots\cdots\Big) = H^n(U).$$

This proves the formulas ⊛.

At this point duality should be a combinatorial type result for the poset X of simplices. In any case we have learned that the basic ingredients of the Weil proof can be expressed in terms of sheaves on the poset of simplices.

The next step is to find combinatorial analogues of the complexes $\Omega^i(U)$ and $\Omega^i_c(U)$ calculating $H^i(U)$ and $H^i_c(U)$ respectively. Let's begin with

$$H^i_c(U) = H^i(X, \mathbb{C}_U) = \operatorname{Ext}^i_X(\mathbb{C}, \mathbb{C}_U)$$

We can calculate this using a projective resolution $P_\bullet$ of $\mathbb{C}$. We then find

$$\mathrm{Ext}^i_X(\mathbb{C},\mathbb{C}_U) = H^i\{\mathrm{Hom}_X(P_\cdot,\mathbb{C}_U)\}$$

$$= H^i\{\check{P}_\cdot \otimes_X \mathbb{C}_U\}$$

where $\check{P}_\cdot$ is a complex of projective contravariant functors (cosheaves) on X. Specifically for $P_\cdot$ we can take

$$\longrightarrow \bigoplus_{x_0<x_1} \mathbb{C}[h^{x_1}] \longrightarrow \bigoplus_{x_0} \mathbb{C}[h^{x_0}]$$

whence $\check{P}_\cdot$ is

$$\prod_{x_0} \mathbb{C}[h_{x_0}] \longrightarrow \prod_{x_0<x_1} \mathbb{C}[h_{x_1}] \longrightarrow \cdots$$

We can calculate the homology of $\check{P}_\cdot$ at the point y by [redacted] using

$$\check{P}_\cdot \otimes_X \underbrace{\mathbb{C}[h^y]}_{\mathbb{C}_{\geq y}} = \check{P}_\cdot(y)$$

Thus we find (from the above)

$\mathcal{H}^i(\check{P}_\cdot)(y) =$ [redacted] $\mathrm{Ext}^i_X(\mathbb{C},\mathbb{C}_{\geq y})$

$$= H^i(X,\mathbb{C}_{\geq y}) = H^i_c(f^{-1}(\geq y))$$

What should be clear is that

$$\Omega^\cdot_c(U) = \check{P}_\cdot \otimes_X \mathbb{C}_U$$

[redacted] is an acyclic complex of cosheaves calculating $H^i_c(f^{-1}U)$, where acyclic means with respect to $\varinjlim$ in some sense.

Next we want to consider $\mathrm{Ext}^i_X(\mathbb{C}_U,\mathbb{C})$ $U \mapsto H^i(U)$. Now the idea here is that the duality theorem says that $H^i_c(U)$ and $H^{n-i}(U)$ should be dual. But we know that $H^{n-i}(U)$ is dual to $H_{n-i}(U)$. Thus the duality thm. says $H^i_c(U) = H_{n-i}(U)$. So what we want to do is write $H_i(U)$ in the same form as $H^i_c(U)$.

Now
$$H_i(U) = H_i(\varinjlim_X P_\cdot)$$
where $P_\cdot$ is a projective resolution of $\mathbb{C}_U$. More generally we have $H_i(F) = H_i(\varinjlim_X P_\cdot(F))$. Here's how to construct $P_\cdot(F)$, a projective resolution of F. Consider

$$\circledast \qquad \longrightarrow \bigoplus_{x_0 < x_1} \mathbb{C}[h^{x_1}] \otimes \mathbb{C}[h_{x_0}] \longrightarrow \bigoplus_{x_0} \mathbb{C}[h^{x_0}] \otimes \mathbb{C}[h_{x_0}]$$

This should be a resolution of the bifunctor $(y,z) \longmapsto \mathbb{C}[\mathrm{Hom}(z,y)]$. Now
$$A = \bigoplus_{y,z} \mathbb{C}[\mathrm{Hom}(z,y)]$$
is the **incidence** algebra of the **poset** and
$$\bigoplus_{y,x_0,z} \mathbb{C}[\mathrm{Hom}(x_0,y)] \otimes \mathbb{C}[\mathrm{Hom}(z,x_0)] \text{ is } A \otimes_S A$$
(draw **arrows** $y \longleftarrow x_0 \longleftarrow z$). So $\circledast$ is the bimodule resolution
$$\longrightarrow A \otimes_S A \otimes_S A \longrightarrow A \otimes_S A \xrightarrow{\text{aug}} A \longrightarrow 0$$

So we get a projective resolution $P_\cdot(F)$ of F by
$$P_\cdot(F) = \mathbb{P}_\cdot \otimes_X F \qquad \text{where } \mathbb{P}_\cdot \text{ is } \circledast$$
above. Then $H_i(F)$ is computed by
$$\mathbb{C} \otimes_X P_\cdot(F) = \mathbb{C} \otimes_X \mathbb{P}_\cdot \otimes_X F$$

In more detail $P_\cdot(F)$ is

$$\longrightarrow \bigoplus_{x_0<x_1} \mathbb{C}[h^{x_1}]\otimes F(x_0) \longrightarrow \bigoplus_{x_0} \mathbb{C}[h^{x_0}]\otimes F(x_0)$$

and $\mathbb{C} \otimes_X P_\cdot$ is the complex $Q_\cdot$ of March 16:

$$\longrightarrow \bigoplus_{x_0<x_1} \mathbb{C}[h_{x_0}] \longrightarrow \bigoplus_{x_0} \mathbb{C}[h_{x_0}]$$

Thus $H_i(F) = H_i(\underbrace{\mathbb{C}\otimes_X P_\cdot}_{Q} \otimes_X F)$

In order to have the duality theorem we want $\mathbb{C}\otimes_X P_\cdot \simeq \check{P}_\cdot[\pm n]$, i.e. $\check{P}_\cdot$ resolves $\mathbb{C}[n]$.

I guess what this means is that we have come full circle and found that our previous analysis of duality for a finite poset really is natural from the viewpoint of Weil's method.

Before going on let's list some <u>ideas</u>.

Finite distributive lattice $\equiv$ poset of irreducibles

$$L \longmapsto \mathcal{P}(L)$$

$$L \xrightarrow{\sim} \text{closed subsets of } \mathcal{P}(L)$$

$$L^{op} \xrightarrow{\sim} \text{open subsets of } \mathcal{P}(L)$$
$$= \text{closed subsets of } \mathcal{P}(L)^{op}.$$

$$\therefore\ \mathcal{P}(L^{op}) = \mathcal{P}(L)^{op}$$

$H^i(U)$, $H^i_c(U)$ is a cohomology (resp. homology) theory in the sense that one has long exact Mayer-Vietoris sequences. Does such a thing make sense on arbitrary space, and might the classification be simpler than for spectra? Combinatorial formula for the K-theory of a finite complex?

$$H^i_c(U) = H^i(M, M-U) \quad \text{if } U \text{ open in } M \text{ compact}$$

This uses $\varinjlim_{\substack{V \supset Z \\ V \text{ open}}} H^i(V) = H^i(Z)$.

The map $f: M \longrightarrow X$ if M is a finite simplicial complex with poset X of simplices. $\forall\, U$ open in X

$$H^i(f^{-1}U) = \mathrm{Ext}^i_X(\mathbb{C}_U, \mathbb{C})$$

$$H^i_c(f^{-1}U) = \mathrm{Ext}^i_X(\mathbb{C}, \mathbb{C}_U)$$

$f^{-1}U$ deforms to the full subcomplex in the barycentric subdivision with U as set of vertices.

Homology: $\mathrm{Ext}^i_X(F, \mathbb{C}) = \mathrm{Hom}_{\mathbb{C}}(H_i(X,F), \mathbb{C})$

$$H_i(X,F) = L_i \varinjlim_X F$$

Basic bifunctor standard resolution

$$P_\cdot : \quad \longrightarrow \bigoplus_{x_0 < x_1 < x_2} \mathbb{C}[h^{x_2}] \otimes \mathbb{C}[h_{x_0}] \longrightarrow \bigoplus_{x_0 < x_1} \mathbb{C}[h^{x_1}] \otimes \mathbb{C}[h_{x_0}] \longrightarrow \bigoplus_{x_0} \mathbb{C}[h^{x_0}] \otimes \mathbb{C}[h_{x_0}]$$

of $(y,z) \longmapsto \mathbb{C}[\mathrm{Hom}(z,y)]$. (This is the standard resolution $\longrightarrow A \otimes_S (A/S) \otimes_S A \longrightarrow A \otimes_S A \dashrightarrow A \longrightarrow 0$.)

Representation of two "homology" theories via complexes:

$$\mathrm{Ext}^i_X(\mathbb{C}, F) = H^i\big(\underbrace{(P_\cdot \otimes_X \mathbb{C})^\vee}_{\check{P}_\cdot} \otimes_X F\big)$$

$H_i(X,F)$ [redacted] $= H_i\big(\underbrace{(\mathbb{C} \otimes_X P_\cdot)}_{Q_\cdot} \otimes_X F\big)$

Duality results when $\check{P}_\cdot$ is a resolution of $\mathbb{C}[u]$.

Different forms of the duality thm.

a) $H^i(K)$ dual to $H^{n-i}(M, M-K)$

b) $H^i(W)$ dual to $H_c^{n-i}(W) = H^{n-i}(M, M-W)$

Observe these are handled by the sheaf theory.

$$H^i(f^{-1}(Z)) = H^i(Z) = \mathrm{Ext}^i_X(\mathbb{C}, \mathbb{C}_Z) \qquad Z \text{ closed in } X$$

$$H^{n-i}(M, M-f^{-1}Z) = H^{n-i}_Z(X, \mathbb{C}) = \mathrm{Ext}^{n-i}_X(\mathbb{C}_Z, \mathbb{C})$$

$$H^i(f^{-1}(U)) = H^i(U, \mathbb{C}) = \mathrm{Ext}^i_X(\mathbb{C}_U, \mathbb{C})$$

$$H_c^{n-i}(f^{-1}(U)) = \mathrm{Ext}^{n-i}_X(\mathbb{C}, \mathbb{C}_U)$$

Thus a), b) are both handled by the sheaf theory but rather differently.

Under what conditions does the duality thm. hold?

(i) $H^i(F) = H_{n-i}(F)$

(ii) $H^i(\mathbb{C}_{\geq y}) = \mathbb{C}[n]$

(iii) $H_i(\mathbb{C}_{\leq y}) = \mathbb{C}[n]$

A better question is what are equivalent formulas of the duality thm? I think the above three are equivalent (up to making precise the isomorphisms). What about

$$\text{(iv)} \quad \begin{cases} H^i(\leq y, < y) = \mathbb{C}[\dim y] & (H^i(\mathbb{C}_y)) \\ H_i(\geq y, > y) = \mathbb{C}[\operatorname{codim} y] & (H_i(\mathbb{C}_y)) \end{cases}$$

These are the groups which occur when we use the "skeletal" filtration $F \supseteq F_{\geq 0} \supset F_{\geq 1} \supset \cdots$

$$H^0(F_{\geq 0}/F_{\geq 1}) \xrightarrow{\delta} H^1(F_{\geq 1}/F_{\geq 2}) \xrightarrow{\delta}$$

$$= \prod_{x \in X_0} H^0(\leq x, < x; F(x))$$

$$H_n(F_{\geq 0}/F_{\geq 1}) \xrightarrow{\partial} H_{n-1}(F_{\geq 1}/F_{\geq 2}) \xrightarrow{\partial}$$

Suppose y is a face of x, whence $\{y,x\} = \mathbb{C}_{\geq y} \cap \mathbb{C}_{\leq x}$ is locally closed. We have an exact sequence

$$0 \longrightarrow \mathbb{C}_x \longrightarrow \mathbb{C}_{yx} \longrightarrow \mathbb{C}_y \longrightarrow 0$$

whence

$$\begin{tikzcd} \delta : H^i(\mathbb{C}_y) \arrow[r] \arrow[d, equal] & H^{i+1}(\mathbb{C}_x) \arrow[d, equal] \\ H^i(\leq y, < y) \arrow[dr, dashed] & H^{i+1}(\leq x, < x) \\ & H^i(< x) \arrow[u, "\wr"'] \end{tikzcd}$$

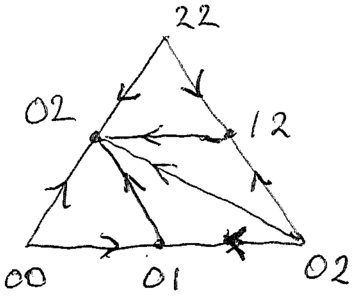
The dotted arrow is ~~[illegible]~~ induced by the inclusion $\mathbb{C}_y \subset \mathbb{C}_{<x}$:

$$\begin{tikzcd} 0 \arrow[r] & \mathbb{C}_x \arrow[r] & \mathbb{C}_{\leq x} \arrow[r] & \mathbb{C}_{<x} \arrow[r] & 0 \\ 0 \arrow[r] & \mathbb{C}_x \arrow[r] \arrow[u, equal] & \mathbb{C}_{yx} \arrow[r] \arrow[u, hook] & \mathbb{C}_y \arrow[r] \arrow[u, hook] & 0 \end{tikzcd}$$

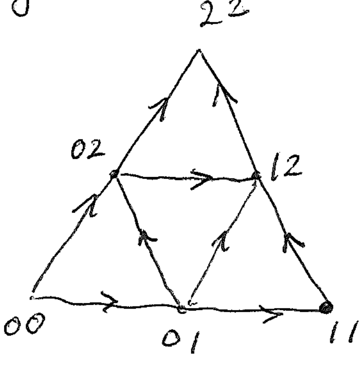
Philosophical idea: We can study posets up to homotopy or up to homeomorphism. This allows a poset to be replaced by its poset of chains, or even intervals. Intervals form the arrow category; there are two different kinds in which $f \to g$ is either

$$\begin{tikzcd} \cdot \arrow[r, "f"] & \cdot \arrow[d] \\ \cdot \arrow[u] \arrow[r, "g"'] & \cdot \end{tikzcd} \qquad \begin{tikzcd} \cdot \arrow[r, "f"] \arrow[d, dash, "f"'] & \cdot \arrow[d, dash, "f"] \\ \cdot \arrow[r, "g"'] & \cdot \end{tikzcd}$$

where the diagram is supposed to be commutative. In the case of $0<1<2$ we get



or



The pictures relate to the different types of edgewise subdivision mentioned in Madsen et al on the cyclotomic trace.

Basic geometric fact: The closed star of a simplex σ is the join $L_\sigma * \sigma$, where L_σ is the link of σ. The boundary = closed star − open star is $L_\sigma * \dot{\sigma}$, and the closed star is the cone on the boundary with vertex the barycenter of σ (since $L_\sigma * \sigma = (L_\sigma * \dot{\sigma}) * pt$). The cohomology with compact support of the open star is the reduced cohomology of the $(\dim\sigma + 1)$th suspension of L_σ.

Homology of a join:

$$\to \tilde{H}_i(X * Y) \longrightarrow \tilde{H}_i(X) \oplus \tilde{H}_i(Y) \longrightarrow \tilde{H}_i(X \# Y) \to H_{i-1}(X * Y) \longrightarrow \cdots$$

$$\therefore \quad \tilde{H}_{n+1}(X \# Y) = \bigoplus_{i+j=n} \tilde{H}_i(X) \otimes \tilde{H}_j(Y) = \tilde{H}_n(X \wedge Y)$$

Geometrically the join is h.eq. to the reduced suspension of the smash product, because one can collapse in $X \# Y$ the contractible space $(* \# Y) \underset{* \# *}{\cup} (X \# *)$, and one obtains the reduced suspension of $X \wedge Y$.

March 30, 1992

Analysis of duality: X finite poset, U open in X.

$$H^i_c(U) = H^i(X, \mathbb{C}_U) = \mathrm{Ext}^i_X(\mathbb{C}, \mathbb{C}_U)$$
$$= H^i(\mathrm{Hom}_X(P, \mathbb{C}_U)) = H_{-i}(\check{P} \otimes_X \mathbb{C}_U)$$

$$H^i(U) = \mathrm{Hom}(H_i(U), \mathbb{C})$$

$$H_i(U) = H_i(X, \mathbb{C}_U) = H_i(Q \otimes_X \mathbb{C}_U).$$

Here P (resp. Q) is a resolution of the constant functor $\mathbb{C}$ by projective left (resp.) right X-modules (i.e. covariant (resp. contravariant) functors on X).

The duality theorem [redacted] holds when [redacted] Q and $\check{P}[n]$ are quasi-isomorphic, and this means they are homotopy equivalent. [redacted] A homotopy equivalence is given by a pair of maps of complexes

$$\check{P} \longrightarrow Q \qquad \text{and} \qquad Q \longrightarrow \check{P}$$

of degree $-n$ and n respectively for the [redacted] lower indexing. Now

$$\mathrm{Hom}_X(\check{P}, Q) = Q \otimes_X \check{\check{P}} = Q \otimes_X P$$

$$H_n(Q \otimes_X P) = H_n(Q \otimes_X \mathbb{C}) = H_n(X, \mathbb{C})$$

so the map [redacted] $\check{P} \longrightarrow Q$ is given by an n-dim'l homology class μ. This fits with the fact that [redacted] such a homology class gives a cup product pairing

$$\begin{tikzcd}
\mathrm{Ext}^{n-i}_X(\mathbb{C}, \mathbb{C}_U) \arrow[d, phantom, "\parallel"] & \times & \mathrm{Ext}^i_X(\mathbb{C}_U, \mathbb{C}) \arrow[d, phantom, "\parallel"] \arrow[r] & \mathrm{Ext}^n_X(\mathbb{C}, \mathbb{C}) = H^n(X) \arrow[d, "\mu"] \\
H^{n-i}_c(U) & \times & H^i(U) \arrow[r] & \mathbb{C}
\end{tikzcd}$$

i.e. $H^{n-i}_c(U) \longrightarrow (H^i(U))^* = H_i(U)$.

Consider now maps $Q \longrightarrow \check{P}$. We should have

$$\mathrm{Hom}_{X^{op}}(Q, \check{P}) = \mathrm{Hom}_{X^{op}}(Q, \mathrm{Hom}_X(P, A))$$

$$= \mathrm{Hom}_{X \times X^{op}}(P \otimes Q, A)$$

where $A = \bigoplus_{x \geq y} \mathbb{C}$ is the incidence algebra of X, i.e. the algebra such that left (resp. right) A-modules are covariant (resp. contravariant) functors on X. Now A is the functor on $X \times X^{op}$ which is $\mathbb{C}$ on $U = \{(x,y) \mid x \geq y\}$ and 0 outside U. Note that U is open in $X \times X^{op}$, since $(x,y) \geq (x',y')$ in $X \times X^{op}$ and $x' \geq y'$ implies $x \geq x' \geq y' \geq y$, so $(x,y) \in U$. Thus $A = \mathbb{C}_U$.

Also $P \otimes Q$ is a resolution of the constant functor $\mathbb{C}$ on $X \times X^{op}$ by projective left $X \times X^{op}$-modules. Thus $\mathrm{Hom}_{X \times X^{op}}(P \otimes Q, A)$ computes $H^i(X \times X^{op}, \mathbb{C}_U)$. Another point is that U is the smallest open subset of $X \times X^{op}$ containing the diagonal $\{(x,x)\}$. Thus a map $Q \longrightarrow \check{P}[n]$ is an element of $H^n(X \times X^{op}, \mathbb{C}_U) = H^n_c(U)$.

Thus we have related the duality theorem to a cohomology class with compact support of the smallest nbd. of the diagonal in $X \times X^{op}$. (Note also that the diagonal is not closed.)

March 31, 1992

Let X be a finite poset, let $|X|$ denote the simplicial complex whose vertices are the points in X, and whose simplices are the non-empty totally ordered subsets of X. Let $f : |X| \longrightarrow X$ be the map sending interior points of a simplex $x_0 < \dots < x_p$ to the largest vertex x_p.

If Z is closed in X, then $f^{-1}Z = |Z|$, because if $x_0 < \dots < x_p$ and $x_p \in Z$, then all $x_i \in Z$.

If S is any subset of X, then S is a poset and $|S|$ can be identified with the full subcomplex of $|X|$ having the vertices S.

If $X = Z \amalg U$ with Z closed, equivalently U open, then $|X| - |Z|$ deforms strongly to $|U|$, and conversely $|X| - |U|$ deforms strongly to $|Z|$. Note that $f^{-1}(U) = f^{-1}(X - Z) = |X| - |Z|$.

Consequences. We know that

$$H^i(X, \mathbb{C}) \xrightarrow{\sim} H^i(|X|, \mathbb{C}).$$

(This results from the standard resolution method for calculating $R^i \varprojlim_X$ and the usual way of calculating the cohomology of a simplicial complex.) Thus we have

$$\begin{tikzcd}
H^i(X, \mathbb{C}_Z) \arrow[r] \arrow[dd, equal] & H^i(|X|, f^*\mathbb{C}_Z) \arrow[d, equal] \\
 & H^i(|X|, \mathbb{C}_{f^{-1}Z}) \arrow[d, equal] \\
H^i(Z, \mathbb{C}) \arrow[r, "\sim"] & H^i(|Z|, \mathbb{C})
\end{tikzcd}$$

It follows that we have for all sheaves F on X

$$\begin{tikzcd}
H^i(X, F) \arrow[r, "\sim"] \arrow[d, equal] & H^i(|X|, f^*F) \arrow[d, equal] \\
\mathrm{Ext}^i_X(\mathbb{C}, F) \arrow[r, "\sim"] & \mathrm{Ext}^i_{|X|}(\mathbb{C}, f^*F)
\end{tikzcd}$$

In effect we have $\forall y \in X$

$$0 \longrightarrow \mathbb{C}_y \longrightarrow \mathbb{C}_{\underbrace{\le y}_{\text{closed}}} \longrightarrow \mathbb{C}_{\underbrace{<y}} \longrightarrow 0$$

Next we have for U open in X

$$\circledast \qquad H^i(f^{-1}(U), \mathbb{C}) = H^i(|U|, \mathbb{C}) = H^i(U, \mathbb{C})$$

i.e.

$$\operatorname{Ext}^i_X(\mathbb{C}_U, \mathbb{C}) \xrightarrow{\sim} \operatorname{Ext}^i_{|X|}(\mathbb{C}_{f^{-1}(U)}, \mathbb{C})$$

It follows that for all sheaves F on X

$$\operatorname{Ext}^i_X(F, \mathbb{C}) \xrightarrow{\sim} \operatorname{Ext}^i_{|X|}(f^*F, \mathbb{C})$$

In effect we use

$$0 \longrightarrow \mathbb{C}_{\underbrace{>y}} \longrightarrow \mathbb{C}_{\underbrace{\ge y}} \longrightarrow \mathbb{C}_y \longrightarrow 0 \ ,$$

$$\leftarrow \text{open} \rightarrow$$

Next let Y, Z be closed subsets of X, let $W = X - Y$. Then

$$\operatorname{Ext}^i_X(\mathbb{C}_W, \mathbb{C}_Z) = H^i(W, \mathbb{C}_Z)$$
$$= H^i(W, \mathbb{C}_{Z \cap W}) = H^i(Z \cap W, \mathbb{C})$$
$$= H^i(Z - Y \cap Z, \mathbb{C})$$

$$\operatorname{Ext}^i_{|X|}(\mathbb{C}_{\underbrace{f^{-1}W}_{|X|-|Y|}}, \mathbb{C}_{\underbrace{f^{-1}Z}_{|Z|}}) = H^i(|X| - |Y|, \mathbb{C}_{|Z|})$$
$$= H^i((|X| - |Y|) \cap |Z|, \mathbb{C})$$
$$= H^i(|Z| - |Y| \cap |Z|, \mathbb{C})$$
$$= H^i(|Z| - |Y \cap Z|, \mathbb{C})$$

But these two agree ~~by~~ applying $\circledast$ ~~to~~ Z and the open set $Z - Y \cap Z$. Thus we conclude

$$\boxed{\operatorname{Ext}^i_X(F, G) \xrightarrow{\sim} \operatorname{Ext}^i_{|X|}(f^*F, f^*G)}$$

April 1, 1992

Review. X finite poset, $|X|$ associated complex, $f: |X| \longrightarrow X$ the largest vertex map. If S is a subset of X, then $|S|$ cbiw the full subcomplex of $|X|$ with vertices in S. Then

a) Z closed $\subset X \implies f^{-1}(Z) = |Z|$ and $f^{-1}(X-Z)$ deforms to $|X-Z|$.

b) If Z', Z are closed in X and $Z' \subset Z$, then $f^{-1}(Z-Z') = |Z| - |Z'|$ deforms to $|Z-Z'|$.

Now if $i: Z \hookrightarrow X$, $j: U \hookrightarrow X$ are inclusions of a closed and open subset respectively of a topological space X we have

$$\mathbb{C}_U = j_! \mathbb{C} \qquad \mathbb{C}_Z = i_* \mathbb{C}$$

and

$$\mathrm{Ext}^i_X(j_! F, G) = \mathrm{Ext}^i_U(F, j^* G)$$

$$\mathrm{Ext}^i_X(F, i_* G) = \mathrm{Ext}^i_Z(i^* F, G)$$

Thus

$$\mathrm{Ext}^i_X(\mathbb{C}_U, \mathbb{C}_Z) = \mathrm{Ext}^i_U(\mathbb{C}, j^* \mathbb{C}_Z)$$

$$= \mathrm{Ext}^i_U(\mathbb{C}, \mathbb{C}_{j^{-1}Z}) = H^i(U \cap Z, \mathbb{C})$$

Now apply this in the case of $X, |X|$ as above. We have

$$\mathrm{Ext}^i_X(\mathbb{C}_U, \mathbb{C}_Z) = H^i(U \cap Z, \mathbb{C})$$

$$\mathrm{Ext}^i_{|X|}(\mathbb{C}_{f^{-1}U}, \mathbb{C}_{f^{-1}Z}) = H^i(f^{-1}(U \cap Z), \mathbb{C})$$

But $U \cap Z = Z - Z'$ where $Z' = (X-U) \cap Z$, so by b) above $f^{-1}(U \cap Z)$ deforms to $|U \cap Z|$.

Now use ⊛ $H^i(X) \xrightarrow{\sim} H^i(|X|)$ for any poset and we conclude

$$\mathrm{Ext}^i_X(\mathbb{C}_U, \mathbb{C}_Z) \xrightarrow{\sim} \mathrm{Ext}^i_{|X|}(\mathbb{C}_{f^{-1}U}, \mathbb{C}_{f^{-1}Z})$$

from which it follows that

$$\mathrm{Ext}^i_X(F, G) \xrightarrow{\sim} \mathrm{Ext}^i_{|X|}(f^*F, f^*G)$$

for any sheaves F, G on X.

Why is ⊛ above true? More generally one proves

$$H^i(X, F) \xrightarrow{\sim} H^i(|X|, f^*F)$$

by using devissage on F; this means induction on the dimension and number of irreducible components of the support. This reduces one to the case $F = \mathbb{C}_{\leq y}$.

Analogy to keep in mind. There's a similarity between a triangulation of a manifold and a Morse function. Ultimately we have to understand stratified spaces. Can you find a combinatorial model for K theory based on the spectral deformation of the unitary group and Grassmannian that you encountered in superconnections.

April 2, 1992

Return to $H^n(X \times X^{op}, \mathbb{C}_{\mathcal{U}_\Delta})$. [redacted]

[redacted] Here $\mathcal{U}_\Delta = \{(x,y) \in X \times X^{op} \mid x \geq y\}$ is the smallest open set containing the diagonal. Suppose X such that all maximal chains have length n. We want to apply the filtration $F = F_{\geq 0} \supset F_{\geq 1} \supset \cdots$ to $\mathbb{C}_{\mathcal{U}_\Delta}$. Observe the dimension of (x,y) in $X \times X^{op}$ is

[redacted] $$|(x,y)| = |x| + n - |y|$$

so that $|(x,y)| \geq n$ for points of $\mathcal{U}_\Delta$ with equality if $x=y$. Also we have

$$H^\bullet(\leq(x,y), <(x,y)) = H^\bullet((\leq x)\times(\geq y), (\leq x)\times(>y) \cup (<x)\times(\geq y))$$

$$= H^\bullet(\leq x, <x) \otimes H^\bullet(\geq y, >y)$$

so if X satisfies the CM conditions

$$H^i(\leq x, <x) = 0 \qquad i \neq |x|$$

$$H^i(\geq y, >y) = 0 \qquad i \neq n-|y|$$

then so does $X \times X^{op}$. Assume these conditions hold. [redacted]

In this case we can calculate $H^\bullet(X \times X^{op}, \mathbb{C}_{\mathcal{U}_\Delta})$ via the complex

$$\cdots \longrightarrow H^j(F_{\geq j}/F_{\geq j+1}) \xrightarrow{\delta} H^{j+1}(F_{\geq j+1}/F_{\geq j+2}) \longrightarrow \cdots$$

with $F = \mathbb{C}_{\mathcal{U}_\Delta}$. In degree [redacted] j this is

$$\bigoplus_{\substack{x \geq y \\ |x|-|y|+n=j}} H^{|x|}(\leq x, x) \otimes H^{n-|y|}(\geq y, >y)$$

so we find that $H^n(X\times X^{op}, \mathcal{O}_{\mathcal{U}_\Delta})$ is the kernel of

$$\bigoplus_{x\in X} H^{|x|}(\le x, <x)\otimes H^{n-|x|}(\ge x, >x) \xrightarrow{\ \delta\ } \bigoplus_{\substack{x>y\\ |x|=|y|+1}} H^{|x|}(\le x, <x)\otimes H^{n-|y|}(\ge y, >y)$$

What can we say about δ? Given (x,y) with $x>y$ and $|x|=|y|+1$ (say x covers y), there are two elements of $\mathcal{U}_\Delta$ below (x,y), namely (x,x) and (y,y), so given $\nu=\sum \nu_x \in \bigoplus_{x\in X} H^{|x|}(\le x,<x)\otimes H^{n-|x|}(\ge x,>x)$, the component of $\delta\nu$ at (x,y) should be the image under the arrows

$$\begin{tikzcd}
H^{|x|}(\le x,<x)\otimes H^{n-|x|}(\ge x,>x) \arrow[dr, "1\otimes\delta\ \text{(maybe with sign)}"] & \\
& H^{|x|}(\le x,<x)\otimes H^{n-|y|}(\ge y,>y) \\
H^{|y|}(\le y,<y)\otimes H^{n-|y|}(\ge y,>y) \arrow[ur, "\delta\otimes 1"'] &
\end{tikzcd}$$

So ν gives an elt. of $H^n(X\times X, \mathcal{O}_{\mathcal{U}_\Delta})$ iff $\boxed{(1\otimes\delta)\nu_x=(\delta\otimes 1)\nu_y}$ for each pair (x,y) with x covering y.

Now recall for a sheaf F on X we have complexes computing $H^i(X,F)$ and $H_{n-i}(X,F)$:

$$H^0(F_{\ge 0}/F_{\ge 1}) \xrightarrow{\delta} H^1(F_{\ge 1}/F_{\ge 2}) \xrightarrow{\delta}$$

$$H_n(F_{\ge 0}/F_{\ge 1}) \xrightarrow{\partial} H_{n-1}(F_{\ge 1}/F_{\ge 2}) \xrightarrow{\partial}$$

i.e. $$\bigoplus_{x\in X_0} H^0(\le x,<x)\otimes F(x) \longrightarrow \bigoplus_{x\in X_1} H^1(\le x,<x)\otimes F(x) \longrightarrow \cdots$$

$$\bigoplus_{x\in X_0} H_n(\ge x,>x)\otimes F(x) \longrightarrow \bigoplus_{x\in X_1} H_{n-1}(\ge x,>x)\otimes F(x) \longrightarrow \cdots$$

Now a natural map $H_{n-\bullet}(X,F) \to H^{\bullet}(X,F)$ would be given by a family of maps

$$\mu_x : H_{n-|x|}(\geq x, > x) \longrightarrow H^{|x|}(\leq x, < x)$$

compatible with faces maps in the sense that when x covers y the following commutes:

$$\begin{tikzcd} H^{|y|}(\leq y, < y) \arrow[r, "\delta"] & H^{|x|}(\leq x, < x) \\ H_{n-|y|}(\geq y, > y) \arrow[u, "\mu_y"] \arrow[r, "\partial"] & H_{n-|x|}(\geq x, > x) \arrow[u, "\mu_x"'] \end{tikzcd}$$

$*$

But μ_x [redacted] can be identified with an elt. of $H^{|x|}(\leq x, < x) \otimes H^{n-|x|}(\geq x, > x)$, and it seems fairly certain that this commutativity is the same as the condition $(1 \otimes \delta)\nu_x = (\delta \otimes 1)\nu_y$ encountered before. The reason is that we know an element $\mu \in H^n(X \times X^{op}, \mathbb{C}_{U_\Delta})$ determines (and maybe is equivalent to) a natural map $H_{n-\bullet}(X,F) \to H^{\bullet}(X,F)$.

It should [redacted] be possible to push these ideas through to a proof that, with [redacted] hypotheses [redacted] like

$$H^{\bullet}(\leq x, < x) \cong \mathbb{C}[|x|]$$

⊛ $$H_{\bullet}(\geq x, > x) \cong \mathbb{C}[n-|x|]$$

and appropriate behavior for the face maps δ, ∂ as in [redacted] $*$ above (these should be $\cong$ at least), one has the duality thm.

Note that the face maps are isomorphisms when

(⁎⁎) $$H^{\bullet}(\mathbb{C}_{yx}) = H_{\bullet}(\mathbb{C}_{yx}) = 0$$

for any pair (x,y) with x covering y. This implies [redacted] ⊛ since we can move to an x of degree 0 or n.

One problem is to properly understand how hypotheses on $<x$ and $>x$ yield the key requirements: $H^i(\mathbb{C}_{\not\geq y}) = 0$ $i \neq n$ (or $H_i(\mathbb{C}_{\leq x}) = 0$ for $i \neq n$) of the duality theorem. I attempt to understand this by looking at the easiest cases:

If $|y| = n$, then $\mathbb{C}_{\geq y} = \mathbb{C}_y$ and $H^\bullet(\mathbb{C}_y) =$ $H^\bullet(\leq y, <y) = \mathbb{C}[n]$ by assumption. Next consider $|y| = n-1$. We have

$$0 \longrightarrow \bigoplus_{\substack{x \text{ covers} \\ y}} \mathbb{C}_x \longrightarrow \mathbb{C}_{\geq y} \longrightarrow \mathbb{C}_y \longrightarrow 0$$

giving

$$\begin{tikzcd}
 & & {} \arrow[r] & 0 \arrow[dll] \\
 & 0 \arrow[r] & H^{n-1}(\mathbb{C}_{\geq y}) \arrow[r] & \overset{\cong \mathbb{C}}{H^{n-1}(\leq y, <y)} \arrow[dll] \\
 & \bigoplus_{\substack{x \text{ covers} \\ y}} \underbrace{H^n(\leq x, <x)}_{\cong \mathbb{C}} \arrow[r] & H^n(\mathbb{C}_{\geq y}) \arrow[r] & 0
\end{tikzcd}$$

To get $H^\bullet(\mathbb{C}_{\geq y}) \simeq \mathbb{C}[n]$, we need there to be exactly 2 x's covering y, i.e. $>y$ to be a 0-sphere. We also need at least one of the face maps $H^{n-1}(\leq y, <y) \to H^n(\leq x, <x)$ to be an isomorphism (non zero).

April 4, 1992

On the preceding page we considered the problem of deriving $H^\bullet(\mathbb{C}_{\geq y}) = \mathbb{C}[n]$ from assumptions about $<x$ and $>x$. The dual problem for

$$H_\bullet(\mathbb{C}_{\leq y}) = H_i(|X|, |X| - |\leq y|)$$

was considered earlier, see 3-19-92. The motivating idea was to dedece the duality theorem by applying the Eilenberg-Steenrod uniqueness result to the cohomology theory $F^\bullet(Z) = H_{n-\bullet}(|X|, |X|-|Z|)$. This uses the skeletal filtration of Z and leads to <u>the complex</u>

$$\circledast \qquad \bigoplus_{x \in Z_0} \underbrace{F^0(x)}_{H_n(\geq x, >x)} \longrightarrow \bigoplus_{x \in Z_1} \underbrace{F^1(\leq x, <x)}_{H_{n-1}(\geq x, >x)} \longrightarrow \bigoplus_{x \in Z_2} \underbrace{F^2(\leq x, <x)}_{H_{n-2}(\geq x, >x)} \to$$

<u>for computing</u> $F^\bullet(Z)$. To establish this <u>one only needs the CM conditions</u>

$$H_i(\mathbb{C}_x) = H_i(\geq x, >x) = 0 \qquad i \neq n-|x|$$

From $\circledast$ we see that a "fundamental class" $\mu_X \in F^0(X)$ exists iff it exists over the 1-skeleton $X_{\leq 1}$. Assuming a fundamental class is given, one should then have a map of cohomology "theories"

$$(**) \qquad H^i(Z) \longrightarrow F^i(Z) = H_{n-i}(X, X-Z)$$

(there should be no problem with this if we set things up properly using complexes). The question of whether $(**)$ is an isomorphism should then reduce to whether the induced map

$$\begin{array}{ccc} H^i(\mathbb{C}_x) & \longrightarrow & H_{n-i}(\mathbb{C}_x) \\ \| & & \| \\ H^i(\leq x, <x) & & H_{n-i}(\geq x, >x) \end{array}$$

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is an isomorphism for all x.

The conclusion I draw from this is that for the duality theorem one does not have to assume $H^{|x|}(\leq x, < x)$, $H_{n-|x|}(\geq x, > x)$ are 1-dimensional, only that there is an isomorphism compatible with face maps. Also it would seem likely that when these are 1-dimensional for all x and the face maps are isomorphisms (this is equivalent to $H^{\cdot}(\mathbb{C}_{yx}) \simeq H_{\cdot}(\mathbb{C}_{yx}) = 0$ whenever y is a face of x), then duality holds.

Points to develop further

Sheaves on a finite poset are the same as <u>modules over the incidence algebra</u> A. The constant sheaf $\mathbb{C}$ (and also the constant cosheaf $\mathbb{C}$) are not the same as the quotient $A/\text{rad}(A) \simeq \bigoplus_x \mathbb{C}$. A is a graded algebra, but the constant sheaf and its filtration are not compatible with the grading. To what extent can the duality theory be generalized to algebras.

Dual cell complex idea, relation with Morse theory — stable and unstable submanifolds associated to **critical** points (submanifolds). Perhaps the Morse theory is the proper setting for all this poset stuff. Example of unitary groups + Grassmannians and the flow  on the eigenvalues.

Weil proof of P.D. and construction of diagonal class. In the C^∞ manifold situation I know how to construct (using a tubular nbd of the diagonal) a Thom class for the diagonal $\Delta M \subset M \times M$.

Actually it is a construction of the Thom class of a vector bundle, which proceeds via the double complex

$$C^\bullet(\mathcal{U}, \pi_! \Omega^\bullet_E)$$

$\uparrow$ sections with proper support over M

using a partition of unity subordinate ~~to~~ $\mathcal{U}$ and [redacted] explicit homotopies for $\pi_! \Omega^\bullet_E$ over the finite intersections U_σ. It seems the an explicit Thom form with support in a very small neighborhood of the diagonal gives a homotopy inverse for the map

$$\Omega^\bullet(U) \longrightarrow \Omega_c^{n-\bullet}(U)'$$

' = topological dual.

of complexes [redacted] "up to shrinking the covering". [redacted] The sheave $\Omega^\bullet$ and $(\Omega_c^{n-\bullet})'$ are fine, which means that one has partitions of unity to glue. Is there any connection between these partitions and having projective modules over the incidence algebra? Original idea: F projective (Nakayama). $\Longrightarrow \mathrm{Tor}_+^A(S, F) = 0$, $S = A/\mathrm{rad}(A)$. However S is not the same as the constant functor which gives the homology: $\mathrm{Tor}_i^A(\mathbb{C}, F) = H_i(F)$. What is the relation between flask and injective for sheaves of $\mathbb{C}$-modules over a poset? [redacted] ~~On~~ On a manifold we have $\mathrm{Ext}^i_M(\mathbb{C}_U, \Omega^\bullet) = H^i(U, \Omega^\bullet) = 0$, $i \neq 0$, whereas flask means $\mathrm{Ext}^i_M(\mathbb{C}_Z, \Omega^\bullet) = 0$ for all closed Z. This somehow means one has a projective dimension 1 situation (Singer's thesis.)

Problem: Correlate the diagonal class as constructed for posets and for manifolds (both via a Weil covering and a Morse function.)