September 15, 1991

On mixed complexes.

The problem is to find a suitable notation for dealing with the cyclic theory of a mixed complex. What is this?

Consider a mixed complex (M, b, B) where $M = \bigoplus_{n\in\mathbb{Z}} M_n$, $M_n = 0$ for $n<0$, b has degree -1, B has degree $+1$. We think of it as the chain complex M with differential b, on which one is given an endom. B of degree $+1$ compatible with the differential and having square 0. $\therefore$ b is the primary differential.

Define a quasi-isom of mixed complexes to be a map $M \longrightarrow M'$ compatible with b, B which is a quis of the underlying complexes:

$$H(M,b) \xrightarrow{\sim} H(M',b)$$

The cyclic theory attached to M should consist of all homology constructed from M which is quasi-isomorphism invariant.

Recall the Husemoller - Kassel idea. A mixed complex is a DG module, zero in negative degrees, over the DG algebra $\mathbb{C}[\varepsilon]$ where $\deg(\varepsilon) = 1$ and $d(\varepsilon) = 0$. So one is looking at the derived category of these DG modules in some sense.

It is natural to consider the bar construction of $\mathbb{C}[\varepsilon]$ which is the DG coalgebra $\mathbb{C}[\eta] = \bigoplus_{n \geq 0} \mathbb{C}\eta^n$ where η has degree 2 and

$$d(\eta^n) = 0 \qquad \Delta(\eta^n) = \sum_{i+j=n} \eta^i \otimes \eta^j$$

A DG comodule X over $\mathbb{C}[\eta]$ can be identified with a complex X having an endomorphism S of degree -2. Lets consider right comodules; the coproduct is a map

$$X \xrightarrow{\Delta} X \otimes \mathbb{C}[\eta]$$

and because $\mathbb{C}[\eta]$ is a tensor coalgebra the coproduct is equivalent to a linear map

$$X \longrightarrow X \otimes \mathbb{C}\eta \simeq \Sigma^2 X$$

which is just S.

Now the point is that the derived category of left DG modules over an augmented algebra $A = \mathbb{C} \oplus \bar{A}$ (zero in negative degrees) is equivalent to the derived category of right DG comodules over $B(A)$. This equivalence is given by the acyclic complex

$$B(A) \otimes_\tau A$$

where τ is the canonical twisting cochain. In our example this is

$$\mathbb{C}[\eta] \otimes \mathbb{C}[\varepsilon]$$

[redacted] The differential is

$$C \otimes A \xrightarrow{\Delta \otimes 1} C \otimes C \otimes A \xrightarrow{1 \otimes \tau \otimes 1} C \otimes A \otimes A \xrightarrow{1 \otimes m} C \otimes A$$

in general, where $\tau : C \to A$ is a ~~twisting~~ cochain. In our case

$$\tau(\eta^n) = \begin{cases} 0 & n \neq 1 \\ \varepsilon & n = 1. \end{cases}$$

So we find

$$d(\eta^n \otimes 1) = (1 \otimes m)(1 \otimes \tau \otimes 1)\Big(\sum_{i+j=n} \eta^i \otimes \eta^j \otimes 1\Big)$$

$$= (1 \otimes m)(\eta^{n-1} \otimes \varepsilon \otimes 1) = \eta^{n-1} \otimes \varepsilon$$

Given a mixed complex M the corresponding complex with S is

$$\mathbb{C}[\eta] \otimes M = \bigoplus_{p \geq 0} \eta^p M$$

with differential

$$d(\eta^p m) = \eta^p \overset{b}{d_M} m + \eta^{p-1} B m$$

Thus the differential is $\boxed{d = b + SB}$

On the other hand given X with S, the corresponding mixed complex is

$$X \otimes_{\mathbb{C}[\eta]} \mathbb{C}[\eta] \otimes \mathbb{C}[\varepsilon] \simeq X \otimes \mathbb{C}[\varepsilon]$$

where B is multiplication by ε. What is b? We have

$$X \otimes \mathbb{C}[\varepsilon] \xhookrightarrow{\Delta} X \otimes \mathbb{C}[\eta] \otimes \mathbb{C}[\varepsilon]$$

where the differential on the latter is $d_X \otimes 1 + 1 \otimes d_{\mathbb{C}[\eta] \otimes \mathbb{C}[\varepsilon]}$.

We use

$$
\begin{tikzcd}
X \otimes \mathbb{C}[\varepsilon] \arrow[r, "\Delta\otimes 1"] \arrow[d, "b"] & X \otimes \mathbb{C}[\eta] \otimes \mathbb{C}[\varepsilon] \arrow[d, "\text{diff'l } d"] \\
X \otimes \mathbb{C}[\varepsilon] & X \otimes \mathbb{C}[\eta] \otimes \mathbb{C}[\varepsilon] \arrow[l, "\text{kills}"', "\eta^n \ n>0"']
\end{tikzcd}
$$

$$\Delta x = x \otimes 1 + Sx \otimes \eta + \dots$$

[redacted]

$$x \otimes 1 \xmapsto{\Delta\otimes 1} (x \otimes 1 + Sx \otimes \eta + \dots) \otimes 1$$

$$\xmapsto{d} dx \otimes 1 \otimes 1 + \underset{\text{dies}}{dSx \otimes \eta \otimes 1} + Sx \otimes \underbrace{d\eta}_{\varepsilon} \otimes 1$$

$$\xmapsto{\text{projection back}} dx \otimes 1 + Sx \otimes \varepsilon$$

So we get $\boxed{b(x \otimes 1) = dx \otimes 1 + Sx \otimes \varepsilon}$.

In other words, we get the complex

$$
\begin{tikzcd}
X_1 \varepsilon \arrow[d] & X_3 \arrow[l] \arrow[d] \\
X_0 \varepsilon \arrow[d] & X_2 \arrow[l, "S"'] \arrow[d] \\
0 \arrow[d] & X_1 \arrow[l] \arrow[d] \\
0 & X_0 \arrow[l]
\end{tikzcd}
$$

[redacted]

Nice cases: If B is exact on M, then we have a quis

$$\mathbb{C}[\eta]\otimes M \longrightarrow M/BM$$

in pictures:

$$\begin{tikzcd}
\vphantom{M} & \vphantom{M} & \vphantom{M} & \vphantom{M} \\
M_2/BM_1 \arrow[u] & M_2 \arrow[l] \arrow[u] & M_1 \arrow[l, "B"'] \arrow[u] & M_0 \arrow[l] \arrow[u] \\
M_1/BM \arrow[u] & M_1 \arrow[l] \arrow[u] & M_0 \arrow[l, "B"'] \arrow[u] & \\
M_0 \arrow[u] & M_0 \arrow[l, dashed] \arrow[u] & &
\end{tikzcd}$$

So up to homotopy we have a well-defined endomorphism S on M/BM of degree -2.

If S is surjective on X, then we have a quis

$$\mathrm{Ker}(S) \longrightarrow X \otimes \mathbb{C}[\varepsilon]$$

whence a B map on $\mathrm{Ker}(S)$ defined up to homotopy.

<u>September 29, 1991</u>

On HC^-. The problem is to understand HC^- from the $\mathbb{Z}/2$-graded complex viewpoint. Recall the bicomplex notation

$$(\mathcal{C}M)_n = \prod_{p\geq 0} M_{n-2p}$$

$$(\hat{\mathcal{C}}^{per}M)_n = \prod_{p\in\mathbb{Z}} M_{n-2p} \qquad M \text{ a mixed complex}$$

$$(\hat{\mathcal{C}}^{-}M)_n = \prod_{p\leq 0} M_{n-2p}$$

These are all complexes with diffl $b+B$, and we have exact sequences

$$0 \longrightarrow \hat{\mathcal{C}}^- M \longrightarrow \hat{\mathcal{C}}^{per}(M) \longrightarrow \Sigma^2 \mathcal{C}M \longrightarrow 0$$

$$0 \longrightarrow \underbrace{\Sigma^{-2}\hat{\mathcal{C}}^- M}_{S\hat{\mathcal{C}}^- M} \longrightarrow \hat{\mathcal{C}}^- M \longrightarrow (M, b) \longrightarrow 0$$

The second gives a Connes exact sequence

$$\longrightarrow H^{c-}_{n+2}M \longrightarrow H^{c-}_n M \longrightarrow H^b_n M \longrightarrow H^{c-}_{n+1} M \longrightarrow$$

and the first gives an exact sequence

$$\hookrightarrow H^{c-}_{n+2} M \longrightarrow H^P_{n+2\mathbb{Z}} M \longrightarrow H^c_n M \hookrightarrow$$

$$\hookrightarrow H^{c-}_{n+1} M \longrightarrow H^P_{n+1+2\mathbb{Z}} M \longrightarrow H^c_{n-1} M \hookrightarrow$$

$$\hookrightarrow H^{c-}_n M$$

I feel that $\{H^{c-}_n M\}$ should be calculable from $\{H_i(F^n\hat{M})\}$ $i \in \mathbb{Z}/2$. We have an exact sequence

$$0 \to F^{n+1}\hat{M} \to F^n\hat{M} \to F^nM/F^{n+1}\hat{M} \to 0$$

which gives a 6 term exact sequence

$$\begin{tikzcd}
H^{c-}_{n+2}M \arrow[r, dash] & H_{n+2\mathbb{Z}}(F^{n+1}\hat{M}) \arrow[r] & \overset{H^{d-}_n}{H_{n+2\mathbb{Z}}(F^n\hat{M})} \arrow[r] & 0 \arrow[d] & \\
& H^b_{n+1}M \arrow[u] & \overset{H^{c-}_{n+1}M}{H_{n+1+2\mathbb{Z}}(F^n\hat{M})} \arrow[l] & H_{n+1+2\mathbb{Z}}(F^{n+1}\hat{M}) \arrow[l] & H^{d-}_{n+1}
\end{tikzcd}$$

One can therefore hope that

$$* \quad \boxed{\begin{aligned} H_{n+1+2\mathbb{Z}}(F^n\hat{M}) &= H^{c-}_{n+1}M \\ H_{n+2\mathbb{Z}}(F^n\hat{M}) &= H^{d-}_{n}M \end{aligned}}$$

(Observe - to get get H^{c-}_n you take $H_{n+2\mathbb{Z}}(F^{n-1}\hat{M})$. To keep things straight, think of HC^- as calculated from the $p \le 0$ part of the bicomplex. Thus there is an obvious canonical map

$$HC^-_n \to HH_n$$

Proof of $*$. We have ~~Recall that~~ $H^{c-}_n M =$ $Z_n(\hat{C}^-M)/B_n(\hat{C}^-M)$ where

$$Z_n(\hat{C}^-M) = \{(x_n, x_{n+2}, \dots) \mid bx_n = Bx_n + bx_{n-2} = \dots = 0\}$$

$$B_n(\hat{C}^-M) = \{(by_{n+1}, By_{n+1} + by_{n+3}, \dots)\}$$

$$(F^n\hat{M})_{n+2\mathbb{Z}} = \{(x_n, x_{n+2}, \cdots) \mid x_n \in bM_{n+1}\}$$

$$(F^n\hat{M})_{n+1+2\mathbb{Z}} = \{(y_{n+1}, y_{n+3}, \cdots)\}$$

$$Z_{n+1+2\mathbb{Z}}(F^n\hat{M}) = \{(y_{n+1}, y_{n+3}, \cdots) \mid by_{n+1} = By_{n+1} + by_{n+3} = \cdots = 0\}$$

$$= Z_{n+1}(\hat{C}^- M)$$

$$B_{n+1+2\mathbb{Z}}(F^n\hat{M}) = \{(Bx_n + bx_{n+2}, \cdots) \mid x_n \in bM_{n+1}\}$$

$$= \{(\underbrace{Bby_{n+1} + bx_{n+2}}_{b(-By_{n+1}+x_{n+2})}, \underbrace{Bx_{n+2}}_{B(-By_{n+1}+x_{n+2})} + bx_{n+4}, \cdots)\}$$

$= B_{n+1}(\hat{C}^- M)$. Thus we have $\boxed{H_{n+1+2\mathbb{Z}}(F^n\hat{M}) = H^{c^-}_{n+1} M}$.

Next

$$Z_{n+2\mathbb{Z}}(F^n\hat{M}) = \left\{(x_n, x_{n+2}, \cdots) \;\middle|\; \begin{array}{l} x_n \in bM_{n+1} \\ Bx_n + bx_{n+2} = Bx_{n+2} + bx_{n+4} = \cdots = 0 \end{array}\right\}$$

$$= \mathrm{Ker}\left\{ \begin{array}{ccc} Z_n(\hat{C}^- M) & \longrightarrow & H^b_n M \\ (x_n, x_{n+2}, \cdots) & \longmapsto & [x_n] \end{array} \right\}$$

$$\ni bx_n = Bx_n + bx_{n+2} = \cdots = 0$$

$$B_{n+2\mathbb{Z}}(F^n\hat{M}) = \{(by_{n+1}, By_{n+1} + by_{n+3}, \cdots)\} = B_n(\hat{C}^- M)$$

Thus $\boxed{H_{n+2\mathbb{Z}}(F^n\hat{M}) = \mathrm{Ker}\{H^{c^-}_n M \longrightarrow H^b_n M\}}$

October 3, 1991

Differential operators.

Let's recall the program for universal enveloping algebras and algebras of differential operators.

Generalized enveloping algebras can be described the following ways.

1) An algebra A with increasing filtration $F_n A$ such that $0 = F_{-1}A \subset \cdots \; \bigcup F_n A = A$, $F_pA \cdot F_qA \subset F_{p+q}A$ and such that $\mathrm{gr}\, A$ is the symmetric algebra on $\mathrm{gr}_1 A$.

2) An $\overset{\text{central}}{\text{extension}}$ of Lie algebras

$$(*) \qquad 0 \longrightarrow \mathbb{C} \longrightarrow \tilde{\mathfrak{g}} \longrightarrow \mathfrak{g} \longrightarrow 0$$

3) An affine space equipped with a Poisson manifold structure such that the Poisson bracket of (affine) linear functions is a linear function. (Technically this description requires $\mathfrak{g} = F_1A/F_0A$ to be finite dimensional.)

The affine space in 3) is the splittings of $(*)$ as vector spaces. Then $\tilde{\mathfrak{g}} = F_1A$ appears as the linear (degree ≤ 1 might be better terminology) functions [redacted] the affine space. The differential forms with polynomial coefficients on this affine space:

$$\underbrace{S_{\Gamma}(\tilde{\mathfrak{g}}) \qquad \left(S(\tilde{\mathfrak{g}})/1_S = 1_{\tilde{\mathfrak{g}}}\right)} \otimes \Lambda \mathfrak{g}$$

form a mixed complex with d as usual and b from the Poisson structure. The basic result to understand properly is [redacted] the fact that the cyclic theory of A is given by this mixed complex.

One presumably starts with the basic A-bimodule resolution of A:

$$\longrightarrow A\otimes\Lambda^2\mathfrak{g}\otimes A \longrightarrow A\otimes\mathfrak{g}\otimes A \longrightarrow A\otimes A \longrightarrow A \longrightarrow 0$$

which is a Koszul type resolution, except there are some subtleties such as the bracket in $\mathfrak{g}$ and the fact that only $\tilde{\mathfrak{g}}$ acts by left + right multiplication on A.

Applying $\otimes_A$ gives the complex

$$\longrightarrow A\otimes\Lambda^2\mathfrak{g} \longrightarrow A\otimes\mathfrak{g} \longrightarrow A$$

computing the Lie algebra homology $H.(\mathfrak{g}, A)$. We have to identify this (I think) with $(S_n(\tilde{\mathfrak{g}})\otimes\Lambda\mathfrak{g}, b)$, which means we bring in a PBW type linear isomorphism $S_n(\tilde{\mathfrak{g}}) \cong A$.

After we understand the Hochschild homology, we still have to worry about the cyclic theory, i.e. how to link B to the d on the above de Rham complex.

Let's consider differential operators. $\mathcal{D}$ = differential operators on M, $\mathcal{O}$ = functions, $\mathcal{T} = \Gamma(M, TM)$ = vector fields. Consider the increasing filtration $F^n\mathcal{D}$ = operators of order $\leq n$. Exact sequence of Lie algebras

$$0 \longrightarrow \mathcal{O} \longrightarrow F_1\mathcal{D} \longrightarrow \mathcal{T} \longrightarrow 0$$

which splits canonically since $\mathcal{T} \subset F_1\mathcal{O}$ as the subspace of operators which kill $1 \in \mathcal{O}$.

However, these Lie algebras have $\mathcal{O}$ bimodule structures to be taken into account, so it seems not a good idea to think of $F_1\mathcal{D}$ as $\mathcal{O} \oplus \mathcal{T}$. Anyway the goal (I think) is to show that the cyclic theory of $\mathcal{D}$ is given by the mixed complex of polynomial coefficient differential forms on T^*M where the b operator comes from the symplectic structure on T^*M.

Presumably one starts with calculating the Hochschild homology. It seems that we have a bimodule resolution

$$\to \mathcal{D} \otimes_{\mathcal{O}} \Lambda^2\mathcal{T} \otimes_{\mathcal{O}} \mathcal{D} \longrightarrow \mathcal{D} \otimes_{\mathcal{O}} \mathcal{T} \otimes_{\mathcal{O}} \mathcal{D} \longrightarrow \mathcal{D} \otimes_{\mathcal{O}} \mathcal{D} \longrightarrow \mathcal{D} \longrightarrow 0$$

of Koszul type. Given $v \in \mathcal{T}$ we assign to v the operator $x \otimes y \longmapsto xv \otimes y - x \otimes vy$ on $\mathcal{D} \otimes_{\mathcal{O}} \mathcal{D}$.

Check well-defined

$$xf \otimes y \longmapsto \overbrace{xfv}^{x[f,v] + xvf} \otimes y - xf \otimes vy$$

$$x \otimes fy \longmapsto xv \otimes fy - x \otimes \underbrace{vfy}_{[v,f]y + fvy}$$

$$\begin{matrix} xf \otimes y \\ -x \otimes fy \end{matrix} \longmapsto \begin{matrix} x[f,v] \otimes y + \cancel{xvf \otimes y} - \cancel{xf \otimes vy} \\ - \cancel{xv \otimes fy} + x \otimes [v,f]y - \cancel{x \otimes fvy} \end{matrix} = 0$$

since $[f,v] \in \mathcal{O}$.

A better proof might be to consider the map

$$\Phi : \mathcal{D} \otimes F_1\mathcal{D} \otimes \mathcal{D} \longrightarrow \mathcal{D} \otimes_{\mathcal{O}} \mathcal{D}$$

$$x \otimes \tilde{v} \otimes y \longmapsto x\tilde{v} \otimes y - x \otimes \tilde{v} y$$

It's clear that $\Phi(x, \tilde{v}, y) = 0$ if $\tilde{v} \in \mathcal{O}$, so $\Phi(x, v, y)$ is defined for $v \in F_1\mathcal{D}/\mathcal{O} \cong \mathcal{T}$. Now

$$\Phi(xf, v, y) = xf\tilde{v} \otimes y - xf \otimes \tilde{v} y$$

$$= \Phi(x, fv, y)$$

and

$$\Phi(x, v, fy) = x\tilde{v} \otimes fy - x \otimes \tilde{v} f y$$

$$= \Phi(x, vf, y)$$

But $fv = vf$ in $F_1\mathcal{D}/\mathcal{O} = \mathcal{T}$. Thus Φ induces a well-defined map

$$\mathcal{D} \otimes_{\mathcal{O}} \mathcal{T} \otimes_{\mathcal{O}} \mathcal{D} \longrightarrow \mathcal{D} \otimes_{\mathcal{O}} \mathcal{D}$$

Let's assume we really do have our resolution

$$\to \mathcal{D} \otimes_{\mathcal{O}} \Lambda^2\mathcal{T} \otimes_{\mathcal{O}} \mathcal{D} \longrightarrow \mathcal{D} \otimes_{\mathcal{O}} \mathcal{T} \otimes_{\mathcal{O}} \mathcal{D} \longrightarrow \mathcal{D} \otimes_{\mathcal{O}} \mathcal{D} \longrightarrow \mathcal{D} \longrightarrow 0$$

Then we get a spectral sequence

$$E^1_{pq} = \mathrm{Tor}_q^{\mathcal{D}\otimes\mathcal{D}}\left(\mathcal{D}, \mathcal{D} \otimes_{\mathcal{O}} \Lambda^p\mathcal{T} \otimes_{\mathcal{O}} \mathcal{D}\right) \Longrightarrow HH_{p+q}(\mathcal{D}).$$

$$\parallel$$

$$\mathrm{Tor}_q^{\mathcal{O}\otimes\mathcal{O}}\left(\mathcal{D}, \Lambda^p\mathcal{T}\right)$$

$$\parallel$$

$$\mathrm{Tor}_q^{\mathcal{O}\otimes\mathcal{O}}(\mathcal{D}, \mathcal{O}) \otimes_{\mathcal{O}} \Lambda^p\mathcal{T}$$

Thus we seem to get

$$E^1_{pq} = H_q(\mathcal{O}, \mathcal{D}) \otimes_{\mathcal{O}} \Lambda^p_{\mathcal{O}} \mathcal{T}$$

Now it seems that

$$\circledast \qquad H_q(\mathcal{O}, \mathcal{D}) = \begin{cases} 0 & q \neq n \\ \Lambda^n \mathcal{T}^* = \Omega^n & q = n \end{cases}$$

Example. $\mathcal{D} = \mathbb{C}[x, p]$, $[x, p] = 1$, $\mathcal{O} = \mathbb{C}[x]$.

Then $\mathcal{D} = \bigoplus \mathbb{C}[x]\, p^n$

$$[\mathcal{O}, \mathcal{D}] = \bigoplus \mathbb{C}[x]\, [x, p^n] = \bigoplus \mathbb{C}[x]\, p^n = \mathcal{D}.$$

and in degree 1 you get a kernel $\simeq \mathcal{O}$.
Assume $\circledast$ holds. Then our spectral lives on the row $q = n$ and

$$E^1_{pn} = \Lambda^n \mathcal{T}^* \otimes_{\mathcal{O}} \Lambda^p \mathcal{T} = \Lambda^{n-p}_{\mathcal{O}} \mathcal{T}^* = \Omega^{n-p}$$

Let's guess that $d^1 : E^1_{pn} \longrightarrow E^1_{p-1, n}$ is the de Rham $d : \Omega^{n-p} \longrightarrow \Omega^{n-p+1}$. If so then we have

$$HH_k(\mathcal{D}) = E^2_{k-n, n} = H^{2n-k}_{DR}(M)$$

The answer for the Hochschild homology is supposed to be the b-homology of $\Omega_{\substack{\text{poly} \\ \text{coeff}}}(T^*M)$. Because T^*M is symplectic one has a symplectic $*$ operator interchanging b, d. Thus the b homology in degree k is isomorphic to the d homology in the complementary degree $2n-k$.

<u>October 10, 1991</u>

Suppose A smooth commutative. We are concerned with constructing an A-bimodule resolution of A which is minimal. Such a resolution is probably not unique but becomes unique if we complete along the diagonal. Thus we look at

$$(A\otimes A)^\wedge = \varprojlim \Big(\underbrace{A\otimes A/I_\Delta^{n+1}}_{J_n}\Big)$$

Now this coincides with the infinite jets J_∞. Recall that we have a canonical exact sequence

$$1)\quad 0 \longrightarrow A \xrightarrow{j_\infty} J_\infty \longrightarrow \Omega^1_A \otimes_A J_\infty \longrightarrow \Omega^2_A\otimes_A J_\infty \longrightarrow \cdots$$

(Spencer sequence). We can understand this Spencer sequence as follows. Recall one has an exact sequence

$$2)\quad 0 \longrightarrow A \xrightarrow{a\,\mapsto\, 1\otimes a} A\otimes A \xrightarrow{d\otimes 1} \Omega^1 A\otimes A \xrightarrow{d\otimes 1} \Omega^2 A\otimes A \longrightarrow$$

where the exactness is clear since $0\to\mathbb{C}\to A \xrightarrow{d} \Omega^1 A \to \cdots$ is exact; also we have b' such that $[b', d\otimes 1] = 1$. Let's use the map

$$\Omega^n A\otimes A \longrightarrow \Omega^n A\otimes_A (A\otimes A) \longrightarrow \Omega^n_A \otimes_A J_\infty$$

Then 1) should be a quotient of 2). Note that $?\otimes_A J_\infty$ should be $\varprojlim_n\, ?\otimes_A J_n$ in the above.

Now

1) $$J_\infty \xrightarrow{d} \Omega^1_A \otimes_A J_\infty \xrightarrow{d} \Omega^2_A \otimes_A J_\infty$$

should be the DR complex associated to the local system given by J_∞ with a certain left connection. Let's work this out in local coordinates using the fact that $j_\infty : A \to J_\infty$ gives the flat sections. This is the map

$$\begin{array}{ccccc} A & \longrightarrow & A \otimes A & \longrightarrow & J_\infty \\ a & \longmapsto & 1 \otimes a & & \end{array}$$

Local coordinates $x = (x^i)$ so that roughly $A = \mathbb{C}[x]$ and $J_\infty = \mathbb{C}[x][[\delta x]]$ and

$$\begin{array}{ccc} A \otimes A & \longleftrightarrow & J_\infty \\ x \otimes 1 & \longmapsto & x \\ 1 \otimes x & \longmapsto & x + \delta x \end{array}$$

Thus $j_\infty(x) = x + \delta x$, so

$$j_\infty(f(x)) = f(x+\delta x) = \sum \frac{1}{n!} f^{(n)}(x)\, \delta x^n$$

(n multiindex in general).

~~[illegible]~~ Now $J_\infty \xrightarrow{d} \Omega^1_A \otimes_A J_\infty$ should be of the former $d = dx^i \partial_i$ and it should kill $j_\infty(A)$. Thus

$$\partial_i f(x, \delta x) = 0 \qquad \text{if} \qquad f(x, \delta x) = f(x + \delta x)$$

This gives

$$\partial_i f(x, \delta x) = \left(\partial_{x^i} - \partial_{\delta x^i}\right) f(x, \delta x).$$

and so

$$d = dx^i \left(\partial_{x^i} - \partial_{\delta x^i}\right) : J_\infty \longrightarrow \Omega^1_A \otimes_A J_\infty .$$

Next consider the problem of finding a b' operator on $\Omega_A \otimes A$. We note that $\Omega_A \otimes A = \bigwedge_{A\otimes A} \left(\Omega^1_A \otimes A\right)$ is an exterior algebra, hence we get an interior product operator $\iota(\xi)$ on $\Omega_A \otimes A$ associated to any $A\otimes A$ - module map

$$\Omega^1_A \otimes A \xrightarrow{\ \xi\ } A \otimes A .$$

Now ξ is equivalent to a left A-module map

$$\Omega^1_A \xrightarrow{\ \xi_0\ } A \otimes A .$$

In order to obtain an $A\otimes A$ resolution of A from the Koszul complex, we want the image of ξ to be $\Omega^1 A = \mathrm{Ker}\{A\otimes A \longrightarrow A\}$. In fact we really want ξ to come from a lifting:

$$\begin{tikzcd} & \Omega^1 A \arrow[d] \\ \Omega^1_A \arrow[ur, "\xi_0"] \arrow[r, equal] & \Omega^1_A \end{tikzcd}$$

so that we get the correct identification of Ω^1_A with $\Omega^1 4 \otimes_A$.

Let's analyze exactly what ξ is. First an $A\otimes A$-module map

$$\Omega^1_A \otimes A \longrightarrow A \otimes A$$

by duality corresponds to a section of

$$\mathcal{T} \otimes A = pr_1^* \mathcal{T}$$

where $\mathcal{T}$ is the tangent bundle on $M = \mathrm{Var}(A)$. Thus we have a vector field on $M \times M$ which is horizontal, or if we want a field over $M \times M$ which gives at each point (x,y) a tangent vector to M at x. The properties of this field are that it vanishes on the diagonal and that for y approaching x it agrees up to first order with $y \longmapsto y - x$. What this means is that the map $(x,y) \longmapsto g(x,y) \in T_x M$ should vanish on the diagonal and the 1-jet along the diagonal, which is a map from TM to TM, should be the identity.

The typical picture for this map is

$$g(x,y) = \exp_x^{-1}(y)$$

where $\exp_x : T_x M \longrightarrow M$ is the exponential map at x associated to a connection on M.

Here's another viewpoint

Let's look at the left A-module map.

$$\xi_0 : \Omega^1_A \longrightarrow A \otimes A$$

This associates to a cotangent vector $\xi \in T^*_x M$ a functor $g(x,\xi)(y)$ on M, which should vanish [redacted] when $y = x$ and [redacted] whose 1-jet at $y = x$ is the linear function on T_x given by ξ. This is therefore a way of assigning to a cotangent vector ξ at x an actual [redacted] function near x extending ξ [redacted] considered as an infinitesimal function.

But we have some new ideas about how to do these tubular neighborhood things. In particular we want to make use of the multiplicative group.

Let's recall the tubular nbd. business. Suppose Y is a submanifold of M, $I \subset \Gamma\mathcal{O}_M$ the ideal of functions vanishing on Y. One has exact sequences

$$0 \longrightarrow I \longrightarrow \underbrace{\Gamma\mathcal{O}_M}_{A} \longrightarrow \underbrace{\Gamma\mathcal{O}_Y}_{B} \longrightarrow 0$$

$$0 \longrightarrow I/I^2 \longrightarrow \Omega^1_A \otimes_A B \longrightarrow \Omega^1_B \longrightarrow 0$$

One chooses a splitting of the bottom sequence. This gives an A-module map $\Omega^1_A \longrightarrow I/I^2$

which, since Ω^1_A is projective, can be lifted to a A-module map $\Omega^1_A \to I$. This gives a derivation $A \xrightarrow{D} I \subset A$ which we know leads to an isomorphism

$$\hat{A} = \prod_{n\geq 0} I^n/I^{n+1} = \prod_{n \geq 0} \mathrm{Sym}^n_B (I/I^2)$$

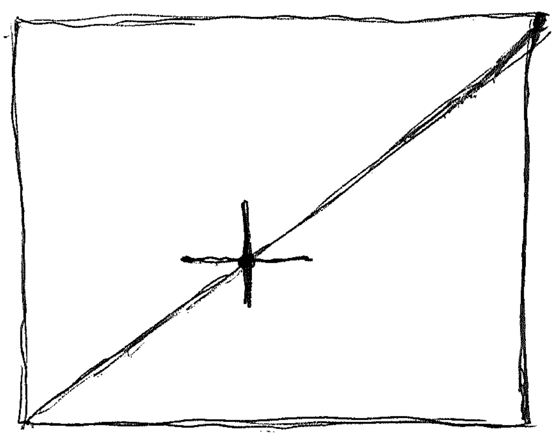
The derivation D is a vector field on M which vanishes on Y and whose 1-jet along Y viewed as an endomorphism of $TM|_Y$ is the projection onto the normal bundle (considered as a subbundle of $TM|_Y$.)

Let's now look at the case of the diagonal

$$0 \longrightarrow I \longrightarrow A \otimes A \longrightarrow A \longrightarrow 0$$

$$0 \to I/I^2 \longrightarrow \Omega^1_A \oplus \Omega^1_A \xrightarrow{+} \Omega^1_A \longrightarrow 0$$

There are two obvious splittings of the second sequence. Picture.



Let's identify the normal bundle of $\Delta M \subset M \times M$ with $pr_1^*(TM)$. Now we want a vector field on $M \times M$ which vanishes on ΔM and whose 1-jet is the projection of $T(M\times M)|_{\Delta M}$ onto the horizontal space

7 with kernel $T(\Delta M)$. [redacted]

Then we want to extend ~~the~~ this 1-jet to a vector field on $M \times M$. Along the diagonal this vector field is horizontal, so the obvious thing to do is to extend it keeping it horizontal. Then for each $y \in M$ we have a vector field $X(y)$ on M which vanishes at y and whose [redacted] 1-jet at y is the identity on $T_y M$.

If we use this sort of extension it is clear that the corresponding lifting of A into $A \otimes A$ which is killed by [redacted] the vector field is $a \longmapsto 1 \otimes a$.

It now appears that we have constructed $b' = i(X)$ on [redacted] $\Omega_A \otimes A$ as well as $d \otimes 1$. Then b' and $d \otimes 1$ should [redacted] commute with multiplication by elts of $pr_2^* A = 1 \otimes A$. Roughly speaking everything [redacted] happens on $M \times \{y\}$ for each $y \in M$. The same should be true for $[b', d \otimes 1]$. This should be the Lie derivative on forms Ω_A associated to the vector field $X(y)$ for each leaf $M \times \{y\}$.

Notice that if we could arrange $X(y)$ to vanish only at y and [redacted] this vector field has flow defined for all t, then M has to be diffeomorphic to $T_y M$.

[redacted]

October 11, 1991

Consider the blowup of $(0,0) \in R \times M$. This is described in alg. geom. by Proj of the graded algebra

$$\circledast \qquad \bigoplus_{n \geq 0} \left(h\mathcal{O}[h] + m[h] \right)^n$$

which appears

$$\begin{array}{llllllll} \text{deg } 0: & \mathcal{O} & \oplus & \mathcal{O}h & \oplus & \mathcal{O}h^2 & \oplus & \cdots \\ \text{deg } 1: & m & \oplus & \mathcal{O}h & \oplus & \mathcal{O}h^2 & \oplus & \cdots \\ & m^2 & \oplus & mh & \oplus & \mathcal{O}h^2 & \oplus & \cdots \end{array}$$

Take the h located in degree 1 and localize; more precisely take the direct limit under multiplication by h. This gives the algebra

$$R = \cdots \oplus m^2h^{-2} \oplus mh^{-1} \oplus \mathcal{O} \oplus \mathcal{O}h \oplus \cdots$$

Note the R is graded; this is related to $\circledast$ being bigraded.

Let us find the variety of R. If $\chi : R \longrightarrow \mathbb{C}$ is a homomorphism, the $\chi(h)$ is non-zero or zero. For $\chi(h) \neq 0$ we have a point of $R_h = \mathcal{O} \otimes \mathbb{C}[h, h^{-1}]$, i.e. we have $\mathbb{C}^\times \times \operatorname{Var}(\mathcal{O})$. If $\chi(h) = 0$, then χ is a point of

$$R/Rh = \cdots \oplus (m/m^2)h^{-1} \oplus \mathcal{O}/m \simeq \operatorname{Sym}_{\mathcal{O}/m}(m/m^2)$$

i.e. a point of the tangent space to M at 0.

Thus R describes the part of the blowup $\widetilde{R\times M}$ complementary to $\tilde{M}$.

Another want to look at R is that it is obtained by adjoining to $\mathcal{O}$ element h, v^i such that $v^i h = x^i$ is a system of parameters generating m. Thus the ratios $\frac{x^i}{h} = v^i$ are well-defined as $h \to 0$.

The next point is to understand to what extent R gives $\mathcal{O}$ as a deformation of $\mathrm{Sym}(m/m^2)$. We have

$$
\begin{aligned}
R &= \quad \oplus\, m^2 h^{-2} \oplus m h^{-1} \oplus \mathcal{O} \oplus \mathcal{O}h \oplus \mathcal{O}h^2 \\
Rh &= \quad \cdots \oplus m^2 h^{-1} \oplus m \oplus \mathcal{O}h \oplus \cdots \\
Rh^2 &= \quad m^2 \oplus mh \oplus
\end{aligned}
$$

Thus $\mathrm{gr}\, R = \bigoplus_{n\geq 0} Rh^n/Rh^{n+1}$ is

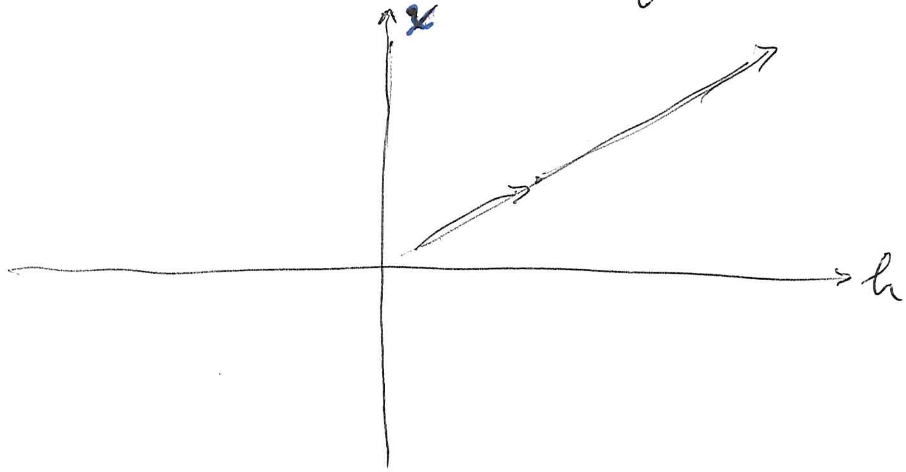
$$
\begin{aligned}
&(m^2/m^3)\, h^{-2} \oplus (m/m^2)\, h^{-1} \oplus \mathbb{C} \\
&\qquad \oplus (m^2/m^3)\, h^{-1} \oplus m/m^2 \oplus \mathbb{C}h \\
&\qquad\qquad \oplus m^2/m^3 \oplus (m/m^2)\, h \oplus \mathbb{C}h^2
\end{aligned}
$$

which is $\left(\bigoplus m^n/m^{n+1}\right) \otimes \mathbb{C}[h]$. Obviously we should now look into applying the tubular nbd. thm, ideas. This means looking for a derivation $D: R \to hR$ such that D induces the identity on hR/h^2R. It suffices that D extends $h\partial_h$ on $\mathbb{C}[h]$. Then D induces a derivation on $R_h = \mathcal{O}\otimes\mathbb{C}[h,h^{-1}]$

which is determined by a derivation

$$\mathcal{O} \longrightarrow \mathcal{O} \otimes \mathbb{C}[h, h^{-1}]$$

Geometrically we want something like a radial vector field



i.e. $x\partial_x + h\partial_h$. When we change to h, v we get the vector field $h\partial_h$:

$$\begin{array}{ccc} f(h,x) & \longleftrightarrow & f(h, vh) \\ \big\downarrow h\partial_h + x\partial_x & & \big\downarrow h\partial_h \\ h\partial_h(h,x) + x\partial_x f(h,x) & \longleftrightarrow & h f_1(h, vh) + hv f_2(h, vh) \end{array}$$

Thus the derivation D of R I am after should be of the form $h\partial_h + D'$ where $D' : \mathcal{O} \longrightarrow \mathcal{O}$ is independent of h. Recall

$$R = \cdots \oplus m h^{-1} \oplus \mathcal{O} \oplus \mathcal{O} h \oplus \cdots$$

We need $D'(m) \subset m$ in order that D be defined on R. We also want D to be zero on R/hR which means $D'\mathcal{O} \subset m$ and $D' = 1$ on m/m^2 so that D is 0 on $(m/m^2)h^{-1}$.

Thus we reach the kind of

derivation $D' : \mathcal{O} \longrightarrow \mathfrak{m}$ associated to the tubular nbd thm. for $O \in M$.

Let's now state the conclusions. We have been studying the tubular nbd. theorem for an embedding of nonsingular varieties of manifolds. Suppose we have $B = A/I$ with A, B smooth. The formal tubular nbd. theorem uses a derivation $D : A \to I \subset A$ such that $D = 1$ on I/I^2. Next consider the "deformation to the normal bundle" which is the algebra

$$R = \bigoplus_{n>0} I^n h^{-n} \oplus \bigoplus_{n \geq 0} A h^n$$

over $\mathbb{C}[h]$. Given D as above on A, let

$$\tilde{D} = h\partial_h + D \qquad \text{on } R.$$

Claim then that $\tilde{D}$ gives the tubular nbd. thm. for $R/hR = \bigoplus_{n \geq 0} (I^n/I^{n+1}) h^{-n}$.

The next stage is to understand the case of Lie algebras. I need to see how b appears

October 18, 1992

Poisson Lie groups: (conversation with Weinstein and Lu).

A Poisson Lie group turns out to be a Lie group G with a 1-cocycle $G \longrightarrow \Lambda^2\mathfrak{g}$ where G acts on $\Lambda^2\mathfrak{g}$ via the adjoint action. (The definition is group object in the category of Poisson manifolds.) A Poisson-Lie algebra is a Lie algebra $\mathfrak{g}$ with a 1-cocycle $\mathfrak{g} \longrightarrow \Lambda^2\mathfrak{g}$. It turns out that there is a natural duality operator on Poisson-Lie algebras $\mathfrak{g} \longmapsto \mathfrak{g}^*$. It would seem that the adjoint of the 1-cocycle for $\mathfrak{g}$ gives the Lie bracket on $\mathfrak{g}^*$ and that the adjoint of the bracket on $\mathfrak{g}$ is the 1-cocycle for $\mathfrak{g}^*$.

<u>Example</u>: If $\mathfrak{g}$ is a Lie algebra with 0 Poisson structure, then the dual is $\mathfrak{g}^*$ with zero bracket and the familiar Poisson manifold structure.

Manin construction. A Poisson Lie algebra $\mathfrak{g}$ determines a symplectic Lie algebra structure on $\mathfrak{g} \oplus \mathfrak{g}^*$ such ~~that~~ $\mathfrak{g}$ and $\mathfrak{g}^*$ are both Lie subalgebras and isotropic in some way. Conversely if we have a symplectic Lie algebra which is a direct sum of isotropic subalgebras, then the two subalgebras are dual Poisson Lie algebras. Example: Complex-Lie group and Gram-Schmidt decomposition (or Iwasawa decomposition) $G = K \cdot AN$. Then K and AN are dual Poisson Lie groups.

Duality for Poisson-Lie groups so linked to Lie algebras that it is meanigful only for 1-connected groups.

Weinstein mentions symplectic groupoid. This is a groupoid $\Gamma \overset{\alpha}{\underset{\beta}{\rightrightarrows}} Ob$ such that both maps are Poisson I think. A key point is that Lagrangian bisections (simultaneously sections of α and β) form a group, infinite-dimensional, whose Lie ~~~~ algebra is 1-forms under Poisson bracket.

Example of such a Γ is $G \times \mathfrak{g}^*$ with G a Lie group & 0 Poisson structure, where G acts on $\mathfrak{g}^*$ via coadjoint repn. $\therefore \mathfrak{g}^* = Ob$. More generally one could take $G \times G^*$, ~~~~ with G acting on G^*. (Also $G^* \times G$ and there is some sort of compatibility.)

Quantizing Γ. The composition is a relation which is a Lagrangian submanifold of $\bar{\Gamma} \times \bar{\Gamma} \times \Gamma$, so if V is the vector space obtained by quantizing Γ one should associate

$$V^* \otimes V^* \otimes V \quad \text{to} \quad \bar{\Gamma} \times \bar{\Gamma} \times \Gamma$$

and an element of $\uparrow$ assoc. to the Lagrangian submanifold, whence we get an algebra structure on V.

October 26, 1991

On cyclic objects.

Motivation: Let P be a projective A-bimodule resolution of A. Then

$$F : [n] \longmapsto [P\otimes_A]^{(n+1)}$$

is a contravariant functor from the "cyclic category without degeneracies" $\overline{Cyc}$ to complexes. $\overline{Cyc}$ is the category of finite nonempty cyclically ordered sets and injective maps preserving the cyclic order. The above functor F takes maps to quasi-isms, so one expects a spectral sequence

$$E^2_{pq} = H_p(\overline{Cyc}, HH_q(A)) \Longrightarrow H(\mathbb{L}\varinjlim_{\overline{Cyc}} F)$$

Also $B\overline{Cyc}$ should be BS^1 up to homotopy equivalence. Thus $\mathbb{L}\varinjlim F$ should give the cyclic homology of A.

Example: A separable, $P = A$ then we have the constant functor $[n] \longmapsto A_\natural$

I would like to understand how to ~~prove~~ prove that $B\overline{Cyc} \sim BS^1$. The idea should be to construct a "circle bundle" over $B\overline{Cyc}$ and show the total space is contractable.

Here's an attempt: Let $\overline{\Delta}$ be the simplicial category without degeneracies, i.e. finite nonempty totally ordered sets + inclusions preserving the order. We have a functor $f : \overline{\Delta} \longrightarrow \overline{Cyc}$ which sends a totally ordered set into the corresponding cyclic ordered set, where the successor of the last element is the first element. Then we get a cofibred category over $\overline{Cyc}$ with fibres $f/[n]$, $n \geq 0$.

What is $f/[n]$? It is the poset whose elements are non-empty subsets $\sigma \subset [n] = \{0, \dots, n\}$ equipped with total order compatible with the cyclic order on σ induced by the cyclic order on $[n]$; the ordering is given by inclusion compatible with the total ordering.

$n = 0$. Then $f/[n]$ is a point

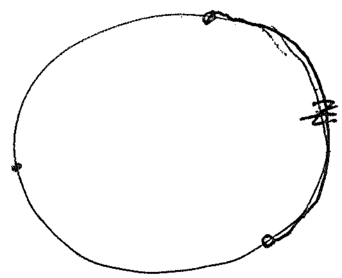
$n = 1$. Then $f/[1]$ has four elements: the subsets $\{0\}, \{1\} \subset \{0, 1\}$, and the subset $\{0,1\}$ with the two total orderings. The poset is this

$$\begin{tikzcd} & \{0<1\} & \\ \{0\} \arrow[ur] \arrow[dr] & & \{1\} \arrow[ul] \arrow[dl] \\ & \{0>1\} & \end{tikzcd}$$

so the homotopy type is S^1.

For $n \geq 1$. $f/[n] \sim S^1$ as follows. Embed $[n]$ inside S^1 as cyclic subset. For each subset $\sigma \subset [n]$ with total order, we associate the interval α_σ (or arc) of S^1 starting with the first element of σ and ending with the last. If $|\sigma| = 1$, this arc α_σ reduces to a point. Then we have a functor from $f/[n]$ to contractible subspaces of S^1.

Given [redacted] $z \in S^1$, consider the [redacted] subposet of σ such that $z \in \alpha_\sigma$. This poset has a least element which reduces to a point if $z \in [n]$, and otherwise to a pair of consecutive points



It is clear this gives a homotopy equivalence of $f/[n]$ with S^1 for $n \geq 1$. It's also clear that any map $[n] \longrightarrow [n']$ induces a heq $f/[n] \longrightarrow f/[n']$, again for $n \geq 1$.

So we have to deal with the problem that $f/[0] = pt$.

November 1, 1991

Some ideas from Goodwillie. Fibre square:

$$\begin{array}{ccc} HC^- & \longrightarrow & HP \\ \downarrow & & \downarrow \\ HH & \longrightarrow & HC \end{array}$$

HC for a cyclic module is $\mathbb{L}\,\underrightarrow{\lim}$

HC^- should be $R\,\underleftarrow{\lim}$

These correspond to the homotopy orbit space and homotopy fixpt space for a circle action. There ought to be a natural way to see that the Dennis trace $K \longrightarrow HH$ factors through the homotopy fixpoint space, and thus becomes a map $K \to HC^-$.

HP should be some kind of Tate homology with respect to the circle action. Apparently for G spectra, G compact there is a notion of Tate (maybe Tate-Farrell) cohomology.

There is a cyclic bar construction for a category in which simplices are loops . When one realizes it one gets the space whose elements are subdivisions of S^1 with objects corresponding to arcs and arrows corresponding to vertices. This seems strange.

Apparently the self maps on the circle act on the realization of cyclic set

November 15, 1991

Dan Freed's top QFT. This is a baby model of Witten's Chern-Simons QFT where the compact conn. Lie group G is replaced by a finite group. One supposes given a class in $H^3(G,\mathbb{R}/\mathbb{Z}) = H^4(BG,\mathbb{Z})$. If X is a closed (oriented) 3-manifold with principal G-bundle P, one gets a number in $\mathbb{Q}/\mathbb{Z}$ by pairing with the fundamental class; this gives the action $S(P)$ and one sums $e^{2\pi i S(P)}$ over the possible P.

The object of the QFT is to do something for closed 2-manifolds Y, more precisely to attach a line to Y and an element of the line when Y is given as the boundary of of 3-manifold. Then later one tries to reduce to codimension 2.

November 15, 1991 (cont.)

After Tsygan.

Let $\mathfrak{g}$ act on A_0, let τ be an invariant trace on A_0. He defines a map of complexes

$$C_\cdot(\mathfrak{g}) \longrightarrow CC^*_{per}(A_0)$$

which is given roughly by

$$D_1 \wedge \cdots \wedge D_n \longmapsto \sum_{\sigma}{}' (-1)^{\sigma}\, \tau(a_0 D_{\sigma 1} a_1 \cdots D_{\sigma n} a_n)$$

If this map is denoted $c \longmapsto \tau_c$, then we have $b\tau_c = 0$, $B\tau_c = \tau_{\partial c}$.

What's going on here is perhaps that we form the DG algebra $C^\cdot(\mathfrak{g}, A_0)$ and use τ as a trace

$$C^\cdot(\mathfrak{g}, A_0) \longrightarrow C^\cdot(\mathfrak{g})$$

November 16, 1991

Return to the derivation of RA given by

$$Da = \phi a$$

where $-\delta\phi = d \circ d$. Then D is of degree 2 relative to the grading of $RA = \Omega^{ev}A$ by degree, so it is possible to describe the lifting homomorphism $l : A \longrightarrow \hat{R}A$ such that $Dl = 0$. Observe that

$$D(da_1 da_2) = D(a_1 a_2 - a_1 \circ a_2)$$

$$= \phi(a_1 a_2) - \phi a_1 \circ a_2 - a_1 \circ \phi a_2$$

$$= \underbrace{\phi(a_1 a_2) - \phi a_1 a_2 - a_1 \phi a_2}_{da_1 da_2} + d\phi a_1 da_2 + da_1 d\phi a_2$$

In fact recall that D extends to $QA = (\Omega A, \circ)$ via $D(da) = \frac{1}{2} da + d\phi a$. Thus

$$D(a_0 da_1 \cdots da_n)$$

$$= (Da_0) \circ da_1 \cdots da_n + \sum_{j=0}^{n} (a_0 da_1 \cdots da_{j-1}) \circ Dda_j \circ (da_{j+1} \cdots da_n)$$

$$= (\phi a_0) da_1 \cdots da_n + \sum_{j=0} a_0 da_1 \cdots da_{j-1} \left(\tfrac{1}{2} da_j + d\phi a_j\right) da_{j+1} \cdots da_n$$

$$= \frac{n}{2}(a_0 da_1 \cdots da_n) + \underbrace{\left\{ \phi a_0 da_1 \cdots da_n + \sum_{j=0}^{n} a_0 da_1 \cdots da_{j-1} d\phi a_j da_{j+1} \cdots da_n \right\}}_{\text{Call this } L(a_0 da_1 \cdots da_n) \text{ in analogy with Lie derivative}}$$

Then

$$D = \frac{N}{2} + L \quad \text{as operators on } \Omega A.$$

Now

$$la = a + l_1 a + \cdots$$

with $l_n a \in \Omega^{2n} a$, so

$$0 = D(la) = \sum_{n \geq 0} n l_n a + \sum_{n \geq 0} L l_n a$$

which gives the recursion relation

$$n l_n a + L l_{n-1} a = 0$$

or $$\boxed{l_n a = \frac{(-1)^n}{n!} L^n a}$$

Thus $$\boxed{la = e^{-L} a}$$

Notice that $L^n : \bar{A} \longrightarrow \Omega^{2n} A$ is an interesting cochain constructed from ϕ, which means that your statement that the only linear maps constructed from ϕ in the universal lifting section is wrong.

The question is whether there might be a nicer derivation than $Da = \phi a$. We would like to preserve the fact that D extends to $\hat{Q}A$. This means that we want a derivation $D\theta : A \longrightarrow \hat{Q}A$ relative to θ, i.e.

1) $D(a_1 a_2 + d(a_1 a_2)) = D(a_1 + da_1) \circ (a_2 + da_2) + (a_1 + da_1) \circ D(a_2 + da_2)$

2) $D(a_1 a_2) = Da_1 \circ a_2 + a_1 \circ Da_2 + Dda_1\, da_2 + da_1\, Dda_2$

3) $Dd(a_1 a_2) = Dda_1 \circ a_2 + a_1 \circ Dda_1 + Da_1\, da_2 + da_1\, Da_2$

suite

4) $$dD(a_1a_2) = dDa_1\, a_2 + Da_1\, da_2 + da_1\, Da_2 + a_1\, dDa_2 + \cancel{(dDd)}a_1\, da_2 - da_1\, \cancel{(dDd)}\, a_2$$

subtracting 4) from 3) we have

$$[D,d](a_1a_2) = [D,d](a_1)\, a_2 + d\underbrace{(\cancel{Dda_1})}_{\text{odd}}\, da_2 + a_1\,[D,d]a_2 - da_1\,\cancel{dDd}\, a_2$$

Thus we see that $[D,d](a) = D(da) - dDa$ is a derivation $A \longrightarrow \Omega^{odd}A$.

Let's write

$$D(a+da) = f(a) = \sum_n f_n(a)$$

where $f_n : \bar{A} \longrightarrow \Omega^n A$. Then the derivation condition is

$$f(a_1a_2) = (a_1+da_1)\circ f(a_2) + f(a_1)\circ(a_2+da_2)$$
$$= a_1 f(a_2) + f(a_1)a_2 - da_1\, df(a_2) - (-1)^{|f|} df(a_1)\, da_2 + da_1 f(a_2) + f(a_1)\, da_2$$

i.e.

5) $$\boxed{+\delta f = -d\cup f - f\cup d + d\cup df + (-1)^{|f|} df\cup d}$$

Recall in general that

6) $$\boxed{(\delta d - d\delta)f = -d\cup f - (-1)^{|f|} f\cup d}$$

From 5) we get

$$d(\delta f) = d\cup df - df\cup d$$

Note that $Da = f_+(a)$ $D(da) = f_-(a)$

so that

$$[D,d](a) = f_-(a) - df_+(a).$$

Let's check that $[D,d]$ is a derivation

$$\delta f_- = -\overset{(1)}{d \circ f_+} - \overset{(2)}{f_+ \circ d} + \overset{(3)}{d \circ df_-} + \overset{(4)}{df_- \circ d}$$

$$\delta(df_+) = d(\delta f_+) + (\delta d - d\delta) f_+$$

$$= \overset{(3)}{d \circ df_-} - \overset{(4)}{df_- \circ d} - \overset{(1)}{d \circ f_+} - \overset{(2)}{f_+ \circ d}$$

$$\therefore \ \delta(f_- - df_+) = 0$$

Recursive construction of f's:

Start with $f_{2n-1} \ni \delta f_{2n-1} = 0$. Then

$$\delta(d \circ f_{2n-1} - f_{2n-1} \circ d) = 0$$

and since A is quasi-free, we can find $f_{2n} \ni$

$$\delta f_{2n} = -d \circ f_{2n-1} - f_{2n-1} \circ d$$

Then you can check that one can continue with

$$f_{2n+1} = df_{2n}, \quad f_{2n+2} = f_{2n+3} = \cdots = 0.$$

Ambiguity in the choice of f_{2n-1} is a derivation w. values in $\Omega^{2n-1}A$; ambiguity in choice of f_{2n} is a derivation w. values in $\Omega^{2n}A$.

Note that if we take D of the form $Da = f_+ a$, $D(da) = \frac{1}{2}da + df_+(a)$, then we can remove the derivation with $a \mapsto \phi a$, $da \mapsto \frac{1}{2}da + d\phi a$ where $\phi a = f_2(a)$. Then we have $f_-(a) = df_+(a)$, and the relations are

$$\delta f_+ = -d\cup df_+ - df_+\cup d + d\cup df_+ + df_+\cup d = 0$$

$$\delta f_- = -d\cup f_+ - f_+\cup d + \cancel{d\cup df_-} - \cancel{df_-\cup d}$$

and the latter is satisfied automatically

since

$$\delta f_- = \delta df_+ = (\delta d - d\delta) f_+$$
$$= -d\cup f_+ - f_+\cup d$$

Thus derivations of the form

$$Da = f_+(a)$$
$$D(da) = \tfrac{1}{2}da + df_+(a)$$

with f_+ of order 2 are the same as a family of derivations $\bar{A} \longrightarrow \Omega^{2n}A$ for $n \geq 2$.

Consider constructing a lifting homomorphism inductively:

$$\begin{tikzcd}
 & & & A \arrow[d, "a \mapsto a-\phi a"] & \\
0 \arrow[r] & \Omega^4 A \arrow[r] & RA/IA^3 \arrow[r] & RA/IA^2 \arrow[r] \arrow[l, dashed, bend right] & 0
\end{tikzcd}$$

This gives a 2-cocycle

$$a_1a_2 - \phi(a_1a_2) - (a_1 - \phi a_1)\circ(a_2 - \phi a_2)$$

$$= \cancel{a_1a_2} - \phi(a_1a_2) - \cancel{a_1a_2} + da_1da_2 + \cancel{\phi a_1 a_2} - d\phi a_1 da_2$$
$$+ \cancel{a_1\phi a_2} - da_1 d\phi a_2 + \phi a_1 \phi a_2$$

$$= \phi a_1 \phi a_2 - d\phi a_1 da_2 - da_1 d\phi a_2$$

We thus have a bimodule map $\Omega^2 A \longrightarrow \Omega^4 A$ which can be composed with

$\phi : \bar{A} \longrightarrow \Omega^2 A$ to obtain a connection ~~xx~~ yielding a lifting homomorphism $A \longrightarrow RA/IA^3$.
However this is apparently different from the canonical lifting homomorphism

$$a \longmapsto a - \underbrace{\phi a}_{L} + \frac{1}{2} L^2 a .$$

Recall

$$\bar{A} \xrightarrow{L=\phi} \Omega^2 A \xrightarrow{\quad L \quad} \Omega^4 A$$

$$a_0 da_1 da_2 \longmapsto \phi a_0 \, da_1 da_2 + a_0 \, d\phi a_1 \, da_2 + a_0 \, da_1 \, d\phi a_2$$

This L on $\Omega^2 A$ does not seem to be a bimodule map.

November 20, 1991

$$D(da_1 da_2) = D(a_1 a_2 - a_1 \circ a_2)$$
$$= \phi(a_1 a_2) - \phi a_1 \circ a_2 - a_1 \circ \phi a_2$$
$$= \phi(a_1 a_2) - \phi a_1 a_2 - a_1 \phi a_2 + d\phi a_1\, da_2 + da_1\, d\phi a_2$$
$$= da_1 da_2 + d\phi a_1\, da_2 + da_1\, d\phi a_2 .$$

$\therefore$ Observe this is consistent with
$$D(da_1 da_2) = \left(\tfrac{1}{2} da_1 + d\phi a_1\right) da_2 + da_1 \left(\tfrac{1}{2} da_2 + d\phi a_2\right)$$

Thus
$$D(a_0 da_1 \cdots da_n) = \frac{n}{2}(a_0 da_1 \cdots da_n) + L(a_0 da_1 \cdots da_n)$$

where
$$L(a_0 da_1 \cdots da_n) = \phi a_0\, da_1 \cdots da_n + \sum_{j=1}^{n} a_0 da_1 \cdots da_{j-1}\, d\phi a_j\, da_{j+1} \cdots da_n$$

In doing this calculation you should note that $D(da) = \frac{1}{2} da + d\phi a$ is closed hence

$$D(a_0 da_1 \cdots da_n) = D(a_0 \circ da_1 \circ \cdots \circ da_n)$$
$$= Da_0 \circ da_1 \circ \cdots \circ da_n + \sum_{j=1}^{n} a_0 \circ da_1 \circ \cdots \circ da_{j-1} \circ Dda_j \circ da_{j+1} \circ \cdots \circ da_n$$
$$= \underbrace{Da_0}_{\phi a_0} da_1 \cdots da_n + \sum_{j=1}^{n} a_0 da_1 \cdots da_{j-1} \underbrace{Dda_j}_{\frac{1}{2} da_j + d\phi a_j} da_{j+1} \cdots da_n$$

etc. Next put $H = \frac{1}{2} n$ on $\Omega^n A$. Then since L is of degree 2 we have $[H, L] = L$:
$$[H, L]\omega = \tfrac{1}{2}|L\omega| L\omega - \tfrac{1}{2}|\omega| L\omega = L\omega .$$

Thus we have

$$e^{-L}He^{L} = e^{-L}\Big(e^{L}H + \underbrace{[H,e^{L}]}\Big)$$

$$\Big(\int_0^1 e^{(1-t)L}\underbrace{[H,L]}_{L}\,e^{tL}\,dt = Le^{L}$$

so $$\boxed{e^{-L}He^{L} = H+L = D}$$

Thus e^{-L} carries $\mathrm{Ker}(H-\tfrac{1}{2}n)$ to $\mathrm{Ker}(D-\tfrac{1}{2}n)$. Now be careful and recall that we have an algebra isomorphism

$$\hat{\Omega}A \xrightarrow{\sim} \hat{Q}A$$

given by sending $\omega \in \Omega^n A$ to the unique eigenvector of D with eigenvalue $\frac{1}{2}n$ whose leading term in ω. Recall that

$$\begin{tikzcd}
\mathrm{Ker}(D-\tfrac{1}{2}n) \arrow[r, phantom, "\subset"] \arrow[dr, bend right, "\simeq"'] & \widehat{qA^n} \arrow[d] \\
& qA^n/qA^{n+1} = \Omega^n A
\end{tikzcd}$$

Now $e^{-L}\omega$ belongs to $\widehat{qA^n}$ and

$$D(e^{-L}\omega) = e^{-L}H\omega = \frac{n}{2}e^{-L}\omega$$

so it's all clear.

November 17, 1991

Question: Consider the algebra

$$M_\infty \mathbb{C} = \varinjlim M_n \mathbb{C}$$

of matrices with finite support, and adjoin an identity. This is not separable, but is it quasi-free?

What are modules over $(M_\infty \mathbb{C})^\sim$?

Dec 9, 1991

Notes from various sheets of paper:

G group. Consider universal extension: $G = F(G)/N(G)$, where $F(G)$ = free group on $G - \{1\}$. Do there exist universal G-modules for cocycles? What is $FG/(NG, NG)$ the universal abelian extension?

Problem: In the setup of Kadison's theorem: $A \otimes_S A$ projective A bimodule, to construct an explicit homotopy equivalence of the resolutions $(\Omega A \otimes A, b')$ and $(\Omega_S A \otimes_S A, b')$. (It should be easy but I got tied up in knots.)

December 13, 1991

Madsen's talk about the cyclotomic traces. The end result is (for p odd)

$$TC(\mathbb{Z}_p, p)^\wedge_p \sim (\mathrm{Im}\, J \times B\mathrm{Im}\, J \times SU)^\wedge_p$$

and $$K(\mathbb{Z}_p)^\wedge_p \xrightarrow[\text{trace}]{\text{cyclotomic}} TC(\mathbb{Z}_p, p)^\wedge_p$$ is split surjective. Now one knows

$$K_{et}(\mathbb{Z}_p)^\wedge_p = (\mathrm{Im}\, J \times B\mathrm{Im}\, J \times SU)^\wedge_p$$

so one has

$$\begin{tikzcd} K(\mathbb{Z}_p)^\wedge_p \arrow[r] \arrow[dr] & TC(\mathbb{Z}_p, p)^\wedge_p \arrow[d, phantom, "\wr"] \\ & K_{et}(\mathbb{Z}_p)^\wedge_p \end{tikzcd}$$

although the commutativity is not clear at the moment. Thus, the "topological cyclic homology at p": $TC(\mathbb{Z}_p, p)^\wedge_p$ of $\mathbb{Z}_p$ is supposed to calculate the K-theory of $\mathbb{Z}_p$.

There is something called topological Hochschild homology due to Bökstedt. This is defined for any ring spectrum R and denoted $T(R)$. One forms a simplicial spectrum

$$\cdots \;\substack{\rightarrow\\\rightarrow\\\rightarrow\\\rightarrow}\; R\wedge R\wedge R \;\substack{\rightarrow\\\rightarrow\\\rightarrow}\; R\wedge R \rightrightarrows R$$

analogous to what one does for an algebra, then takes the colimit (or realization). Perhaps the good way to say it is that one has a [redacted] cyclic spectrum

$$[n] \longmapsto R \wedge \overset{n+1}{\cdots} \wedge R \,.$$

Suppose we take $R = \mathbb{Z}$, more precisely, $R =$ [redacted] Eilenberg-Maclane spectrum. Note that $R \wedge R$ is $K(\mathbb{Z}) \wedge K(\mathbb{Z})$, which is not $K(\mathbb{Z})$, because $\pi(K(\mathbb{Z}) \wedge K(\mathbb{Z})) = H(K(\mathbb{Z}), \mathbb{Z})$. So even in the case $T(\mathbb{Z})$, the top Hoch homology is nontrivial. I believe Bökstedt has computed $\pi(T(\mathbb{Z}_p)$ (mod p ?)) and found the answer to be a polynomial ring on generator of degree $2p$ tensored with an exterior algebra with generator of degree $2p-1$ (?)

The next thing is the topological cyclic homology, which is probably a negative cyclic theory, i.e. taking homotopy fixpts for the circle actions. There is a subdivision process on $T(R)$ which allows one to concentrate on the subgroup C_{p^n} of the circle, and by taking suitable fixpts, to define $T(R, p)$.

December 14, 1991

Gunnar Carlsson's talk:

There is an assembly map

$$B\Gamma_+ \wedge \underline{K}A \longrightarrow \underline{K}A[\Gamma]$$

One would like to prove it is a split injection of spectra, and ultimately an equivalence in good cases. The strategy is to realize $\underline{K}A[\Gamma]$ as the fixpt spectrum for a spectrum M with Γ-action. Then we have a canonical map

$$\underline{K}A[\Gamma] = M^{\Gamma} \longrightarrow \underbrace{M^{h\Gamma}}_{\substack{\text{homotopy} \\ \text{fixpt spectrum}}} = \underset{\text{Maps}}{\overset{\shortparallel}{\mathrm{Hom}}}(E\Gamma, M)^{\Gamma}$$

For some reason, because homotopy fixpts can be analyzed via spectral sequences, one might be able to define a map from $M^{h\Gamma}$ to $B\Gamma_+ \wedge \underline{K}A$. In the case Γ finite of order N we have an embedding

$$A[\Gamma] \subset M_n A$$

given by the left regular representation, and

$$A[\Gamma] = (M_n A)^{\Gamma}$$

where Γ acts on matrices via right multiplication. This gives what we want

$$\underline{K}A[\Gamma] = \underline{K}(M_n A)^{\Gamma}$$

Now $\underline{K}(M_n A) \sim \underline{K}A$ by Morita equivalence, so it seems that

$$\underline{K}(M_n A)^{h\Gamma} \approx \underline{K}A^{h\Gamma} = \mathrm{Map}(B\Gamma, \underline{K}A)$$

since Γ acts trivially on $\underline{K}A$. The question is why should there be a map

$$\mathrm{Map}(B\Gamma, \underline{K}A) \longrightarrow B\Gamma_+ \wedge \underline{K}A\,?$$

The Tate idea is to use $N : H_0(G,M) \longrightarrow H^0(G,M)$, so the map seems to go in the opposite direction.

5

Madsen's constructions

6 Cyclotomic trace maps

$$\mathrm{Trc}: \quad K(R) \longrightarrow TC(R,p) \qquad R \text{ ring spectrum}$$

$$\mathrm{Trc}: \quad A(X) \longrightarrow TC(X,p) \qquad \text{Waldhausen}$$

Big Thm about cyclotomic trace is $\mathrm{Trc}: \tilde{A}(X)^\wedge_p \xrightarrow{\sim} \tilde{TC}(X,p)^\wedge_p$

Basic properties

$$T(R,p)^\wedge_p \simeq TC(R \otimes \mathbb{Z}_p, p)^\wedge_p$$

$$TC(M_k R, p) \simeq TC(R,p).$$

Topological Hochschild homology:

$$T(R) = \left| B^{cy}_{spec}(R,R) \right|$$

by subdivision $\quad \underset{D}{\simeq} \left| B^{cy}_{spec}({}_t R^{\wedge p^n}, R^{\wedge p^n}) \right|$

t ↑ cyclic shift

This gives action of $C_{p^n} = \mathbb{Z}/p^n$.

<u>FACTS:</u>

1. $T(R)$ $\quad C_{p^n}$-equivariant spectrum
2. $\Phi: T(R)^{C_{p^n}} \longrightarrow T(R)^{C_{p^{n-1}}}$ $\quad$ (Some sort of transfer??)
3. $D: T(R)^{C_{p^n}} \longrightarrow T(R)^{C_{p^{n-1}}}$ $\quad$ inclusion of fixpt subspaces
4. $\Phi D = D \Phi$, $\quad \mathrm{hfibre}(\Phi) = T(R) \wedge_{C_{p^n}} EC_{p^n +}$

$$TC(R,p) = \left(\underset{\underset{D}{\longleftarrow}}{\mathrm{holim}}\; T(R)^{C_{p^n}} \right)^{h\Phi} \quad \text{(This is a total } \underleftarrow{\mathrm{holim}} \text{ wrt } \Phi, D \text{ maps)}$$

Example $R = QS^0$

$$T(R)^{C_{p^n}} = Q(BC_{p^n +}) \times \cdots \times Q(BC_{p+}) \times Q(S^0)$$

$$\Phi(x_n, \dots, x_0) = (x_{n-1}, \dots, x_0)$$

$$D(x_n, \dots, x_0) = (\operatorname{trf}(x_n), \dots, \operatorname{trf}(x_1) + x_0)$$

$$\operatorname{trf} : Q(BC_{p^k +}) \longrightarrow Q(BC_{p^{k-1} +})$$

transfer wrt $BC_{p^{k-1}} \to BC_{p^k}$.

Φ is related to the geometric operation

$$(f : S^V \longrightarrow S^V) \longmapsto (f^{C_p} : S^{V^{C_p}} \longrightarrow S^{V^{C_p}})$$

Note $\underset{\overleftarrow{D}}{\operatorname{holim}}\ T(QS^0)^{C_{p^n}} = (QS^0)^{S^1} = \prod_{-\infty}^{\infty} Q(\Sigma_+ \mathbb{C}P^\infty)$

by a version of Segal's calculation

$$(QS^0)^G = \prod_{\substack{H < G \\ \text{conj classes}}} Q(BN_G H) \quad \text{if } G \text{ finite}$$

$TC(R,p)$

$T(R)$

$TC(R)$?

$T(R)^{C_{p^n}}$ $\quad$ $T(R)^{hC_{p^n}}$

$\to T(R)^{C_{p^n}} \xrightarrow{\text{incl}=D} T(R)^{C_{p^{n-1}}}$

$T(R)^{C_{p^n}} \xrightarrow{\Phi} TC(R)^{C_{p^{n-1}}}$

$Q(S^0)^{C_{p^n}} = T(QS^0)^{C_{p^n}}$

$\downarrow\downarrow$

$Q(S^0)^{C_{p^{n-1}}}$

$T_k(R)^{C_p} = \varinjlim_{n_0,\dots,n_k} \mathrm{Map}(S^{n_0 p} \wedge \dots \wedge S^{n_k p}, (RS^{n_0})^{\wedge p} \wedge \dots \wedge (RS^{n_k})^{\wedge p})^{C_p}$

$\downarrow \Phi$ $\quad$ $\downarrow$

$T_k(R)$ $\quad$ $\varinjlim \mathrm{Map}(S^{n_0} \wedge \dots \wedge S^{n_k}, RS^{n_0} \wedge \dots \wedge RS^{n_k})$

$f: S^V \to S^V$

$\Phi(f) = f^{C_p}: S^{V^{C_p}} \to S^{V^{C_{p^n}}}$

$\operatorname{holim}_{\overleftarrow{D}} T(QS^0)^{C_{p^n}} = (QS^0)^{S^1}$

$= \prod_{-\infty}^{\infty} Q(\Sigma_+ \mathbb{C}P^\infty)$

$\operatorname{holim}_{\overleftarrow{D}} T(R)^{C_{p^n}} \circlearrowleft \Phi$

$TC(R,$

$(QS^0)^G = \prod_{(H)\subset G} Q(BN_G H)$

G finite

December 16, 1991

End homology theory of Pedersen-Weibel.

$\mathcal{M}$ = category of metric spaces and eventual Lipschitz maps. $f: M \to M'$ is eventual Lipschitz where $\exists\, k, l$ depending on f such that

$$d(f(x), f(y)) \leq k\, d(x,y) + l \qquad \forall x, y \in M$$

such an f need not be continuous.

$\mathcal{X}$ = category of compact subsets of S^∞ contained in S^n for some n and Lipschitz maps. There is a functor $\mathcal{X} \longrightarrow \mathcal{M}$, $X \mapsto O(X)$ where $O(X)$ is the cone on X.

Key point: If $\mathfrak{a}$ is a permutative category, one can define a K-spectrum $\underline{K}(M, \mathfrak{a})$ by using chains $\bigoplus_{x \in M} A_x$ which are locally finite and a suitable kind of bounded equivalence. (Carlsson described this in the case of finitely generated free A-modules as follows. Take infinitely generated free modules with given basis S and a map $S \to M$ which is locally-finite. Use operators that are like infinite matrices with finitely many entries in each row & column; maybe it's more like having finite band around the diagonal.

Basic result is that $\underline{K}(\mathbb{R}^n, \mathfrak{a})$, $n \geq 0$, which are connected spectra (so view them as infinite loop spaces) is a non connected delooping of $\underline{K}\mathfrak{a}$

as in Wagoner–Karoubi delooping.

The way to say this maybe is that the Pedersen–Weibel bounded homology theory is a far reaching generalization of the Wagoner–Karoubi result.

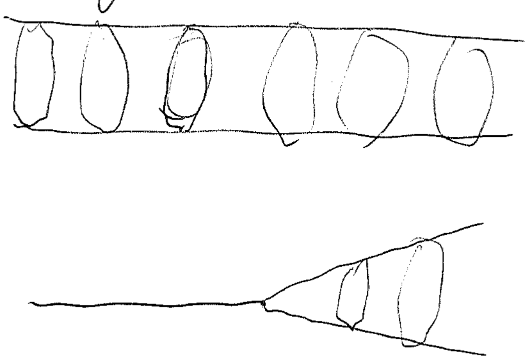
In certain cases one has (or would like)

$$h^{\text{locally finite}}(M, \underline{K}\mathcal{O}) \sim \underline{K}(M, \mathcal{O})$$

One needs M to be locally contractible in some sense.

Important picture related to surgery (or L-theory)

$$\begin{array}{c} \mathbb{R} \times M \\ \downarrow \\ O(pt \amalg M) \end{array}$$



December 22, 1991

We have embeddings

$$\left\{\begin{matrix}\text{nonunital}\\ \text{algs.}\end{matrix}\right\} \hookrightarrow \left\{\begin{matrix}\text{unital}\\ \text{algebras}\end{matrix}\right\} \hookrightarrow \left\{\begin{matrix}\text{nonunital}\\ \text{algs.}\end{matrix}\right\}$$

$$\mathcal{A} \longmapsto \tilde{\mathcal{A}} = \mathbb{C}\oplus\mathcal{A}$$

$$A \longmapsto A.$$

Given a functor F on unital algebras, let

$$F'(\mathcal{A}) = \operatorname{Ker}\{F(\tilde{\mathcal{A}}) \to F(\mathbb{C})\} = \operatorname{Coker}\{F(\mathbb{C}) \to F(\tilde{\mathcal{A}})\}.$$

Given a functor G on nonunital algebras we just restrict it to unital algebras.

We've seen that there is a canonical map:

$$\begin{tikzcd}
F'(A) \arrow[r] \arrow[d] & F(A) \arrow[d] \\
F(\tilde{A}) \arrow[r] \arrow[d] & F(A)\times F(\mathbb{C}) \arrow[d, "pr_2"] \\
F(\mathbb{C}) \arrow[r, equal] & F(\mathbb{C})
\end{tikzcd}$$

so that $F'(A) \xrightarrow{\sim} F(A) \iff F(A\times\mathbb{C}) \xrightarrow{\sim} F(A)\times F(\mathbb{C})$.

I have adopted the view that cyclic theory, say HC, is defined for unital algs., and that because of the direct ~~product~~ theorem it extends to non-unital algebras. But suppose you took the other view that cyclic theory is defined for nonunital algebras primarily, say via Connes-Tsygan complex, or maybe by cyclic object theory. Then the issue is to see that

$$\textcircled{*} \quad 0 \to HC(\mathcal{A}) \to HC(\tilde{\mathcal{A}}) \to HC(\mathbb{C}) \to 0$$

is exact. This is a type of excision result.

without any hypothesis on $\mathcal{A}$.

Actually if HC is defined via the C-T bicomplex, then one might be able to establish $\otimes$ by the method used in LQ. (removing the b'- columns, then normalizing).

Perhaps what is worthwhile in this situation to observe is that it might be possible to understand degeneracies for cyclic objects, (better: get some insight about degeneracies) using the cyclic objects $\mathcal{A}^{\otimes n}$, $n \geq 0$ for $\mathcal{A}$ non-unital.