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August 30, 1990

Suppose E is an algebra, S and Q are subalgebras, and that $S\otimes Q \xrightarrow{\sim} E$, $s\otimes x \longmapsto sx$. (This is the algebra analogue of a group being the product of two subgroups).

We have the exact sequence of Q-bimods

$$0 \longrightarrow \Omega^1 Q \longrightarrow Q\otimes Q \longrightarrow Q \longrightarrow 0$$

which splits as either left or right Q-modules. Thus in the diagram below the first row is exact

$$\begin{tikzcd}
0 \arrow[r] & E\otimes_Q \Omega^1 Q \arrow[r] \arrow[d] & \overbrace{E\otimes Q}^{S\otimes Q\otimes Q} \arrow[r, "\text{mult}"] \arrow[d] & E \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & \Omega^1(E/S) \arrow[r] & \underbrace{E\otimes_S E}_{} \arrow[r, "\text{mult}"] & E \arrow[r] & 0
\end{tikzcd}$$

$$S\otimes Q\otimes_S (S\otimes Q) = S\otimes Q\otimes Q$$

Thus the middle vertical map $\xi\otimes x \longmapsto \xi\otimes_S x$ is bijective. We conclude therefore that we have a bijection

$$E\otimes_Q \Omega^1 Q \xrightarrow{\sim} \Omega^1(E/S) \qquad \xi\otimes x\,dy \mapsto \xi x\,dy$$

Also

$$\underbrace{(E\otimes_Q \Omega^1 Q)\otimes_E (E\otimes_Q \Omega^1 Q)}_{\begin{array}{c}\| \\ E\otimes_Q \Omega^2 Q\end{array}} \xrightarrow{\sim} \Omega^2(E/S)$$

and similarly we have

$$\Omega^n(E/S) = \Omega^1(E/S) \otimes_E \Omega^{n-1}(E/S)$$
$$= \left(E \otimes_Q \Omega^1 Q\right) \otimes_E \left(E \otimes_Q \Omega^{n-1} Q\right)$$
$$= E \otimes_Q \Omega^1 Q \otimes_Q \Omega^{n-1} Q = E \otimes_Q \Omega^n Q$$

Actually this is not very precise. Better is

$$\Omega^n(E/S) = \Omega^1(E/S) \otimes_E \Omega^{n-1}(E/S)$$
$$= \Omega^1(E/S) \otimes_E \left(E \otimes_Q \Omega^{n-1} Q\right) \qquad \text{(induction)}$$
$$= \Omega^1(E/S) \otimes_Q \Omega^{n-1} Q$$
$$= E \otimes_Q \Omega^1 Q \otimes_Q \Omega^{n-1} Q = E \otimes_Q \Omega^n Q$$

So we have

$$\Omega(E/S) \cong E \otimes_Q \Omega Q \cong S \otimes \Omega Q$$

and the interesting thing is that the left side is a differential graded algebra. Thus from the fact that we have a product on $S \otimes Q$ we conclude there is a product on $S \otimes \Omega Q$.

Observe S, Q play symmetrical roles so that we should also have

$$\Omega(E/Q) \cong \Omega S \otimes_S E \cong \Omega S \otimes Q$$

I have the feeling that we have something analogous to transverse foliations.

Let's go back to

$$\Omega(E/S) = E \otimes_Q \Omega Q = S \otimes \Omega Q$$

and try to derive this in another way. The idea is to ~~make $S \otimes \Omega Q$ into a complex, which is easy, and then define a left action by E~~

define an action of $\Omega(E/S)$ on $E \otimes_Q \Omega Q$ in the same sort of way we make ΩA act on $\bigoplus_{n\geq 0} A \otimes \bar{A}^{\otimes n}$. First we note that because $E = S \otimes Q$, E is free as a right Q-module and so it has a ! flat "connection" $\nabla : E \longrightarrow E \otimes_Q \Omega^1 Q$ such that $\nabla s = 0$ for $s \in S$. This makes $E \otimes_Q \Omega Q$ into a complex. In fact it is just the complex $S \otimes \Omega Q$ with the differential $1 \otimes d$.

But we also have left multiplication by E on $E \otimes_Q \Omega Q$. We consider the DG algebra of operators on $E \otimes_Q \Omega Q$, denote this $\mathrm{End}(E \otimes_Q \Omega Q) = R$. We have a map $E \longrightarrow R$ given by left mult., and elements of S commute with the differentials. Thus by the universal property of $\Omega^\bullet(E/S)$ we should get a DG homorphism $\Omega(E/S) \longrightarrow R$, whence a map $\Omega(E/S) \longrightarrow E \otimes_Q \Omega Q$ by acting on 1. The fact this is an isomorphism ought to be similar to the result for ΩA.

August 31, 1990

Relative theory. Given a map of algs $S \longrightarrow A$ we have relative constructions

$$R(A;S) \qquad Q(A;S) = A *_S A \qquad \Omega(A;S).$$

$R(A;S)$ is a universal algebra R equipped with an algebra hom. $S \longrightarrow R$ and an S-bimodule map $A \xrightarrow{\rho} R$ such that $\rho(s) = s$ for all $s \in S$. Thus

$$R(A;S) = T_S(A) / (\rho(s) - s)$$

$\Omega(A;S)$ is a universal DGA Ω equipped with an alg homomorphism $A \longrightarrow \Omega^0$ such that $ds = 0 \ \forall s \in S$.

These constructions are easily seen to be the same as those where S is replaced by its image in A. So we can suppose S is a subalgebra of A.

Then we have an exact sequence of S-bimodules

$$(*) \quad 0 \longrightarrow S \longrightarrow A \longrightarrow A/S \longrightarrow 0$$

and we can form relative normalized chains

$$A \otimes_S T_S^n(A/S) = A \otimes_S (A/S) \otimes_S \overset{n}{\cdots} \otimes_S (A/S)$$

This is what one obtains by normalizing the simplicial S-bimodule $[n] \longmapsto A \otimes_S \overset{n+1}{\cdots} \otimes_S A$. It seems clear that we get differentials b, d on these chains, where

$$d(a_0 \otimes \bar{a}_1 \otimes \cdots \otimes \bar{a}_n) = 1 \otimes \bar{a}_0 \otimes \cdots \otimes \bar{a}_n$$

and $\bar{a}$ denotes the image of $a \in A$ in A/S.

Note that in general one does not expect the d homology to be trivial since if we tensor $\circledast$ with $T^n_S(A/S)$ we get only

$$T^n_S(A/S) \longrightarrow A \otimes_S T^n_S(A/S) \longrightarrow T^{n+1}_S(A/S) \longrightarrow 0$$

Thus for the exactness of d (except degree 0) we want to assume $\circledast$ splits as a sequence of <u>right</u> S-modules (or weaker that A/S is right S-flat).

At this point we expect all of Ch. I to extend to this relative situation. Let's write it out carefully for $R(A;S) = R$. Define

$$\Omega^{ev} = \bigoplus_{n \geq 0} A \otimes_S T^{2n}_S(A/S) \xrightarrow{\quad \Phi \quad} R$$

$$(a_0, \cdots, a_{2n}) \longmapsto \rho(a_0)\,\omega(a_1, a_2) \cdots \omega(a_{2n-1}, a_{2n})$$

Here $\omega(a_1, a_2) = \rho(a_1 a_2) - \rho(a_1)\rho(a_2)$. Since $\rho : A \to R$ is assumed to be an S-bimodule map which restricts to the algebra homom. $S \to R$, it is clear that

$$\omega : (A/S) \otimes_S (A/S) \longrightarrow R$$

is an S-bimodule morphism. Thus Φ is well-defined. By Ch. I arguments Φ is surjective. By the same formulas one defines an R-module structure on Ω^{ev} which gives $R \longrightarrow \Omega^{ev}$ by acting on 1. Check Φ is an R-module homom.

by looking at generators. Clear.

Corresponding to these relative $\rho : A \to B$, i.e. S-bimodule maps such that $\rho(1) = 1$. is a relative GNS construction

$$\Gamma_S(\rho) = A \oplus A \otimes_S B \otimes_S A .$$

Here the canonical idempotent e commutes with S.

Suppose that S, Q are subalgebras of E such that multiplication gives a bijection $S \otimes Q \xrightarrow{\sim} E$. We wish to consider the relative complex

$$E/[S,E] \rightleftarrows \Omega^1(E;S)_\natural$$

Now we have $E = S \otimes Q$ and $\Omega^1(E;S) = S \otimes \Omega^1 Q$. In the case where E is the tensor product algebra one has

$$E/[S,E] = S \otimes Q/[S, S \otimes Q] = S_\natural \otimes Q$$

$$\Omega^1(E;S)_\natural = S_\natural \otimes \Omega^1 Q_\natural$$

and we get an isomorphism

$$S_\natural \otimes \left(Q \rightleftarrows \Omega^1 Q_\natural \right) \xrightarrow{\sim} \left(E/[S,E] \rightleftarrows \Omega^1(E;S)_\natural \right)$$

However in the general case there doesn't even seem to be a map. Thus if $s \in S$, $x\,dy \in \Omega^1 Q$ then we might try to send $s \otimes x\,dy$ on the left to

$s x\, dy$ on the right. But for this to be well-defined on $S \otimes \Omega^1 Q_\natural$ we need to have

$$s\, zx\, dy \overset{?}{=} s\, x\, dy\, z \quad \text{in } \Omega^1(E;S)_\natural$$
$$\qquad\qquad\quad \| $$
$$\qquad\qquad\quad zs\, x\, dy$$

which means effectively that we have to assume $s \in S$, $z \in Q$ commutes.

Note that in the case $E = EA$, $Q = QA$, $S = k[F]$ and the superalgebra setting we have a <u>tricky isomorphism</u>

$$\left(Q \rightleftarrows \Omega^1 Q_{\natural s}\right)^{\pm} \simeq \left(E/[S,E]_s \rightleftarrows \Omega^1(E;S)_{\natural s}\right)^{\mp}$$

using F for the (+ left − right) case and not using F in the other. Now $S_{\natural s} = kF$, so this tricky isomorphism is not in any obvious way of the form $S_\natural \otimes X^s(Q) \longrightarrow X^s(E;S)$.

September 3, 1990

$$0 \to \Omega^1 S \to S \otimes S \to S \to 0$$

splits as a sequence of either left or right S-modules. If M is an S-bimodule then we have exact sequence of S-bimods

$$0 \to M \otimes_S \Omega^1 S \to M \otimes S \to M \to 0$$

hence

$$0 \to H_1(S,M) \to M \otimes_S \Omega^1 S \otimes_S \xrightarrow{b} M \to M/[S,M] \to 0$$

since $M \otimes S$ is acyclic for $H_\cdot(S,?)$. Thus get exact seq.

$$\boxed{0 \to H_1(S,M) \to M \otimes_S \Omega^1 S \otimes_S \to [S,M] \to 0}$$

also

$$\begin{tikzcd}
 & & & 0 \arrow[d] & 0 \arrow[d] & \\
 & & & \Omega^1 E \arrow[r] \arrow[d] & \Omega^1(E;S) \arrow[d] & \\
0 \arrow[r] & \mathrm{Tor}_1^S(E,E) \arrow[r] & E \otimes_S \Omega^1 S \otimes_S E \arrow[r] & E \otimes E \arrow[r] \arrow[d] & E \otimes_S E \arrow[r] \arrow[d] & 0 \\
 & & & E \arrow[r, equal] \arrow[d] & E \arrow[d] & \\
 & & & 0 & 0 &
\end{tikzcd}$$

gives an exact sequence

$$\boxed{0 \to \mathrm{Tor}_1^S(E,E) \to E \otimes_S \Omega^1 S \otimes_S E \to \Omega^1 E \to \Omega^1(E;S) \to 0}$$

Consider the right part of this sequence and of the preceding one :

$$\begin{tikzcd}
E\otimes_S\Omega^1S\otimes_S E \arrow[r] \arrow[d, "\wr"] & \Omega^1E \arrow[r] \arrow[d] & \Omega^1(E;S) \arrow[r] \arrow[d] & 0 \\
\cdots \arrow[r] & E\otimes E \arrow[r] & E\otimes_S E \arrow[r] & 0
\end{tikzcd}$$

and apply $H_0(S,?)$ to obtain

$$\begin{tikzcd}
 & E\otimes_S\Omega^1S\otimes_S \arrow[r] \arrow[d, "b"] & \Omega^1E_\natural \arrow[r] \arrow[d, "b"] & \Omega^1(E;S)_\natural \arrow[r] \arrow[d, "b"] & 0 \\
0 \arrow[r] & {[S,E]} \arrow[r] & E \arrow[r] & E/[S,E] \arrow[r] & 0
\end{tikzcd}$$

If $H_1(S,E)=0$ the ~~right~~ left b is an isomorphism. Since $d: E\to\Omega^1E_\natural$ kills $[E,E]$ we get an exact sequence of $\mathbb{Z}/2$-graded cxs.

$$\begin{tikzcd}
0 \arrow[r] & E\otimes_S\Omega^1S\otimes_S \arrow[r] \arrow[d, shift right, "b"', "\simeq"] & \Omega^1E_\natural \arrow[r] \arrow[d, shift right] & \Omega^1(E;S)_\natural \arrow[r] \arrow[d, shift right] & 0 \\
0 \arrow[r] & {[S,E]} \arrow[r] \arrow[u, shift right, "0"'] & E \arrow[r] \arrow[u, shift right] & E/[S,E] \arrow[r] \arrow[u, shift right] & 0
\end{tikzcd}$$

We therefore obtain:

Prop. If S is a subalgebra of E such that $H_1(S,E)=0$, then one has an exact sequence

(*) $\quad 0 \longrightarrow K \longrightarrow X(E) \longrightarrow X(E;S) \longrightarrow 0$

where K: $[S,E] \underset{0}{\overset{b}{\leftrightarrows}} E\otimes_S\Omega^1S\otimes_S$ has zero homology.

Next we discuss splitting the exact sequence $(\ast)$. This should employ a strengthening of the condition $H_1(S,E)=0$. The natural condition is that one has a section of $E\otimes S \xrightarrow{b'} E$ which is an S-bimodule map. Thus we want $\varphi_i : E \longrightarrow E$ $\quad s_i \in S$ satisfying

$$\sum s_i\varphi_i(\xi) = 1 \qquad\qquad \varphi_i(s\xi) = s\varphi_i(\xi)$$

$$\sum \varphi_i(\xi s)\otimes s_i = \sum \varphi_i(\xi)\otimes s_i s$$

When $E=S$, then $\varphi_i(\xi) = \xi s_i'$, so we have $Y = \sum s_i'\otimes s_i \in S\otimes S$ such that $b'Y = \sum s_i' s_i = 1$ and $sY = Ys$, $\forall s\in S$.

We then have the splitting

$$\begin{array}{ccccccccc}
0 & \longrightarrow & E\otimes_S \Omega^1 S & \rightleftarrows & E\otimes S & \rightleftarrows & E & \longrightarrow & 0 \\
 & & & & \sum\varphi_i(\xi)\otimes s_i & \longleftarrow & \xi & & \\
 & & \sum\varphi_i(\xi)ds_i & \longleftarrow & \xi\otimes 1 & & & &
\end{array}$$

Check: $\quad \xi\, ds \longmapsto \xi(s\otimes 1 - 1\otimes s)$

$$\sum\varphi_i(\xi s)ds_i - \sum\varphi_i(\xi)ds_i s$$

$$\parallel$$

$$\sum\varphi_i(\xi)d(s_i s) - \sum\varphi_i(\xi)ds_i s$$

$$\parallel$$

$$\sum\varphi_i(\xi)s_i\, ds = \xi\, ds$$

$$\sum\varphi_i(\xi)ds_i \longleftarrow \xi\otimes 1 \longrightarrow \xi$$

$$\longmapsto \sum\varphi_i(\xi)(s_i\otimes 1 - 1\otimes s_i) \;\Big|\; \sum\varphi_i(\xi)\otimes s_i$$

these add to
$\sum\varphi_i(\xi)s_i\otimes 1$
$= \xi\otimes 1$

So the result is the splitting

$$\begin{tikzcd} 0 \arrow[r] & E \otimes_S \Omega^1 S \otimes_S \arrow[r] \arrow[d, "b"', "\cong"] & \Omega^1 E_\natural \arrow[r] \arrow[d] \arrow[l, dotted, bend right, leftarrow] & \Omega^1(E;S) \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & {[S,E]} \arrow[r] & E \arrow[r] \arrow[ul, dotted, "\alpha"'] & E/[S,E] \arrow[r] \arrow[l, dotted, bend right, "\beta"'] & 0 \end{tikzcd}$$

$$\alpha(\xi) = \sum \varphi_i(\xi)\, ds_i \qquad \beta(\xi) = \sum s_i \varphi_i(\xi)$$

$$\Big\downarrow b$$

$$\sum [\varphi_i(\xi), s_i] = \xi - \beta(\xi).$$

Maybe an important thing is the map

$$\begin{tikzcd} \Omega^1 E_\natural \arrow[r] \arrow[d, shift right, "b"'] & {[S,E]} \arrow[d, shift right, "1"'] \\ E \arrow[r, "\bar\alpha"] \arrow[u, shift right, "d"'] & {[S,E]} \arrow[u, shift right, "0"'] \end{tikzcd}$$

$$\xi \overset{\bar\alpha}{\longmapsto} \sum [\varphi_i(\xi), s_i]$$

September 5, 1990

Consider an alg morphism $S \to E$. We have seen there is a comm. diagram with exact rows

$$\begin{tikzcd}
0 \arrow[r] & {[S,E]} \arrow[r] & E \arrow[r] & E/[S,E] \arrow[r] & 0 \\
 & E\otimes_S \Omega^1 S \otimes_S \arrow[u, "b"] \arrow[r] & \Omega^1 E_\natural \arrow[u, "b"] \arrow[r] & \Omega^1(E;S)_\natural \arrow[u, "b"] \arrow[r] & 0
\end{tikzcd}$$

and that one has an exact seq

$$0 \to H_1(S,E) \to E\otimes_S \Omega^1 S \otimes_S \xrightarrow{b} [S,E] \to 0$$

From this we conclude

<u>Prop.</u> $H_1(S,E) = 0 \implies$ the canonical surjection $X(E) \to X(E;S)$ is a quasi-isom. The kernel is

$$[S,E] \underset{d=0}{\overset{b\,\cong}{\rightleftarrows}} E\otimes_S \Omega^1 S \otimes_S \qquad \text{(Call this } K\text{)}$$

We have seen that if we strengthen the condition $H_1(S,E)=0$ to requiring that $E\otimes S \xrightarrow{b'} E$ has an S-bimodule section $\xi \mapsto \sum \varphi_i(\xi)\otimes s_i$, then we ~~[illegible]~~ have a splitting of

$$0 \to K \to X(E) \to X(E;S) \to 0$$

Specifically consider

$$\begin{tikzcd}
0 \arrow[r] & {[S,E]} \arrow[r] & E \arrow[r] \arrow[l, dashed, bend right] & E/[S,E] \arrow[r] \arrow[l, dashed, bend right] & 0 \\
0 \arrow[r] & E\otimes_S \Omega^1 S \otimes_S \arrow[u] \arrow[r] & \Omega^1 E_\natural \arrow[u, "b"] \arrow[r] \arrow[l, dashed, bend right] & \Omega^1(E;S)_\natural \arrow[u] \arrow[r] \arrow[l, dashed, bend right] & 0
\end{tikzcd}$$

$\sum [\varphi_i(\xi), s_i] \xleftarrow{\alpha} \xi$ $\qquad \sum s_i \varphi_i(\xi) \xleftarrow{\beta} \xi$

$\sum_i \varphi_i(\xi)\, ds_i$

$\sum \varphi_i [\xi,\eta]\, ds_i \xleftarrow{\alpha} \xi\, d\eta$

$\xi\, d\eta - \sum \varphi_i [\xi,\eta]\, ds_i \xleftarrow{\beta} \xi\, d\eta$

Now the important thing is the lifting $\beta : X(E;S) \longrightarrow X(E)$, which seems to be given by

$$\beta(\xi) = \sum s_i \varphi_i(\xi) \qquad \text{on } E$$

$$\beta(\xi\, d\eta) = \xi\, d\eta - \sum \varphi_i[\xi,\eta]\, ds_i \qquad \text{on } \Omega(E;S)/?$$

We should check that $\beta(\xi\, ds) = 0$. But by assumption $\sum_i \varphi_i(s\xi) \otimes s_i = s \sum \varphi_i(\xi) \otimes s_i$

$$\sum_i \varphi_i(\xi s) \otimes s_i = \left(\sum \varphi_i(\xi) \otimes s_i\right) s$$

and $\sum \varphi_i(\xi) s_i = \xi$

so

$$\beta(\xi\, ds) = \xi\, ds - \sum \varphi_i(\xi s)\, ds_i + \sum_i \varphi_i(s\xi)\, ds_i$$

$$= \xi\, ds - \sum \varphi_i(\xi)\, d(s_i s) + s \sum \varphi_i(\xi)\, ds_i$$

$$= \xi\, ds - \underbrace{\sum \varphi_i(\xi) s_i\, ds}_{\xi\, ds} + \underbrace{\left[s, \sum \varphi_i(\xi)\, ds_i\right]}_{0}$$

$$= 0$$

and this is OK.

Let's now consider the situation where $\varphi_i(\xi) = \xi x_i \qquad x_i \in E$. Then

$$\beta(\xi) = \sum s_i \xi x_i$$

$$\beta(\xi\, d\eta) = \text{[illegible]}$$

$$= \xi\, d\eta - [\xi,\eta] \sum_i x_i\, ds_i$$

So if $X = \sum x_i ds_i$, then

$$\beta(\xi d\eta) = \xi d\eta - [\xi, \eta]X$$
$$= \xi d\eta - \xi[\eta, X]$$
$$= \xi\big(d\eta - [\eta, X]\big)$$

Check this. Assume that $\sum x_i \otimes s_i \in E \otimes S$ defines a left E, right S-bimodule ~~section~~ section of $E \otimes S \xrightarrow{b'} E$. This means that

$$s\left(\sum x_i \otimes s_i\right) = \left(\sum x_i \otimes s_i\right)s \qquad \sum x_i s_i = 1.$$

Put $X = \sum x_i ds_i \in \Omega^1 E$. Then consider $\Delta : E \longrightarrow \Omega^1 E$ given by

$$\Delta(\xi) = d\xi - [\xi, X]$$

One has

$$[s, X] = \underbrace{s \sum x_i ds_i} - \sum x_i ds_i s$$

$$\sum x_i d(s s_i [redacted])$$

as $\sum s x_i \otimes s_i = \sum x_i \otimes s_i s$ in $E \otimes S$

$$= \left(\sum x_i s_i\right) ds = ds$$

Thus $\Delta = 0$ on S, ~~thus~~ we get a E-bimodule map $\tilde{\Delta} : \Omega^1(E; S) \longrightarrow \Omega^1 E$, which gives our [redacted] section $\beta : \Omega^1(E;S)_\natural \longrightarrow \Omega^1 E_\natural$ defined by $\beta(\xi d\eta) = \xi(d\eta - [\eta, X])$.

Here is an improvement. Let us ask for

an E-bimodule section of $E \otimes E \longrightarrow E \otimes_S E$. Such a section is given by $\sum x_i \otimes y_i \in E \otimes E$ satisfying $\sum x_i \underset{S}{\otimes} y_i = 1 \underset{S}{\otimes} 1$, $\sum s x_i \otimes y_i = \sum x_i \otimes y_i s$.

In effect the section is $\xi \underset{S}{\otimes} \eta \longmapsto \xi \left(\sum x_i \otimes y_i \right) \eta$ and this is well-defined by the second condition.

So now put $X = \sum x_i \, dy_i$ and consider $\Delta : E \longrightarrow \Omega^1 E$ given by $\Delta(\xi) = d\xi - [\xi, X]$.

Then $[s, \sum x_i dy_i] =$

$$\sum s x_i \, dy_i - \sum x_i \, dy_i \, s$$

$$= \sum x_i \, d(y_i s) - \sum x_i \, dy_i \, s$$

$$= \sum x_i y_i \, ds = ds$$

so $\Delta = 0$ on S, and we obtain $\tilde{\Delta} : \Omega^1(E; S) \to \Omega^1 E$.

September 6, 1990

Consider the ~~map~~ surjection of exact sequences of E-bimodules

$$\begin{array}{ccccccccc} 0 & \to & \Omega^1 E & \xrightarrow{\partial} & E\otimes E & \longrightarrow & E & \longrightarrow & 0 \\ & & \downarrow & & \downarrow \uparrow \alpha & & \| & & \\ 0 & \to & \Omega^1(E;S) & \xrightarrow{\partial} & E\otimes_S E & \longrightarrow & E & \longrightarrow & 0 \end{array}$$

Suppose given a bimodule section α of the middle map. This is the same as an element $z = \sum x_i \otimes y_i \in E\otimes E$ such that

$$s\sum x_i \otimes y_i = \sum (x_i \otimes y_i)s \qquad \forall s \in S$$

$$\sum x_i \otimes_S y_i = 1\otimes_S 1 \quad \in E\otimes_S E$$

Then it induces a bimodule section of the map at the left, which we now compute. Take $d_S y \in \Omega^1(E;S)$. Then $\alpha\,\partial(d_S y) = \alpha(y\otimes_S 1 - 1\otimes_S y) = yz - zy$. Also viewing $dy \in \Omega^1(E)$ one has $\partial(dy) = y(1\otimes 1) - (1\otimes 1)y$ so

$$\partial(dy - \alpha(d_S y)) = y(1\otimes 1 - z) - (1\otimes 1 - z)y$$

$$\begin{aligned} 1\otimes 1 - z &= 1\otimes 1 - \sum x_i\otimes y_i \\ &= \sum x_i(y_i\otimes 1 - 1\otimes y_i) \\ &= \partial\left(\sum x_i\, dy_i\right) \end{aligned}$$

$$X = \sum x_i dy_i$$

$$\therefore \quad dy - \alpha(d_S y) = [y, \sum x_i dy_i]$$

$$\boxed{\alpha(x d_S y) = x(dy - [y, X])}$$

Next pass to commutator quotient spaces. This gives

$$\begin{tikzcd}
\Omega^1 E_\natural \arrow[r, "b"] \arrow[d] & E \arrow[d] \\
\Omega^1(E;S)_\natural \arrow[r] \arrow[u, bend left, dashed, "\alpha"] & E/[S,E] \arrow[u, bend left, dashed, "\alpha"]
\end{tikzcd}$$

where
$$\alpha(x\,d_S y) = x(dy - [y, X])$$
$$= x\,dy - [x,y]X \quad \text{in } \Omega^1 E_\natural$$
and
$$\alpha(x) = \sum y_i x x_i.$$
Check compatibility of α with b, d.

$$b\alpha(x\,d_S y) = [x,y] - \underbrace{\sum_i [[x,y]x_i, y_i]}_{[x,y]\underbrace{\sum x_i y_i}_{1} - \sum y_i [x,y] x_i}$$

$$= \sum y_i [x,y] x_i = \alpha b(x\,dy)$$

Note that $\alpha(x) = \sum y_i x x_i = \underbrace{\sum x x_i y_i}_{x} - \sum [x x_i, y_i]$ is congruent to x modulo $[E,E]$ and hence $d\alpha(x) = dx$ in $\Omega^1 E_\natural$. Also $\alpha(d_S x) = dx - [x, X] = dx$, so $\alpha d_S = d\alpha$

So we have constructed a section of the canonical surjection

$$X(E) \underset{\overset{\alpha}{\dashleftarrow}}{\twoheadrightarrow} X(E;S)$$

September 12, 1990

Let $R = T_A(M)$ where M is an A-bimodule. We have

$$\Omega^1(R;A) \Longleftarrow R \otimes_A M \otimes_A R$$

First proof: An A-derivation D with values in N is the same as a lifting homom. $R \xrightarrow{u} R \oplus N$ whose restriction to A is $a \mapsto a \oplus 0$. By the univ. prop of R, a homo $R \to R\oplus N$ is equivalent to an A-bimodule map $M \longrightarrow R \oplus N$ whose first component is the inclusion of M in R. In general, when the homomorphism $A \to R \oplus N$ is not $a \longmapsto a \oplus 0$, the A-bimodule structure on $R \oplus N$ is tricky, however, when $a \mapsto a \oplus 0$ then ~~the~~ we are concerned with just an A-bimod morphism $M \longrightarrow N$, or equivalently an R-bimodule morphism $R \otimes_A M \otimes_A R \longrightarrow N$.

2nd proof: Use

$$0 \longrightarrow \Omega^1(R;A) \longrightarrow R \otimes_A R \longrightarrow R \longrightarrow 0$$

$$R \otimes_A R = \bigoplus_{i,j \geq 0} \underbrace{T_A^i M \otimes_A T_A^j N}_{T_A^{i+j} M}$$

$$R \otimes_A M \otimes_A R = \bigoplus_{i,j \geq 0} T_A^{i+j+1} M$$

and do some combinatorics.

We now consider the relative complex

$$X(R;A): \qquad R/[A,R] \rightleftarrows \Omega^1(R;A)_\natural$$

One has $R/[A,R] = \bigoplus_{n\geq 0} [M\otimes_A]^n$

and

$$\Omega^1(R;A)_\natural = R\otimes_A M\otimes_A = \bigoplus_{n\geq 0} [M\otimes_A]^{n+1}$$

The b, d maps should be just as the case of a tensor algebra: $b = 1-\sigma$ and $d = N$ on the cyclic tensor products. Thus the homology of $X(R;A)$ should reduce to $A/[A,A]$ on the even side.

Next we consider $X(R)$. Let's suppose to simplify that M is flat as both a left and a right A-module. Then we have $\mathrm{Tor}_1^A(R,R)=0$ hence a diagram with exact rows

$$\begin{array}{ccccccccc} 0 & \to & R\otimes_A \Omega^1 A\otimes_A R & \to & R\otimes R & \to & R\otimes_A R & \to & 0 \\ & & \| & & \cup & & \cup & & \\ 0 & \to & R\otimes_A \Omega^1 A\otimes_A R & \to & \Omega^1 R & \to & \Omega^1(R;A) & \to & 0 \end{array}$$

Now we want to take commutator ~~quotient~~ spaces, From the top row we get an exact sequence

$$* \quad 0 \to H_1(R, R\otimes_A R) \to R\otimes_A \Omega^1 A\otimes_A \to R \to R/[A,R] \to 0$$

$$\| $$

$$H_1\big((R\otimes_A R)\overset{\mathbb{L}}{\otimes}_R\big) = H_1\big((R\otimes_A R)\otimes_R P\otimes_R\big) = H_1(P\otimes_A)$$

where P is a free R-bimodule resolution of R. Notice that a free R-bimodule $R\otimes V\otimes R$ is a flat A-bimodule as we are assuming M left & right A-flat. Thus

$$H_1(P\otimes_A) = H_1(R\overset{\mathbb{L}}{\otimes}_A) = H_1(A,R)$$

Actually this is obviously from the exact sequence $*$. The argument should have been given for the other sequence

$$\longrightarrow H_1(R, R\otimes_A M\otimes_A R) \longrightarrow R\otimes_A \Omega^1 A\otimes_A \longrightarrow \Omega^1 R_\natural \longrightarrow R\otimes_A M\otimes_A \longrightarrow 0$$

$$\|$$

$$H_1\big((R\otimes_A M\otimes_A R)\otimes_R P\otimes_R\big) = H_1\big(M\otimes_A P\otimes_A\big)$$

$$= H_1(A, M\otimes_A R) = H_1(A, R^{>0})$$

$$= \mathrm{Ker}\left\{R^{>0}\otimes_A \Omega^1 A\otimes_A \longrightarrow R^{>0}\right\}$$

Let's review & start with

$$\begin{tikzcd}
0 \arrow[r] & R\otimes_A \Omega^1 A\otimes_A R \arrow[r] & R\otimes R \arrow[r] & R\otimes_A R \arrow[r] & 0 \\
0 \arrow[r] & \cdots\cdots \arrow[u, equal] \arrow[r] & \Omega^1 R \arrow[u, hook] \arrow[r] & \Omega^1(R;A) \arrow[u, hook] \arrow[r] & 0 \\
 & & & R\otimes_A M\otimes_A R \arrow[u, equal] &
\end{tikzcd}$$

Apply $\otimes_R$ and we obtain

$$\begin{tikzcd}
0 \arrow[r] & H_1(R, R\otimes_A R) \arrow[r] & R\otimes_A \Omega^1 A\otimes_A \arrow[r] & R \arrow[r] & R/[A,R] \arrow[r] & 0 \\
\arrow[r] & H_1(R, R\otimes_A M\otimes_A R) \arrow[u] \arrow[r, "\delta"] & R\otimes_A \Omega^1 A\otimes_A \arrow[u, equal] \arrow[r] & \Omega^1 R_\natural \arrow[u] \arrow[r] & \Omega(R;A)_\natural \arrow[u] \arrow[r] & 0
\end{tikzcd}$$

with exact rows, hence a short exact sequence of complexes

$$**\qquad\begin{tikzcd}
0 \arrow[r] & {[A,R]} \arrow[r] \arrow[d, shift right, "d=0"'] & R \arrow[r] \arrow[d, shift right, "d"'] & R/[A,R] \arrow[r] \arrow[d, shift right, "d"'] & 0 \\
0 \arrow[r] & \mathrm{Cok}(\delta) \arrow[r] \arrow[u, shift right, "b"'] & \Omega^1 R_\natural \arrow[r] \arrow[u, shift right, "b"'] & \Omega^1(R;A)_\natural \arrow[r] \arrow[u, shift right, "b"'] & 0
\end{tikzcd}$$

What is $\operatorname{Coker}(\delta)$? We have isomorphisms

$$H_1(R, R\otimes_A R) \xleftarrow{\cong} H_1(A,R) = \operatorname{Ker}\left\{ R\otimes_A \Omega^1 A \twoheadrightarrow [A,R] \right\}$$

$$H_1(R, R\otimes_A M \otimes_A R) \xleftarrow{\cong} H_1(A, \underbrace{R\otimes_A M}_{\cong R^{>0}}) = \operatorname{Ker}\left\{ R^{>0}\otimes_A \Omega^1 A \twoheadrightarrow [A,R^{>0}] \right\}$$

Consider the $\mathbb{N}$ grading given by M-degree. $R\otimes_A M \otimes_A R$ is supported in degrees >0. So we know $(R\otimes_A \Omega^1 A \otimes_A)(0) = (\operatorname{Cok}(\delta))(0) = \Omega^1 A \otimes_A$.

A natural <u>conjecture</u> is

$$\operatorname{Cok}(\delta) = \Omega^1 A_\natural \oplus [A, R^{>0}]$$

If so, the 6-term exact sequence of homology associated to $**$ is

$$0 \to \underset{B}{H^{DR}_0(R)} \to A/[A,A] \to HH_1(A) \to HC_1(R) \to 0$$

I am assuming the b-map $\operatorname{Coker}(\delta) \to [A,R]$ is the map $b : \Omega A_\natural \to [A,A]$ direct sum the obvious inclusion $[A,R^{>0}] \hookrightarrow [A,R]$. If this is so we get the Conjecture:

$$H^{DR}_0(A) = H^{DR}_0(R)$$
$$HC_1(A) = HC_1(R)$$

assuming $R = T_A(M)$ where M is left + right A-flat. The first is clearly true since H^{DR}_0 is a homotopy invariant and R deforms to A.

Notice that because the b,d homology at the point $\Omega^1(R;A)_\natural$ is zero, and because $H_0^{DR}(R) = H_0^{DR}(A)$ injects into $A/[A,A]$, we know that the b,d homology at the point $[A,R]$ is zero. This shows b maps $\mathrm{Cok}(\delta)$ onto $[A,R]$.

To prove it one has to compute the map $H_1(R, R\otimes_A M \otimes_A R) \longrightarrow H_1(R, R\otimes_A R)$. Maybe a better way is the following:

$$\begin{array}{ccccc}
R\otimes_A \Omega^1 A \otimes_A & \twoheadrightarrow & [A,R] & \hookrightarrow & R \\
\Vert & & \uparrow b & & \uparrow b \\
R\otimes_A \Omega^1 A \otimes_A & \twoheadrightarrow & \mathrm{Cok}(\delta) & \hookrightarrow & \Omega^1 R_\natural
\end{array}$$

What you want to show is that $\mathrm{Cok}(\delta)_{>0}$ injects via b into $[A,R]_{>0}$. So if we take an element $\gamma \in R\otimes_A \Omega^1 A \otimes_A$ of degree >0 which goes to zero in $[A,R]$, then perhaps the reason it vanishes can be used to show γ becomes zero in $\Omega^1 R_\natural$.

Incidently Feigin + Tsygan should tell me what $HC_1(R)$ is in terms of some Nil-term.

September 19, 1990

Fredholm modules over A naturally leads to $EA = A * \mathbb{C}[F] = (QA) \rtimes \mathbb{Z}/2$.

Consider $A \xrightarrow{\theta} L$ a homom. and $F \in L$ an involution. This pair is the same as a homomorphism $E = EA \longrightarrow L$. Suppose Tr is a trace on L. Pulling back gives a trace on E, which we again denote Tr. This trace can be split into even and odd traces. Now an odd trace on E is equivalent to an even supertrace on Q. It turns out that an even trace on E is equivalent to an even trace on Q.

Lets introduce notation. Let τ be a trace on E and $\tau^{\pm}$ its even and odd components. One has

$$\tau^{+}(x + Fy) = \tau(x)$$
$$\tau^{-}(x + Fy) = \tau(Fy)$$

We know $y \mapsto \tau(Fy)$ is an even supertrace on Q. Since

$$
\begin{aligned}
[E,E] &= [F,E] + [Q,E] \\
&= [F, Q + FQ] + [Q, Q + FQ] \\
&= FQ^{-} + Q^{-} + [Q,Q] + F[Q,\gamma Q] \\
&= \left(Q^{-} + [Q,Q]\right) \oplus F\left(Q^{-} + [Q,\gamma Q]\right)
\end{aligned}
$$

$$E/[E,E] = \underbrace{\left(Q/Q^{-} + [Q,Q]\right)}_{\text{universal for even traces}} \oplus F\underbrace{\left(Q/Q^{-} + [Q,\gamma Q]\right)}_{\text{universal for even } \gamma\text{-traces}}$$

we know $\tau^{\pm}$ are equivalence to even traces and even supertraces on Q. The former is described by $\tau^{+}_{2n} = \tau(\rho q^{2n})$, $n \geq 0$ which satisfy $b\tau^{+}_{2n} = 0$, $\kappa\tau^{+}_{2n} = -\tau^{+}_{2n}$ and which are arbitrary subject to these conditions. The latter (even supertrace) is described by $\tau^{-}_{2n} = \tau(F\rho q^{2n})$ which satisfy $b\tau^{-}_{2n} = \frac{1}{n+1} B\tau^{-}_{2n+2}$ $\kappa\tau^{-}_{2n} = \tau_{2n}$, $n \geq 0$ and are arbitrary satisfying these conditions.

The important point is that a trace on E ~~[illegible]~~ has an even part equivalent to an even trace on Q and an odd part equivalent to an odd supertrace on Q. The former is needed for positivity and latter is cohomologically interesting.

Now suppose we are given $\theta : A \to L$ $F, \varepsilon \in L$ involutions, anticommuting, and $\varepsilon a = a \varepsilon$ $\forall a \in A$. This is the graded setup. Suppose Tr given on L. The universal algebra for this setup is $E \times \mathbb{Z}/2$. Denote this algebra R; we have $R = E + \varepsilon E = Q + FQ + \varepsilon(Q + FQ)$. By "duality" $R \simeq M_2 Q$. Tr pulls back to a trace τ on R, which by Morita equivalence is equivalent to a trace on Q. We can decompose this trace on Q into even and odd components, so we get an even trace and an

odd supertrace on Q which carries the cohomological info.
Here's the way the Morita equiv. works.

Lemma: Let e be an idempotent in an algebra R such that $ReR = R$. Then eRe is canonically Morita equivalent to R.

Proof. I guess we really want to work more generally first. Suppose e idempotent in R. One has a projective right R-module eR, whose endomorphisms is the algebra eRe. Note: $eR = R/(1-e)R$, so that $\mathrm{Hom}_{R^o}(eR, N) = \{n \mid ne = n\} = Ne$, in particular $\mathrm{Hom}_{R^o}(eR,$ [redacted] $eR) = eRe$. The issue in Morita equivalence in this case is whether eR is a [redacted] generator for $P(R)$, i.e. whether there is a surjection $(eR)^{\oplus n} \longrightarrow\!\!\!\!\to R$.
Such a surjection has to be of the form $(er_1, \dots, er_n) \longmapsto \sum x_i r_i$ where $x_i e = x_i \;\; \forall i$.
Thus one has such a surjection exactly when $ReR = R$. In this case we have the explicit bimodules giving the M-equivalence:

$$ {}_R Re_{eRe} \qquad {}_{eRe} eR_R $$

$$ eR \otimes_R Re \xrightarrow{\sim} eRe $$

$$ Re \otimes_{eRe} eR \xrightarrow{\sim} R $$

(Direct proof of latter: The map is surjective, hence we have a split exact R-bimodule sequence

$$0 \longrightarrow K \longrightarrow Re \otimes_{eRe} eR \longrightarrow R \longrightarrow 0$$

One has $0 \to eK \to eRe \otimes_{eRe} eR \to eR \to 0$

$$eRe \otimes_{eRe} eR = eR$$

so $eK = 0$, so

$$\left(Re \otimes_{eRe} eR\right) \otimes_R K \twoheadrightarrow R \otimes_R K = K$$

$$\left(Re \otimes_{eRe} eR\right) \otimes_R K = Re \otimes_{eRe} (eK) = 0.\ )$$

Now go back to $R \cong Q + FQ + \varepsilon Q + \varepsilon FQ$ and consider the idempotent $e = \frac{1+\varepsilon}{2}$. One has $e + FeF = e + 1 - e = 1$, so $ReR = R$. Also $e(x + Fy + \varepsilon z + \varepsilon Fz)e = ex + \cancel{(eFe)}y + ez + \varepsilon\cancel{(eFe)}z$ since ε (hence e) commutes with Q. Thus $eRe = eQ$, and so we see that the map from traces τ on R to traces on Q given by $\tau \longmapsto \left(x \longmapsto \tau(ex) = \tau\left(\frac{1+\varepsilon}{2}x\right)\right)$ is bijective. Split the trace into even + odd parts:

$$\tau\left(\frac{1+\varepsilon}{2}x\right) = \underbrace{\frac{1}{2}\tau(x)}_{\text{even}} + \underbrace{\frac{1}{2}\tau(\varepsilon x)}_{\text{odd}}$$

since γ on Q is conjugating by F.

Here's why $R \cong M_2(Q)$. Consider

the homomorphism $Q \longrightarrow R$ given by

$$x \longrightarrow ex + (1-e)x^\gamma$$
$$= ex + F(ex)F$$

Think of a graded Fredholm module $H = H^+ \oplus H^-$ where $\varepsilon = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ $F = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ and we taking the action of Q on H^+ and transporting it to H^- via F. It is clear that the image of Q under this homomorphism commutes with ε, F which generate $M_2\mathbb{C}$.

To summarize we learn that a trace on E (ungraded) (resp $E \times \mathbb{Z}/2$ (graded)) is equivalent to an even trace on Q and an even (resp odd) supertrace on Q. The supertr carries the cohomological info. and the even trace is the "contractible" positivity part.

September 20, 1990

Yesterday's lesson: A trace on EA (resp. $EA[\varepsilon]$) is the same as an even trace on QA and an even (resp. odd) supertrace on QA.

Note that E and $E[\varepsilon]$ are the universal sort of algebras arising from Fredholm modules over A.

It should be possible to refine the above lesson to include homotopy. One varies θ, F but not ε so the complexes of interest are $X(E)$ (resp. $X(E[\varepsilon];\varepsilon)$). Perhaps these split in a similar way as traces.

However, I am really interested only in the supertrace part, and I can't restrict immediately from E to Q when homotopy is concerned since F varies. So it seems it should be possible to work out a satisfactory theory using $X^s(E)$.

[Digression: We know traces on $E[\varepsilon]$ and Q are equivalent via

$$\mathrm{Tr} \longmapsto \mathrm{Tr}(ex) = \tfrac{1}{2}\big(\mathrm{Tr}(x) + \mathrm{Tr}(\varepsilon x)\big) \qquad x \in Q$$

$$\begin{matrix} \| \\ \tau \end{matrix} \qquad \simeq \qquad \tau^{+} + \tau^{-}$$

So Tr corresponds to an even (resp. odd) trace on Q $\Longleftrightarrow$ $\mathrm{Tr}(\varepsilon Q) = 0$ (resp $\mathrm{Tr}(Q) = 0$). Since Tr kills $F(Q + \varepsilon Q)$ this means that Tr is even (resp odd) $\Longleftrightarrow$ it is invariant (anti-invariant) under the automorphism $= 1$ on Q, F and sending $\varepsilon \mapsto -\varepsilon$. This is the dual $\mathbb{Z}/2$ action on $E[\varepsilon] = E \times \mathbb{Z}/2$.]

Return to the important matter: handling the supertrace side of Fredholm modules by means of E alone. Let's go over first the supertrace scene then try to include homotopy.

We consider $\theta : A \longrightarrow L \ni F$, Tr on L with ε present or not. We then get a homomorphism $E \longrightarrow L$, which in the graded case is a superalgebra homomorphism, where the involution on L is ε-conjugation. In the ungraded case we pull back Tr and make it odd $\tau(x+Fy) = \mathrm{Tr}(Fy)$; in the graded case we pull back the even supertrace $\mathrm{Tr}\,\varepsilon$ to get $\tau(x+Fy) = \mathrm{Tr}(\varepsilon(x+Fy)) = \mathrm{Tr}(\varepsilon x)$.

The goal is to include homotopy. In practice this means keeping L, Tr, (and ε in the graded case) fixed, but allowing θ, F to vary. As θ, F vary (this is the same as varying the homomorphism $E \longrightarrow L$) the supertrace τ varies.

① September 28, 1990.

$$R = S \otimes Q$$

First you want to understand

$$X(R;S) + K = X(R).$$ So we have something simpleminded. The idea is that we have to do something soon.

You have this decomposition but you want the homotopy formula which means taking $\overbrace{T}^{\text{a trace}}$ on $\Omega^1 R$ and understanding Td. Decomposition

$$0 \longrightarrow RdSR \underset{i}{\overset{\beta}{\rightleftarrows}} \Omega^1 R \underset{\pi}{\overset{\alpha}{\rightleftarrows}} \Omega^1(R;S) \longrightarrow 0$$

BASIC DECOMPOSITION

$$\boxed{\begin{aligned} T(xdy) &= T(x(dy - [y,\cancel{Y}])) \\ &\quad + T([x,y]\cancel{Y}) \end{aligned}}$$

Then $T(dy) = T(dy - [y,Y])$

So how does this look when $R = S \otimes Q$ and S is separable

$$RdSR = R \otimes_S \Omega^1 S \otimes_S R$$

$$= Q \otimes \Omega^1 S \otimes Q$$

$$(RdSR)_\natural = Q \otimes \Omega^1 S_\natural$$

② Anyway

$$0 \longrightarrow R \otimes_S \Omega^1 S \otimes_S R \longrightarrow \Omega^1 R \longrightarrow \Omega^1(R;S) \to 0$$

$$\Omega^1(R;S) = S \otimes \Omega^1 Q$$

OKAY next what

If this sequence splits, then

$$\begin{tikzcd}
0 \arrow[r] & R \otimes_S \Omega^1 S \otimes_S \arrow[r] \arrow[d, "s"] & \Omega^1 R_\flat \arrow[r] \arrow[d, "b"] & \Omega^1(R;S)_\flat \arrow[r] & 0 \\
0 \arrow[r] & {[R,S]} \arrow[r, hook] & R \arrow[r] & R/[R,S] \arrow[r] & 0
\end{tikzcd}$$

So what do I learn at this point?

$$\Omega^1(R;S) = S \otimes \Omega^1 Q$$

$$\Omega^1(R;S)_\flat = S_\flat \otimes \Omega^1 Q_\flat$$

$$R = S \otimes Q$$

$$R \otimes_S \Omega^1 S \otimes_S = \Omega^1 S_\flat \otimes Q$$

$$\Omega^1(R)_\flat = \frac{\Omega^1(S)_\flat \otimes Q \oplus S \otimes \Omega^1 Q_\flat}{\Omega^1 S_\flat \otimes \Omega^1 Q_\flat}$$

$$= \Omega^1 S_\flat \otimes Q \text{ ``}\oplus\text{'' } S_\flat \otimes \Omega^1 Q_\flat$$

So what is the point

③ So what do I learn?

Lesson seems to be that $\Omega^1 R_\natural$ is $S_\natural \otimes \Omega^1 Q_\natural \oplus \underbrace{\Omega^1 S_\natural \otimes Q}$

$$\underbrace{\Omega^1 S_\natural \otimes Q} = [S,S] \otimes Q.$$

So I have problems understanding what is going on. Thus I want to find something out about homotopy. All I can say is that a trace T on $\Omega^1 R$ splits

$$T(x\,dy) = T(x(dy - [y,Y])) + T([x,y]Y)$$

and that $T([x,y]Y) = (bf)(x\,dy)$ where $f(x) = T(xY)$ is a linear functional on R.

There are still things which I don't understand. Thus one has this projection $\beta(x\,dy) = [x,y]Y = x[y,Y]$

$$\begin{tikzcd}
\Omega^1 R_\natural \arrow[r, "b"] \arrow[d, "g"'] & R \arrow[dl, dashed, "\beta"'] \\
(RdSR)_\natural \arrow[r] & {[R,S]} \arrow[u, hook]
\end{tikzcd}$$

$\beta(x\,ds) = [x,s]Y = x\underbrace{[s,Y]}_{ds}$

Thus one has this projection

For some reason $T''(x\,dy) = T([x,y]Y)$ is same as a linear fnl on $(RdSR)_\natural \simeq [R,S]$

④ This appears to be an extra point
You really want somehow to say that ?

So the idea is that I have T on $(\Omega^1 R_q)^*$ and I propose to decompose it

$$T = (Ti)\beta + (T\alpha)\pi$$

where $T\alpha \in \Omega^1(R;S)_q^*$

and Ti equiv. to ~~s~~something in $[R,S]^*$

So ~~it~~ it's getting clear.

Sunday. At this point I have done the ~~E section~~ Fredholm section ~~in~~ near final form. What I need now is to work on the earlier ones.

Outline

$X(R;S)$

E section

F section

Yes.

E section. First part on $E = \cancel{S \otimes Q}\ Q \times \mathbb{Z}/2$ for any superalgebra – you prove

$$X^s(E;F) \simeq X^s(Q)$$

$$X^s(E) = X^s(E;F) \oplus K$$

You give formulas which can be checked – motivation is done in the ~~rel~~-section of Ch 3.

After this you state what this gives ~~[illegible]~~ for $E = EA$, and you end with the homotopy formula.

(5) What's happening is that

$x \longmapsto T(xY)$ is a linear fn on R vanishing on $\operatorname{Im} \alpha : R/[R,S] \longrightarrow R$.

So what we have to make clear is the direct sum decomposition

$$R = [R,S] \oplus \alpha\big(R/[R,S]\big)$$

$$\Omega^1 R_\natural = (RdSR)_\natural \oplus \alpha\big(\Omega^1(R;S)_\natural\big)$$

$$\begin{aligned} \alpha\pi(x) &= x - i\beta x \\ &= x - b(xY) \end{aligned}$$

$$\begin{aligned} b([r,s]Y) &= b(r[s,Y]) = b(r\,ds) \\ &= [r,s] \end{aligned}$$

$$\beta(x) = b(xY)$$

$$\beta(x\,dy) = b([x,y]Y)$$

So the first thing we want to understand is the splitting of

$$\begin{tikzcd} 0 \arrow[r] & (RdSR)_\natural \arrow[r, shift right] & R \arrow[l, shift right, "\beta"'] \arrow[r] & R/[R,S] \arrow[r] & 0 \\ 0 \arrow[r] & RdSR \arrow[r, shift right] & R\otimes R \arrow[l, shift right, "\beta"'] \arrow[u] \arrow[r] & R\otimes_S R \arrow[r] & 0 \\ & xYy & x\otimes y \arrow[l, mapsto] & & \end{tikzcd}$$

$$\beta(x) = xY$$

⑥

$$\begin{tikzcd} RdSR \arrow[r, phantom, "\subset"] & \Omega^1 R \arrow[l, bend right, "\beta I"'] \arrow[r, phantom, "\overset{I}{\subset}"] & R\otimes R \arrow[ll, bend right=40, "\beta"'] \end{tikzcd}$$

$$\begin{tikzcd} (RdSR)_\natural \arrow[r] & \Omega^1 R_\natural \arrow[r, "b"] \arrow[l, bend right, "\beta b"'] & R \arrow[ll, bend right=40, "\bar\beta"'] \end{tikzcd}$$

$$R/\mathrm{Im}\,(RdSR)_\natural = R/[R,S]$$

$$\Omega^1 R_\natural/\mathrm{Im}(RdSR)_\natural = \Omega^1(R;S)_\natural$$

So what this means is that if I am carefully

Sept 29

Let's summarize the situation. Let us consider the goals. I have to handle ~~EA~~ describe supertraces on EA and prove a homotopy formula. And I propose to the relative route:

$$X^s(E,F) \cong X^s(Q)$$

$$X^s(E) = X^s(E,F) \oplus K$$

Now I need to work this all out in the ~~case~~ ordinary algebra case.

Setting: $S \Rightarrow R$ ~~say~~

$$RdSR \subset \Omega^1 R \subset R\otimes R$$

$$\boxed{\begin{aligned} R\otimes R/\Omega^1 R &= R \\ R\otimes R/RdSR &= R\otimes_S R \\ \Omega^1 R/RdSR &= \Omega^1(R;S). \end{aligned}}$$

7 Consider splitting

$$\begin{tikzcd} R\,dS\,R \arrow[r, phantom, "\subset"] & \Omega^1 R \arrow[r, "I"] \arrow[l, bend right, "\beta"'] & R\otimes R \end{tikzcd}$$

Assume $\exists\, Y \in R\,dS\,R \ni ds = [s, Y]$

Define $\beta(x \otimes y) = xYy$. Induces

$$\begin{tikzcd} (R\,dS\,R)_\natural \arrow[r, "i"] & \Omega^1 R_\natural \arrow[r, "b"] \arrow[l, bend right, "\bar\beta"'] & R \end{tikzcd}$$

$$\bar\beta(x) = xY \in (R\,dS\,R)_\natural .$$

So what? ~~Fro~~

~~From the proof~~

What to say: You maybe should work backwards. Start with $T \in (\Omega^1 R_\natural)^*$

Then ~~$T \quad T(\bar\beta b i) \quad T$ splits~~

$$T = (T i \bar\beta b) + \underbrace{T(1 - i\bar\beta b)}_{(T\alpha)\pi}$$

$$T(xdy) = T(x(dy - [y, Y])) + \overbrace{T([x,y]Y)}^{T(i\bar\beta b)(xdy)}$$

~~Maybe one should start out with gilch.~~

Logic? Ultimately you need some way to hold this in your mind. Consider

$$\underbrace{R\,dS\,R \longrightarrow \Omega^1(R) \longrightarrow R\otimes R} \longrightarrow R\,dS\,R$$

~~Ag~~ I am confused

Let's try again to get through.

The idea is: We have injections

$$RdSR \longrightarrow \Omega^1 R \longrightarrow R\otimes R$$

such that $R\otimes R/\Omega^1 R = R$, $R\otimes R/RdSR = R\otimes_S R$, $\Omega^1 R/RdSR = \Omega^1(R;S)$, whence we have

$$R\otimes_S R/\Omega^1(R;S) = R$$

Next ~~assuming~~ let Y be an element on $RdSR$ and ~~define~~ $\beta: R\otimes R \longrightarrow RdSR$, be the corresp. bimodule map. $\beta(x\otimes y) = xYy$. Then β is a projection on $RdSR \iff ds = \beta(s\otimes 1 - 1\otimes s) \iff ds = [s,Y]$, for all $s\in S$. Assume this holds. Then we get ~~a decomposition an~~ isoms.

$$R\otimes R \xleftarrow{\sim} RdSR \oplus R\otimes_S R$$

$$\Omega^1 R \xleftarrow{\sim} RdSR \oplus \Omega^1(R;S)$$

$\mathrm{Ker}(\beta)$ complement for $RdSR$ in $R\otimes R$.

Pass to comm. quot. spaces:

$$\begin{tikzcd} (RdSR)_\natural \arrow[r, "i"] & \Omega^1 R_\natural \arrow[r, "b"] & R \arrow[ll, bend right, "\beta"'] \end{tikzcd}$$

$$\beta(x) = xY \qquad\qquad \beta b i = id$$

Know ~~$\mathrm{Ind}(bi) \cong$~~ $(RdSR)_\natural \xrightarrow{\sim} \cancel{\mathrm{Im}(bi)} = [R,S]$

$\mathrm{Im}(bi) = [R,S]$ and we have an isom. maps bi, β. ~~Thus~~ Also we have

R

One further topic. We should go back

$$
\begin{tikzcd}
0 \arrow[d] & & 0 \arrow[d] \\
\Omega^1 R \arrow[r, "\sim"] \arrow[d] & R\otimes R/RZR & \oplus\ \Omega^1(R;S) \arrow[d] \\
R\otimes R \arrow[r, "\sim"] \arrow[d] & R\otimes R/RZR & \oplus\ R\otimes_S R \arrow[d] \\
R \arrow[rr, equal] \arrow[d] & & R \arrow[d] \\
0 & & 0
\end{tikzcd}
$$

<u>OKAY</u>

Go back to derivations. $D: R \longrightarrow M$.

want $m \ni Ds = [s,m]$. $\forall s$.

$\cancel{\tilde{D}Y = \tilde{D}\sum_i r_i' ds_i r_i''}$

$\cancel{= \sum r_i [s_i, m] r_i''}$

Set $m = \tilde{D}Y$. Then

$$[s,m] = \tilde{D}[s,Y] = \tilde{D}ds = Ds.$$

and $m = \tilde{D}Y = \sum r_i' Ds_i r_i''$

$$= \sum r_i' [s_i, m] r_i'' = Ym$$

$$T(x\,dy) = T(x[y,Y]) + T(x\Delta y)$$
$$= T([x,y]Y) + T(x\Delta y)$$

$$T = fb + T' \qquad f(x) = T(xY)$$

$$f(x) = T(xY) \underset{\text{as } T'(xY)=0.}{=} f(b(xY))$$

$$\Omega^1 R_q \xrightarrow[\sim]{b,\pi} R/(1-p)R \oplus \Omega(R;S)_q.$$

~~All this tells~~ This tells that

$$R/(1-p)R \xrightarrow{\sim} (R\,dS\,R)_q$$

$$x \longmapsto xY$$

so $T = fb + T'$ $\quad ! \ni \ f(x) = f(b(xY))$

$\therefore \ T(xY) = f(x)$ $\qquad \therefore \ T(xY) = T(b(xY)Y)$

Let's reformulate

$$\Omega^1 R \xrightarrow[\sim]{(b',\pi)} \left(R\otimes R/RZR\right) \oplus \Omega^1(R;S)$$

$$\xleftarrow{Y\cdot,\ \Delta}$$

$$\Omega^1 R_q \xrightarrow{\sim} R/(1-p)R \oplus \Omega^1(R;S)_q$$

$$s\otimes[x,y] \longmapsto (s - b(sY))[x,y]$$

$$s\otimes x\,dy \longmapsto s\,x\,\Delta y$$

$$b(sx\Delta y) = b(sx\,dy - sx[y,Y])$$

$$= [sx,y] - \underbrace{b([sx,y]Y)}_{b(sY[x,y])}$$

$$= (s - b(sY))[x,y].$$

(11) ~~Suspicion~~ General situation

$$\begin{array}{ccccccccc} 0 & \to & RdSR & \to & R\otimes R & \to & R\otimes_S R & \to & 0 \\ & & \| & & \uparrow & & \uparrow & & \\ 0 & \to & RdSR & \to & \Omega^1 R & \to & \Omega^1(R;S) & \to & 0 \end{array}$$

better maybe is: have inclusions of R-bimods

$$RdSR \subset \Omega^1 R \hookrightarrow R\otimes R$$

$$dy \longmapsto y\otimes 1 - 1\otimes y$$

[redacted] Induced maps on comm quot spaces

$$(RdSR)_\natural \xrightarrow{\iota^1} (\Omega^1 R)_\natural \xrightarrow{b} R$$

$\operatorname{Im}(b\iota^1) = [R,S]$. Define K $\mathbb{Z}/2$ gr cx with $K^0 = K^1 = (RdSR)_\natural$, $b = id : K^1 \to K^0$, $d = 0$. Then have map $\boxed{\iota^0 = b\iota^1 :}$

$$i : K \longrightarrow X(R)$$

~~Also then~~ Why?

$$\begin{array}{ccc} (RdSR)_\natural & \xrightarrow{\iota^0} & R \\ 0=d \downarrow \uparrow b=1 & & d \downarrow \uparrow b \\ (RdSR)_\natural & \xrightarrow{\iota^1} & \Omega^1 R_\natural \end{array}$$

$d\iota^0 = \overset{0}{\overset{\|}{\iota^1 d}}$ because $\operatorname{Im}\iota^0 = \operatorname{Im}(b\iota^1) = [R,S]$ and d vanishes on $[R,R]$. Clear that have r.e.s.

$$K \xrightarrow{\iota} X(R) \xrightarrow{\pi} X(R;S) \longrightarrow 0$$

Next ~~discuss~~ gives a criterion to be a split exact seq. Return to $RdSR \subset R\otimes R$; let $Y \in RdSR$ and $\beta : R\otimes R \to RdSR$, $x\otimes y \mapsto xYy$ corresp Bimod map. $\beta(dx) = \beta(x\otimes 1 - 1\otimes x) = [x,Y]$ so β a projection onto $RdSR \iff ds = [s,Y],\ \forall s$.

(12)

β induces $\quad p^0 : R \to (R\,dS\,R)_\natural$, $x \mapsto xY$

One has $p^0 b \iota^1 = \mathrm{id}$. Put $\boxed{p^1 = p^0 b}$:

$\Omega^1 R_\natural \to (R\,dS\,R)_\natural$. Claim $p = (p^0, p^1) : X(R) \to K$

is a map of complexes such that $pi = \mathrm{id}$

$$\left(\begin{tikzcd} R \arrow[r, "p^0"] \arrow[d, shift right, "d"'] & (R\,dS\,R)_\natural \arrow[d, shift right, "d"'] \\ \Omega^1 R_\natural \arrow[r, "p^1"] \arrow[u, shift right, "b"'] & (R\,dS\,R)_\natural \arrow[u, shift right, "\iota"'] \end{tikzcd}\right.$$

$p^0 b = p^1$ ✓

$p^1 d = 0$

$p^0 b d = 0$ ✓

$$p^0 \iota^0 = p^0 b \iota^1 = 1$$
$$p^1 \iota^1 = p^0 b \iota^1 = 1.$$

Clear.

Corresponding section of π is

$$\alpha\pi(x) = x - \iota^0 p^0 x = x - b\iota^1(xY)$$

$$\begin{aligned} \alpha(x\,d_S y) &= x\,dy - \iota^1 p^1(x\,dy) \\ &= x\,dy - \iota^1([x,y]Y) \\ &= x\,dy - [x,y]Y \\ &= x(dy - [y,Y]) \end{aligned}$$

$$\alpha : X(R;S) \to X(R)$$

$$\alpha(x \bmod [R,S]) = x - b(xY)$$

Useful to mention that ιp is the ~~projection~~ idempotent

$$(\iota p)^0(x) = b\iota^1(xY) = b(xY)$$

$$\begin{aligned} (\iota p)^1(x\,dy) &= \iota^1 p^0 [x,y] = \iota^1([x,y]Y) \\ &= \iota^1(x[y,Y]) = \text{"}\ [x,y]Y \end{aligned}$$

(13) Clear things up visually. Injections

$$RdSR \longrightarrow \Omega^1 R \longrightarrow R\otimes R$$
inclusion $\quad dx \longmapsto x\otimes 1 - 1\otimes x$

$$\Omega^1 R / RdSR = \Omega^1(R;S)$$
$$R\otimes R / RdSR = R\otimes_S R$$
$$R\otimes R / \Omega^1 R = R$$

proves exactness of
$$0 \to \Omega^1(R;S) \to R\otimes_S R \to R \to 0$$

Induced maps on comm. quot sp

$$(RdSR)_\natural \xrightarrow{\iota'} \Omega^1 R_\natural \xrightarrow{b} R$$
$$x\,dy \overset{b}{\longmapsto} [x,y]$$

Let $K^0 = K^1 = (RdSR)_\natural \qquad b = 1,\ d = 0$

$$\cancel{K^\cdot\ (RdSR)_\natural \underset{d=0}{\overset{b=1}{\rightleftarrows}} (RdSR)_\natural}$$

~~[illegible]~~ Right exactness of $\natural$

$$\cancel{(RdSR)_\natural \xrightarrow{\iota'} \Omega^1 R_\natural \xrightarrow{\pi'} \Omega^1(R;S)_\natural \to 0}$$

$$\begin{tikzcd}
(RdSR)_\natural \arrow[r, "\iota^0"] \arrow[d, shift right, "0"'] & R \arrow[r, "\pi^0"] \arrow[d, shift right, "d"'] & R/[R,S] \arrow[r] \arrow[d, shift right, "d_S"'] & 0 \\
(RdSR)_\natural \arrow[r, "\iota^1"] \arrow[u, shift right, "1"'] & \Omega^1 R_\natural \arrow[r, "\pi^1"] \arrow[u, shift right, "b"'] & \Omega^1(R;S)_\natural \arrow[r] \arrow[u, shift right, "b"'] & 0
\end{tikzcd}$$

$$K \xrightarrow{\iota} X(R) \longrightarrow X(R;S) \longrightarrow 0$$

October 9, 1990

$\Omega^1 R$ arises in two different ways which are not obviously the same. On one hand it is the target of the universal derivation. If we interpret derivations as lifting homomorphisms, this means that

$$\underbrace{Q/J^2}_{\substack{\text{universal}\\ \text{square zero}\\ \text{extension of } R \text{ with two section homomorphisms}}} = R \oplus \Omega^1 R$$

On the other hand it occurs as the kernel in the bimodule exact sequence

$$0 \longrightarrow \Omega^1 R \xrightarrow{\ i\ } R \otimes R \xrightarrow{\ m\ } R \longrightarrow 0$$

$$dx \longmapsto x \otimes 1 - 1 \otimes x$$

The transpose of i for bimodule maps, i.e. applying $\mathrm{Hom}_{\mathrm{bim.}}(\ ,M)$ is the map sending an element of m to the corresponding inner derivation. This shows right exactness of the above sequence, but the injectivity of i needs another proof. This injectivity is equivalent to knowing that any square zero extension with two section homomorphisms can be enlarged so they become conjugate.

Apparently there is an isomorphism

$$R \otimes R \xrightarrow{\ \sim\ } Q/J^2$$

which means a different bimodule structure on Q/J^2 other than θ, θ. In fact it is θ, θ^γ up to sign. Thus identifying: $Q/J^2 = R \oplus \Omega^1 R$

so that $\theta x = x$, $\theta^\delta x = x + dx$
then using θ on the left and θ^δ
on the right gives an isom.

$$R \otimes R \longrightarrow R \oplus \Omega^1 R$$
$$x \otimes y \qquad\qquad x(y + dy) = xy + x\,dy$$

This corresponds exactly to splitting

$$R \otimes R \longrightarrow R \oplus \Omega^1 R$$
$$x \otimes y \longmapsto xy + \underbrace{x \otimes y - xy \otimes 1}_{-x\,dy}$$

October 11, 1990

Recall R^e is the tensor product of R and the opposed algebra R^o. Thus we have a canonical homom. $R \to R^e$, $x \mapsto x$ and a canonical anti-hom. $R \longrightarrow R^e$, $x \mapsto x^o$ such that $xy^o = y^o x$ and no further relations. Bimodules over R are the same as (left) R^e modules via

$$(xy^o)(m) = xmy$$

Of course the bimodule $R \otimes R$ is the free R^e module generated by $1 \otimes 1$. This means we have a bijection

$$R^e \simeq R \otimes R$$

$$xy^o \longleftrightarrow x \otimes y$$

Under this bijection we have

$$x - x^o \longleftrightarrow x \otimes 1 - 1 \otimes x = dx$$

so that $\Omega^1 R$ corresponds to the left ideal in R^e generated by the elements $x - x^o$. Similarly $RdSR$ corresponds to the left ideal generated by the elts $s - s^o$.

Let us now interpret the conditions $ds = [s, Y]$ for an element $Y \in R \otimes R$. This means

$$s \otimes 1 - 1 \otimes s = sY - Ys \qquad \text{in } R \otimes R$$

or

$$s - s^o = (s - s^o) Y \qquad \text{in } R^e$$

Thus the conditions 1) $Y \in RdSR$ 2) $ds = [s, Y]$ $\forall s \in S$ means that Y is in the left ideal generated by the elements $s - s^o$ and that $s - s^o = (s - s^o)Y$ $\forall s$

whence we conclude

$$R^e dS = R^e\{s - s^\circ \mid s \in S\} = R^e Y, \qquad Y^2 = Y.$$

Notice also that via the bijection $R^e \simeq R \otimes R$, elements of $R \otimes R$ operate on any bimodule M. Thus $dx = x \otimes 1 - 1 \otimes x$ operates as $ad(x) : m \longmapsto xm - mx$. So we can talk about Ym, and Zm, where $Z = 1 - Y$ is the complementary idempotent.

Left ideals in R^e correspond to bimodules with one generator. [redacted] Note $\mathrm{Hom}_{R^e}(R^e/L, M) = \{m \in M \mid \mathrm{Ann}(m) \supset L\}$. For example $R^e/R^e dS \xrightarrow{\sim} R \otimes R/RdSR = R \otimes_S R$ represents the centralizer of S.

Let us consider the complementary idempotents Y, Z. We have

$$R^e \xrightarrow{\sim} R^e/R^e Z \times R^e/R^e Y$$

$$\Omega^1 R \simeq R^e dR \xrightarrow{\sim} R^e/R^e Z \times R^e dR/R^e Y$$

Thus any derivation $D : R \longrightarrow M$ is equivalent to an element m such that $Zm = 0$ together with a derivation $\Delta : R \longrightarrow M$ which is an S-derivation: $\Delta s = 0 \quad \forall s$.

October 12, 1990 (Becky is 24)

A key mystery is why derivations in the noncommutative setup are so closely related to bimodules. In general derivations are infinitesimal automorphisms; it is not immediately clear why these should be related to $R^e = R \otimes R^\circ$.

So this is why it might be useful to study the equivalence between defining $\Omega^1 R$ via derivations: $Q(R)/J^2 = R \oplus \Omega^1 R$ and via the kernel of $R \otimes R \xrightarrow{m} R$. The question is why the kernel has the required universal property relative to derivations.

We show that derivations have a bimodule meaning. Consider a bimodule extension

$$\circledast \quad 0 \longrightarrow M \longrightarrow E \xrightarrow{\pi} R \longrightarrow 0$$

together with an element $\xi \in R$ with $\pi(\xi) = 1$. Then we get a derivation $D: R \to M$ given by $Dx = [x, \xi]$. Observe that there is a canonical isomorphism $R \oplus M \xrightarrow{\sim} E$ compatible with left R-module structure, which sends 1 to ξ, and that D is exactly what is needed to describe a right module structure on $R \oplus M$ compatible with the given right multiplication on M as subobject and R as quotient.

$\therefore$ These "extensions together with ξ" form a category equivalent to the cofibred over bimodules associated to the functor $M \longmapsto$ $\mathrm{Der}(R, M)$. On the other hand $0 \to \Omega^1 R \to R \otimes R \to R \to 0$ with $1 \otimes 1$ is obviously an initial object, which yields the desired universal property.

November 1, 1990

In the past few days I have been reviewing higher homotopies for traces. Consider two algebras A, B and let τ be a trace on B. Let U be the 'space' of homomorphisms $u : A \longrightarrow B$. (More precisely, we should consider a family of homomorphisms $A \longrightarrow B$ parametrized by an affine variety U over $\mathbb{C}$. This amounts to a homomorphism

$$A \longrightarrow S \otimes B$$

where S is the coordinate ring of U. When phrased this way, the generalization possible in the noncommutative direction becomes clear.)

So we have a family of traces τu on A for $u \in U$. We ~~consider a path~~ $u = u_t$. Recall One has

$$\tau u_1 - \tau u_0 = \int_0^1 dt \; \partial_t (\tau u) = \int_0^1 dt \; \tau(\partial_t u)$$

$$= \left(\int_0^1 dt \; \tau\big(\widetilde{\partial_t u}\big) \right) d$$

In other words if two traces can be joined by a path in the family, then they differ by a trace of the form Td with T a trace on $\Omega^1 A$. The idea which I pursued before was to find a suitable generalization for two parameter families. Thus if to a path we associate a trace on $\Omega^1 A$ what do we associate to a path of paths, i.e. a square?

Let us observe that we should be thinking in terms of currents on U. Thus the trace $\tau u_1 - \tau u_0$ is the pairing of the function $u \longmapsto \tau u$ with the distribution $\chi_{u_1} - \chi_{u_0}$ (χ = char fn.). Similarly the path u_t, $0 \le t \le 1$, is a 1-chain.

This viewpoint using distributions on S suggests differential forms on S.

Thus we have the function τu on S with values in traces on A, its differential $d(\tau u) = \tau(du)$ which is a 1-form on S with values in $HC^0(A)$. We have also $\tau(\widetilde{du})$ which is a 1-form on S whose values are traces on $\Omega^1 A$:

$$\tau(\widetilde{du})(a_0 da_1) = \tau(u(a_0)\, du(a_1))$$

We thus might write $\tau(u\,du)$ for $\tau(d\tilde{u})$. More accurately $\tau(u\,du)(a_0, a_1) = \tau(u(a_0) du(a_1))$, i.e. $\tau(u\,du)$ is a 1-form on S with values in $Z^1_{norm}(A, A^*)$. We have

$$d\tau(u) = B\tau(u\,du)$$

Let's continue in the obvious way. We take $d\tau(u\,du) = \tau(du\,du)$ which is a 2 form having values in $Z^1_N(A, A^*)$. On the other hand

$$\tau(u\,du^2)(a_0, a_1, a_2) = \tau(u(a_0)\, du(a_1)\, du(a_2))$$

is a 2-form with values in $Z^2_N(A, A^*)$. In fact I claim its values are K-invariant Hochschild 2-cocycles. To see this fix a point u_0 and two tangent vectors $\delta_1 u_0$, $\delta_2 u_0$ at that point; these are derivations $A \to B$ relative to u_0. Suppressing u_0 from the notation (i.e. regarding u_0 as an inclusion map) ~~and taking~~ we have

$$\iota_{\delta_2} \iota_{\delta_1} \tau(u\,du\,du)(a_0\, da_1 da_2)$$

$$= \tau(a_0\, \delta_1 a_1\, \delta_2 a_2) - \tau(a_0\, \delta_2 a_1\, \delta_1 a_2)$$

Now recall

$$\tau(a_0\,\delta_2 a_1\,\delta_1 a_2) = \tau(\delta_1 a_2\, a_0\, \delta_2 a_1)$$
$$= \tau(\delta_1(a_2 a_0)\,\delta_2 a_1) - \tau(a_2\,\delta_1 a_0\,\delta_2 a_1)$$

So if $\psi(a_0, a_1, a_2) = \tau(a_0\,\delta_1 a_1\,\delta_2 a_2)$, then

$$\tau(a_0\,\delta_2 a_1\,\delta_1 a_2) = -\psi(a_2, a_0, a_1) + \psi(1, a_2 a_0, a_1)$$
$$= -(K\psi)(a_0, a_1, a_2).$$

Thus $\iota_{\delta_2}\iota_{\delta_1}\tau(u\,du\,du) = \psi + K\psi$, which is K-invariant.

A generalization of this calculation yields that if $\psi(a_0, \ldots, a_n) = \tau(u(a_0)\,\delta_1 u(a_1)\cdots\delta_n u(a_n))$, then

$$\tau(u(a_0)\,\delta_2 u(a_1)\cdots\delta_n u(a_{n-1})\,\delta_1 u(a_n))$$
$$= (-1)^{n-1}(K\psi)(a_0, \ldots, a_n)$$

This shows that cyclically permuting with sign the tangent vectors $\delta_1 u \ldots \delta_n u$ corresponds to the action of K. From this one can deduce that the values of the n-form on S given by $\tau(u\,du^n)$ are in K-invariant Hochschild n-cocycles.

Here is the good way to look at the above calculations. We have a homomorphism

$$A \longrightarrow B \otimes S$$

which induces a homomorphism

$$\Omega A \longrightarrow \Omega(B \otimes S)$$

But we also have a homom.

$$\Omega(B\otimes S)\longrightarrow \Omega B\otimes \Omega S .$$

So we get a map of complexes

$$\Omega A/[\ ,\ ]_s \longrightarrow \Omega B/[\ ,\ ]_s \otimes \Omega S/[\ ,\ ]_s$$

This enables us to [redacted] get a map of complexes

$$\Omega A/[\ ,\ ]_s \longrightarrow \Omega S/[\ ,\ ]_s$$

from a map of complexes

$$\Omega B/[\ ,\ ]_s \longrightarrow \mathbb{C}[0]$$

i.e.

$$\begin{tikzcd} B_\natural \arrow[r] \arrow[d] & \Omega^1 B_\natural \arrow[r] & {} \\ \mathbb{C} & 0 & \end{tikzcd}$$

i.e. from any trace on B.

Another idea here:

$$\Omega(B\otimes S) \longrightarrow \Omega(B\otimes S; B) = B\otimes \Omega S$$

Relative mixed complex: [redacted] On

$$\Omega(R;S)\otimes_S$$

we expect to have the b, d, K, P game. [redacted] Moreover ~~these~~ operators should be compatible with the canonical surjection

$$\Omega R \longrightarrow \Omega(R;S)\otimes_S$$

So in the case of just traces on B we should therefore have a nice map

$$\Omega(B\otimes \mathbb{C}) \longrightarrow \Omega(B\otimes \mathbb{C}; B)\otimes_B = B_\natural \otimes [redacted]\ \Omega \mathbb{C}$$

compatible with all the operators.

This is not very clearly **said**. The point seems to be ~~that~~ there is a basic problem lifting results about $\Omega/[\Omega,\Omega]_s$ to results about the good mixed complex $P\Omega$, and that we have run into it again in the context of the Künneth formula. We have Connes map

$$\Omega(B\otimes C)/[\,,\,]_s \longrightarrow \Omega B/[\,,\,]_s \otimes \Omega C/[\,,\,]_s$$

but we have only managed so far to obtain a map

$$P\Omega(B\otimes C) \longrightarrow B_4 \otimes P\Omega C$$

Let's make a summary of ideas for future reference and then see if we can get back and finish the paper this month.

1. <u>Künneth thm.</u> in cyclic homology. It is not clear whether you have any useful techniques to handle this. The key point is that it is a result that is true on the level of Hochschild homology and which then you can improve to a statement for cyclic homology.

Recall earlier work that if $R = A*B$ $R/I = A\otimes B$, then

$$X(A)\otimes X(B) \simeq \left\{ R/I^2+[R,I] \rightleftarrows \Omega^1 R/I\Omega^1 R+[R,\Omega^1 R] \right\}$$

<u>Useful project</u>: Does this lead to a proof of Künneth for periodic cyclic homology?

The central problem from a formula point of view is that the Hochschild chains on $A\otimes B$, call this $C(A\otimes B)$ is not the tensor product $C(A)\otimes C(B)$. This might be true if one works semi-simplicially (or maybe with cyclic sets). Loday treats the problem using cyclic shuffles which should be some explicit version of Eilenberg-Zilber.

The approach I can think of is the following. It is clear from the projective resolution business that Hochschild homology satisfies Kunneth, so what one would like is a way to define the B operator and the whole cyclic machine in the same generality. This leads to the old question of a <u>derived category approach to cyclic homology.</u> You have something valid for compact groups which should extend to loop groups.

Any formula type approach to Kunneth has to be at least as complicated at the most degenerate case: $A=\tilde V$, $B=\tilde W$ where V, W are algebras with zero multiplication. Recall

$$\overline{HC}_n(\tilde V) = V_\lambda^{\otimes n+1} \qquad n\geq 0$$

$$HH_n(\tilde V)\begin{cases} = \overline{HC}_n(\tilde V)\oplus \overline{HC}_{n-1}(\tilde V) = V_\lambda^{\otimes n+1}\oplus V_\lambda^{\otimes n} & \text{for } n>0 \\ = \tilde V = V\oplus \mathbb{C} & \text{for } n=0.\end{cases}$$

I think

$$\overline{HC}_n(\tilde V\otimes\tilde W) = \bigoplus_{\substack{i+j=n+1\\ i,j\geq 0}} V_\lambda^{\otimes i}\otimes W_\lambda^{\otimes j} \oplus \bigoplus_{\substack{i+j=n+2\\ i,j>0}} V_\lambda^{\otimes i}\otimes W_\lambda^{\otimes j}$$

where $V_\lambda^{\otimes 0}=\mathbb{C}$

Method used: square zero extension
$$0 \to \tilde{V} \otimes W \longrightarrow \tilde{V} \otimes \tilde{W} \longrightarrow \tilde{V} \to 0$$
which splits, so can use Goodwillie

---

2. Deformation theory of $P\Omega(A)$. Recall $P\Omega(A)$ is roughly an extension of $\overline{CC}(A)$ by itself shifted. Goodwillie gives $\overline{CC}(A \oplus M)$ so perhaps there's a way to understand $P\Omega(A \oplus M)$.

Another variant of this problem: We have the map
$$P\Omega(S \otimes A) \longrightarrow S_\natural \otimes P\Omega(A)$$
Can we refine this map so $S_\natural$ is replaced by $X(S)$? This might be very close to what you did for derivations on A acting on $P\Omega(A)$.

3. What sort of things induce maps from the periodic ~~cyclic~~ theory of A to that of B? This should relate to asymptotic morphisms.

Example:
$$\begin{tikzcd} E \arrow[r] \arrow[d] & B \\ A & \end{tikzcd}$$
where E is a nilpotent extension of A.

Variant: $\rho : A \to B$ such that the induced hom $RA \longrightarrow B$ kills IA^m.

Example: $A \longrightarrow S \otimes B$ together with a trace on S.

To what extent might these examples be related, e.g. Can one ~~[illegible]~~ find an S with tr and a canonical map
$$A \longrightarrow S \otimes RA/IA^m$$

whose effect on K or period cyclic homology is inverse to the canonical map $RA/IA^m \longrightarrow A$? This seems unlikely since we would then have

$$X(A) \longrightarrow S_q \otimes X(RA/IA^m) \longrightarrow X(RA/IA^m)$$

which we don't expect in general (only for A with $\Omega^1 A$ proj.)

4. Is it possible to use $X(A)$ to establish per cyclic homology effectively? Perhaps a good question is to ask what sort of information, or data, is needed to be able to claim that some $\mathbb{Z}/2$ graded complex gives $HP(A)$. We run into this problem already with $X(A) \otimes X(B)$. And it clearly is important in the case of gen. enveloping algebras.

5. Something not to forget is the idea of exploring the result that $(P\Omega, b, B)$ gives the cyclic homology. In the paper with Cuntz I referred to [LQ] but in some sense there are things to be understood by looking at the double complexes in LQ. The standard cyclic double cx. with b, b' columns is the double complex associated to the mixed complex $(\Omega(\tilde{A}), b, B)$. But then we replace this by a subcomplex where the b' columns are gone and this maps down onto the double complex assoc. $(\Omega(A), b, B)$. Perhaps this would help one to understand whether one has reduced cyclic cohomology classes associated to fine Fredholm modules. Also the fundamental stabilization mystery behind K-theory?

November 3, 1990

Let us consider the I-adic filtration on $R*S$ where I is the ideal generated by $[r,s]$, $\forall r\in R$, $\forall s\in S$. We have

$$gr_0 = R*S/I = R\otimes S$$

Consider $gr_1 = I/I^2$ which is a bimodule over $R\otimes S$. For r fixed the map $s\longmapsto [r,s]$ is a derivation $S\longrightarrow I/I^2$ hence it induces an S-bimodule map

$$\Omega^1 S \longrightarrow I/I^2 \qquad s_1 ds_2 \longmapsto s_1[r,s_2]$$

For $s_1 ds_2$ fixed the map $r\longmapsto s_1[r,s_2]$ is a derivation $R\longrightarrow I/I^2$:

$$s_1[r_1r_2,s_2] = (s_1[r_1,s_2])r_2 + \underbrace{s_1r_1[r_2,s_2]}_{r_1(s_1[r_2,s_2])}$$

(these are congruent mod I^2)

so we have an R-bimodule map

$$\Omega^1 R \longrightarrow I/I^2 \qquad r_1dr_2 \longmapsto r_1s_1[r_2,s_2]$$

Thus we have a canonical $R\otimes S$-bimodule map

$$\boxed{\begin{array}{ccc}\Omega^1 R\otimes\Omega^1 S & \longrightarrow & I/I^2 \\ r_1dr_2\otimes s_1ds_2 & \longmapsto & r_1s_1[r_2,s_2]\end{array}}$$

This induces a canonical homomorphism of graded algs.

$$(*)\qquad \bigoplus_{n\geq 0}\Omega^n R\otimes\Omega^n S \longrightarrow gr^I(R*S).$$

where on the left we use the ordinary (non-super) tensor product algebra structure.

From previous work it is clear that $(*)$ is an isomorphism. In effect we showed there is a vector space isomorphism

$$\bigoplus_{n\geq 0} R\otimes S\otimes(\bar{R}\otimes\bar{S})^{\otimes n} \xrightarrow{\sim} R * S$$

$$(r_0, s_0, \ldots, r_n, s_n) \longmapsto r_0 s_0 [r_1, s_1] \cdots [r_n, s_n]$$

We ask now whether there is an increasing filtration on $R*S$ which is complementary to the decreasing I-adic filtration. First we need to find a complement to I, that is, lift $R\otimes S$ into $R*S$. There are two choices $r\otimes s \longmapsto rs$ and sr and possibly their averages. (Maybe one should use the linear path. I think Wodzicki did this. ~~[illegible]~~ The average ~~[illegible]~~ gives the Weyl calculus.)

So the natural inclusion filtrations to try are $(RS)^n$ and $(SR)^n$. Lets work out the complement of $RS \subset (RS)^2$ given by $(RS)^2 \cap I$. One has

$$\underset{\rho(r_0s_0)\rho(r_1s_1)}{\overset{\parallel}{r_0s_0r_1s_1}} = \underset{\rho(r_0s_0r_1s_1)}{\overset{\parallel}{r_0r_1s_0s_1}} - \underbrace{r_0[r_1, s_0]s_1}_{\omega(r_0s_0,\, r_1s_1)}$$

What is the image of ω in $I/I^2 \simeq \Omega^1R\otimes\Omega^1S$?

$$\begin{aligned} r_0[r_1, s_0]s_1 &= r_0[r_1, s_0s_1] - r_0s_0[r_1, s_1] \\ &\leftarrow r_0\,dr_1\, d(s_0s_1) - r_0\,dr_1\, s_0ds_1 \\ &= r_0dr_1\, ds_0\, s_1 \end{aligned}$$

This is confused. ~~[illegible]~~ You know that $r_0dr_1\, s_0ds_1 \in \Omega^1R\otimes\Omega^1S$ corresponds to $r_0s_0[r_1, s_1]$ in I/I^2. But observe

$$\begin{aligned} r_0s_0[r_1, s_1] &= r_0\, s_0r_1s_1 - r_0(s_0s_1)r_1 \\ &\in RSRS + RSR = (RS)^2 \end{aligned}$$

So the map

$$\Omega^1R\otimes\Omega^1S \longrightarrow I \qquad r_0dr_1\, s_0ds_1 \mapsto r_0s_0[r_1, s_1]$$

gives the lifting of I/I^2 into I corresponding to $(RS)^2 \cap I$.

Next we notice that

~~$r_0 s_0 [r_1, s_1] \cdots [r_n, s_n] \in (RS)^n \cap I^{n-1}$ by induction~~

$$r_0 s_0 [r_1, s_1] \cdots [r_n, s_n] \in (RS)^{n+1} \cap I^n$$

by induction on n. In effect assuming $\omega = r_0 s_0 \cdots [r_{n-1}, s_{n-1}] \in (RS)^n$, then

$$\omega [r_n, s_n] = \omega r_n s_n - \omega s_n r_n$$
$$\in (RS)^n RS + (RS)^n SR = (RS)^{n+1}$$

Thus we conclude that one has an isom.

$$(**) \qquad \bigoplus_{n \geq 0} \Omega^n R \otimes \Omega^n S \xrightarrow{\sim} R * S$$

$$r_0 dr_1 \cdots dr_n \otimes s_0 ds_1 \cdots ds_n \longmapsto r_0 s_0 [r_1, s_1] \cdots [r_n, s_n]$$

and that it is the isomorphism obtained from the canonical isomorphism on the gr level by using the filtration $(RS)^n$ to split the I-adic filtration. Better to say: The increasing filtration $(RS)^n$ is complementary to the I-adic filtration.

Let's now ask what the product in the algebra $R*S$ corresponds to on the left side of $(**)$. We look for the operators on the left corresponding to left mult by r, s on $R*S$. Clearly left mult by r on $R*S$ corresponds to left mult by r on $\Sigma = \bigoplus_{n \geq 0} \Omega^n R \otimes \Omega^n S$

Next we have in $R*S$

$$s\, r_0 s_0 = [s, r_0] s_0 + r_0 s s_0$$
$$= -[r_0, s] s_0 + r_0 s s_0$$
$$= r_0 s s_0 - [r_0, s s_0] + s [r_0, s_0]$$

which means

$$s*(r_0 s_0) = r_0 s s_0 - dr_0\, d(s s_0) + s\, dr_0\, ds_0$$
$$= r_0 s s_0 - dr_0\, ds\, s_0$$
$$= s(r_0 s_0) - (d^R \otimes ds) r_0 s_0$$

Thus $s* = s - d^R ds$ on Σ

where s and ds are treated as multiplication operators on forms. Check:

$$(s_1 - d^R ds_1)(s_2 - d^R ds_2) = s_1 s_2 - d^R ds_1 s_2 - d^R s_1 ds_2$$
$$= (s_1 s_2) - d^R d(s_1 s_2)$$

Commutator

$$[r, s - d^R ds] = rs - r d^R ds - (s - d^R ds) r$$
$$= \cancel{rs} - r d^R ds - \cancel{sr} + (d^R \circ r) ds = dr\, ds$$

Thus the operator $[r, s - d^R ds]$ is left mult by $dr\, ds$. Finally the formula for the $*$ product is found as follows

$$(r_0 s_0 [r_1, s_1] \cdots [r_n, s_n]) * (\alpha' \beta') \qquad \alpha' \in \Omega^n R,\ \beta' \in \Omega^n S$$
$$= r_0 (s_0 - d^R ds_0)(dr_1 \cdots dr_n \alpha' \cdot ds_1 \cdots ds_n \beta')$$
$$= r_0 dr_1 \cdots dr_n \alpha' s_0 ds_1 \cdots ds_n \beta' - (-1)^n r_0 dr_1 \cdots dr_n d\alpha' ds_0 \cdots ds_n \beta'$$

In other words

$$\boxed{(\alpha\beta)*(\alpha'\beta') = \alpha\alpha'\beta\beta' - (-1)^{|\alpha|} \alpha\, d^R\alpha'\, d^S\beta\, \beta'}$$

November 4, 1990.

Let A be an algebra such that $\Omega^1 A$ is a projective bimodule, let R/I be a nilpotent extension of order m, that is, $I^{m+1} = 0$. We know that lifting homom: $A \to R$ exist. We claim that any two can be joined by a polynomial family $u_t : A \to R[t]$ of lifting homomorphisms where u_t has degree $\leq m$ in t.

The universal ~~situation~~ occurs where $R = A * A / J^{m+1}$ with the two lifting homos. θ and θ^γ. Consider

$$\rho_t = (1-t)\theta + t\theta^\gamma = \theta\text{[redacted]} + t\overbrace{(\theta^\gamma - \theta\text{[redacted]})}^{-2q}$$

$$\omega_t = \text{[redacted]}\ b'\rho_t - \rho_t^2 = t(1-t)(\theta^\gamma - \theta)^2$$

$$= 4t(1-t)\,q^2$$

Corresponding to this family of linear liftings is a homomorphism

$$RA \longrightarrow \left(A*A/J^{m+1}\right) \otimes \mathbb{C}[t]$$

$$\rho\omega^n \longmapsto \rho_t\omega_t^n = (\theta - 2tq)\left(4t(1-t)q^2\right)^n$$

Actually I want to center this, so put $t = \frac{1+s}{2}$

$$4t(1-t) = 4\left(\tfrac{1+s}{2}\right)\left(\tfrac{1-s}{2}\right) = 1 - \text{[redacted]}s^2$$

$$\theta - (1+s)q = \rho - sq$$

$$\therefore\ \rho\text{[redacted]}\omega\text{[redacted]}^n \longmapsto \rho q^{2n}(1-s^2)^n - \text{[redacted]}\,q^{2n+1}s(1-s^2)^n$$

Thus I observe that the degree in q matches the degree in s. More precisely, modulo J^{m+1} we have a polynomial in s of degree m. Check this carefully. Suppose $m=2k$. Then we have $\rho q^{2k}(1-s^2)^k$ from $\rho\omega^k$ and lower powers of q, hence s. If $m=2k+1$ we have $q^{2k+1}s(1-s^2)^k$ and $\rho q^{2k}(1-s^2)^k$ and lower powers of q, hence s.

Thus it seems [redacted] all right. So we have proved

Prop. Let A be an algebra such that $\Omega^1 A$ is a proj bimodule over A, and let $A = R/I$ be an algebra extension such that $I^{m+1}=0$. Then lifting homomorphisms $A \to R$ exist and any two of ~~them~~ can be joined by a polynomial family $u_t : A \to R[t]$ of lifting homomorphism, where u_t has degree $\leq m$ in t.

November 8, 1990

I want to give a treatment of square zero extensions without choosing a linear lifting.

First suppose we have a general extension $A = R/I$. Then we have an exact sequence

$$(*) \qquad 0 \to I/I^2 \to R/I \otimes_R \Omega^1 R \otimes_R R/I \to \Omega^1(R/I) \to 0$$

The right exactness is clear by considering derivations, and only the injectivity at the left is obscure. It is equivalent to the assertion that one can find a derivation $D: R \to M$ where M is an A-bimodule such that $I/I^2 \to R/I^2 \xrightarrow{D} M$ is injective.

We can prove $(*)$ by a Tor calculation.

$$0 \to \Omega^1 R \to R \otimes R \to R \to 0$$

$$0 \to \mathrm{Tor}_1^{R^e}(R, A^e) \to A \otimes_R \Omega^1 R \otimes_R A \to A \otimes A \to A \to 0$$

so the point is to see this Tor_1 is I/I^2. But

$$0 \to I \otimes I \to R \otimes I \oplus I \otimes R \to R \otimes R \to A \otimes A \to 0$$

and $R \otimes I$, $I \otimes R$, $R \otimes R$ are acyclic for $\mathrm{Tor}_1^{R^e}(R, ?)$. Thus the Tor_1 we want is the homology at

$$0 \to I \otimes_R I \to I \oplus I \to R \to A \to 0$$

which is clearly I/I^2. In fact I notice now that I can use the isomorphism

$$\mathrm{Tor}_n^{R^e}(R, M \otimes N) = \mathrm{Tor}_n^R(N, M)$$

and obtain

$$\mathrm{Tor}_n^{R^e}(R, A^e) = \mathrm{Tor}_n^R(A, A) = \begin{cases} I/I^2 & n = 1 \\ \mathrm{Ker}\, I \otimes_R I \to I^2 & n = 2 \end{cases}$$

Here's how to go between square zero extensions $A = E/M$ of A and A-bimodule extension $\Omega^1 A = N/M$ of $\Omega^1 A$. Given the bimodule extension we form the semi-direct product $A \oplus N$ which is a square zero extension of $A \oplus \Omega^1 A$ by M and then we form the pull back:

$$\begin{tikzcd}
0 \arrow[r] & M \arrow[r] \arrow[d, equal] & E \arrow[r, "\pi"] \arrow[d, "\pi + D"] & A \arrow[r] \arrow[d, "1+d"] & 0 \\
0 \arrow[r] & M \arrow[r] & A \oplus N \arrow[r] & A \oplus \Omega^1 A \arrow[r] & 0
\end{tikzcd}$$

One then has a derivation $D : E \to N$ restricting to the inclusion $M \subset N$, and this induces an isom of bimodule exact sequences

$$\begin{tikzcd}
0 \arrow[r] & M \arrow[r] \arrow[d, equal] & A \otimes_E \Omega^1 E \otimes_E A \arrow[r] \arrow[d, "\tilde{D}"] & \Omega^1 A \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & M \arrow[r] & N \arrow[r] & \Omega^1 A \arrow[r] & 0
\end{tikzcd}$$

Conversely given $A = E/M$ the associated bimodule extension is $N = A \otimes_E \Omega^1 E \otimes_E A$. We note that there is a canonical derivation $D : E \to N$ i.e. homomorphism $E \to A \oplus N$, etc.

In concrete terms, the fact that square zero extensions split when $\Omega^1 A$ is projective is proved as follows. Starting from $A = E/M$, define $\Omega^1 E$ as $\mathrm{Ker}(E \otimes E \to E)$ and calculate by the Tor business that one has the bimodule extension $0 \to M \to A \otimes_E \Omega^1 E \otimes_E A \to \Omega^1 A \to 0$ from which E results as above. Then use projectivity of $\Omega^1 A$.

November 10, 1990

It is necessary to make a sustained effort to finish the section on traces and homotopy. So I want to make a list of ideas to develop later.

1) Index thm. on a torus and Fedosov. The idea here is to work with traces on suitable nilpotent extensions in order to get from the index trace to the integral

2) Morita: Recall in thinking about Lundell's deformation of the Bott map the idea that the space of embeddings $U_n \longrightarrow U_N$ such that the induced repn. of U_n is standard repn. + trivial is the space $U_N/S^1\times U_{N-n}$ which is of the homotopy type BS^1 as $N \longrightarrow \infty$. The matrix version is: The space of nonunital algebra homomorphisms $M_n \longrightarrow M_N$ such that the induced representation of M_n on $\mathbb{C}^N$ is the simple module + trivial module is the same space - the principal bundle of the canonical subbundle over the Grassmannian of n planes in $\mathbb{C}^N$ divided by $\mathbb{C}^\times$. (Should be more careful because you need idempotents and not just n-planes.) A particular case is $U_2/S^1\times S^1 = S^2$ which appears inside the nonunital embeddings $A \longrightarrow M_2A$

Keep in mind the idea that there are "K" maps $A \dashrightarrow B$ not obtained from homomorphisms $A \to B$.

Example: $A \longrightarrow S\otimes B$, together with a trace on S.

Also nilpotent extension

$$\begin{array}{ccc} R/I & \longrightarrow & B \\ \uparrow & & \\ A & & \end{array} \qquad I^N = 0$$

Also Morita type map where A acts on a finite projective B module E. In general we expect a trace type "K" map

$$\mathrm{End}_B(E) \dashrightarrow B$$

for example, the usual trace map $M_n B \dashrightarrow B$ when $E = B^n$. In general writing $E = eB^n$ with $e^2 = e$ we have a nonunital homomorphism

$$\mathrm{End}_B(E) = e(M_n B)e \hookrightarrow M_n B$$

So a concrete problem to work on is to construct a map

$$X(\mathrm{End}_B(E)) \longrightarrow X(B)$$

when E is a finite projective B-module. Possible method: Use the fact that an idempotent e in R generates a separable subalgebra, and we have a reduction of $X(R)$ to $X(R;e)$. Perhaps we have a canonical map

$$X(eRe) \longrightarrow X(R;e)$$

3) Homotopy: I feel that the theory of derivations and cyclic formalism should extend from the case of a derivation $D: A \to A$ to a pair $(u, \dot{u}): A \longrightarrow B \oplus B$. Thus associated to $(u, \dot{u})$ should be operators $L: \Omega A \longrightarrow \Omega B$ compatible with the operators $b, d, K, \ldots$ and $I: P\Omega A \longrightarrow P\Omega B$ satisfying $[b, I] = 0$ $[B, I] = PL$

It might be possible to prove this from the case of derivations already treated by using various tricks. For example observe that on $R[t]$, t an indeterminate we have the derivation $D = \partial_t$ which is locally nilpotent. Hence it, and its effect on Ω can be exponentiated, and this coincides with the automorphism $f(t) \longmapsto e^{s\partial_t} f(t) = f(t+s)$. So the effect of e^{sL_D} on $P\Omega(R[t])$ should be homotopic to the identity, and the two evaluation maps

$$P\Omega(R[t]) \underset{t \mapsto 1}{\overset{t \mapsto 0}{\rightrightarrows}} P\Omega(R)$$

should be homotopic.

A related idea is that one should have enough now to extend

$$P\Omega(A \otimes S) \longrightarrow P\Omega(A) \otimes S_4$$

to the first order:

$$P\Omega(A \otimes S) \longrightarrow P\Omega(A) \otimes X(S)$$

Here's a lemma I've forgotten

<u>Lemma:</u> Suppose we have an exact sequence

$$0 \longrightarrow E' \underset{i}{\overset{k}{\rightleftarrows}} E \underset{p}{\overset{h}{\rightleftarrows}} E'' \longrightarrow 0$$

and maps h, k satisfying $1 = ik + hp$.
Then we have $ki = 1$, $ph = 1$, $kh = 0$.

Proof. $p = \cancel{pik} + php = php$, so p surjective $\Rightarrow ph = 1$
Similarly $i = iki + \cancel{hpi}$, so i injective $\Longrightarrow ki = 1$.
$ikhp = (1-hp)hp = hp - h\underbrace{php}_{p} = 0$, so i inj $+ p$ surj $\Longrightarrow kh = 0$.

November 11, 1990

We observe that we have a natural homomorphism

$$* \qquad \boxed{K_1^{alg}A \longrightarrow \mathrm{Ker}\{\Omega^1 A_\natural \longrightarrow \Omega^2 A_{\natural,K}\}}$$

given by $g \in GL_n A \longmapsto tr(g^{-1}dg)$ where tr is the matrix trace. In effect

$$tr\,(g_1g_2)^{-1}d(g_1g_2) = \underbrace{tr(g_2^{-1}g_1^{-1}dg_1\,g_2)}_{\overset{\shortparallel}{tr(g_1^{-1}dg_1) \quad \text{in } \Omega^1 A_\natural}} + tr(g_2^{-1}dg_2)$$

So we get a homomorphism $GL_nA \longrightarrow \Omega^1 A_\natural$ which of course factors through $(GL_nA)_{ab}$ giving a homom. $K_1^{alg}A \longrightarrow \Omega^1 A_\natural$. Next observe that

$$d\,tr(g^{-1}dg) = -tr(g^{-1}dg\,g^{-1}dg)$$

and this should be zero in $(\Omega A)_{\natural s}$ by the usual argument.

Let us notice that the homomorphism $*$ is not compatible with homotopy. More precisely suppose $g_t \in GL_nA$ is a family of invertible matrices Then

$$\partial_t\, tr(g^{-1}dg) = tr(-g^{-1}\dot{g}g^{-1}dg) + tr(g^{-1}d\dot{g})$$

and

$$d\,tr(g^{-1}\dot{g}) = tr(-g^{-1}dg\,g^{-1}\dot{g}) + tr(g^{-1}d\dot{g}).$$

Thus if you want something which is homotopy invariant you want to divide by the image of $A_\natural$ which gives

$$\mathrm{Ker}\{\Omega^1 A_\natural/dA_\natural \longrightarrow \Omega^2 A_{\natural,K}\} = H_1^{DR}(A).$$

Recall

$$H_1^{DR}(A) = \mathrm{Ker}\{\bar{HC}_1(A) \xrightarrow{B} HH_2(A)\}$$
$$= \mathrm{Im}\{S : \bar{HC}_3(A) \longrightarrow \bar{HC}_1(A)\}$$

Anyway, I think the point to observe is that your map $K_1^{alg}A \longrightarrow H_1(X(A)) = HC_1(A)$ is not the best possible map for detecting elements of $K_1^{alg}(A)$. We actually have a better map

$$* \quad K_1^{alg}(A) \longrightarrow \mathrm{Ker}\{\Omega^1 A_\natural \xrightarrow{d} \Omega^2 A_{\natural, K}\}.$$

Put another way not just cyclic 1-cocycles pair with $K_1^{alg}(A)$, but Hoch 1-cocycles do.

(There's some confusion here with $\bar{HC}_1$ and HC_1 but observe $0 \to HC_1 \to \bar{HC}_1 \to \mathbb{C} \to HC_0 \to \bar{HC}_0 \to 0$ so that maybe the difference between HC_1 and $\bar{HC}_1$ doesn't matter. K_1 has to map to $HC_1 \subset \bar{HC}_1$.)

The conclusion of the above discussion is that the map $K_1^{alg}A \longrightarrow HC_1(A)$ can be refined to the map $*$. This suggests we look at $HC^-(A)$. Supposedly higher algebraic K-theory maps to $HC^-(A)$. The idea is that we have a triangle

$$\begin{tikzcd} HC^- \arrow[rr] & & HP \arrow[dl] \\ & HC^+ \arrow[ul] & \end{tikzcd}$$

with HC^+ the usual cyclic homology HC. This triangle is perhaps related to the van Est and relative, alg, top triangles of Karoubi.

Let's recall the picture:

$$\begin{array}{ccccccc}
 & & K^{rel} & \longrightarrow & K^{alg} & \longrightarrow & K^{top} \\
 & & \downarrow & & \downarrow & & \downarrow \\
G & \longrightarrow & B\mathfrak{g} & \longrightarrow & B_{cont}G & \longrightarrow & BG \\
 & & \downarrow & & \downarrow & & \downarrow \\
 & & HC^{+} & \longrightarrow & HC^{-} & \longrightarrow & HP
\end{array}$$

Let's work out the formulas in the smooth commutative case where we have Connes formula for the cyclic homology:

$$K_3^{rel} \to K_3^{alg} \to K_3^{top} \to K_2^{rel} \to K_2^{alg} \to K_2^{top} \to K_1^{rel} \to$$

$$\begin{array}{cccccccccccccccc}
HP_2 & \to & HC_2^{+} & \to & HC_1^{-} & \to & HP_1 & \to & HC_1^{+} & \to & HC_0^{-} & \to & HP_0 & \to & HC_0^{+} & \to \\
H^0 & & H^0 & & & & H^1 & \hookrightarrow & \Omega^1/d\Omega^0 & \to & Z^2 & & H^0 & \hookrightarrow & \Omega^0 & \\
\oplus & & \oplus & & & & \oplus & & & & & \searrow & \oplus & & & \\
H^2 & & \Omega^2/d\Omega^1 & \to & Z^3 & \twoheadrightarrow & H^3 & & & & & \twoheadrightarrow & H^2 & & & \\
\oplus & & & & \oplus & & \oplus & & & & & & \oplus & & & \\
\vdots & & & & H^5 & = & H^5 & & & & & & \vdots & & & \\
 & & & & \oplus & & \oplus & & & & & & & & & \\
 & & & & \vdots & & \vdots & & & & & & & & &
\end{array}$$

$$\begin{array}{ccccccccccccc}
K_2^{top} & \longrightarrow & K_1^{rel} & \longrightarrow & K_1^{alg} & \longrightarrow & K_1^{top} & \longrightarrow & K_0^{rel} & \longrightarrow & K_0^{alg} & \xrightarrow{\sim} & K_0^{top} \\
HP_0 & \longrightarrow & HC_0^{+} & \longrightarrow & HC_{-1}^{-} & \longrightarrow & HP_{-1} & & 0 & & HC_{-2}^{-} & & HP_{-2} \\
H^0 & \longrightarrow & \Omega^0 & \longrightarrow & Z^1 & \twoheadrightarrow & H^1 & & & & H^0 & = & H^0 \\
\oplus & & & & \oplus & & \oplus & & & & \oplus & & \oplus \\
H^2 & & & & H^3 & = & H^3 & & & & H^2 & = & H^2 \\
\oplus & & & & \oplus & & \oplus & & & & \oplus & & \oplus \\
\vdots & & & & \vdots & & \vdots & & & & \vdots & & \vdots
\end{array}$$

Here $Z^n = \operatorname{Ker}\{\Omega^n \xrightarrow{d} \Omega^{n+1}\}$

· Notice that

$$K_1^{alg} \longrightarrow HC^-_{-1} = \underbrace{Z_1}_{\mathrm{Ker}\{\Omega^1 \xrightarrow{d} \Omega^2\}} \oplus H^3 \oplus \cdots\cdots$$

The moral is that our map $*$ can be viewed as the map into HC^-_{-1}, followed by some sort of truncation.

Nov 12, 1990:

Here is something we missed yesterday. Recall the map

$$K_1 A \longrightarrow \Omega^1 A_\natural$$

$$g \longmapsto \mathrm{tr}\, g^{-1}dg \qquad g \in GL_n A$$

In fact this map has its image contained in

$$HH^1 A = \mathrm{Ker}\{\Omega^1 A_\natural \xrightarrow{b} A\}$$

Also

$$d\,\mathrm{tr}(g^{-1}dg) = -\mathrm{tr}(dg^{-1}dg)$$

$$B\,\mathrm{tr}(g^{-1}dg) = \mathrm{tr}(dg^{-1}dg) - \mathrm{tr}(dg\,dg^{-1})$$

$$= -\mathrm{tr}(g^{-1}dg\,g^{-1}dg) + \mathrm{tr}(dg\,g^{-1}dg\,g^{-1})$$

$$= 0$$

Here we think of B as going

$$\begin{array}{ccc} \Omega^1 A_\natural & \xrightarrow{B} & \Omega^2 A_\natural \\ \cup & & \cup \\ HH^1(A) & \xrightarrow{B} & HH^2(A) \end{array}$$

So something we can remember is that we have a canonical map

$$* \quad \boxed{K_1 A \longrightarrow \operatorname{Ker}\{HH_1(A) \xrightarrow{B} HH_2(A)\}}$$

which I guess is the Dennis trace on K_1.

Recall also that

$$H_1^{DR}(\tilde{A}) = \operatorname{Ker}\{HC_1(A) \xrightarrow{B} HH_2(A)\}$$

and

$$HC_0(A) \xrightarrow{B} HH_1(A) \longrightarrow HC_1(A) \longrightarrow 0$$

Putting these together gives the basic Dennis trace

$$* \quad \boxed{\begin{array}{ccc} K_1 A & \longrightarrow & \operatorname{Ker}\{HH_1(A) \xrightarrow{B} HH_2(A)\} \\ & & \downarrow \\ H_1^{DR}(\tilde{A}) & = & \operatorname{Ker}\{HC_1(A) \xrightarrow{B} HH_2(A)\} \end{array}}$$

and the cruder map to $H_1^{DR}(\tilde{A})$. Recall the cruder map is needed if we want invariants for elements of K_1 under homotopy.

Next write $A = R/I$ with R free and recall our candidate for the periodic cyclic homology

$$R^\wedge \underset{d}{\overset{b}{\rightleftarrows}} \Omega^1 R^\wedge_\natural$$

In particular
$$HP_1(A) = \operatorname{Ker}\{\Omega^1 R^\wedge_\natural \xrightarrow{b} R^\wedge\}/d\hat{R}$$
$$= HC_1(R^\wedge) = HH_1(\hat{R})/d\hat{R}$$

where $R^\wedge$ has to be treated as a topological alg and HC_1 HH_1 computed accordingly.

~~Then the~~ We have a canonical map (Dennis)

$$K_1 R^\wedge \longrightarrow HH_1(R^\wedge)$$

in fact to $\mathrm{Ker}\{HH_1(\hat{R}) \xrightarrow{B} HH_2(\hat{R})\}$, but I feel that $HH_n(\hat{R}) = 0$ for $n \geq 2$. If we want an invariant for $K_1 A$ we need to kill dI in $HH_1(\hat{R}) \subset \Omega^1 R^\wedge_\natural$. So it seems we get a map

$$\begin{tikzcd}
 & K_1 A \arrow[d] & & \\
A_\natural \arrow[r, "d"] & \dfrac{\mathrm{Ker}\{\Omega^1 \hat{R}_\natural \xrightarrow{b} R^\wedge\}}{d\hat{I}} \arrow[r] & HP_1(A) \arrow[r] & 0
\end{tikzcd}$$

which is in complete agreement with

$$HC_0(A) \longrightarrow HC^-_{-1}(A) \longrightarrow HP_{-1}(A) \longrightarrow 0$$

<u>November 15, 1990</u>

Fedosov proof of the index theorem:

I think what he does can be described by the following discussion.

Let us recall what the index theorem says. Let M be a compact manifold, and consider the K-group with compact support of its cotangent bundle T^*M. There are two index maps from $K(T^*M)$ to $\mathbb{Z}$, one defined analytically by lifting a K-class to an elliptic operator and taking the index, the other defined topologically (more precisely using characteristic classes) by multiplying the Chern character of the K-class by the Todd genus of T^*M (it is symplectic $\therefore$ almost complex) and integrating over T^*M. The index theorem states that the analytic and topological index maps coincide.

Analytical proofs of the index theorem use some sort of asymptotic method to evaluate the analytic index. This index is expressed as the trace of some operator depending on a parameter. The index doesn't change as the parameter goes to zero, but the asymptotic theory expresses the leading term of the trace as an integral. ~~The Most proofs use heat kernel asymptotics~~ Proofs for Dirac type operators use the McKean-Singer formula for the index as a trace and heat kernel asymptotics.

I want to do the asymptotics using Connes' tangent groupoid associated to M. This is a differentiable groupoid having $\mathbb{R}\times M$ for the manifold of "objects". ~~The manifold~~ If $(h,x)\in\mathbb{R}\times M$ is an object h is to be thought of as Planck's constant.

The manifold Γ of "morphisms" is a suitable blowup of $R \times M \times M$. This blowup maps to the $R \times M \times M$ and is a diffeomorphism where $h \neq 0$, i.e. over $(R - \{0\}) \times M \times M$. Over $h = 0$ we have the tangent bundle TM and things are glued together so that (h, x, y) converges to $(0, v)$ with $v \in TM$ when $\frac{x-y}{h} \to v$. If one has an embedding $M \subset R^N$ then Γ is the closure of the space of $(h, x, y, \frac{x-y}{h})$, $h \neq 0$, in $R \times M \times M \times R^N$. Thus for $h = 0$ we get the groupoid where objects are pts of M, where a morphism $x \to x$ is a tangent vector at x, and where composition is addition of tangent vectors. For $h \neq 0$ we get the groupoid with the same objects, where a morphism $x \mapsto y$ is the pair (x, y).

I should go on and describe the convolution algebra of this groupoid. This consists of smooth functions (or maybe some kind of densities) on Γ. In particular these functions depend on h. For $h = 0$ we have smooth kernels, i.e. smoothing operators, hence a canonical trace given by integrating over the diagonal. For $h = 0$ we have Schwartz fns. on TM which by F.T. is the same as the algebra of Schwartz functions on T^*M under multiplication.

Now the canonical trace for $h \neq 0$ blows up as $h \to 0$, but it seems one can define a trace on the convolution algebra $S(\Gamma)$ having values in $h^{-n} C^\infty(R)$ where $n = \dim M$. One can work formally around $h = 0$.

$$\varprojlim_{k} S(\Gamma) / h^k S(\Gamma)$$

is a deformation of $\mathcal{S}(T^*M)$ on which one has a trace with values in $h^{-n}\mathbb{C}[[h]]$.

This seems to be the important fact, namely that there is a canonical algebra $\mathcal{A}(M)$ over $\mathbb{C}[[h]]$ together with a trace having values in $h^{-n}\mathbb{C}[[h]]$ such that $\mathcal{A}(M)/h\mathcal{A}(M) = \mathcal{S}(T^*M)$. A first problem is how to describe this algebra intrinsically.

There should be an action of $\mathbb{R}^{\times}_{>0}$ on this algebra which rescales h and also cotangent vectors. What this should mean is that the trace on $\mathcal{A}(M)$ given by the coefficient of h^k is null homotopic for $k \neq 0$.

Let's be more specific. Suppose M is a torus.