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March 13, 1988

Let's go over the Beilinson - Schechtman account of Tate's approach to residues.

Let $F = E(\mathring{\mathcal{U}}) \cong \mathbb{C}((t))^n \supset E(\mathcal{U}) \cong \mathbb{C}[[t]]^n$.

A subset S of F is called open if $S \supset t^N E(\mathcal{U})$ and bounded if $S \subset t^{-N} E(\mathcal{U})$ for $N \gg 0$. Let

$$R = \mathrm{End}_{cont}(F) = \{\alpha \in \mathrm{End}_{\mathbb{C}} F \mid \alpha^{-1} \text{ open is open}\}$$

$$I_0 = \{\alpha \mid \mathrm{Ker}\,\alpha \text{ open}\}$$

$$I_b = \{\alpha \mid \mathrm{Im}\,\alpha \text{ bdd}\}$$

These are 2-sided ideals in R. One has $I_0 + I_b = R$ since if e is a projector of F onto $E(\mathcal{U})$, then $1 = (1-e) + e \in I_0 + I_b$. If $\alpha \in I_0 \cap I_b$, then we can choose lattices (bounded open subspaces) L_0, L_1 such that

$$L_0 \subset \mathrm{Ker}\,\alpha, \quad L_1 \supset \mathrm{Im}\,\alpha, \quad L_0 \subset L_1$$

Then the filtration $0 \subset L_0 \subset L_1 \subset F$ is stable under α; in fact, $\alpha(L_0) = 0$, $\alpha(F) \subset L_1$, so α induces zero on L_0, F/L_1. Thus we can define

$$\mathrm{tr}(\alpha) = \mathrm{tr}_{L_1/L_0}(\alpha)$$

since L_1/L_0 is finite dimensional. This is obviously independent of the choice of L_0, L_1. This trace defined on $I_0 \cap I_1$ is obviously $\mathbb{C}$-linearly. It satisfies

$$\mathrm{tr}([\alpha, \beta]) = 0 \qquad \text{if } \alpha \in I_0, \beta \in I_b .$$

To see this, I first must go back and check that any continuous operator maps bounded sets to bounded sets. Let $\rho \in \mathcal{R}$ and let L be a lattice. Because ρ is continuous we know $\rho^{-1}L_1$ (L_1 another lattice) is open, hence $L/L\cap\rho^{-1}L_1$ is finite-dimensional. Let $x_1,\dots,x_n \in L$ span $L/L\cap\rho^{-1}L_1$, then $L \subset \mathbb{C}x_1+\dots+\mathbb{C}x_n+\rho^{-1}L_1$, so $\rho(L) \subset \mathbb{C}\rho(x_1)+\dots+\mathbb{C}\rho(x_n)+L_1$; we can enlarge L_1 to contain $\rho(x_1),\dots,\rho(x_n)$, so $\rho(L)$ is bounded.

Now let $\alpha, \beta \in \mathcal{R}$ be such that $\mathrm{Ker}(\alpha)$ is open and $\mathrm{Im}\,\beta$ is bounded. Then $\mathrm{Ker}(\alpha\beta) = \beta^{-1}\mathrm{Ker}(\alpha)$ and $\mathrm{Ker}(\beta\alpha) \supset \mathrm{Ker}(\alpha)$ are both open, so we can find a lattice L_0 with

$$L_0 \subset \mathrm{Ker}\,\alpha,\ \mathrm{Ker}\,\beta\alpha,\ \mathrm{Ker}\,\alpha\beta$$

Similarly $\mathrm{Im}\,\alpha\beta = \alpha(\mathrm{Im}\,\beta)$ and $\mathrm{Im}\,\beta\alpha \subset \mathrm{Im}\,\beta$ are bounded, so we can find a lattice L_1 such that

$$L_1 \supset L_0,\ \mathrm{Im}(\beta),\ \mathrm{Im}(\alpha\beta),\ \mathrm{Im}(\beta\alpha)$$

Then $\alpha\beta$, $\beta\alpha$ induce maps on L_1/L_0 and $\mathrm{tr}(\alpha\beta)$, $\mathrm{tr}(\beta\alpha)$ are the ordinary traces of the induced maps on L_1/L_0.

Unfortunately I have no control over $\mathrm{Im}\,\alpha$ or $\mathrm{Ker}\,\beta$. Look at $0 \subset L_0 \subset L_1 \subset F$

$$\alpha : \begin{pmatrix} 0 & & ? \\ 0 & & \\ 0 & & \end{pmatrix}$$

March 14, 1988

Consider a square zero extension

$$0 \longrightarrow M \longrightarrow Q \xrightarrow[\pi]{} A \longrightarrow 0$$

We know this leads to a Connes homomorphism $\gamma : HC_2(A) \longrightarrow HC_0(Q)$ which is completely canonical. On p. 310 we found a formula for γ, specifically a cyclic 2-cocycle on A with values in $HC_0(Q)$, depending on a linear section $\rho : A \to Q$ of π. The formula is

$$\varphi(a_0, a_1, a_2) = \rho(a_0)\rho(a_1)\rho(a_2) - \tfrac{1}{2}\rho(a_0a_1)\rho(a_2) - \tfrac{1}{2}\rho(a_1a_2)\rho(a_0) - \tfrac{1}{2}\rho(a_2a_0)\rho(a_1)$$

It would be nice to derive this formula by naturality conditions, using that it has to be compatible with the S-operator and also for A. This S-operator $HC_2(Q) \to HC_0(Q)$ is given by the 2-cocycle $(x, y, z) \longmapsto xyz$.

Naturality means that the diagram

$$\begin{tikzcd} HC_2(Q) \arrow[r, "S"] \arrow[d] & HC_0(Q) \arrow[d] \\ HC_2(A) \arrow[r, "S"] \arrow[ur, dashed, "\gamma"] & HC_0(A) \end{tikzcd}$$

commutes. Thus if γ is represented by $\varphi(a_0, a_1, a_2)$ we must have

$$xyz - \varphi(\pi x, \pi y, \pi z) = (b\psi)(x, y, z)$$

where $\psi : \Lambda^2 Q \longrightarrow HC_0(Q)$ is a cyclic 1-cochain. Similarly

$$a_0a_1a_2 - \pi\varphi(a_0, a_1, a_2) = \delta\chi(a_0, a_1, a_2)$$

with $\chi : \Lambda^2 A \longrightarrow HC_0(A)$.

March 15, 1988

We consider a square zero extension
$$0 \longrightarrow M \longrightarrow Q \xrightarrow{\pi} A \longrightarrow 0$$
and we would like to find a formula for the Connes homomorphism γ

$$\begin{tikzcd} HC_2(Q) \arrow[r, "S"'] \arrow[d] & HC_0(Q) \arrow[d] \\ HC_2(A) \arrow[r, "S"'] \arrow[ur, dashed, "\gamma"] & HC_0(A) \end{tikzcd}$$

The S operator is given by the cyclic cocycle $(a_0,a_1,a_2) \longmapsto a_0a_1a_2 \mod [A,A]$. We want to find a cyclic 2-cocycle $\varphi(a_0,a_1,a_2)$ on A with values in $HC_0(Q)$ representing γ. The commutativity of the top triangle says

$$(*) \qquad x_0x_1x_2 - \varphi(\pi x_0, \pi x_1, \pi x_2) = (b\psi)(x_0,x_1,x_2)$$

where $\psi : \Lambda^2 Q \longrightarrow HC_0(Q)$ is a cyclic 1-cochain. It seems reasonable to require
$$\pi\varphi(a_0,a_1,a_2) = a_0a_1a_2$$
which guarantees $\pi\gamma = S_A$.

Notice that ψ determines φ; thus all we need to do is find ψ so that the difference of the cyclic cocycles $x_0x_1x_2$ and $b\psi$ vanishes if one of the x_i is in M.

We expect φ,ψ to depend upon a choice of linear cross-section $\rho: A \longrightarrow Q$ of π. One way to obtain the formulas might be as follows. <u>First suppose</u> the extension splits, i.e. ρ can be chosen as a homomorphism. Then the obvious choice for φ is
$$\varphi(a_0,a_1,a_2) = \rho(a_0)\rho(a_1)\rho(a_2) = \rho(a_0a_1a_2)$$
and then $\quad x_0x_1x_2 - \varphi(\pi x_0,\pi x_1,\pi x_2) = (1-\rho\pi)(x_0x_1x_2)$

But $1, \rho\pi$ are two liftings

$$\begin{tikzcd} & & Q \arrow[dll, shift left, "1 = id"'] \arrow[dll, shift right, "\rho\pi"] \arrow[d, "\pi"] \\ M \arrow[r] & Q \arrow[r, "\pi"'] & A \end{tikzcd}$$

and their difference $\partial = 1 - \rho\pi : Q \longrightarrow M$ is a derivation. Now I want to use that $\partial S = S\partial = 0$. We have

$$(**) \qquad \partial(x_0 x_1 x_2) = (\partial x_0) x_1 x_2 + (\partial x_1) x_2 x_0 + (\partial x_2) x_0 x_1 .$$

This is the composition

$$HC_2(Q) \xrightarrow{\partial} HC_2(Q) \xrightarrow{S} HC_0(Q)$$

which is zero, which means that $(**)$ is in the form $b\psi$ with $\psi : \Lambda^2 Q \longrightarrow HC_0(Q)$. (One should maybe think of an arbitrary trace on Q as given, so as to have cocycles with values in $\mathbb{C}$.)

There are very few possibilities for ψ, i.e. it has to be bilinear and skew-symmetric hence a multiple of the following

$$\psi(x,y) = (\partial x) y - x (\partial y) \qquad \text{mod } [Q,Q]$$

$$\begin{aligned} b\psi(x,y,z) &= \psi(xy,z) + \psi(yz,x) + \psi(zx,y) = (\partial x\, y + x\, \partial y) z - xy\, \partial z + \text{cyclic} \\ &= \partial x\, yz + zx\, \partial y - \partial z\, xy + \text{cyclic versions} \\ &= (\partial x) yz + \text{cyclic versions} \end{aligned}$$

Next suppose that our extension splits, so there is a homomorphism $s : A \to Q$ such that $\pi s = id_A$, but that $\rho(a) = s(a) + g(a)$ with $g : A \longrightarrow M$. We are trying to find ψ so that $xyz - b\psi(x,y,z)$ vanishes when any of these elements lie in M. We know this is the case for

$$\psi_0(x,y) = ((1-s\pi)(x))y - x(1-s\pi)(y)$$

$$= -(s\pi(x))y + x(s\pi(y))$$

But if we add to this

$$\psi_1(x,y) = -(g(\pi x))y + x\,g(\pi y)$$

$$= -g(\pi x)\,\pi y + \pi x \cdot g(\pi y)$$

then

$$xyz - (b\psi_0 + b\psi_1)(x,y,z)$$

$$= \underbrace{xyz - b\psi_0(x,y,z)}_{s(\pi x\ \pi y\ \pi z)} - \underbrace{b\psi_1(x,y,z)}_{\text{2-cocycle depending on } \pi x,\ \pi y,\ \pi z}$$

Thus we get the desired formula by using

$$\boxed{\psi(x,y) = -\rho(\pi x)y + x\rho(\pi y)}$$

Now we should check this works in general.

Set $h(x) = x - \rho(\bar{x})$. Then

$$h(xy) = xy - \rho(\bar{x}\bar{y})$$

$$= xy - \rho(\bar{x})\rho(\bar{y}) + \rho(\bar{x})\rho(\bar{y}) - \rho(\bar{x}\bar{y})$$

$$= (x-\rho(\bar{x}))y + \rho(\bar{x})(y-\rho(\bar{y})) + f(\bar{x},\bar{y})$$

$$\boxed{h(xy) = h(x)y + xh(y) - \underbrace{h(x)h(y)}_{\in M^2 = 0} + f(\bar{x},\bar{y})}$$

Do another stage:

$$h(xyz) = h(xy)z + xy\,h(z) - h(xy)h(z) + f(\overline{xy},\bar{z})$$

$$= \big(h(x)y + xh(y) + h(x)h(y) + f(\bar{x},\bar{y})\big)\big(z + h(z)\big) + xyh(z) + f(\overline{xy},\bar{z})$$

$$= \left.\begin{array}{l} h(x)yz + xh(y)z + xyh(z) \\ \quad + f(\bar{x},\bar{y})z + f(\overline{xy},\bar{z}) \end{array}\right\} \in I$$

signs wrong

$$\left.\begin{array}{l} + h(x)h(y)z + h(x)y\,h(z) + x\,h(y)h(z) \\ \quad + f(\bar{x},\bar{y})h(z) \end{array}\right\} \in I^2$$

$$\left. + h(x)h(y)h(z) \right\} \in I^3$$

So now go back to $h(x) = x - \rho(\bar{x})$ and $\psi(x,y) = -\rho(\bar{x})y + x\rho(\bar{y}) = h(x)y - xh(y)$. Then

$$(b\psi)(x,y,z) = \psi(xy,z) + \text{cyc.}$$

$$= h(xy)z - xyh(z) + \text{cyc.}$$

$$= \big(h(x)y + xh(y) + f(\bar{x},\bar{y})\big)z - xyh(z) + \text{cyc.}$$

Modulo $[Q,Q]$ this is

$$= h(x)yz + f(\bar{x},\bar{y})\bar{z} + \text{cyc}$$

On the other hand (using $M^2=0$)

$$xyz - \rho(\bar{x})\rho(\bar{y})\rho(\bar{z}) = (x-\rho(\bar{x}))yz + \rho(\bar{x})(y-\rho(\bar{y}))z + \rho(\bar{x})\rho(\bar{y})(z-\rho(\bar{z}))$$

$$= h(x)yz + xh(y)z + xyh(z)$$

$$\therefore \quad (b\psi)(x,y,z) = xyz - \rho(\bar{x})\rho(\bar{y})\rho(\bar{z}) + \big(f(\bar{x},\bar{y})\bar{z} + \text{cyc.}\big)$$

So

$$\boxed{\varphi(a_0,a_1,a_2) = \rho(a_0)\rho(a_1)\rho(a_2) - \big(f(a_0,a_1)a_2 + \text{cyc}\big)}$$

is the formula for the Connes homomorphism

Now there are ~~two~~ several projects which seem worthwhile. First of all I would like to evaluate the cyclic 2-diml class on $a = \mathcal{S}(T^*(S^1))$ that is obtained from the trace on $\tilde{a}$. Secondly I want to find general formulas for the Connes homomorphism in higher degrees. Thirdly there is the circle of ideas related to Toeplitz operators, by which I mean the way the cyclic cocycles become non-commutative "cycles" when the lifting is given by a Toeplitz projection.

Let's look at $A = a = \mathcal{S}(T^*(S^1))$, and take $Q = a + ha \mod h^2$ with the twisted multiplication. In this case the Hochschild 2-cocycle describing the extension is

$$f(a_0, a_1) = a_0 * a_1 - (a_0 a_1)$$

$$= \frac{h}{i} \partial_p a_0 \, \partial_x a_1$$

What is the problem? Associated to this extension is a Connes homomorphism γ from $HC_2(A)$ to $HC_0(Q)$.

$$HC_0(Q) = A \oplus hA/\{A,A\}$$

$$\begin{tikzcd} & HC_0(Q) \arrow[d] \\ HC_2(A) \arrow[ur, "\gamma"] \arrow[r, "S"'] & HC_0(A) = A \end{tikzcd}$$

since A is commutative

The map S is zero because there are faithful derivations on A. So any trace on Q which reduces to $\int \frac{dx\,dp}{2\pi} \beta(x,p)$ on $\alpha + h\beta$ will give the same 2-diml cyclic cohomology class. The problem is to show that the cyclic 2-~~class~~ cocycle

$$\varphi(a_0, a_1, a_2) = \int_{T^*S^1} a_0 \, da_1 \, da_2$$

represents this class up to some numerical factor.

Now in p.591 we obtained a representative φ for γ which depends on a choice of lifting ρ. Set

$$\psi(x,y) = h(x)y - x\,h(y) \qquad h(x) = x - \rho(\bar{x})$$

$$= -\rho(\bar{x})y + x\rho(\bar{y})$$

Then

$$\varphi(\bar{x},\bar{y},\bar{z}) = xyz - b\psi(x,y,z)$$

$$= xyz - \big(h(xy)z - xy\,h(z) + \text{cyc.}\big)$$

$$= xyz - \big\{\big(h(x)y + x\,h(y) + f(\bar{x},\bar{y})\big)z - xy\,h(z) + \text{cyc}\big\}$$

$$= xyz - \big\{h(x)yz + f(\bar{x},\bar{y})z + \text{cyc.}\big\}$$

But

$$xyz - \rho(\overline{xyz}) = h(xyz)$$

$$= h(x)yz + xh(y)z + xy\,h(z) + f(\bar{x},\bar{y})z + f(\overline{xy},z)$$

$$= \big(h(x)yz + \text{cyc.}\big) + f(\bar{x},\bar{y})z + f(\overline{xy},z)$$

$$\therefore\ \varphi(\bar{x},\bar{y},\bar{z}) + xyz - \rho(\overline{xyz}) = xyz - \big\{f(\bar{x},\bar{y})z + f(\bar{y},\bar{z})x + f(\bar{z},\bar{x})y\big\} + f(\bar{x},\bar{y})z + f(\overline{xy},z)$$

which gives the formula

$$\boxed{\varphi(\bar{x},\bar{y},\bar{z}) = \rho(\overline{xyz}) + f(\overline{xy},\bar{z}) - \bar{x}f(\bar{y},z) - \bar{y}f(\bar{z},\bar{x})}$$

Now take $f(a_0,a_1) = \frac{h}{i}(\partial_p a_0)(\partial_q a_1)$. Then

$$f(a_0a_1,a_2) - a_0 f(a_1,a_2) - a_1 f(a_2,a_0) =$$

$$\frac{h}{i}\left\{\partial_p(a_0a_1)\,\partial_q a_2 - a_0\partial_p a_1\,\partial_q a_2 - a_1\partial_p a_2\,\partial_q a_0\right\}$$

$$= \frac{h}{i}\, a_1\left(\partial_q a_2\,\partial_p a_0 - \partial_p a_2\,\partial_q a_0\right) = \frac{h}{i}\, a_1\{a_2, a_0\}$$

Now apply the trace which takes $\int \frac{dq\,dx}{2\pi}$ of the coefficient of h. This kills $\rho(A)$ and gives the cyclic 2-cocycle

$$\frac{1}{2\pi i}\int a_1\, da_2\, da_0 = \frac{1}{2\pi i}\int a_0\, da_1\, da_2 .$$

I propose now to work a little toward motivating Connes' approach to cyclic theory via differential calculus.

At first sight this is a puzzle, because we know $HC_2(A)$ has to do with traces on square zero extensions, whereas differential 1-forms and derivations have to do with split extensions. But it seems there is an interesting formalism to be understood.

Suppose we have a diagram

$$\begin{tikzcd} 0 \arrow[r] & M \arrow[r] & Q \arrow[r, "\pi"] & A \arrow[r] & 0 \\ & & B \arrow[u, shift left, "u_1"] \arrow[u, shift right, "u_2"'] \arrow[ur] & & \end{tikzcd}$$

where u_1, u_2 are alg. homomorphisms congruent mod M and $M^2=0$. Then we get the commutative diagram

$$\begin{tikzcd} HC_2(B) \arrow[r, shift left, "u_1"] \arrow[r, shift right, "u_2"'] \arrow[dr] & HC_2(Q) \arrow[r, "S"] \arrow[d, "\pi"] & HC_0(Q) \\ & HC_2(A) \arrow[ur, "\gamma"'] & \end{tikzcd}$$

showing that $Su_1 = Su_2 : HC_2(B) \to HC_0(Q)$.

But this comment should hold much more generally for higher cyclic groups. First we can pull-back the extension

$$
\begin{tikzcd}
0 \arrow[r] & M \arrow[r] & Q \arrow[r] & A \arrow[r] & 0 \\
0 \arrow[r] & M \arrow[r] \arrow[u, equal] & Q \times_A B \arrow[r, shift left] \arrow[u] & B \arrow[r] \arrow[u] \arrow[ul, shift left] \arrow[ul, shift right] \arrow[l, bend left] \arrow[l, bend left=50] & 0
\end{tikzcd}
$$

and so reduce to two sections of a square zero extension of B. These differ by a derivation with values in M, which means means we have a unique map

$$
\begin{tikzcd}
0 \arrow[r] & M \arrow[r] & Q \times_A B \arrow[r] & B \arrow[r] \arrow[l, bend right, "u_1"'] \arrow[l, bend right=50, "u_2"'] & 0 \\
0 \arrow[r] & \Omega^1_B \arrow[r] \arrow[u] & B \oplus \Omega^1_B \arrow[r] \arrow[u] & B \arrow[r] \arrow[u, equal] \arrow[l, bend right, "i"'] \arrow[l, bend right=50, "\bar{i}"'] & 0
\end{tikzcd}
$$

where $\bar{i} = i + d$. Thus the two maps

$$u_1, u_2 : HC_n(B) \rightrightarrows HC_n(Q)$$

come from the maps

$$i,\ i+d : HC_n(B) \rightrightarrows HC_n(B \oplus \Omega^1_B)$$

We have seen that these two maps agree on $S\,HC_{n+2}(B)$. Thus we can conclude that in

$$
\begin{tikzcd}
HC_{n+2}(Q) \arrow[r, "S"] & HC_n(Q) \\
HC_{n+2}(B) \arrow[r, "S"] \arrow[u, shift left, "u_2"'] \arrow[u, shift right, "u_1"] \arrow[ur, dotted, "?"] & HC_n(B) \arrow[u, shift left, "u_2"'] \arrow[u, shift right, "u_1"]
\end{tikzcd}
$$

$u_1 S = u_2 S \implies Su_1 = Su_2$. This is not quite as strong as having a canonical map

$$? : HC_{n+2}(B) \longrightarrow HC_n(Q)$$

attached to a square-zero extension which might not exist.

(Recall that if $B = R/I$ with R free and $Q = R/I^2$, then we have

$$
\begin{tikzcd}
HC_{2n}(Q) \arrow[r, hook] & HC_0(R/I^{2n+2}) \\
HC_{4n+2}(B) \arrow[ur] &
\end{tikzcd}
$$

)

Question: Assume $u_1, u_2 : B \rightrightarrows Q$ are congruent modulo a nilpotent ideal of Q. Can you ~~[illegible]~~ find a fixed k depending on the degree of nilpotense such that $S^k(u_{1*} - u_{2*}) = 0$ on cyclic homology?

Problem. Can we deduce Goodwillie's theorem that a nilpotent extension induces an isomorphism on periodic cyclic homology from the isomorphisms in my paper? Consider the even case:

$$HC_{2n}(A) = \varprojlim_{\substack{Q \text{ nilp. extension} \\ \text{of } A \text{ of order } n}} HC_0(Q)$$

$$HC^{per}_{even}(A) = \varprojlim_{\substack{Q \text{ nilp} \\ \text{extension of } A}} HC_0(Q)$$

(I am fudging the $R^1\varprojlim$ problem.)

Let $\mathcal{N}(A)$ be the category of nilpotent extensions of A. Given a map $A' \longrightarrow A$ we have pull-back $Q \longrightarrow A' \times_A Q$ from $\mathcal{N}(A)$ to $\mathcal{N}(A')$ and this induces a map

$$1) \qquad \Phi : \varinjlim_{Q' \in \mathcal{N}(A')} F(Q') \longrightarrow \varinjlim_{Q \in \mathcal{N}(A)} F(Q)$$

for any functor on algebras such as HC_0.

On the other hand if $A' \to A$ is a nilpotent extension of A, then we have a functor $\mathcal{N}(A') \longrightarrow \mathcal{N}(A)$ given by composition $(Q' \to A') \longmapsto (Q' \to A' \to A)$, whence a map Ψ in the opposite direction to 1). We want to show the composition is an isomorphism.

Let $\xi \in \varprojlim_{\mathcal{N}(A)} F(Q)$. What is $\Phi\Psi(\xi)$?

$\Psi(\xi)$ assigns to $Q' \to A'$ the element $\xi(Q' \to A' \to A)$

and $\Phi(\Psi(\xi))$ assigns to $Q \to A$ the element

$$\Psi(\xi)(A'\times_A Q \to A') = \xi(A'\times_A Q \to A' \to A) \in F(A'\times_A Q)$$

pushed into $F(Q)$ via $pr_2 : A'\times_A Q \to Q$.

Actually I should have been more careful about Φ.

$$\Phi : \varprojlim_{Q'\in \mathcal{N}(A')} F(Q') \longrightarrow \varprojlim_{Q\in \mathcal{N}(A)} F(Q)$$

$\Phi(\eta)$ assigns to $Q \to A$, the image of the element $\eta(A'\times_{A'} Q \to A')$ $\in F(A'\times_A Q)$ in $F(Q)$.

So therefore we have to see that

$$\xi(Q\to A) = (pr_2)_*(A'\times_A Q \to A' \to A)$$

which is OKAY because $pr_2 : A'\times_A Q \to Q$ is a map in $\mathcal{N}(A)$.

$$\begin{tikzcd}
A'\times_A Q \arrow[r, "pr_2"] \arrow[d] \arrow[dr] & Q \arrow[d] \\
A' \arrow[r] & A
\end{tikzcd}$$

On the other hand let's start with $\eta \in \varprojlim_{\mathcal{N}(A')} F(Q')$

Then $\Psi(\Phi(\eta))$ assigns to $Q' \to A'$ the element $\Phi(\eta)(Q' \to A' \to A)$ which is $\eta(A'\times_A Q' \xrightarrow{pr_1} A')$ pushed via $(pr_2)_* : F(A'\times_A Q') \to F(Q')$ into $F(Q')$.

$$\begin{tikzcd}
A'\times_A Q' \arrow[r, shift right, "pr_2"'] \arrow[d] & Q' \arrow[l, shift right, "\sigma"'] \arrow[d] \arrow[dl, dashed] \\
A' \arrow[r] & A
\end{tikzcd}$$

One has to be careful because pr_2 is not a map over A'. However the map $Q' \to A'$ gives a section $\sigma : Q' \to A'\times_A Q'$ which is a map in $\mathcal{N}(A')$, and so by the property of η

$$\sigma_*\eta(Q'\to A') = \eta(A'\times_A Q' \xrightarrow{pr_1} A')$$

Now apply $(pr_2)_*$ and we win.

Next let's check this argument for the odd case where

$$\bar{H}C_{2n+1}(A) = \varprojlim_{R/I=A} I^{n+1}/[I,I^n]$$

Here we have a functor $F(R\to A)$ which is not just a functor of R.

Again given $A' \to A$ we define

$$\Phi : \varprojlim F(R'\to A') \longrightarrow \varprojlim F(R\to A)$$

by letting $\Phi(\eta)(R\to A) = pr_{2*}\{\eta(A'\times_A R \xrightarrow{pr_1} A') \in F(A'\times_A R \to A')\}$ $\in F(R\to A)$. ~~For a nilpotent extension $A'\to A$, we define $\Phi(\xi)(R'\to A') = \xi(R'\to A'\to A)$.~~

~~What is $\Phi(\Phi(\xi))$? $\boxed{\text{Given } R\to A}$ It assigns~~

~~$\xi(A'\times_A R \to A' \to A) \in F$.~~

Notice that when $F(R\to A) = \varprojlim_n I^{n+1}/[I,I^n]$ that $pr_{2*} : F(A'\times_A R \to A') \xrightarrow{\sim} F(R\to A)$ since the ideal doesn't change as an algebra.

Actually we might be a bit more precise about the functor F. It assigns to an extension $R\to A$ the pro-object $J \mapsto J/[J,J]$, where J ranges over all ideals in R such that R/J is a nilpotent extension of A.

(Why:

$$I^{4n}/[I^{2n},I^{2n}] \longrightarrow I^{2n+1}/[I,I^{2n}] \longrightarrow I^{2n}/[I^n,I^n] \longrightarrow I^{n+1}/[I,I^n]$$

because $[I,I^{2n}] \subset [R,I^{2n}] \subset [I^n,I^n]$)

Suppose that $A' \to A$ is a nilpotent extension. Given $R \twoheadrightarrow A'$, let $R/J = A'$, $R/I = A$ so that $I^N \subset J \subset I$ for some N. If K is an ideal of R contained in J, then clearly $R/K \to R/J = A'$ is a nilpotent extension iff $R/K \to R/I = A$ is a nilpotent extension ($(J/K)^n \subset (I/K)^n$ and $(I/K)^{Nn} \subset (J/K)^n$)

Thus we have an isomorphism

$$F(R \to A') \xleftarrow{\sim} F(R \to A)$$

Let now consider

$$\varprojlim_{R' \twoheadrightarrow A'} F(R' \to A') \underset{\Psi}{\overset{\Phi}{\rightleftarrows}} \varprojlim_{R \twoheadrightarrow A} F(R \to A)$$

$$\Phi(\eta)(R \to A) = \eta(A' \times_A R \to A') \in \begin{array}{c} F(A' \times_A R \to A') \\ \beta \downarrow \wr \\ F(R \to A) \end{array}$$

$$\Psi(\xi)(R' \to A') = \xi(R' \to A' \to A) \in F(R' \to A) \underset{\alpha}{\overset{\sim}{\to}} F(R' \to A')$$

What is $\Psi(\Phi(\eta))$? Applied to $R' \to A'$, it gives $\alpha\, \Phi(\eta)(R' \to A' \to A)$ which is $\alpha\beta\eta(A' \times_A R' \to A')$ in

$$\begin{tikzcd} A' \times_A R' \arrow[r, shift left] \arrow[d] & R' \arrow[l, shift left] \arrow[d] \arrow[dl] \\ A' \arrow[r] & A \end{tikzcd} \qquad \begin{array}{c} F(A' \times_A R' \to A') \\ \beta \downarrow \wr \\ F(R' \to A) \underset{\alpha}{\overset{\sim}{\to}} F(R' \to A') \end{array}$$

It seems OK but it's not very clear.

March 17, 1988

New approach: The Connes homomorphism at least in the odd degree case results naturally by examining the cyclic complex for the DG algebra $R \leftarrow I$. Thus one has a quis

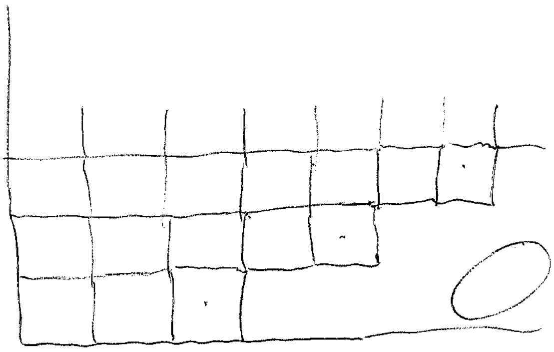
$$CC(A) \longleftarrow CC(R \leftarrow I)$$

because CC preserves quis, and on the other hand one can see explicit cocycles on the latter complex because of the edge structure.

We can generalize this as follows: ~~Suppose we~~ take a DG algebra resolution of A of length r. To fix the ideas take r=2.

$$0 \leftarrow A \leftarrow R_0 \leftarrow R_1 \leftarrow R_2 \leftarrow 0 \leftarrow$$

Then $CC(R_\cdot)$ is a double complex with the shape



and we will get [redacted] edge homomorphisms

$$HC_2(A) \longrightarrow (R_2)/\text{something}$$

$$HC_5(A) \longrightarrow (R_2)^{\otimes 2}_{\lambda} / \text{[redacted] something [redacted]}$$

$$HC_8(A) \longrightarrow (R_2)^{\otimes 3}_{\lambda} / \text{something}.$$

In fact the something is the image of b in the appropriate bidegree. So we are dealing with the non-commutative differentials on the DGA. There seems to be a definite "integral" or "cycle" (in the spirit of Connes) which gives rise to a well-defined cyclic class on A.

Actually we can describe the cokernel of b easily in the top degree. We want the cokernel of

$$R_\circ^{\otimes(n+1)} \xrightarrow{b} R_\circ^{\otimes n}$$

in the highest degree $2n$ where

$$\left(R_\circ^{\otimes n}\right)_{2n} = R_2^{\otimes n}$$

Look at a typical homogeneous element of $\left(R_\circ^{\otimes(n+1)}\right)_{2n}$ say

$$(x_0, \cdots, x_n) \quad \text{where} \quad \sum_{i=0}^{n} \deg(x_i) = 2n$$

There are two cases: 1) where exactly one of the $x_i \in R_0$ and the rest are in R_2, and 2) where two of the x_i are in R_1 and the rest are in R_2. The first gives relations leading to the n-fold cyclic tensor product of R_2 over R_0. The second forces us to divide out by R_1^2 inside each factor of R_2, once $n \geq 2$.

$$\therefore \ (R_2)^{\otimes n}_\lambda / \text{something} = \left((R_2/R_1^2)^{\otimes}_{R_0}\right)^n_\lambda \qquad n \geq 2.$$

Notice that one has a map

$$HC_*(A) \longrightarrow H_*(R_\circ/[R_\circ, R_\circ])$$

which is essentially an isomorphism for $R_\circ$ free

The central goal remains to find a really good approach to cyclic theory. Up to now I have accepted the cyclic complex as some kind of natural object. But I want a way to interpret this complex in a way which explains its significance. Why should <u>cyclic</u> cocycles be good objects of study?

Can one find a better description of the even Connes homomorphism?

Idea: Let's consider the graded vector space $V_0 \oplus V_1$ and form its super symmetric algebra, namely $S(V_0) \otimes \Lambda(V_1)$. We consider maps $V_0 \underset{k}{\overset{h}{\rightleftarrows}} V_1$. These are elements of the Lie superalgebra $\mathrm{End}(V)$ and hence extend as (super) derivations ~~to~~ $S(V_0) \otimes \Lambda V_1$. Then $[h,k] = hk + kh$ is the derivation of $S(V_0) \otimes \Lambda V_1$ which extends hk on V_1, kh on V_0. Next suppose $hk = id$ on V_1 whence kh is the projector on V_0 with kernel $= \mathrm{Ker}\, h$ (call this W) and $\mathrm{Im} = \mathrm{Im}(k)$. We have

$$S(V_0) \otimes \Lambda V_1 = S(W) \otimes \left(S(kV_1) \otimes \Lambda V_1 \right)$$

The derivation $[h,k]$ acts trivially on $S(W)$ and by multiplication by the total degree in $S(kV_1) \otimes \Lambda V_1$. Therefore if we view $S(V_0) \otimes \Lambda(V_1)$ as a DGA with the differential k we obtain a contraction of this DGA to its homology $S(W)$ by using h divided by the degree in $S(kV_1) \otimes \Lambda(V_1)$.

Let's now take an extension

$$0 \longrightarrow I \xhookrightarrow{\partial} R \xrightarrow[\pi]{} A \longrightarrow 0$$

and choose a lifting $\rho : A \longrightarrow R$ which is $\mathbb{C}$ linear and set

$$h = 1 - \rho\pi : R \longrightarrow I$$

~~Then consider the tensor product~~ ~~$(R \oplus I)^{\otimes n}$~~ ~~the inclusion map $I \hookrightarrow R$ when extended to~~

Consider the complex $(R \longleftarrow I)^{\otimes n}$. The differential ∂ in this complex is obtained by extending the odd map given by the inclusion of I in R to the tensor algebra as a (anti) derivation.

A nice way (in the spirit of physics) to think is to view $R \oplus I$ as a supervector space acted on by the Lie superalgebra $\mathrm{End}(R \oplus I)$. Then the tensor product $(R \oplus I)^{\otimes n}$ is also acted on by this Lie superalgebra.

If $X \in \mathrm{End}(R \oplus I)$, let $\rho(X) = X \otimes 1 \otimes \cdots \otimes 1 + 1 \otimes X \otimes 1 \otimes \cdots + 1 \otimes 1 \otimes \cdots \otimes X$ be the induced map on the tensor product. Then ρ is a Lie superalgebra homomorphism

$$\rho([X,Y]) = [\rho(X), \rho(Y)].$$

Now if ∂ is the odd endom of $R \oplus I$ which is 0 on R and the inclusion on I, then the differential in the complex $(R \longleftarrow I)^{\otimes n}$ is $\rho(\partial)$. The reason it's a differential is

$$\rho(\partial)^2 = \tfrac{1}{2}[\rho(\partial), \rho(\partial)] = \rho(\partial^2) = 0$$

Similarly, let $h \in \mathrm{End}^-(R \oplus I)$ be 0 on I and $1 - \rho\pi$ on R. Then we get the operator $\rho(h)$

on the tensor product such that

$$\rho(h)^2 = 0$$

and $$[\rho(h), \rho(\partial)] = \rho(h\partial + \partial h)$$

Now $h\partial + \partial h$ is the identity on I and on ∂I and it's zero on $\rho(A)$.

Let split our complex

$$R \leftarrow I \;=\; \rho(A) \;\oplus\; (I \xleftarrow{1} I)$$

Then the n-fold tensor product splits into 2^n summands each stable under $\rho(\partial)$ and $\rho(h)$. Provided we divide by the degree that $(I \xleftarrow{1} I)$ occurs, $\rho(h)$ gives a contracting homotopy on each subcomplex except by $\rho(A)^{\otimes n}$.

The beautiful thing about this construction is that it is compatible with cyclic permutations and so having made our choice of ρ we now have canonical contractions on the rows of $CC(R \leftarrow I)$. Now all we have to do is to see that we indeed get Connes formulas.

Let's now try to get his formula for the map $HC_3(A) \longrightarrow (I \otimes_R)^2_\sigma \longrightarrow I^2/[I,I]$.

Let's discuss the general setup first. We have a double complex which is $CC(R \leftarrow I)$ together with the augmentation to $CC(A)$. This double has acyclic rows. (Now actually I prefer to work with the Hochschild complex to do the calculations, and then use the natural surjection to the cyclic double complex).

So we have this double complex with horizontal boundary ∂ and vertical boundary b and we have a contracting homotopy in the horizontal direction which we denote H. It is $\{\rho^{\otimes n}\}$ on the first column $\{A^{\otimes n}\}$, and then it is the "derivation" extending $h : R \longrightarrow I$ divided by the appropriate integers. To do our calculations we want to use

$$R \oplus I = \Big(\rho(A) \oplus \underbrace{(\mathrm{Ker}\,\pi)}_{I}\Big) \oplus I$$

Let's write $\rho(a) = \tilde{a}$ and write J instead $\mathrm{Ker}\,\pi$. Thus the n-th row of our double complex is

$$A^{\otimes n} \oplus (\underbrace{\tilde{A} \oplus J}_{R,\ \deg 0} \oplus \underbrace{I}_{\deg 1})^{\otimes n}$$

To find H on a tensor $(\tilde{a}_\lambda, \, , j_\mu, \, , x_\mu)$ we use the derivation rule for extending h. Given an element $x \in I$ let us write x for the element x in $R =$ degree 0 part of $R \oplus I$ and let $\bar{x}$ denote x in $I =$ degree 1 part. Then

$$h(\tilde{a}) = 0 \quad , \quad h(x) = \bar{x} \quad , \quad h(\bar{x}) = 0 .$$

Finally H is this derivation applied to $(\ldots \tilde{a} \ldots x \ldots \bar{x} \ldots)$ divided by the number of x's and $\bar{x}$'s. $H = 0$ on $\tilde{A}^{\otimes n}$, and $H = \rho^{\otimes n}$ on $A^{\otimes n}$.

Now $[b, H]$ will be a map of complexes lowering degree by 1 between adjacent columns of our double complex. But

$$\partial[b,H] + [b,H]\partial = [b, \partial H + H\partial] = 0$$

so $[b,H]^n$ will map the first column, call it K_0, to the kernel of $\partial : K_n \to K_{n-1}$.

So for example $[b,H]^2$

$$\begin{tikzcd}
A_\lambda^{\otimes 4} \arrow[d] \arrow[dr, dashed] & R_\lambda^{\otimes 4} \arrow[l] \arrow[d] & & & \\
A_\lambda^{\otimes 3} \arrow[d] & R_\lambda^{\otimes 3} \arrow[l] \arrow[d] \arrow[dr, dashed] & {} \arrow[l] \arrow[d] & {} \arrow[l] \arrow[d] & I_\sigma^{\otimes 3} \arrow[l] \\
A_\lambda^{\otimes 2} \arrow[d] & R_\lambda^{\otimes 2} \arrow[l] \arrow[d] & R \otimes I \arrow[l] \arrow[d] & I_\sigma^{\otimes 2} \arrow[l] & \\
A_\lambda & R \arrow[l] & I \arrow[l] & &
\end{tikzcd}$$

will map $CC(A)$ to $\mathrm{Ker}(K_2 \xrightarrow{\partial} K_1)$, which is a ~~quotient~~ of K_3 ~~and~~ which agrees with K_3 at the beginning group $I_\sigma^{\otimes 2}$. Thus

$$H_1(\mathrm{Ker}(K_2 \xrightarrow{\partial} K_1)) = H_1(K_3)$$

Now we proceed to calculate $[b,H]^2(a_0,a_1,a_2,a_3)$

$$\begin{aligned}
b\,H(a_0,a_1,a_2,a_3) &= b(\tilde a_0, \tilde a_1, \tilde a_2, \tilde a_3) \\
&= (\tilde a_0 \tilde a_1, \tilde a_2, \tilde a_3) - (\tilde a_0, \tilde a_1 \tilde a_2, \tilde a_3) + (\tilde a_0, \tilde a_1, \tilde a_2 \tilde a_3) - (\tilde a_3 \tilde a_0, \tilde a_1, \tilde a_2)
\end{aligned}$$

$$\begin{aligned}
&H b(a_0,a_1,a_2,a_3) \\
&= (\widetilde{a_0 a_1}, \tilde a_2, \tilde a_3) - (\tilde a_0, \widetilde{a_1 a_2}, \tilde a_3) + (\tilde a_0, \tilde a_1, \widetilde{a_2 a_3}) - (\widetilde{a_3 a_0}, \tilde a_1, \tilde a_2)
\end{aligned}$$

$$\therefore [b,H](a_0,a_1,a_2,a_3) = (f_{01}, \tilde a_2, \tilde a_3) - (\tilde a_0, f_{12}, \tilde a_3) + (\tilde a_0, \tilde a_1, f_{23}) - (f_{30}, \tilde a_1, \tilde a_2)$$

where $f_{i,i+1} = \tilde a_i \tilde a_{i+1} - \widetilde{a_i a_{i+1}} \in J$

Next $$\begin{aligned} bH(f_{01}, \tilde a_2, \tilde a_3) &= b(\bar f_{01}, \tilde a_2, \tilde a_3) \\ &= (\bar f_{01} \tilde a_2, \tilde a_3) - (\bar f_{01}, \tilde a_2 \tilde a_3) + (\tilde a_3 \bar f_{01}, \tilde a_2) \end{aligned}$$

haven't calculated $H(f_{01}, \tilde a_2 \tilde a_3)$ correctly see p609

$$\begin{aligned}
Hb(f_{01}, \tilde a_2, \tilde a_3) &= H\{(f_{01}\tilde a_2, \tilde a_3) - (f_{01}, \tilde a_2 \tilde a_3) + (\tilde a_3 f_{01}, \tilde a_2)\} \\
&= (\overline{f_{01}\tilde a_2}, \tilde a_3) - (\bar f_{01}, \tilde a_2 \tilde a_3) - (f_{01}, h(\tilde a_2 \tilde a_3)) + (\overline{\tilde a_3 f_{01}}, \tilde a_2)
\end{aligned}$$

Now since $f_{01} \in J$ so does $f_{01}\tilde{a}_2$ and $\overline{f_{01}\tilde{a}_2} = \bar{f}_{01} \cdot \tilde{a}_2$ in the sense of the product in $R \oplus I$. Also

$$h(\tilde{a}_2 \tilde{a}_3) = \tilde{a}_2 \tilde{a}_3 - \widetilde{a_2 a_3} = \bar{f}_{23}$$

so we find

$$[b,H](f_{01}, \tilde{a}_2, \tilde{a}_3) = (f_{01}, \bar{f}_{23})$$

× see 609

Similarly when you do the other terms the only non-zero contribution comes when two $\tilde{a}$'s are multiplied.

$$-[b,H](\tilde{a}_0, f_{12}, \tilde{a}_3) = +(\bar{f}_{30}, f_{12})$$

$$[b,H](\tilde{a}_0, \tilde{a}_1, f_{23}) = -(\bar{f}_{01}, f_{23})$$

× see p. 609

$$-[b,H](f_{30}, \tilde{a}_1, \tilde{a}_2) = -(f_{30}, \bar{f}_{12})$$

Thus

$$[b,H]^2(a_0, \dots, a_3) = (f_{01}, \bar{f}_{23}) + (\bar{f}_{30}, f_{12}) - (\bar{f}_{01}, f_{23}) - (f_{30}, \bar{f}_{12})$$

$$= \partial\big((\bar{f}_{01}, \bar{f}_{23}) - (\bar{f}_{30}, \bar{f}_{12})\big)$$

Let's consider now the general case. We want to start with $(a_0, \dots, a_{2n-1})$ apply $[b,H]^n$ and write the result as ∂ of something in $I_\lambda^{\otimes n}$. Since we know ∂ anti-commutes with $[b,H]$, we can do the last step by applying ∂H. Thus the element of $I^{\otimes n}$ (in fact!) we are after is

$$H[b,H]^n(a_0, \dots, a_{2n-1})$$

$$= H(bH - \cancel{Hb})(bH - \cancel{Hb}) \cdots (a_0, \dots, a_{2n-1}) = (Hb)^n(\tilde{a}_0, \dots, \tilde{a}_{2n-1})$$

Actually I should be more careful with the previously calculation (p. 607) where I computed

$$H(f_{01}, \tilde{a}_2\tilde{a}_3)$$

This should be written

$$H(f_{01}, \widetilde{a_2a_3}) + H(f_{01}, f_{23})$$

$$\overset{\parallel}{(\bar{f}_{01}, \widetilde{a_2a_3})} + \overbrace{\tfrac{1}{2}\left((\bar{f}_{01}, f_{23}) + (f_{01}, \bar{f}_{23})\right)}$$

Then we have to remove from this $(\bar{f}_{01}, \tilde{a}_2\tilde{a}_3)$ which should give

$$[b, H](f_{01}, \tilde{a}_2, \tilde{a}_3) = \tfrac{1}{2}\left(-(\bar{f}_{01}, f_{23}) + (f_{01}, \bar{f}_{23})\right)$$

$$= +\tfrac{1}{2}\,\partial\,(\bar{f}_{01}, \bar{f}_{23})$$

and this should lead to the same result on p. 608.

Let check this alternatively

$$(Hb)(\tilde{a}_0, \ldots, \tilde{a}_3) = H\{(\tilde{a}_0\tilde{a}_1, \tilde{a}_2, \tilde{a}_3) - (\tilde{a}_0, \tilde{a}_1\tilde{a}_2, \tilde{a}_3)$$
$$+ (\tilde{a}_0, \tilde{a}_1, \tilde{a}_2\tilde{a}_3) - (\tilde{a}_3\tilde{a}_0, \tilde{a}_1, \tilde{a}_2)\}$$

Now you split $\tilde{a}_i\tilde{a}_j = \widetilde{a_ia_j} + f_{ij}$ and find

$$Hb(\tilde{a}_0, \ldots, \tilde{a}_3) = (\bar{f}_{01}, \tilde{a}_2, \tilde{a}_3) - (\tilde{a}_0, \bar{f}_{12}, \tilde{a}_3)$$
$$+ (\tilde{a}_0, \tilde{a}_1, \bar{f}_{23}) - (\bar{f}_{30}, \tilde{a}_1, \tilde{a}_2)$$

Apply Hb and notice that when we multiply an $\tilde{a}$ by an $\bar{f}$ we get an element whose entries are from $\tilde{A}$ or I, and hence H kills it. So the only terms surviving are

$$-H(\bar{f}_{01}, \widetilde{a_2a_3}) - H(\tilde{a}_3\tilde{a}_0, \bar{f}_{12}) + H(\tilde{a}_0\tilde{a}_1, \bar{f}_{23}) + H(\bar{f}_{30}, \tilde{a}_1\tilde{a}_2)$$

$$= -H(\bar f_{01}, f_{23}) - H(f_{30}, \bar f_{12}) + H(f_{01}, \bar f_{23}) + H(\bar f_{30}, f_{12})$$

$$= \tfrac{1}{2}\left\{H(\bar f_{01}, \bar f_{23}) - H(\bar f_{30}, \bar f_{12}) + H(\bar f_{01}, \bar f_{23}) - H(\bar f_{30}, \bar f_{12})\right\}$$

Thus $\boxed{(Hb)^2(\tilde a_0, \tilde a_1, \tilde a_2, \tilde a_3) = (\bar f_{01}, \bar f_{23}) - (\bar f_{30}, \bar f_{12})}$

The conjectured formula is then

$$(Hb)^n(\tilde a_0, \dots, \tilde a_{2n-1}) = (\bar f_{01}, \bar f_{23}, \dots, \bar f_{2n-2,2n-1}) - (\bar f_{2n-1\,0}, \bar f_{12}, \dots)$$

The proof is clear except for the signs, and these should work because the extra sign in H in passing thru an $\bar f$ is needed to compensate for the sign in b due to the fact that $\bar f$ stands for 2 places. Hb contracts adjacent pairs of $\tilde a$'s, and the $n!$ possibilities lead to a factor $n!$ cancelling the denominators in H.

Remarks:

1) It seems that we get some sort of Connes homomorphism on the level of <u>Hochschild homology</u>

$$H_{2n-1}(A, A) \longrightarrow I^{\otimes n}/\text{something}$$

We should see if something simple happens on the level of the bar resolution.

2) To what extent can one think of $\rho : A \to R$ as being a "connection" and

$$f(a_1, a_2) = \tilde a_1 \tilde a_2 - \widetilde{a_1 a_2}$$

as being its "curvature"?

Let's consider the double complex obtained by taking the Hochschild complex of $R \leftarrow I$. This should give information about $H_*(A,A)$ in terms of the Hochschild homology of the semi-direct product $R\oplus I$ considered as a superalgebra. Unfortunately I don't ~~[illegible]~~ have ~~[illegible]~~ a Goodwillie theorem describing the Hochschild complex of $R\oplus I$. There are two things we can do.

First there is Wodzicki's remark that the Hochschild complex is quis the fibre of the S-map on the cyclic complex. The S-map is likely to be zero on the positive degree (in I) part of $HC^S_\cdot(R\oplus I)$, hence maybe one knows the Hochschild homology in terms of the cyclic homology.

Secondly we can compute carefully the edge group.

$$\begin{tikzcd}
 \arrow[d] & \arrow[d] \\
\bigoplus_{k=0}^{n} I^{\otimes k}\otimes R\otimes I^{\otimes n-k} \arrow[d, "b"] & I^{\otimes(n+1)} \arrow[l] \\
I^{\otimes n} &
\end{tikzcd}$$

Recall that $I\cdot I=0$ in $R\oplus I$ so that

$$b(x_0,\dots,x_{k-1},r,x_k,\dots,x_n) = (-1)^{k-1}(x_0,\dots,x_{k-1}r,x_k,\dots) + (-1)^k(x_0,\dots,x_{k-1},rx_k,\dots)$$

which means the $k=1,\dots,n-1$ summands lead to $I\otimes_R \overset{n}{\cdots}\otimes_R I$.

$$b(r,x_1,\dots,x_n) = (rx_1,\dots,x_n) + (-1)^{n}(x_nr,x_1,\dots,x_{n-1})$$

$$b(x_1,\dots,x_n,r) = (-1)^{n-1}(x_1,\dots,x_nr) + (-1)^n(rx_1,x_2,\dots,x_n)$$

The last relation leads to $(I\otimes_R)^n$ but the first one forces cyclic symmetry (take $r=1$). Thus the quotient is $(I\otimes_R)^n_\sigma$ which means that the map

$$H_{2n-1}(A,A) \longrightarrow (I\otimes_R)^n$$

is obtained from the Connes homomorphism. In other words, we <u>don't get anything new</u>

Next I want to try to find formulas for the Connes homomorphism in the even case. Recall that this homomorphism results by chasing in the double complex

$$CC(R\leftarrow R)/CC(R\leftarrow I)$$

which also resolves $CC(A)$. Indeed

$$\circledast \qquad CC(R\leftarrow R)/\operatorname{Ker}(CC(R\leftarrow I)\to CC(A))$$

is a quotient of row acyclic double complexes whose beginning column is $CC(R)/\operatorname{Ker}(CC(R)\to CC(A)) = CC(A)$.

Now our problem is to combine the contracting row homotopies in the two complexes in $\circledast$.

First look at the homotopy H in the complexes $(R\leftarrow R)^{\otimes n}_\lambda$. This should give us the S operator. Specifically $H[b,H]^n : R^{\otimes(2n+1)}_\lambda \longrightarrow R^{\otimes(n+1)}_\sigma$ followed by trace of the product should give the S-map.

$$\begin{tikzcd}
R^{\otimes 3}_\lambda \arrow[d] \arrow[dr, dotted] & \arrow[l] \arrow[d, "b"] & \arrow[d] \\
R^{\otimes 2}_\lambda \arrow[d] & R\otimes R \arrow[l] \arrow[d, "b"] & R^{\otimes 2}_\sigma \arrow[l] \\
R & R \arrow[l] &
\end{tikzcd}$$

Note that

$$H[b,H]^n = H(bH)^n = (Hb)^n H$$

Let's try this for $n=1$

$$HbH(x_0,x_1,x_2) = Hb\underset{3}{H}\Big((\bar{x}_0,x_1,x_2)+(x_0,\bar{x}_1,x_2)+(x_0,x_1,\bar{x}_2)\Big)$$

$$= \frac{1}{3}H\left[\begin{array}{lll} (\bar{x}_0x_1,x_2) & +(x_0\bar{x}_1,\bar{x}_2) & +(x_0x_1,\bar{x}_2) \\ -(\bar{x}_0,x_1x_2) & -(x_0,\bar{x}_1x_2) & -(x_0,x_1\bar{x}_2) \\ +(x_2\bar{x}_0,x_1) & +(x_2x_0,\bar{x}_1) & +(\bar{x}_2x_0,x_1) \end{array}\right]$$

$$= \frac{1}{6}\left[\begin{array}{ll} -2(\overline{x_0x_1},\bar{x}_2) & +(\overline{x_0x_1},\bar{x}_2) \\ +(\bar{x}_0,\overline{x_1x_2}) & -2(\bar{x}_0,\overline{x_1x_2}) \\ -2(\overline{x_2x_0},\bar{x}_1) & +(\overline{x_2x_0},\bar{x}_1) \end{array}\right]$$

$$= -\frac{1}{6}\Big\{(\overline{x_0x_1},\bar{x}_2)+(\bar{x}_0,\overline{x_1x_2})+(\overline{x_2x_0},\bar{x}_1)\Big\}$$

which becomes $-\frac{1}{2}(x_0x_1x_2)$ in $R/[R,R]$.

The general case is similar, but the signs are very messy and it's not clear one can do it correctly. Each time one applies H one has a sum of terms which — is applied to the elements in a symbol, and when b is applied one has a term for each comma. The net effect is to produce terms

$$(\overline{x_m\, x_{2n}\, x_0\cdots x_k},\ \overline{x_{k+1}\cdots x_\ell},\ \cdots\cdots,\ \overline{x_{m-1}})$$

with signs + multiplicities which are bewildering. Thus the end map $R_\lambda^{\otimes 2n+1} \longrightarrow R/[R,R]$ is some multiple of $(x_0,\dots,x_{2n}) \longmapsto tr(x_0\cdots x_{2n})$

Next we want to bring in the ideal I. We would like to have a contracting homotopy in the quotient complex

$$(R\xleftarrow{1}R)_\lambda^{\otimes n} \Big/ \mathrm{Ker}\Big\{(R\xleftarrow{\otimes}I)_\lambda^{\otimes n}\longrightarrow A_\lambda^{\otimes n}\Big\}$$

It seems too hard to carry this out, since we can't satisfactorily handle the case $I=0$ this way.

At this point I want to try to make sense of the idea of $\rho : A \to R$ being the analogue of a connection and $f(a_0,a_1) = \rho(a_0)\rho(a_1) - \rho(a_0a_1)$ being the analogue of curvature.

Idea: I regard the fundamental problem in cyclic theory to be to find a good interpretation of $HC_*(A)$ as the universal recipient group for traces of higher order morphisms, whatever these should be. Let's recall a good example:

$$\mathrm{Ext}^i_A(M,M) = H^i\left(\mathrm{Hom}_A(\mathcal{B}^N(A)\otimes_A M, M)\right)$$

$$= H^i\left\{\mathrm{Hom}_{A\otimes A^o}\left(\mathcal{B}^N(A), \mathrm{End}_k(M)\right)\right\}$$

$$\longrightarrow H^i\left\{\mathrm{Hom}\left(\mathcal{B}^N(A)\otimes_A, \mathrm{End}_k(M)\otimes_A\right)\right\}$$

Now if M is finite-dimensional over k, then we have the map

$$\mathrm{End}_k(M)\otimes_A \longrightarrow k$$

given by trace_M. So, to the higher order maps from M to itself over A represented by the elements of $\mathrm{Ext}^i_A(M,M)$, we have associated Hochschild cocycles which might be viewed as their traces.

We can generalize and replace $\mathrm{End}_k(M)$ by an A-bimodule K with a trace $\tau : K/[A,K] \to k$, for example take K = finite rank operators in $\mathrm{End}_k(M)$.

But notice

$$\mathrm{Hom}_k(K/[A,K], k) = \mathrm{Hom}_k(K \otimes_{A\otimes A^\circ} A, k)$$

$$= \mathrm{Hom}_{A\otimes A^\circ}(K, \text{[redacted]}\, A^*)$$

i.e. the bimodule A^* has the trace $\lambda \mapsto \lambda(1)$, and any trace on K is induced from this one by a unique bimodule map $K \to A^*$.

We learn from this discussion that any Hochschild cohomology class $H_n(A,A) \to k$ can be [redacted] represented by an n-extension of A-bimodules

$$0 \to K \to E_{n-1} \to \cdots\cdots \to E_0 \to A \to 0$$

where K has a trace $\tau: K \to k$. It's natural to ask when such a thing gives a cyclic cohomology class. For example, suppose we have an extension of bimodules

$$\circledast \qquad 0 \to K \to E \to A \to 0$$

and a linear functional $\tau : (K\otimes_A)^n \longrightarrow k$. Then we can take the class $\chi \in H^1(A,K)$ represented by $\circledast$, form its nth power in the sense of cup product

$$\chi \cdot \overset{n}{\cdots} \cdot \chi \in H^n(A, K \otimes_A \overset{n}{\cdots} \otimes_A K)$$

and apply the trace τ to get a n-dim'l Hochschild cohomology class. [redacted] Does this Hochschild class come from a cyclic cohomology class?

We [redacted] pick a right A-module splitting h of $\circledast$, whence we have a map $\Omega^1_A \to K$. Then we have that χ^n comes from the canonical class in $H^n(A, \Omega^n_A)$ under the tensor product map $\Omega^n_A \longrightarrow K \otimes_A \overset{n}{\cdots} \otimes_A K$.

Actually we should first look at $n=1$, where we have a bimodule extension together with a trace on the kernel

$$\begin{tikzcd} 0 \arrow[r] & K \arrow[r] \arrow[d, "\tau"] & E \arrow[r] & A \arrow[r] & 0 \\ & k & & & \end{tikzcd}$$

In this case there's an obstruction to obtaining a 1-diml cyclic cohomology class. This obstruction is a trace on A, namely

$$a \longmapsto \tau([a,h])$$

Here h is an element of E over $1 \in A$. This trace vanishes iff the Hochschild cocycle is cyclic. The Hochschild cocycle is

$$\varphi(a_0, a_1) = \tau(a_0[a_1, h])$$

and
$$\varphi(a_0,a_1) + \varphi(a_1,a_0) = \tau(a_0[a_1,h] + [a_0,h]a_1)$$
$$= \tau([a_0a_1, h])$$

Question: Given $0 \to I \to R \to A \to 0$ and $\tau: I/[R,I] \to k$, can I get a bimodule extension of the above sort?

March 19, 1988

Stinespring circle of ideas:

(GNS) Let $\varphi : A \longrightarrow \mathbb{C}$ be a state on a C^* algebra with unit, i.e. $\varphi(a^*a) \geq 0$. Then $(a_1|a_2) = \varphi(a_1^* a_2)$ is an inner product on A such that left multiplication is a $*$ repn. of A. One has

$$A \xrightarrow[\text{mult}]{\text{left}} \mathrm{End}(A) \longrightarrow \mathbb{C}$$
$$\alpha \longmapsto \varphi(\alpha 1)$$

where the first map is a ring homomorphism and the second is $\alpha \longmapsto \pi \alpha i$ with

$$\mathbb{C} \xrightarrow{i1=1} A \xrightarrow{\pi=\varphi} \mathbb{C}$$

(Stinespring). Let $\varphi : A \longrightarrow \mathrm{End}(H)$ be "completely positive" Then we can define an inner product on $A \otimes H$ by

$$(a_1 \otimes h_1 \mid a_2 \otimes h_2) = (h_1 | \varphi(a_1^* a_2) | h_2)$$

We have a $*$ repn of A on $A \otimes H$ given by left multiplication, and we have maps

$$H \xrightarrow{i} A \otimes H \xrightarrow{\pi} H$$
$$h \longmapsto 1 \otimes h$$
$$a \otimes h \longmapsto \varphi(a) h$$

$\pi i = id$ if $\varphi(1) = 1$.

This gives

$$\alpha \longmapsto \pi \alpha i$$

$$\begin{tikzcd} A \arrow[r, "\text{left}", "\text{mult}"'] \arrow[rr, bend right, "\varphi"'] & \mathrm{End}(A \otimes H) \arrow[r] & \mathrm{End}(H) \end{tikzcd}$$

since $(\pi\, a\, i)(h) = \pi\, a(1 \otimes h) = \pi(a \otimes h) = \varphi(a) h$

Next try to replace $\mathrm{End}(H)$ by an algebra B. We start with $\varphi : A \longrightarrow B$ a linear map (let's ignore positivity considerations) and work with $A\otimes B$, B as right B-modules. We have B^o-module maps

$$B \xrightarrow{i = 1\otimes ?} A\otimes B \xrightarrow{\pi} B \qquad \pi(a\otimes b) = \varphi(a) b$$

and $\pi i = 1$ assuming $\varphi(1) = 1$. We have

$$\begin{tikzcd} A \arrow[r, "\text{left}"', "\text{mult}"] \arrow[rr, bend right, "\varphi"'] & \mathrm{End}_{B^o}(A\otimes B) \arrow[r, "\alpha \mapsto \pi\alpha i"] & \mathrm{End}_{B^o}(B) = B \end{tikzcd} \tag{$*$}$$

since $(\pi a_{\otimes} i)([redacted] b) = \pi a(1\otimes b) = \pi(a\otimes b) = \varphi(a) b$.

All of the above amount to ways of representing [redacted] a linear map $\varphi : A \to B$ as a "matrix element" of a homomorphism of algebras from A to a matrix algebra over B.

It seems strange, but note that when A is finite-dimensional the above [redacted] diagram $(*)$ defines a "Morita" map from A to B for each linear map $A \xrightarrow{\varphi} B$ such that $\varphi(1) = 1$. The φ enters only in defining the retraction of the free B^o-module $A\otimes B$ onto B, so it's clear this map is just the composition of the Morita maps

$$A \xrightarrow[\text{mult}]{\text{left}} [redacted]\ \mathrm{End}(A) \dashrightarrow \mathbb{C} \longrightarrow B$$

March 20, 1988

Let's go back to the idea of representing cyclic cocycles using iterated bimodule extensions. We now show how Connes cocycles can be put in this form.

Consider his standard even situation: We have two homomorphisms $u, v : A \Longrightarrow \operatorname{End}(H)$ which are congruent modulo an ideal K such that a suitable trace is defined on K^p. He then considers the superalgebra $\operatorname{End}(H \oplus H)$ and the homomorphism + involutions

$$a \longmapsto \begin{pmatrix} u(a) & 0 \\ 0 & v(a) \end{pmatrix} \qquad F = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \qquad \varepsilon = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Then

$$[F,a] = \left[\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} u(a) & 0 \\ 0 & v(a) \end{pmatrix} \right] = \begin{pmatrix} 0 & -u(a)+v(a) \\ -v(a)+u(a) & 0 \end{pmatrix}$$

and his cocycles are

$$\cancel{\varphi(a_0, \dots, a_{2n}) = \operatorname{tr}\left(\varepsilon [F,a_0] \cdots [F,a_{2n}]\right)}$$
$$\cancel{= 2 \operatorname{tr}\left(\varepsilon a_0 [F,a_1] \cdots [F,a_{2n}]\right)}$$

$$\varphi(a_0, \dots, a_{2n}) = \operatorname{tr}\left(\varepsilon a_0 [F,a_1] \cdots [F,a_{2n}]\right)$$
$$= \frac{1}{2} \operatorname{tr}\left(\varepsilon F [F,a_0] \cdots [F,a_{2n}]\right)$$

Set $\delta a = u(a) - v(a)$. Then

$$[F,a_1][F,a_2] = \begin{pmatrix} & -\delta a_1 \\ +\delta a_1 & \end{pmatrix} \begin{pmatrix} & -\delta a_2 \\ +\delta a_2 & \end{pmatrix} = - \begin{pmatrix} \delta a_1 \delta a_2 & 0 \\ 0 & \delta a_1 \delta a_2 \end{pmatrix}$$

$$\varepsilon F [F,a_0] = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} & \delta a_0 \\ -\delta a_0 & \end{pmatrix} = - \begin{pmatrix} \delta a_0 & 0 \\ 0 & \delta a_0 \end{pmatrix}$$

so

$$\boxed{\varphi(a_0, \dots, a_{2n}) = (-1)^{n+1} \frac{1}{2} \operatorname{tr}\left(\delta a_0 \, \delta a_1 \cdots \delta a_{2n}\right)}$$

Notice that $\delta a = u(a) - v(a)$ is a derivation of A with values in K considered as a bimodule over A in two ways.

$$u(a_0a_1) - v(a_0a_1) = u(a_0)\big(u(a_1) - v(a_1)\big) + \big(u(a_0) - v(a_0)\big)v(a_1)$$

$$= v(a_0)\big(u(a_1) - v(a_1)\big) + \big(u(a_0) - v(a_0)\big)u(a_1)$$

This gives us two bimodule maps.

$$\Omega^1_A \longrightarrow {}_uK_v \qquad a_0da_1a_2 \longmapsto u(a_0)\,\delta a_1\,v(a_2)$$

$$\Omega^1_A \longrightarrow {}_vK_u \qquad a_0da_1a_2 \longmapsto v(a_0)\,\delta a_1\,u(a_1)$$

hence by tensoring we get bimodule maps

$$\Omega^{2n}_A \longrightarrow \overbrace{{}_uK_v \otimes_A {}_vK_u \otimes_A \cdots \otimes_A {}_vK_u}^{2n \text{ factors}} \longrightarrow {}_uK^{2n}_u$$

$$a_0da_1\cdots da_{2n} \longmapsto u(a_0)\,\delta a_1\cdots\delta a_{2n}$$

and $\Omega^{2n}_A \longrightarrow {}_vK_u \otimes_A \overset{2n}{\cdots\cdots} \otimes_A {}_uK_v \to {}_vK^{2n}_v$, $\quad a_0da_1\cdots da_{2n} \longmapsto v(a_0)\delta a_1\cdots\delta a_{2n}$

which gives us two Hochschild cocycles

$$(*) \qquad \mathrm{tr}\big(u(a_0)\,\delta a_1\cdots\delta a_{2n}\big), \quad \mathrm{tr}\big(v(a_0)\,\delta a_1\cdots\delta a_{2n}\big)$$

whose difference is $\mathrm{tr}(\delta a_0\cdots\delta a_{2n})$. This is is a cyclic cochain so we get a cyclic $2n$-cocycle.

Let's consider next the odd case where $v(a) = Fu(a)F$, F an involution on H. Then

$$F\,\delta a\,F = F\big(u(a) - v(a)\big)F = -\delta a$$

hence the above two Hochschild cocycles $(*)$ coincide:

$$\mathrm{tr}\big(u(a_0)\,\delta a_1\cdots\delta a_{2n}\big) = \mathrm{tr}\Big(F\big(u(a_0)\,\delta a_1\cdots\delta a_{2n}\big)F\Big) = \mathrm{tr}\big(v(a_0)\delta a_1\cdots\delta a_{2n}\big)$$

Better, let's consider the odd case as follows. One is given a homomorphism $A \xrightarrow{\bullet} \mathrm{End}(H)$ and an involution F on H such that $u(a) = a$ $\quad v(a) = FaF^{-1}$ are congruent modulo an ideal K such that a suitable trace is defined on K^p. Connes cocycles are

$$\varphi(a_0, \dots, a_{2n-1}) = \mathrm{tr}\left(a_0[F,a_1]\cdots[F,a_{2n-1}]\right).$$

Put $\quad \delta a = u(a) - v(a) = a - FaF = F[F,a]$, whence

$$\delta a_1 \, \delta a_2 = F[F,a_1]F[F,a_2] = -[F,a_1][F,a_2].$$

and thus

$$\begin{aligned}
\varphi(a_0, \dots, a_{2n-1}) &= (-1)^{n-1}\,\mathrm{tr}\big(a_0 \overset{\text{move}}{\overrightarrow{F\,\delta a_1 \cdots \delta a_{2n-1}}}\big) \\
&= (-1)^n \,\mathrm{tr}\left(F a_0 \delta a_1 \cdots \delta a_{2n-1}\right) \\
&= (-1)^n \tfrac{1}{2}\,\mathrm{tr}\left(F\,\delta a_0 \cdots \delta a_{2n-1}\right)
\end{aligned}$$

The last formula shows φ is a cyclic cochain. The second formula allows us to see it is a Hochschild cocycle as in the even case: We have a bimodule map

$$\Omega_A^{2n-1} \longrightarrow \underbrace{{}_uK_v \otimes_A {}_vK_u \otimes_A \cdots \otimes_A {}_uK_v}_{2n-1 \text{ factors}} \longrightarrow {}_uK_v^{2n-1}$$

$$a_0\,da_1 \cdots da_{2n-1} \longmapsto a_0\,\delta a_1 \cdots \delta a_{2n-1}$$

Now $\alpha \mapsto \mathrm{tr}(F\alpha)$ from K^{2n-1} to $\mathbb{C}$ has the property

$$\mathrm{tr}(Fu(a)\alpha) = \mathrm{tr}(v(a)F\alpha) = \mathrm{tr}(F\alpha v(a))$$

and so is a linear map $\quad {}_uK_v^{2n-1} \otimes_A \longrightarrow \mathbb{C}$

Moreover $a \longmapsto eae + eKe$ is a homomorphism of A onto $eAe + eKe/eKe$ so we have an algebra extension

$$\begin{tikzcd}
0 \arrow[r] & eKe \arrow[r] \arrow[d] & R \arrow[r] \arrow[d] \arrow[dr, phantom, "\text{cart.}"] & A \arrow[r] \arrow[d] \arrow[l, dashed, bend right, "\rho"'] & 0 \\
0 \arrow[r] & eKe \arrow[r] & eAe + eKe \arrow[r] & eAe + eKe/eKe \arrow[r] & 0
\end{tikzcd}$$

with $eKe = I$.

There is a lifting ρ given by $\rho(a) = eae$ and we have from the previous calculation

$$\rho(a_1)\rho(a_2) - \rho(a_1a_2) = e\,a_1[e,a_2]e$$

$= e a_1 (1-e)[e,a_2]$ [redacted] $=$ [redacted] $[e,a_1](1-e)[e,a_2]$ [redacted]

There are various ways to write this. Perhaps the best way for future insight is

$$\rho(a_1)\rho(a_2) - \rho(a_1a_2) = -e\,a_1(1-e)a_2\,e$$

[redacted] or $= e[e,a_1][e,a_2]$

since this emphasizes the resemblance to curvature.

Return to Stinespring ideas: Let $\rho : A \to B$ be a linear map such that $\rho(1) = 1$. Then [redacted] ρ can be obtained as [redacted] a block in an possibly infinite rank matrix algebra over B as follows. We have [redacted] right B-module maps

$$\begin{array}{ccccc}
B & \xrightarrow{\ i\ } & A \otimes B & \xrightarrow{\ i^*\ } & B \\
b & \longmapsto & 1 \otimes b & & \\
 & & a \otimes b & \longrightarrow & \rho(a)b
\end{array}$$

Thus $tr(F a_0 \delta a_1 \cdots \delta a_{2n-1})$ is a Hochschild cocycle, and since we have seen it is cyclic we have a cyclic cocycle.

Now we would like to understand this business much better. First of all we have lots of ways of producing Hochschild cocycles using cyclic tensor products of lower dimensional Hochschild cocycles. Thus if we have 1-diml cocycles.

$$\varphi_i : \Omega^1_A \longrightarrow M_i \qquad i = 1, \dots, n$$

we can take

$$\varphi_1 \otimes_A \cdots \otimes_A \varphi_n \otimes_A : (\Omega^1_A \otimes_A)^n \longrightarrow M_1 \otimes_A \cdots \otimes_A M_n \otimes_A$$

$$\| $$

$$\Omega^n_A \otimes_A$$

and any linear functional on the latter gives us a Hochschild n-cocycle. In all this I should be saying normalized Hochschild cocycle.

So the issue is when do we obtain a cyclic cocycle? There are two conditions for a (norm) Hochschild cocycle $\varphi : \Omega^n_A \otimes_A = (\Omega^1_A \otimes_A)^n \longrightarrow \mathbb{C}$ to be a cyclic cocycle. First it must vanish on the image of $d\Omega^{n-1}_A$, and secondly φ must factor through $(\Omega^1_A \otimes_A)^n_\lambda$.

We have to discuss this carefully. Let $g : \Omega^n_A \otimes_A = (\Omega^1_A \otimes_A)^n \longrightarrow \mathbb{C}$ and put

$$g(a_0 da_1 \cdots da_n) = \varphi(a_0, \dots, a_n)$$

Then we know that φ is a normalized Hochschild cocycle with values in A^*. Suppose that g

vanishes on exact differentials

$$g(da_1 \cdots da_n) = \varphi(1, a_1, \ldots, a_n) = 0$$

Then

$$0 = (b\varphi)(1, a_0, \ldots, a_n) = \varphi(a_0, \ldots, a_n) + (-1)^{n+1} \varphi(a_n, a_0, \ldots, a_{n-1})$$

which shows φ is a cyclic cocycle. The converse is also true. $\therefore$

<u>Lemma</u>: a normalized Hochschild cocycle $\varphi(a_0, \ldots, a_n)$ (this means $b\varphi = 0$ and $\varphi = 0$ if [redacted] any of $a_1, \ldots, a_n$ are 1) is a cyclic cocycle $\iff \varphi(1, a_1, \ldots, a_n) = 0$

If φ is cyclic, then the map $g : \Omega^n_A \to \mathbb{C}$ defined by $g(a_0 da_1 \cdots da_n) = \varphi(a_0, \ldots, a_n)$ factors through $(\Omega^1_A \otimes_A)^n_A$. In effect

$$g(da_2 \cdots da_n a_0 da_1) = g\begin{pmatrix} da_2 \cdots d(a_n a_0) da_1 \\ - da_2 \cdots d(a_{n-1} a_n) da_0 da_1 \\ (-1)^{n-3} da_2\, d(a_3 a_4) \cdots da_0 da_1 \\ (-1)^{n-2} d(a_2 a_3) da_4 \cdots da_0 da_1 \\ (-1)^{n-1} a_2 da_3 \cdots\cdots da_1 \end{pmatrix}$$

$$= (-1)^{n-1} \varphi(a_2, a_3, \ldots a_n, a_0, a_1)$$

$$= \text{[redacted]} (-1)^{n-1} (-1)^n (-1)^n \varphi(a_0, a_1, \ldots, a_n)$$

$$= (-1)^{n-1} g(a_0 da_1 \cdots da_n)$$

[redacted] Simpler proof:

$$g(da_n a_0 da_1 \cdots da_{n-1}) = -g(a_n da_0 \cdots da_{n-1})$$

$$= -\varphi(a_n, a_0, \ldots, a_{n-1}) = -(-1)^n \varphi(a_0, \ldots, a_n)$$

$$= (-1)^{n-1} g(a_0 da_1 \cdots da_n)$$

However the fact that g factors through the action of $\mathbb{Z}/n$ on $\Omega_A^n \otimes_A = (\Omega_A^1 \otimes_A)^n$ is not sufficient for g to vanish on $d\Omega_A^{n-1}$. This is clear for $n=1$, and seems OK for $n=2$.

Conclusion: We have lots of ways to manufacture normalized Hochschild cocycles via cyclic $\cup_n$ cup products, but to have cyclic cocycles we must know the resulting linear functional on Ω_A^n vanishes on $d\Omega_A^{n-1}$.

March 21, 1988

I want to go over what Beilinson + Schechtman do for differential operators and the Tate residue construction.

We work over the formal punctured disk
$$\mathring{U} = \mathrm{Sp}\ \mathbb{C}((t))$$
and are concerned with a vector bundle E over $\mathring{U}$
$$E(\mathring{U}) \cong \mathbb{C}((t))^n$$
and the ring of differential operators on E:
$$\mathcal{D}_E(\mathring{U}) \cong \mathrm{End}(E)(\mathring{U})\,[\partial_t]$$
$$\cong M_n\big(\mathbb{C}((t))[\partial_t]\big)$$

It turns out that there is a canonical cyclic 1-cocycle defined on $\mathcal{D}_E(\mathring{U})$. In fact this cocycle is defined on the algebra R of all continuous operators on $E(\mathring{U})$ (viewed as a locally linearly compact vector space).

There are two ways to proceed. The first uses the bimodule extension of $\mathcal{D}_E(\mathring{U})$ which comes from the residue representation of differential operators:
$$0 \to P_{E,-1}(\mathring{U}) \longrightarrow P_E(\mathring{U}) \longrightarrow \mathcal{D}_E(\mathring{U}) \longrightarrow 0$$
The second uses a bimodule extension of R defined by Tate
$$0 \to I_{bdd} \cap I_{open} \longrightarrow I_{bdd} \oplus I_{open} \longrightarrow R \longrightarrow 0$$
There are natural traces defined on the kernels of these extensions, so there are Hochschild cocycle classes defined. For some reason these Hochschild

cocycles are cyclic, and for some other reason these are canonical choices for these cocycles. There's also a map from the first extension to the second.

It seems there should also be a version of this example where $\mathring{U}$ is replaced by the circle S^1, E is a smooth vector bundle over S^1, and $\mathcal{D}_E(\mathring{U})$ is the algebra of differential operators on E. One thing that's artificial about $\mathring{U} = \mathrm{Sp}\, \mathbb{C}((t))$ is that $\mathbb{C}[[t]]$ can be recovered from it; it's roughly as if the circle were filled in with a disk.

Consider $P_E(\mathring{U})$. We work on the product $\mathring{U} \hat{\times} \mathring{U}$ which in the present formal situation is obtained from $U \hat{\times} U = \mathrm{Sp}\left(\mathbb{C}[[x,y]]\right)$ $\qquad x = t_1,\ y = t_2$ by localizing:

$$\mathring{U} \hat{\times} \mathring{U} = \mathrm{Sp}\left(\mathbb{C}[[x,y]]\,[x^{-1}, y^{-1}]\right)$$

The rest is the same as for a curve, so that $P_E(\mathring{U})$ consists of formal kernels along the diagonal:

$$\psi(x,y)\,dy = \sum_{n \leq N} \frac{a_n(x)}{(y-x)^{n+1}}\,dy$$

and the corresponding differential operator is obtained by taking the residue at $y = x$.

$$\mathrm{Res}_{\Delta}\left(\psi(x,y)\,dy\, f(y)\right) = \sum_{0 \leq n \leq N} a_n(x) \frac{1}{n!} \left(\partial_x^n f\right)(x)$$

The trouble with such a formula is that it leaves too much obscure. We don't yet understand why $P_E(\mathring{u})$ is a bimodule over $\mathcal{D}_E(\mathring{u})$ although we can obviously write down formulas and check things work.

<u>Question:</u> Supposedly $P = P_E(\mathring{u})$ is a bimodule over $\mathcal{D} = \mathcal{D}_E(\mathring{u})$, and the map Res_Δ from P to $\mathcal{D}$ is a bimodule morphism $\partial : P \to \mathcal{D}$. Is the formula

$$(*) \qquad \partial(\psi_1)\cdot\psi_2 = \psi_1 \cdot \partial(\psi_2)$$

satisfied? (NO, see below)

Recall where this condition comes from: If $\partial : P \longrightarrow \mathcal{D}$ were extendable to a DGA, say

$$\xrightarrow{\partial} P_2 \xrightarrow{\partial} P \xrightarrow{\partial} \mathcal{D}$$

then for $x, y \in P$ we have $xy \in P_2$ and

$$\partial(xy) = \partial x \cdot y - x \cdot \partial y$$

In particular $\to 0 \to P \xrightarrow{\partial} \mathcal{D}$ is a DGA iff ∂ is a $\mathcal{D}$-bimodule homomorphism such that

$$\partial(x)\cdot y = x\cdot \partial(y) \qquad \forall x, y \in P.$$

In our situation ∂ maps P onto $\mathcal{D}$, so if $P \xrightarrow{\partial} \mathcal{D}$ were a DGA, the homology would have to be zero, since the homology is a unital algebra in which $1 = 0$. Specifically if $\partial(y) = 1$, then $x = \partial(x)\cdot y$, so that $\partial(x) = 0 \Rightarrow x = 0$.

Since $\partial : P \to \mathcal{D}$ is not injective, we conclude $(*)$ is not satisfied.

Let's now turn to the map

$$P_E(\mathring{u}) \longrightarrow I_{bdd} \oplus I_{open}$$

Recall that I_{bdd} consists of operators on $\dot{E}(U) = \mathbb{C}((t))^n$ whose images are bounded (contained in $t^{-N}\mathbb{C}[[t]]^n$ for some N) and that I_{open} consists of operators whose kernel is open (contains $t^N\mathbb{C}[[t]]^n$ for some N).

Observe that from the cocycle viewpoint the most interesting element of P is the kernel

$$\frac{1}{y-x}\,dy$$

which gives rise to the identity differential operator. One has

$$\left(\sum_{n\geq 0}^{N} a_n(x)\partial_x^n\right)\circ \frac{1}{y-x}\,dy = \sum_{n\geq 0}^{N} a_n(x)\frac{n!}{(y-x)^{n+1}}\,dy$$

Let's see what happens to $\frac{1}{y-x}dy$ under the B+S map to $I_{bdd}\oplus I_{open}$. We need to find the operator

$$f(t) \longmapsto \operatorname{Res}_{y=0} \frac{f(y)}{y-t}\,dy$$

Now $$\frac{1}{y-t} = -\frac{1}{t}\left(\frac{1}{1-\frac{y}{t}}\right) = -\frac{1}{t} - \frac{y}{t^2} - \cdots - \frac{y^{k-1}}{t^k} - \cdots$$

Thus $$\operatorname{Res}_{y=0} \frac{y^{-k}}{y-t}\,dy = \begin{cases} -t^{-k} & k\geq 1 \\ 0 & k\leq 0 \end{cases}$$

and we get the operator $-e$ $\quad e = $ proj on negative powers of t with kernel $\mathbb{C}[[t]]^n$.

Thus $\operatorname{Res}_0 = -e$, $\operatorname{Res}_\Delta = 1$

So we see that the cocycle I would obtain from the B+S method should in fact be the sort of thing encountered before, namely

$$\varphi(a_0, a_1) = \operatorname{tr}(a_0 [a_1, e])$$

where now one can have $a_0, a_1 \in \mathcal{D}_E(\mathfrak{U})$ or more generally $a_0, a_1 \in R$.

Why is it true that

$$\varphi(1, a) = \operatorname{tr}([a, e]) = 0$$

for $a \in R$? Tate's methods prove this as follows, because he proves $\operatorname{tr}[a,b] = 0$ if $a \in I_{bdd}$ and $b \in I_{open}$. Thus since $e \in I_{bdd}$ one has

$$\operatorname{tr}[a,e] = 0 \quad \text{for} \quad a \in I_{open}$$

and since $1-e \in I_{open}$

$$\operatorname{tr}[a,e] = -\operatorname{tr}[a, 1-e] = 0 \quad \text{for } a \in I_{bdd}.$$

so therefore $\operatorname{tr}[a,e] = 0$ for all a since $R = I_{bdd} + I_{open}$.

Let's return now to the main example. We suppose given a homomorphism $A \longrightarrow B$, an ideal K in B, and a projector e in B such that $[e,a] \in K$ for all $a \in A$. Then the subspace $eAe + eKe$ of eBe is a subalgebra, since

$$ea_1e\, ea_2e - ea_1a_2e = ea_1ea_2e - ea_1a_2ee$$
$$= ea_1[e, a_2]e \quad \in eKe$$

whose composition is the identity. Also A acts by left-multiplication on $A \otimes B$. Thus we have a ring hom.

$$A \longrightarrow \mathrm{End}_{B^o}(A \otimes B)$$

and if we look at the i, i^* block we get

$$b \longmapsto i^*(a \otimes 1 \cdot i(b)) = i^*(a \otimes b) = \rho(a) b$$

so $i^*(a \otimes 1) i = \rho(a)$.

I would like to use this as follows. Suppose I is an ideal in B such that $\rho : A \longrightarrow B \longrightarrow B/I$ is a homomorphism. I would like to find an ideal K in $C = \mathrm{End}_{B^o}(A \otimes B)$ such that if $e = i i^*$, then

$$\begin{cases} eKe = I & \\ [a, e] \in K & \forall a \in A. \end{cases}$$

First take $I = 0$, i.e. suppose ρ is a homomorphism.

March 22, 1988 (Carl is 23)

Let A, B be [redacted] unital algebras, M a [redacted] B°-module. Then $A \otimes M$ is an A-B bimodule equipped with a B°-module map $M \longrightarrow A \otimes M$, $m \longmapsto 1 \otimes m$ having an evident universal property

Similarly $\mathrm{Hom}_{\mathbb{C}}(A, M)$ is an A-B bimodule with

$$(afb)(a_1) = [redacted]\ f(a_1 a)b$$

equipped with a B°-module map

$$\mathrm{Hom}(A, M) \longrightarrow M \qquad f \longmapsto f(1)$$

with the following universal property. Given a B°-module map $u : E \to M$ where E is an A-B bimodule, there is a unique A-B bimodule map $\tilde{u}$ making a commutative triangle

$$\begin{tikzcd} E \arrow[r, "\tilde{u}"] \arrow[dr, "u"'] & \mathrm{Hom}(A, M) \arrow[d] \\ & M \end{tikzcd}$$

(Check: Uniqueness

$$\tilde{u}(\xi)(a) = \tilde{u}(\xi)(1a) = (a\tilde{u}(\xi))(1) = \tilde{u}(a\xi)(1) = u(a\xi)$$

$$\boxed{\tilde{u}(\xi)(a) = u(a\xi)}$$

Now check this formula defines an A-B hom.

$$(\tilde{u}(\xi b))(a) = u(a\xi b) \qquad u\ B^\circ\text{-mod map}$$

$$(\tilde{u}(\xi)b)(a) = (\tilde{u}(\xi)(a))b = u(a\xi)b$$

$$(\tilde{u}(a_1\xi))(a) = u(aa_1\xi)$$

$$(a_1\tilde{u}(\xi))(a) = \tilde{u}(\xi)(aa_1) = u(aa_1\xi)$$

)

Let $\rho : A \to B$ be a $\mathbb{C}$ linear map such that $\rho(1) = 1$. Then the B^o-module map

$$\begin{aligned} A \otimes B &\xrightarrow{u} B \\ a \otimes b &\longmapsto \rho(a) b \end{aligned}$$

extends to a A-B bimodule map

1) $$A \otimes B \xrightarrow{\tilde{u}} \mathrm{Hom}(A, B)$$

given by $$\tilde{u}(a \otimes b)(a_1) = u(a_1 a \otimes b) = \rho(a_1 a) b$$

Similarly the B^o-module map

$$B \xrightarrow{v} \mathrm{Hom}(A, B) \qquad v(b)(a) = \rho(a) b$$

extends to a A-B bimodule map

2) $$A \otimes B \xrightarrow{\tilde{v}} \mathrm{Hom}(A, B)$$

given by $$\tilde{v}(a \otimes b)(a_1) = (a\, v(b))(a_1) = v(b)(a_1 a) = \rho(a_1 a) b.$$

Thus the two A-B bimodule maps $\tilde{u}, \tilde{v}$ coincide.

I think that the image of the canonical map $\tilde{u} = \tilde{v}$ gives the Stinespring representation in the completely positive setup. Let's check this.

First suppose ρ is an algebra homomorphism. Then $\tilde{u} = \tilde{v}$ is the composition

$$A \otimes B \xrightarrow{u} B \xrightarrow{v} \mathrm{Hom}(A, B)$$

Check $(vu)(a \otimes b)(a_1) = v(\rho(a) b)(a_1) = \rho(a_1)\rho(a) b = \rho(a_1 a) b.$

Put another way B is already an A-B bimodule and u, v are bimodule maps.

Secondly consider a state $\rho : A \to \mathbb{C}$ i.e. $\rho(a^*a) \geq 0$. Then

$$A \xrightarrow{\tilde{u}} A^* \qquad \tilde{u}(1) = \rho$$

$$\tilde{u}(a)(a_1) = \tilde{u}(1a)(a_1) = (a\,\tilde{u}(1))(a_1) = \rho(a_1 a)$$

so

$$\operatorname{Ker}(\tilde{u}) = \{a \mid \rho(a_1 a) = 0 \text{ for all } a_1 \in A\} = \{a \mid \rho(a^*a) = 0\}.$$

Thirdly the general case ought to be similar. One must think in terms of Hilbert C^*-module over B. Thus on $A \otimes B$ you want an inner product with values in B. But for $\rho : A \to B$ to be completely positive means by definition that for any (a_i) in A the matrix

$$\rho(a_i^* a_j)$$

over B is ≥ 0 i.e. that for all (b_i)

$$\sum_{i,j} b_i^* \rho(a_i^* a_j) b_j \geq 0$$

as an elt of B. (Think of B as $C(X)$). Thus one has an inner product on $A \otimes B$

$$(a_1 \otimes b_1 \mid a_2 \otimes b_2) = b_1^* \rho(a_1^* a_2) b_2$$

with values in B. $\operatorname{Ker}(\tilde{u})$ consists of $\sum a_i \otimes b_i$ such that

$$\sum_i \rho(a'^* a_i)\, b_i = 0 \qquad \forall a' \in A$$

This implies $\left(\sum a_i \otimes b_i \mid \sum a_i \otimes b_i\right) = 0$; if the usual properties of inner products hold ($(\xi \mid \xi) = 0 \Longrightarrow (\eta \mid \xi) = 0$ for all η), then we find $\|\sum a_i \otimes b_i\|^2 = 0$ implies

$$\sum_i b'^* \rho(a'^* a_i) b_i = 0$$

for all a', b'. Take $b' = 1$ and $a' \mapsto a'^*$ and we see that $\operatorname{Ker}(\tilde{u})$ consists exactly of the $\sum a_i \otimes b_i \in A \otimes B$ killed under completing.

(Finally note that if $(\xi|\xi) = 0$ then from

$$0 \leq (\xi + \eta b | \xi + \eta b) = b^*(\eta|\xi) + (\xi|\eta) b + b^*(\eta|\eta) b$$

Now replace b by εb with ε real > 0, divide by ε, and let $\varepsilon \downarrow 0$, whence

$$b^*(\eta|\xi) + (\xi|\eta) b \geq 0$$

for all $b \in B$. Then take $-b = (\eta|\xi) = (\xi|\eta)^*$ and you get a contradiction unless $(\eta|\xi) = 0$.

Let review the sort of structure we want. We start with two algebras A, B, an ideal I in B, and a linear map $\rho: A \to B$ such that $\rho(1) = 1$, and such that $\rho(a_1)\rho(a_2) - \rho(a_1 a_2) \in I \quad \forall a_1, a_2 \in A$

Let us perform the Stinespring construction on ρ. This means we look for an $A \otimes B^o$-module E together with B^o-module maps

$$B \xrightarrow{i} E \xrightarrow{i^*} B$$

such that $i^* i = id_B$ and such that $i^* \hat{a} i = \widehat{\rho(a)}$ (where $\wedge$ means left multiplication). Then i, i^* induce $A\otimes B^0$-module maps

$$A\otimes B \longrightarrow E \longrightarrow \mathrm{Hom}(A,B)$$
$$a\otimes b \longmapsto a i(b)$$
$$\xi \longmapsto (a_1 \mapsto i^*(a_1 \xi))$$

whose composition is the map

$$a\otimes b \longmapsto a i(b) \longmapsto (a_1 \mapsto i^*(a_1 a i(b)))$$

or
$$a\otimes b \longmapsto (a_1 \mapsto \rho(a_1 a)\, b)$$

i.e. the canonical map associated to ρ.

Thus we have classified all triples $(E,\ B \xrightarrow{i} E \xrightarrow{i^*} B)$ in terms of factorizations of the canonical map

$$\begin{tikzcd} A\otimes B \arrow[rr, "\tilde{\rho}"] \arrow[dr, dashed] & & \mathrm{Hom}(A,B) \\ & E \arrow[ur, dashed] & \end{tikzcd}$$

The "smallest" such factorization is clearly $E = \mathrm{Im}(\tilde{\rho})$, the Stinespring construction. Let's take this choice for E and see where it leads. From now on I guess we work in the algebra $C = \mathrm{End}_{B^0}(E)$

First let's proceed without assuming $E = \mathrm{Im}(\tilde{\rho})$. In the algebra $C = \mathrm{End}_{B^0}(E)$, we have the projector $e = i i^*$ such that eCe can be identified with B. Furthermore we have a homomorphism $A \longrightarrow C$ such that $\rho(a) = eae$.

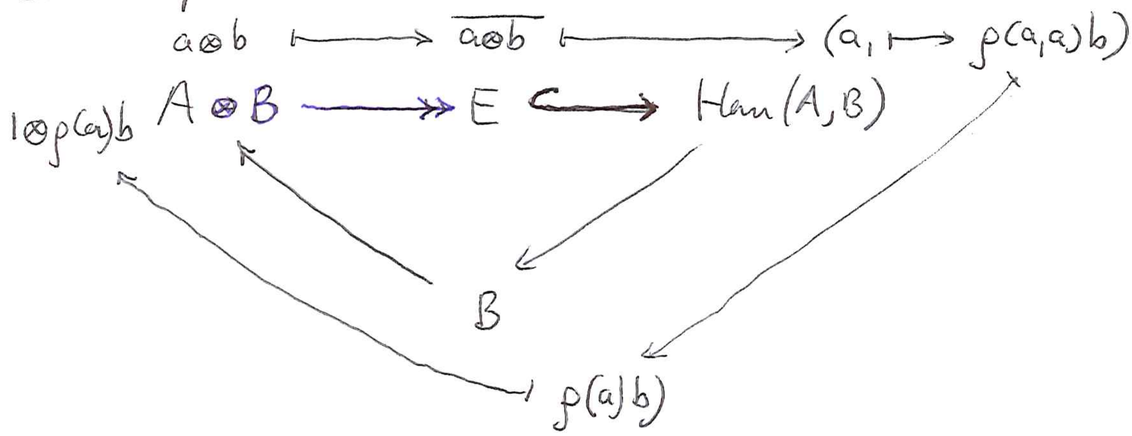
So we have block representation of A on $E = eE \oplus (1-e)E$

$$a \longmapsto \begin{pmatrix} eae & ea(1-e) \\ (1-e)ae & (1-e)a(1-e) \end{pmatrix}$$

such that $\rho(a)$ is the upper left block.

The next stage will be to bring in the fact that ρ is a homomorphism modulo I. Then we want to find an ideal K in C such that $eKe = I$ and $[a,e] \in K$ for all $a \in A$.

Let's compute the commutator $[a,e]$



Thus $\quad e(\overline{a \otimes b}) = \overline{1 \otimes \rho(a)b} = (a_1 \longmapsto \rho(a_1)\rho(a) b)$

so
$$a_2 e(\overline{a \otimes b}) = a_2(\overline{1 \otimes \rho(a)b}) = \overline{a_2 \otimes \rho(a)b} = (a_1 \longmapsto \rho(a_1 a_2)\rho(a)b)$$

$$e a_2(\overline{a \otimes b}) = e\,\overline{a_2 a \otimes b} = \overline{1 \otimes \rho(a_2 a)b} = (a_1 \longmapsto \rho(a_1)\rho(a_2 a)b)$$

Thus
$$[a_2, e](\overline{a \otimes b}) = \overline{a_2 \otimes \rho(a)b} - \overline{1 \otimes \rho(a_2 a)b}$$
$$= (a_1 \longmapsto [\rho(a_1 a_2)\rho(a) - \rho(a_1)\rho(a_2 a)]b)$$

Now let's look a bit at the analytical situation where E is supposed to be a Hilbert module completion of $A \otimes B$.

For a generic ρ I expect

$$\tilde{\rho} : A \otimes B \longrightarrow \mathrm{Hom}(A, B)$$

(this is the unique $A \otimes B^{\circ}$-module map sending $1 \otimes 1$ to ρ) to be injective with dense image. It is then sort of clear what compact operators ought to be, namely, "finite rank" maps of B°-modules from $\mathrm{Hom}(A, B)$ to $A \otimes B$. Such finite rank maps are given by elements of $A \otimes B \otimes A$ by the rule

$$(a_1 \otimes b_1 \otimes a_2)(f) = a_1 \otimes b_1 f(a_2)$$

Notice that $i i^* : \mathrm{Hom}(A, B) \longrightarrow A \otimes B$ is a "finite-rank" operator

$$i i^*(f) = i f(1) = 1 \otimes f(1);$$

it corresponds to $a_1 \otimes b_1 \otimes a_2 = 1 \otimes 1 \otimes 1$.

Let's now look at the ideal I and try to locate a nice ideal of operators on E.

Let's begin with a review. So far we have understood how to use $\rho : A \to B$, $\rho(1) = 1$ to construct

$$a \otimes b \mapsto (a_1 \mapsto (a \rho b) a_1)$$
$$\rho(a_1 a) b$$

$$\begin{tikzcd}
A \otimes B \arrow[rr, "\tilde{\rho}"] & & \mathrm{Hom}(A, B) \arrow[dl, "i^*"] \\
& B \arrow[ul, "i"] &
\end{tikzcd}$$

and we propose to use the $A \otimes B^{\circ}$-module

$$E = \mathrm{Im}(\tilde{\rho})$$

and to work in its algebra $\mathrm{End}_{B^\circ}(E)$ of endomorphisms. Such an endomorphism or operator determines an element of

$$\mathrm{Hom}_{B^\circ}(A\otimes B, \mathrm{Hom}(A,B)) = \mathrm{Hom}(A\otimes A, B)$$

So we can think of our operators as having kernels which are bilinear maps from $A\times A$ to B.

Let's compute kernels.

1) identity map of E: $\longmapsto \tilde\rho(a_0\otimes 1) = (a_1 \mapsto \rho(a_1 a_0))$

2) mult. by a: $\longmapsto \tilde\rho(aa_0\otimes 1) = (a_1 \mapsto \rho(a_1 a a_0))$

3) $e(\overline{a_0\otimes b}) = \overline{1\otimes\rho(a_0)b} = (a_1 \mapsto \rho(a_1)\rho(a_0)b)$

so the kernel for e is $\rho(a_1)\rho(a_0)$

4) $[a,e](\overline{a_0\otimes b}) = \overline{a\otimes\rho(a_0)b} - \overline{1\otimes\rho(aa_0)b}$

$= (a_1 \mapsto [\rho(a_1 a)\rho(a_0) - \rho(a_1)\rho(aa_0)]b)$

so the kernel for $[a,e]$ is

$\rho(a_1 a)\rho(a_0) - \rho(a_1)\rho(aa_0)$

5) The kernel for $1-e$ is $\rho(a_1 a_0) - \rho(a_1)\rho(a_0)$

6) $i\,(\text{left mult by } b)\,i^*\,(\overline{a_0\otimes 1}) = i\,b\,i^*(a_1 \mapsto \rho(a_1 a_0))$

$= i\,b(\rho(a_0)) = i\,b\rho(a_0) = \overline{1\otimes b\rho(a_0)}$

$= (a_1 \mapsto \rho(a_1)b\rho(a_0))$

So we have described a lot of operators on E. I guess what to do now is to make a list of the operators we want or need.

Notice that only by taking $E = \mathrm{Im}(\tilde\rho)$ do we obtain an injective map from operators to kernels. Indeed if $E \longrightarrow \mathrm{Hom}(A,B)$ is not injective, then a[redacted] nonzero operator from E to the kernel of this map would have zero (Schwartz) kernel; similarly if $A \otimes B \longrightarrow E$ is not surjective then a non-zero operator from the cokernel of this map to E would be an operator on E with zero (Schwartz) kernel.

Let's go back to the C^*-situation. What I am trying to understand is the way one goes from extensions to odd Fredholm modules in KK-theory. To fix the ideas suppose $B = \mathcal{B}(H)$ where H is a separable Hilbert space and $I = \mathcal{K}(H)$. An extension

$$0 \longrightarrow \mathcal{K}(H) \longrightarrow R \longrightarrow A \longrightarrow 0$$

is equivalent to a $*$homomorphism $A \longrightarrow \mathcal{B}(H)/\mathcal{K}(H)$ (its Busby invariant). When A is nuclear[redacted] one can lift this homomorphism to a completely positive map $\rho : A \longrightarrow \mathcal{B}(H)$. Then [redacted] Stinespring's thm. gives a larger Hilbert space E [redacted] containing H and on which there's a $*$ repn. of A such that $\rho(a)$ is the compression of a on E to H.

Finally one shows that $[a,e] \in \mathcal{K}(E)$ where e is the projection on H. Thus one has constructed from an extension of A by $\mathcal{K}(H)$ a odd Fredholm module, namely a Hilbert space representation of A with an involution F such that $[F,a]$ is compact for all a.

It is this last step which I have to understand. I have to start with the hypothesis that ρ is a homomorphism mod compacts, and then show [redacted] for the Stinespring representation that $[a,e]$ is compact.

Let's try to see why it might be true. For a generic ρ we expect

Ξ = Hilbert space completion of $A \otimes H$ to roughly an infinite direct sum of copies of H being moved around by A according to its regular representation. Now the compact operators on an infinite direct of Hilbert spaces are restricted in two directions – compact between different factors but then also strong decay in the "A-direction".

Review. A, B unital algebras, I ideal in B, $\rho : A \longrightarrow B$ is a linear map which becomes a homomorphism modulo I.

To ρ we associate the $A \otimes B^o$-module map

$$A \otimes B \xrightarrow{\tilde{\rho}} \mathrm{Hom}(A, B)$$

such that $1 \otimes 1 \longmapsto \rho$. Thus,

$$a \otimes b \longmapsto (a_1 \longmapsto \rho(a_1 a) b)$$

We have B^o-module maps

$$\begin{tikzcd} A \otimes B \arrow[rr, "\tilde{\rho}"] & & \mathrm{Hom}(A,B) \arrow[dl, "\tilde{i}^*"] \\ & B \arrow[ul, "\tilde{i}"] & \end{tikzcd}$$

$$\tilde{i}(b) = 1 \otimes b$$
$$\tilde{i}^*(f) = f(1).$$

Let E be the image of $\tilde{\rho}$; it is an $A \otimes B^o$-module which we can call the Steinspring representation of A dilating ρ. Note that $\tilde{i}, \tilde{i}^*$ identify B is a direct factor of E.

The image of $a \otimes b$ in E will be denoted $\overline{a \otimes b}$. Thus $\overline{a \otimes b} = (a_1 \longmapsto \rho(a_1 a) b)$. Define

$$B \xrightarrow{i} E \xrightarrow{i^*} B$$

by $\quad i(b) = \overline{1 \otimes b} = (a_1 \longmapsto \rho(a_1) b)$ and
$\quad i^*(f) = f(1).$

These are B^o-module maps such that

$$i^* i(b) = \rho(1) b = b.$$

Moreover

$$(\imath^* a \imath)(b) = \imath^*\left(a(\overline{1\otimes b})\right) = \imath^*(\overline{a\otimes b})$$
$$= \imath^*\left(a_1 \mapsto \rho(a_1 a)b\right) = \rho(a)b$$

Thus $\imath^* a \imath = \rho(a)$ if we identify $\mathrm{End}_{B^0}(B)$ with B using left multiplication.

Now we are going to be working with operators θ on E as a B^0-module. The composition

$$\begin{array}{ccccccc} A\otimes B & \twoheadrightarrow & E & \xrightarrow{\theta} & E & \hookrightarrow & \mathrm{Hom}(A,B) \\ a_0\otimes b & \longmapsto & \overline{a_0\otimes b} & \longmapsto & \theta(\overline{a_0\otimes b}) & \longmapsto & \left(a_1\mapsto \theta(\overline{a_0\otimes b})(a_1)\right) \\ & & & & \| & & \\ & & & & \theta(\overline{a_0\otimes 1})b & \longmapsto & \left(a_1 \mapsto \theta(\overline{a_0\otimes 1})(a_1)b\right) \end{array}$$

is a B^0-module map. But

$$\mathrm{Hom}_{B^0}(A\otimes B, \mathrm{Hom}(A,B)) = \mathrm{Hom}(A\otimes A, B).$$

Thus by assigning to θ the bilinear map

$(a_0, a_1) \longmapsto \theta(\overline{a_0\otimes 1})(a_1)$ from $A\times A$ to B

we obtain an injective map from operators to such bilinear maps. So we have attached "kernels" to operators.

Formulas for the kernels of certain operators.

$$\mathrm{id}_E \longleftrightarrow \rho(a_1 a_0)$$

$$e \longleftrightarrow \rho(a_1)\rho(a_0)$$

$$a \longleftrightarrow \rho(a_1 a a_0)$$

$$\imath b \imath^* \longleftrightarrow \rho(a_1) b \rho(a_0)$$

A most important class of operators on E is obtained by taking linear combinations of $\alpha i \beta i^* \alpha'$ where $\alpha, \alpha' \in A$ and $\beta \in B$. Compute the kernel:

$$(\alpha i \beta i^* \alpha')(\overline{a_0 \otimes 1}) = \alpha i \beta i^*(a_1 \mapsto \rho(a_1 \alpha' a_0))$$

$$= \alpha i \beta \rho(\alpha' a_0) = \overline{\alpha \otimes \beta \rho(\alpha' a_0)}$$

$$= (a_1 \longmapsto \rho(a_1 \alpha) \beta \rho(\alpha' a_0))$$

Thus

$$\boxed{\alpha i \beta i^* \alpha' \longleftrightarrow \rho(a_1 \alpha) \beta \rho(\alpha' a_0)}$$

Special cases:

$$ae \longleftrightarrow \rho(a_1 a) \rho(a_0)$$

$$ea \longleftrightarrow \rho(a_1) \rho(a a_0)$$

$$eae = i\rho(a) i^* \Longleftrightarrow \rho(a_1) \rho(a) \rho(a_0).$$

$$ea(1-e) \Longleftrightarrow \rho(a_1)[\rho(a a_0) - \rho(a)\rho(a_0)]$$

$$(1-e)ae \Longleftrightarrow [\rho(a_1 a) - \rho(a_1)\rho(a)]\rho(a_0)$$

The last two we want to lie in the ideal K of "compact operators" on E. Notice that in addition to being in the space of operators $A i B i^* A$, their kernels have values in I. Notice that

$$e \longleftrightarrow \rho(a_1)\rho(a_0) \qquad \text{belongs to } A i B i^* A$$

$$1-e \longleftrightarrow \rho(a_1 a_0) - \rho(a_1)\rho(a_0) \qquad \text{has kernel with values in } I.$$

Thus we perhaps have two A-bimodules whose intersection might be the ideal K of "compact operators".

Let $C = A + A\iota B\iota^* A \subset \operatorname{End}_{B^0}(E)$. This is a subalgebra since

$$(\alpha_1 \iota \beta_1 \iota^* \alpha_1')(\alpha_2 \iota \beta_2 \iota^* \alpha_2') = \alpha_1 \iota(\beta_1 \rho(\alpha_1'\alpha_2)\beta_2)\iota^*\alpha_2'$$

(In fact we can even define an algebra structure on $A \oplus A \otimes B \otimes A = A \oplus M$ because it turns out that $(\alpha_1, \beta_1, \alpha_1')(\alpha_2, \beta_2, \alpha_2') = (\alpha_1, \beta_1\rho(\alpha_1'\alpha_2)\beta_2, \alpha_2')$ is an associative A-bimodule map $M \otimes_A M \longrightarrow M$.)

Clearly $A\iota B\iota^* A$ is an ideal in C. Also C is generated by its subalgebras A and $\iota B\iota^* \simeq B$, except the latter is not unital. In fact $\iota B\iota^* = eCe$

Let K be the subspace of $A\iota B\iota^* A$ consisting of operators $\sum_k \alpha_k \iota \beta_k \iota^* \alpha_k'$ whose kernel is I-valued, i.e. such that

$$\sum_k \rho(a_1\alpha_k)\beta_k\rho(\alpha_k' a_0) \in I \qquad \forall a_0, a_1 \in A.$$

We wish to check this is an ideal in C, and so we check it is closed under left and right multiplication by elements in $A \cup \iota B\iota^*$.

$$a\Big(\sum \alpha_k \iota \beta_k \iota^* \alpha_k'\Big) = \sum_k a\alpha_k \iota \beta_k \iota^* \alpha_k'$$

$$\longleftrightarrow \sum_k \rho(a_1 a\alpha_k)\beta_k\rho(\alpha_k' a_0) \in I \qquad \text{OK.}$$

$$\Big(\sum \alpha_k \iota\beta_k \iota^*\alpha_k'\Big)a \longleftrightarrow \sum \rho(a_1\alpha_k)\beta_k\rho(\alpha_k' a a_0) \in I$$

$$i b \iota^*\Big(\sum_k \alpha_k \iota \beta_k \iota^* \alpha_k'\Big) = \sum_k i b \rho(\alpha_k) \beta_k i^* \alpha_k'$$

$$\longleftrightarrow \sum_k \rho(a_1)\, b \rho(\alpha_k)\beta_k\, \rho(\alpha_k' a_0)$$

$$= \rho(a_1) b \Big(\sum_k \rho(1\alpha_k)\beta_k \rho(\alpha_k' a_0)\Big) \in I \qquad \text{OK.}$$

$$\Big(\sum_k \alpha_k \iota \beta_k \iota^* \alpha_k'\Big) i b i^* = \sum_k \alpha_k \iota(\beta_k \rho(\alpha_k') b)\iota^*$$

$$\longleftrightarrow \sum_k \rho(a_1 \alpha_k)\beta_k \rho(\alpha_k') b\, \rho(a_0)$$

$$= \Big(\sum_k \rho(a_1\alpha_k)\beta_k \rho(\alpha_k' 1)\Big) b \rho(a_0) \in I \qquad \text{O.K.}$$

Thus K is an ideal in $C = A + AiBi^*A$.

Note that if $\xi = \sum \alpha_k \iota \beta_k \iota^* \alpha_k' \in K$, then

$$\iota^*\Big(\sum \alpha_k \iota \beta_k \iota^* \alpha_k'\Big) i = \sum \rho(\alpha_k)\beta_k \rho(\alpha_k') = \xi(1,1) \in I$$

so $\iota^* K \iota \subset I$. But if $\beta \in I$, then $\iota\beta i^* \in K$ (because $\iota\beta i^* \longleftrightarrow \rho(a_1)\beta\rho(a_0)$), and $\iota^*(i\beta i^*) i = \beta$.

Thus $\boxed{\iota^* K i = I}$.

Finally we observe that from the assumption that ρ is a homomorphism mod I, it follows that the kernels for $ea(1-e)$, $(1-e)ae$ found on p644 have values in I. Thus $ea(1-e), (1-e)ae \in K$.

At this point, it follows I think that we have everything we need to assume a lifting is of the form eae in a larger ring. However it will probably be useful to consider whether the space of elements in C with I-valued kernels is an ideal.

Interesting point: Let us consider the space of kernels

$$\mathrm{Hom}_{B^0}(A\otimes B, \mathrm{Hom}(A,B)) = \mathrm{Hom}(A\otimes A, B).$$

We know $\mathrm{End}_{B^0}(E)$ sits inside this space of kernels. But I think we ought to be able to make the operators in A, $\iota B\iota^*$ act on the whole space of kernels. This is clear for A and the formulas are

$$(ak)(a_1, a_0) = k(a_1 a, a_0)$$
$$(ka)(a_1, a_0) = k(a_1, a a_0)$$

But it's also OK for $\iota B\iota^*$. Let $\theta : A\otimes B \to \mathrm{Hom}(A,B)$ be a B^0-module map; the corresponding kernel is $\hat\theta(a_1, a_0) = \theta(a_0\otimes 1)(a_1)$. Then $(ib\iota^*\theta)(a_0\otimes 1) = ib\,\hat\theta(1,a_0)$ $= 1\otimes b\hat\theta(1,a_0) \in A\otimes B$ and this goes to $a_1 \mapsto \rho(a_1)b\hat\theta(1,a_0)$. Thus we have

$$\widehat{ibi^*\theta}(a_1,a_0) = \rho(a_1)\,b\,\hat\theta(1,a_0)$$

Similarly $(\theta\iota b i^*)(a_0\otimes 1) = \theta i b\,\rho(a_0) = \theta(1\otimes b\rho(a_0))$
$= \theta(1\otimes 1)\cdot b\rho(a_0) = (a_1 \mapsto \hat\theta(a_1,1)b\rho(a_0))$.

Thus

$$\widehat{\theta\iota b i^*}(a_1,a_0) = \hat\theta(a_1,1)\,b\,\rho(a_0)$$

Thus we conclude that the I-valued kernels are closed under the bimodule action of A and $iB\iota^*$.

There are still lots of things I don't understand yet about the construction. For example I ought to be able to define C directly as a suitable quotient of $A \oplus A \otimes B \otimes A$. [redacted]

What is the proper [redacted] interpretation of this algebra [redacted] which apparently acts on any $A^{\otimes B^\circ}$-module E' equipped with B° maps $B \xrightarrow{i} E \xrightarrow{i^*} B$ such that $\iota a \iota^* = \rho(a)$.

What is the link between the present construction and the Tate construction as described by Beilinson + Schechtman? In the Tate construction one has an A-module [redacted] H say equipped with a splitting $H = H_+ \oplus H_-$. Let $i : H_+ \to H$ and $i^* : H \to H_+$ be the inclusion and projection respectively. The algebra C is supposed to be generated by the operators A and $\iota B \iota^*$, where B is some algebra of operators on H_+ containing the operators $\rho(a) = \iota^* a \iota$. [redacted] The condition that $A \otimes B \longrightarrow E$ should translate into $H = AH_+$, whereas $E \hookrightarrow \mathrm{Hom}(A,B)$ should translate to $H \longrightarrow \mathrm{Hom}(A, H_+)$, $h \mapsto (a \mapsto \iota^*(a,h))$ being injective. ?

There's a problem with trying to explain the odd Connes homomorphism using the Stinespring construction, namely the powers of the ideal don't seem to work out.

Suppose we have a [redacted] homomorphism $A \longrightarrow C$ an ideal K in C and an idempotent e in C such that $[a,e] \in K$ for all a. Let

$F = 2e-1$ as usual. Then the Connes cyclic cocycle of degree $2n-1$ is

$$\varphi(a_0, \dots, a_{2n-1}) = \frac{1}{2} tr\left(F[F,a_0]\cdots[F,a_{2n-1}]\right) = tr\left(a_0[F,a_1]\cdots[F,a_{2n-1}]\right)$$

and ~~this~~ is well-defined when tr is defined on $K^{2n}/[K, K^{2n-1}]$. This is OK actually because athough $eKe = I$, all we have for the odd part is $eK(1-e)\cdot(1-e)Ke \subset eKe = I$.

One can see some problems arising as follows. I want to think of $a_0, \dots, a_{2n-1} \longmapsto a_0F[F,a_1]\cdots F[F,a_{2n-1}]$ as being a normalized Hochschild cocycle on A with values in ${}_uK_v \otimes_A {}_vK_u \otimes_A \cdots \otimes_A {}_uK_v$, $2n-1$ factors. This is the wrong power of K needed for the Connes homomorphism.

March 25, 1988

Let's recall our method for producing Hochschild classes. We start with classes $\chi_i \in H^{d_i}(A, M_i)$, $i = 1, \dots, k$ take their cup product

$$\chi_1 \cup \dots \cup \chi_k \in H^{\overbrace{d_1 + \dots + d_k}^{d}}(A, M_1 \otimes_A \dots \otimes_A M_k).$$

If we represent this class by a map of A-bimodule complexes

$$B^N(A) \longrightarrow M_1 \otimes_A \dots \otimes_A M_k\, [d]$$

then we have an induced map

$$B^N(A) \otimes_A \longrightarrow M_1 \otimes_A \dots \otimes_A M_k \otimes_A\, [d]$$

which gives a map

$$H_d(A, A) \longrightarrow M_1 \otimes_A \dots \otimes_A M_k \otimes_A$$

i.e. a Hochschild class for each linear fnl. on the latter.

Now consider Connes' standard odd situation. This means we have a homomorphism $A \longrightarrow C$, an involution F in C, an ideal K in C such that $[F, a] \in K$ for all $a \in A$. We have seen how this standard setup arises from the Stinespring construction, where we start from a linear $\rho : A \longrightarrow B$ with $\rho(1) = 1$, and an ideal $I \subset A$ such that ρ is a homom. mod I. Recall that $B = eCe$, $e = \frac{F+1}{2}$, $I = eKe$, $\rho(a) = eae$.

Now the important end-product we want ~~[illegible]~~ from (A, B, ρ, I) are Connes odd cocycles

$$\varphi : \bar{A}^{\otimes(2n)}_\lambda / \mathrm{Im}\, b \longrightarrow (I \otimes_A)^n_\sigma \longrightarrow I^n / [I, I^{n-1}]$$

I noticed yesterday that there's a problem of landing in this power of I if one takes the

Hochschild viewpoint described above

We have an improvement on the discussion of the odd case on p.621. We consider the standard setup $(A \to C, F, K)$. Then

$$a \longmapsto [F,a]$$

is a derivation of A with values in the A-bimodule K. So taking the cup product gives bimodule maps

$$a_0[F,a_1]\cdots[F,a_m] \qquad \Omega^m_A \longrightarrow K \otimes_A \cdots \otimes_A K \to K^m$$

and Hochschild cocycles

$$\operatorname{tr}\left(a_0[F,a_1]\cdots[F,a_m]\right) \qquad \operatorname{tr} \text{ defined on } (K \otimes_A)^m$$

But these are not the right objects. The really good thing is

$$\operatorname{tr}\left([F,a_0]\cdots[F,a_{2n-1}]\right) \qquad \operatorname{tr} \text{ defd on } K^{2n}$$

My feeling is that I want to think of a cyclic cocycle as primarily the trace of a such a product which has the obvious cyclic symmetry properties

---

Here's a possible approach to getting a formula for the even degree cocycles attached to an extension. Given $0 \to I \to R \overset{\rho}{\underset{\leftarrow}{\to}} A \to 0$ we want to construct a cyclic $2n$-cocycle on A with values in $HC_0(R/I^{n+1})$. We have already a cyclic $(2n+1)$-cocycle with values in $I^{n+1}/[I,I^n]$, call this φ. Now if a trace on I^{n+1} vanishing on $[I,I^n]$ is the restriction of a trace on R, it should be the case that the corresponding $2n$-cocycle is a boundary. Thus

$$\bar{A}^{\otimes(2n+2)} \xrightarrow{\varphi} I^{n+1}/[I,I^n] \longrightarrow R/[R,R]$$

should be of the form $\delta\psi$, where ψ is a cyclic $2n$-cochain with values in $R/[R,R]$. It then follows that modulo I^{n+1}, ψ is a cyclic $2n$-cocycle.

Let's return to the Stinespring construction and see if we can work out the analogy with Tate's theory. We have the algebra

$$C = A + AiBi^*A$$

acting on $E = \mathrm{Im}\{A\otimes B \xrightarrow{\tilde{\delta}} \mathrm{Hom}(A,B)\}$. C contains the ideal $I_1 = AiBi^*A$ and the ideal I_2 consisting of those operators whose kernels have values in I. The ideal K of interest is $I_1\cap I_2$. Notice also that $I_1+I_2=C$ because I_2 contains $1-e$, so $a = ae + a(1-e) \in I_1+I_2$.

In analogy with Tate's theory I want to define a trace on K with values in $I/[B,I]$ such that $\mathrm{tr}[I_1,I_2]=0$. This implies that $\mathrm{tr}[K,R] \subset \mathrm{tr}[K,I_1]+\mathrm{tr}[K,I_2] = 0$. Hence any element of K anti-commuting with F such as $ea(1-e)$ or $(1-e)ae$ should have zero trace.

Notice that $eC(1-e) \in I_1CI_2 = I_1I_2 \subset K$ so $eC(1-e) = eK(1-e)$ and similarly for the other off-diagonal block. Note that

$$(1-e)C(1-e) = (1-e)A(1-e) + (1-e)Ae(iBi^*)eA(1-e)$$

so that modulo K one has

$$C \equiv \underbrace{eCe}_{iBi^*} + (1-e)A(1-e)$$

March 26, 1988

In Tate's theory one encounters a ring C with two ideals I_1, I_2 such that $C = I_1 + I_2$ and there is a linear functional $\tau : I_1 \cap I_2 \longrightarrow \mathbb{C}$ such that $\tau[I_1, I_2] = 0$. I want to understand whether this situation occurs in the Stinespring situation.

Let $C = A + A\iota B\iota^* A$ as before and take $I_1 = A\iota B\iota^* A$. Then

$$CeC \subset A\iota B\iota^* A \subset Ae\iota B\iota^* A \subset CeC$$

so we have $\boxed{CeC = A\iota B\iota^* A}$. Previously we took I_2 to consist of elements of C whose kernels have values in I, but it seems simpler now to take $I_2 = C(1-e)C$. With this choice we don't need kernels, and so we should try to do everything in the case where $C \cong A \oplus A \otimes B \otimes A$, whence we can let ρ vary without C jumping around.

We next describe the block structure of C and these ideal with respect to e. In the following just suppose $C = A + A\iota B\iota^* A$ is arbitrary, e.g. the alg. of operators on any $A \otimes B^\circ$ module E equipped with B°-maps $B \xrightarrow{\iota} E \xrightarrow{\iota^*} B$.

Note that $eA\iota B\iota^* = \iota\rho(A)B\iota^* = \iota B\iota^*$ and also $(1-e)A\iota B\iota^* = (1-e)Ae \cdot \iota B\iota^*$. Thus

$$\begin{matrix} CeC \\ \| \\ I_1 = A\iota B\iota^* A \end{matrix} = \begin{pmatrix} \iota B\iota^* & \iota B\iota^* \cdot eA(1-e) \\ & \\ (1-e)Ae \cdot \iota B\iota^* & (1-e)Ae \cdot \iota B\iota^* \cdot eA(1-e) \end{pmatrix}$$

gives the block structure of I_1.

What's the blocks for $I_2 = C(1-e)C$.

$$
\begin{aligned}
eC(1-e)C(1-e) &= eC(1-e) \\
&= eA(1-e) + eI_1(1-e) \\
&= eA(1-e) + iBi^* \cdot eA(1-e) \\
&= iBi^* \cdot eA(1-e) \qquad \text{as } e \in iBi^*.
\end{aligned}
$$

$$
\begin{aligned}
eC(1-e)Ce &= iBi^* eA(1-e) \cdot (1-e)Ae \cdot iBi^* \\
&= iBi^* eA(1-e)Ae\, iBi^*
\end{aligned}
$$

$$
\begin{aligned}
(1-e)C(1-e)C(1-e) &= (1-e)C(1-e) \\
&= (1-e)A(1-e) + (1-e)Ae \cdot iBi^* \cdot eA(1-e).
\end{aligned}
$$

Thus

$$
I_2 = C(1-e)C = \begin{pmatrix} iBi^* eA(1-e)Ae\, iBi^* & iBi^* eA(1-e) \\ (1-e)Ae \cdot iBi^* & \begin{matrix} (1-e)A(1-e) \\ + \\ (1-e)Ae \cdot iBi^* \cdot eA(1-e) \end{matrix} \end{pmatrix}
$$

Now one also has block descriptions for C and $K = I_1 \cap I_2$

$$
C = \begin{pmatrix} iBi^* & iBi^* eA(1-e) \\ (1-e)Ae \cdot iBi^* & \begin{matrix} (1-e)A(1-e) \\ + \\ (1-e)Ae \cdot iBi^* \cdot eA(1-e) \end{matrix} \end{pmatrix}
$$

$$
K = \begin{pmatrix} iBi^* \cdot eA(1-e)Ae \cdot iBi^* & iBi^* \cdot eA(1-e) \\ (1-e)Ae \cdot iBi^* & (1-e)Ae \cdot iBi^* \cdot eA(1-e) \end{pmatrix}
$$

Notice that we have with this choice of I_2 made

$$eKe = \text{ideal gen. by } eA(1-e)Ae \text{ in } iBi^*$$

We can obviously enlarge I_2 so that

$eI_2e = eKe =$ any ideal in iBi^* containing

$eA(1-e)Ae = \{ea_1a_2e - ea_1ea_2e\} = i\{\rho(a_1a_2) - \rho(a_1)\rho(a_2)\}i^*$.

This again allows for variation of ρ at a later stage.

Let's perform this modification so that now I_2, K have upper left block iIi^* where I is an ideal in B containing $\{\rho(a_1a_2) - \rho(a_1)\rho(a_2)\}$.

Now I wish to define a linear map

$$\tau : K/[I_1, I_2] \longrightarrow I/[B,I]$$

There is a problem with defining τ on the lower right block $(1-e)Ae\, iBi^*\, eA(1-e)$. One wants to take $X \in (1-e)Ae \cdot iBi^* = (1-e)Ce$ and $Y \in iBi^* \cdot eA(1-e) = (1-e)Ce$ and define

$$\tau(XY) = \tau(YX) \overset{\text{defn.}}{=} YX \bmod [B,I].$$

However I don't see that τ is well-defined on $(1-e)K(1-e) = (1-e)CeC(1-e)$. Thus we have

$$\begin{tikzcd} (1-e)Ce \otimes_{eCe} eC(1-e) \arrow[r, two heads] \arrow[d] & (1-e)K(1-e) \\ I/[B,I] & \end{tikzcd}$$

but I seem to have no control over the kernel of the horizontal arrow.

However let us take the case

$$C \cong A \oplus A \otimes B \otimes A$$

In this case

$$I_1 \leftarrow\!\!\!\sim\; A \otimes B \otimes A \;\cong\; \begin{pmatrix} B & B\otimes\bar{A} \\ \bar{A}\otimes B & \bar{A}\otimes B\otimes\bar{A} \end{pmatrix}$$

so we should have

$$\begin{array}{ccc}
(1-e)Ce \otimes_{eCe} eC(1-e) & \longrightarrow & (1-e)K(1-e) \\
\| & & \| \\
((1-e)Ae\, iBi^*) \otimes_{iBi^*} (iBi^* eA(1-e)) & \longrightarrow & (1-e)Ae \cdot iBi^* \cdot eA(1-e) \\
\wr\| & & \uparrow\wr \\
(\bar{A}\otimes B) \otimes_B (B \otimes \bar{A}) & \xrightarrow{\ \sim\ } & \bar{A}\otimes B\otimes\bar{A}
\end{array}$$

Thus we do get in this case a well-defined linear map $\tau : K \longrightarrow I/[B,I]$. Write

$$K \cong \begin{pmatrix} I & B\otimes\bar{A} \\ \bar{A}\otimes B & \bar{A}\otimes B\otimes\bar{A} \end{pmatrix}$$

τ is zero on the off-diagonal blocks, it the obvious map to $I/[B,I]$. On the bottom right block we have

$$(\alpha,\beta,\alpha') \longmapsto \tau\big((1-e)\alpha\, i\beta i^* \alpha'(1-e)\big)$$

$$= \tau\big(i\beta i^* \alpha'(1-e)\alpha\big) = \beta\, i^*\alpha'(1-e)\alpha i \quad \text{mod}[B,I]$$

$$= \beta\big[\rho(\alpha'\alpha) - \rho(\alpha')\rho(\alpha)\big] \quad \text{in } I/[B,I]$$

Now we have to check that τ vanishes on brackets. First consider odd elements - notice that $C^{odd} = K^{odd}$. Suppose we want to prove $\tau[\xi,\eta] = 0$

where $\xi, \eta \in K^{odd} = C^{odd}$. We can suppose ξ is in the lower left block and η is in the upper right block. Then $\xi\eta$ is in the lower right block and by definition $\tau(\xi\eta) = \tau(\eta\xi)$. Actually let's be more careful and suppose

$$\xi = (1-e)\alpha e\, i\beta i^* \qquad \eta = i\beta' i^* e\,\alpha'(1-e)$$

Then $\xi\eta = (1-e)\alpha e\, i\beta\beta' i^* e\,\alpha'(1-e)$ and by definition

$$\tau(\xi\eta) = \beta\beta'[\rho(\alpha'\alpha) - \rho(\alpha')\rho(\alpha)] \quad \text{in } I/[B,I]$$

Also

$$\eta\xi = i\beta' i^* \alpha'(1-e)\alpha\, i\beta i^*$$

$$= i\,\beta'[\rho(\alpha'\alpha) - \rho(\alpha')\rho(\alpha)]\beta i^*$$

so

$$\tau(\eta\xi) = \beta'[\rho(\alpha'\alpha) - \rho(\alpha')\rho(\alpha)]\beta \quad \text{in } I/[B,I]$$

and $\tau(\xi\eta) = \tau(\eta\xi)$.

Now suppose $\xi, \eta \in C^{ev}$ and write $\xi = \xi' + \xi''$, $\eta = \eta' + \eta''$, where $'$ denotes upper left and $''$ denotes lower right. Then

$$[\xi, \eta] = [\xi', \eta'] + [\xi'', \eta'']$$

Suppose $\xi \in I_1$, $\eta \in I_2$ and we want to show $\tau[\xi, \eta] = 0$. Then $\xi' \in iBi^*$ and $\eta' \in iIi^*$ and $[\xi', \eta'] \in i[B,I]i^*$ so $\tau[\xi', \eta'] = 0$.

Then $\xi'' \in I_1$, $\eta'' \in I_2$ so we can suppose

$$\xi = \xi'' \in$$

$$(1-e)I_1(1-e) = (1-e)A\iota B\iota^* A(1-e)$$

$$\eta = \eta'' \in (1-e)I_2(1-e) = (1-e)A(1-e) + (1-e)Ai\beta i^* A(1-e)$$

suppose $\xi = \underbrace{(1-e)\alpha_1 \iota\beta_1\iota^*}_{X_1}\underbrace{e\alpha_1'(1-e)}_{Y_1}$

$$\eta = \underbrace{(1-e)\alpha_2 i\beta_2\iota^*}_{X_2}\underbrace{e\alpha_2(1-e)}_{Y_2}$$

Then $\tau(\xi\eta) = \tau((X_1Y_1X_2)Y_2) = \tau(Y_2(X_1Y_1X_2))$

$$\tau(\eta\xi) = \tau(X_2(Y_2X_1Y_1)) = \tau((Y_2X_1Y_1)X_2)$$

Next suppose $\eta = (1-e)a(1-e)$

$$\xi = \underbrace{(1-e)\alpha\iota\beta\iota^*}_{X}\underbrace{e\alpha'(1-e)}_{Y}$$

$$\tau(\xi\eta) = \tau(X(Y\eta)) = \tau(Y\eta X)$$

$$\tau(\eta\xi) = \tau((\eta X)Y) = \tau(Y\eta X)$$

$Y\eta \in C^{odd}$
$X \in C^{odd}$

Check

$$\xi\eta = (1-e)\alpha\, i\beta i^*\, \alpha'(1-e)a(1-e)$$

$$= (1-e)\alpha e \cdot i\beta i^* \cdot (\alpha' a - \alpha' e a)(1-e)$$

$$= \pm(1-e)\alpha e \cdot i\beta i^* \cdot e\alpha' a(1-e)$$

$$\mp(1-e)\alpha e \cdot i(\beta\rho(\alpha'))i^* e a(1-e)$$

$$\tau(\xi\eta) = \beta[\rho(\overset{1}{\alpha'} a\alpha) - \rho(\alpha' a)^{3}\rho(\alpha)]$$

$$- \beta\rho(\alpha')[\rho(\overset{2}{a}\alpha) - \rho(a)^{4}\rho(\alpha)]$$

$$\eta\xi = (1-e)a(1-e)\,\alpha\, i\beta i^*\alpha'(1-e)$$

$$= (1-e)\,a\alpha e\; i\beta i^*\alpha'(1-e)$$

$$-(1-e)\,ae\; i\rho(\alpha)\beta\, i^*\alpha'(1-e)$$

$$\tau(\eta\xi) = \beta\left[\overset{1}{\rho}(\alpha' a\alpha) - \rho(\alpha')\overset{2}{\rho}(a\alpha)\right]$$

$$-\rho(\alpha)\beta\left[\overset{3}{\rho}(\alpha' a) - \rho(\alpha')\overset{4}{\rho}(a)\right]$$

Conclusion: We have seen how to go from the setup A, B, I, ρ, τ on $I/[B,I]$ and construct a larger algebra C with $A \to C$, e, $eCe = B$, I_1, I_2, and to extend τ to $K/[I_1, I_2]$. Thus we have the analogue of Tate's setup.