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Given a Riemann surface X with boundary we want to know everything about functions and 1-forms on ∂X and harmonic functions and 1-forms on X. The Dirichlet theorem for functions says

$$\Gamma(\mathcal{H}) \xrightarrow{\sim} C^\infty(\partial X).$$

What's the corresponding result for harmonic 1-forms?

$$0 \longrightarrow K \longrightarrow \Gamma(\Omega^1) \oplus \Gamma(\bar{\Omega}^1) \longrightarrow \Omega^1(\partial X) \xrightarrow{\int_{\partial X}} \mathbb{C} \longrightarrow 0$$

It says the above sequence is exact.

Proof. We have

$$\begin{tikzcd} H^1(X,\partial X) \arrow[r] & H^1(X) \arrow[r] & H^1(\partial X) \arrow[r, "\delta"] \arrow[dr, "\int_{\partial X}"'] & H^2(X,\partial X) \arrow[r] \arrow[d, equal] & 0 \\ & & & \mathbb{C} & \end{tikzcd}$$

which shows that any form ω on ∂X whose integral is zero represents ~~the~~ the restriction of a class $c \in H^1(X)$. c can be rep. by a harmonic 1-form, so to show ω is the restriction of a harmonic 1-form, we can suppose ω represents zero, i.e. $\omega = df$ on ∂X. But then we extend f harmonically and we see ω is the restriction of a harmonic form. QED.

Another question is whether K and $d(\Gamma\mathcal{H})$ are such that each is the annihilator in $\Gamma(\Omega^1) \oplus \Gamma(\bar{\Omega}^1)$ of the other. Note they do annihilate each other, since if f, ω are harmonic with $\omega|_{\partial X} = 0$, then

$$\int_X df\, \omega = \int_{\partial X} f\omega = 0.$$

Here's a diagram of exact sequences which summarizes the Dirichlet problem for functions and 1-forms.

$$
\begin{tikzcd}
 & & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
 & 0 \arrow[r] & K \cap d(\Gamma\mathcal{H}) \arrow[r] \arrow[d] & d(\Gamma\mathcal{H}) \arrow[r, "\text{restriction to } \partial X"] \arrow[d] & d\{C^\infty(\partial X)/H^0(\partial X)\} \arrow[r] \arrow[d] & 0 \\
 & 0 \arrow[r] & K \arrow[r] \arrow[d] & \Gamma(\Omega^1)\oplus\Gamma(\bar{\Omega}^1) \arrow[r] \arrow[d] & \Omega^1(\partial X)_{\int=0} \arrow[r] \arrow[d] & 0 \\
H^0(\partial X)/\mathbb{C} \arrow[r] & 0 \arrow[r] & H^1(X,\partial X) \arrow[r] \arrow[d] & H^1(X) \arrow[r] \arrow[d] & H^1(\partial X)_{\int=0} \arrow[r] \arrow[d] & 0 \\
 & & 0 & 0 & 0 &
\end{tikzcd}
$$

Returning to the annihilator question for K and $d(\Gamma\mathcal{H})$, let ω be a harmonic 1-form annihilating $d(\Gamma\mathcal{H})$. Thus $\forall f \in \Gamma\mathcal{H}$, we have

$$0 = \int_X df\,\omega = \int_{\partial X} f\,\omega$$

and since $\Gamma\mathcal{H} \xrightarrow{\sim} C^\infty(\partial X)$, this means $\omega|\partial X = 0$, i.e. $\omega \in K$. Thus $(d(\Gamma\mathcal{H}))^0 = K$.

As for K^0 we have by the non-degeneracy of the pairing on $\Gamma\Omega^1 \oplus \Gamma\bar{\Omega}^1$, that its codimension is $\dim K$. In effect non-degeneracy + $\dim K < \infty$ imply

$$(\Gamma\Omega^1 \oplus \Gamma\bar{\Omega}^1) \twoheadrightarrow K^\vee .$$

Now $d(\Gamma\mathcal{H}) \subset K^0$ and $d(\Gamma\mathcal{H})$ has codim = $\dim H^1(X)$. But $\dim K = 2g + n - 1 = \dim H^1(X)$.

Thus we have proved

<u>Prop:</u> $K^0 = d(\Gamma\mathcal{H})$ and $d(\Gamma\mathcal{H})^0 = K$.

Ultimately we are interested in the holomorphic functions $\Gamma\mathcal{O}$ and $\Gamma\mathcal{O}^\times$, and the way they relate to $\boxed{V = C^\infty(\partial X)/H^0(\partial X)}$, which is the symplectic space with real structure in whose Heisenberg repn. we wish to find a line.

Now we have

$$d(\Gamma\mathcal{O}) = (\Gamma\Omega^1) \cap d(\Gamma\mathcal{H})$$

since if df is of type $(1,0)$, f is holomorphic.

Recall what Kronheimer said

$$(\Gamma\Omega^1)^0 = \Gamma\Omega^1 \qquad \text{(obvious)}$$

$$(d(\Gamma\mathcal{O}))^0 = \Gamma\Omega^1 + K \qquad \left(\text{since } \supset \text{ and } \dim K = \dim(\Gamma\Omega^1/d\Gamma\mathcal{O})\right)$$

Now I want the annihilator of the image of $\Gamma\mathcal{O}$ in $V = C^\infty(\partial X)/H^0(\partial X)$. Call this image of W and its annihilator W^0. Let's use

$$0 \to K \cap d(\Gamma\mathcal{H}) \to d(\Gamma\mathcal{H}) \longrightarrow d\left(C^\infty(\partial X)/H^0(\partial X)\right) \to 0$$

$$\cong V$$

The inverse image of W, call it $\tilde{W} \subset d(\Gamma\mathcal{H})$, is the sum of $K \cap d(\Gamma\mathcal{H})$ and $d(\Gamma\mathcal{O})$.

$$\tilde{W} = K \cap d(\Gamma\mathcal{H}) + d(\Gamma\mathcal{O}) \twoheadrightarrow \operatorname{Im}\Gamma\mathcal{O} \text{ in } V$$

Now let $Q = d(\Gamma\mathcal{H}) \cap (\Gamma\Omega^1 + K) = d(\Gamma\mathcal{H}) \cap (d(\Gamma\mathcal{O}))^0$

Then we have

$$\begin{tikzcd} & 0 \arrow[d] & 0 \arrow[d] & & \\ 0 \arrow[r] & d\Gamma\theta \arrow[r] \arrow[d] & \Gamma\Omega^1 \arrow[r] \arrow[d] & H^1(X) \arrow[r] \arrow[d, phantom, "\parallel"] & 0 \\ 0 \arrow[r] & Q \arrow[r] \arrow[d] & \Gamma\Omega^1 + K \arrow[r] \arrow[d] & H^1(X) \arrow[r] & 0 \\ & K \arrow[r, phantom, "="] \arrow[d] & K \arrow[d] & & \\ & 0 & 0 & & \end{tikzcd}$$

so that Q is an extension of K by $d\Gamma\theta$.

~~But we are not interested in $Q = d(\Gamma\mathcal{H}) \cap (d\Gamma\theta)^\circ$, but rather $d(\Gamma\mathcal{H}) \cap \tilde{W}^\circ$ which is smaller. We have to remove from Q those elements pairing non-trivially with $d\Gamma\mathcal{H} \cap K$.~~

~~Let $Q_1 = d(\Gamma\mathcal{H}) \cap \tilde{W}^\circ = \left\{ df \;\middle|\; \begin{array}{l} f \text{ harmonic} \\ df \text{ ann. } K \cap d\Gamma\mathcal{H},\ d\Gamma\theta \end{array} \right\}$~~

~~Then $Q_1 \supset d\Gamma\theta$.~~

Now K annih. $d(\Gamma\mathcal{H})$, so

$$Q = d(\Gamma\mathcal{H}) \cap (\tilde{W})^\circ$$

and the annihilator W° we want is the image of this modulo $K \cap d(\Gamma\mathcal{H})$). Thus we get an extension

$$\begin{tikzcd} 0 \arrow[r] & d(\Gamma\theta) \arrow[r] \arrow[d, phantom, "\cong"] & Q/K \cap d(\Gamma\mathcal{H}) \arrow[r] \arrow[d, phantom, "\parallel"] & K/K \cap d(\Gamma\mathcal{H}) \arrow[r] \arrow[d, phantom, "\cong"] & 0 \\ & W & W^\circ & H^1(X, \partial X) & \end{tikzcd}$$

To summarize Q is the space of differentials of harmonic functions which coincide on the boundary with a holom. 1-form. Q contains such diffls which vanish on the boundary, and the quotient, or really the indefinite integrals of elements of Q give the annihilator W°.

This still isn't very clear, but the important point I feel is to look at indefinite integrals of $\omega \in \Gamma(\Omega^1)$ on the various circles $S_i \subset \partial X$. Such an indefinite integral is multiple-valued unless $\int_{S_i} \omega = 0$ $\quad i = 1, \ldots, r$. When this is the case one has $\omega = df$ on ∂X, and f can be extended harmonically so that $\omega = df + k$ with $k \in K$. ~~Hence~~ $df \in (\Gamma\Omega^1 + K) \cap d(\Gamma\mathcal{H})$.

~~Proposition. $(\Gamma\Omega^1 + K) \cap d(\Gamma\mathcal{H})$ is the [illegible] harmonic 1-forms~~

Let's introduce

$$Z = \{\omega \in \Gamma\Omega^1 \mid \int_{S_i} \omega = 0 \quad \text{all } i = 1, \ldots, r\}.$$

so that $Z/d(\Gamma\mathcal{O}) \simeq H^1(X, \partial X)$. We have just seen that any $\omega \in Z$ can be written

$$\omega = df + k \qquad df \in d(\Gamma\mathcal{H}),\ k \in K$$

and this decomposition is unique up to $K \cap d(\Gamma\mathcal{H})$. f is just the indefinite integral of ω on the boundary and this is a well-defined element of $C^\infty(\partial X)/H^0(\partial X)$.

June 19, 1987

Let X have g handles and r boundary circles, let K be the space of harmonic 1-forms on X vanishing when restricted to ∂X. We have seen there is an exact sequence

$$\begin{tikzcd} 0 \arrow[r] & K\cap d\Gamma\mathcal{H} \arrow[r] \arrow[d, phantom, "\wr"] & K \arrow[r] & H^1(X,\partial X) \arrow[r] & 0 \\ & H^0(\partial X)/\mathbb{C} \end{tikzcd}$$

so K has dimension $2g+r-1$.

We can also consider $*K$ which intersects K trivially, since if ω and $*\omega$ both are vanish when restricted to ∂X, then $\omega \pm i*\omega$ are holomorphic + anti-holomorphic 1-forms vanishing on ∂X hence identically by analytic continuation.

If $X\cup\bar{X}$ is the double of X, then there are on the space of harmonic 1-forms on $X\cup\bar{X}$ the reflection σ and the $*$ operator which anti-commute. K can be identified with the harmonic 1-forms on the double which are odd under σ, and $*K$ with the ones even under σ.

Now K contains the subspace $K\cap d\Gamma\mathcal{H}$ of exact harmonic 1-forms vanishing on ∂X, i.e. diffls of harmonic functions which are loc. constant on ∂X. It's natural to ask about $*(K\cap d\Gamma\mathcal{H})$ and what cohomology classes are represented by forms in this subspace. Thus take f harmonic and locally constant on ∂X and look at $*df|\partial X$. It would be nice if these for $*df|\partial X$ generated the cohomology [redacted] $H^1(\partial X)_{\int=0}$. This would imply that the sequence

$$0 \to H^1(X,\partial X) \to H^1(X) \to H^1(\partial X)_{\int=0} \to 0$$

splits [redacted] canonically. Now this is true in any case because one has a pairing

? $$H^1(X) \otimes H^1(X,\partial X) \to H^2(X,\partial X) \xrightarrow{\int} \mathbb{C}$$

[redacted] which restricts to a non-degenerate skew-form on $H^1(X,\partial X)$. ? Thus one gets a complement to $H^1(X,\partial X)$ in $H^1(X)$ which consists of all $\alpha \in H^1(X)$ such that $\int \alpha\beta = 0$ for all $\beta \in H^1(X,\partial X)$.

[redacted] ---

Let's return to

$$\begin{tikzcd}
0 \arrow[r] & d\Gamma\mathcal{H} \arrow[r] \arrow[d, equal] & \Gamma\Omega^1 \oplus \Gamma\overline{\Omega^1} \arrow[r] & H^1(X) \arrow[r] & 0 \\
0 \arrow[r] & d\Gamma\mathcal{H} \arrow[r] \arrow[d] & d\Gamma\mathcal{H} + K \arrow[r] \arrow[d] \arrow[u, phantom, "\cup"] & H^1(X,\partial X) \arrow[r] \arrow[d, equal] \arrow[u, phantom, "\cup"] & 0 \\
0 \arrow[r] & d\Gamma\mathcal{H} / d\Gamma\mathcal{H}\cap K \arrow[r] & d\Gamma\mathcal{H}+K / d\Gamma\mathcal{H}\cap K \arrow[r] & H^1(X,\partial X) \arrow[r] & 0
\end{tikzcd}$$

$d\Gamma\mathcal{H} / d\Gamma\mathcal{H}\cap K \cong C^\infty(\partial X)/H^0(\partial X)$

$$\underbrace{d\Gamma\mathcal{H}+K / d\Gamma\mathcal{H}\cap K} = d\Gamma\mathcal{H} / d\Gamma\mathcal{H}\cap K \oplus K / d\Gamma\mathcal{H}\cap K \xrightarrow{\sim} H^1(X,\partial X)$$

I claim the canonical polarization on $\Gamma\Omega^1 \oplus \Gamma\overline{\Omega^1}$ induces one on [redacted] $d\Gamma\mathcal{H}+K / d\Gamma\mathcal{H}\cap K$.

Let $(\Gamma\Omega^1)_0 = \{\omega \in \Gamma\Omega^1 \mid \int_{S_i} \omega = 0 \quad i=1,\dots,n\}$

Then we have

$$\begin{tikzcd}
0 \arrow[r] & R \arrow[r] \arrow[d, hook] & (\Gamma\Omega^1)_0 \oplus (\Gamma\bar{\Omega}^1)_0 \arrow[r] \arrow[d, phantom, "\bigcap\ \text{cod } r-1"] \arrow[dr] & H^1(X,\partial X) \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & d\Gamma\mathcal{H} \arrow[r] \arrow[d] & d\Gamma\mathcal{H} + K \arrow[r] \arrow[d, "\text{dim } r-1"] & H^1(X,\partial X) \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & \dfrac{d\Gamma\mathcal{H}}{d\Gamma\mathcal{H}\cap K} \arrow[r] & d\Gamma\mathcal{H}+K \big/ d\Gamma H \cap K \arrow[r] & H^1(X,\partial X) \arrow[r] & 0
\end{tikzcd}$$

Here $R = d\Gamma\mathcal{H} \cap \left(\Gamma\Omega^1 + \Gamma\bar{\Omega}^1\right)_0$ consists of df, f harmonic such that $\int_{S_i} *df = 0 \quad \forall i$. Thus we want to know that $R \cap K = 0$, i.e. that if f is locally constant on ∂X and not constant then $\int_{S_i} *df \neq 0$ for some i.

However if $\int_{S_i} *df = 0$ for all i, then

$$*df = dg + k. \quad ?$$

Let's take an alternative approach. Suppose V symplectic with real structure and polarization

$$V_r \subset V = W \oplus \overline{W}$$

Let N_r be isotropic in V_r, $N = \mathbb{C}N_r$. Then we saw that we could split V into

$$V = (N + JN) \oplus (N + JN)^\circ$$

and this is compatible with V_r and W. Here $J = -i$ on W, $+i$ on $\overline{W}$.

Note that J is a real operator, i.e. defined on V_r, so that $J(v^*) = (Jv)^*$. Thus $N+JN$ and $(N+JN)^\circ$ are defined over the reals, and stable under J. The only thing to be checked is that the symplectic form on $N+JN$ is nondegenerate.

But for any $v \in V = W \oplus \overline{W}$, say $v = w_1 + w_2^*$ we have

$$\frac{1}{i}\left[v, J(v^*)\right] = \frac{1}{i}\left[w_1 + w_2^*,\ i w_1^* - i w_2\right]$$

$$= [w_1, w_1^*] + [w_2, w_2^*] = \| w_1 + w_2^* \|^2$$

Thus N, JN are isotropic subspaces of V which pair non-degenerately against each other, hence $N+JN$ is non-degenerate. (Note that

$$[Jv, Jv'] = [v, v']$$

since this is true for $v \in W$, $v' \in \overline{W}$ and the reverse and the other two possibilities give zero.)

Furthermore as $W = [W \cap (N+JN)] \oplus [W \cap (N+JN)^\circ]$

$$N^\circ = N \oplus (N+JN)^\circ$$

$$W \cap N^\circ = 0 \oplus W \cap (N+JN)^\circ$$
$$\overline{W} \cap N^\circ = 0 \oplus \overline{W} \cap (N+JN)^\circ$$

because $w \in W \cap N \implies [w, w^*] \in [N, \overset{N}{\overset{\shortparallel}{N^*}}] = 0$ so $w = 0$. Thus

$$N^\circ / N \simeq (N+JN)^\circ$$

is polarized by $\underline{W \cap N^\circ \oplus \overline{W} \cap N^\circ}$.

Now let's return to X. Take $V = \underbrace{\Gamma\Omega^1}_{W} \oplus \Gamma\overline{\Omega}^1$

and take $N = (d\Gamma\mathcal{H}) \cap K = \{df \mid$
f is harmonic and locally constant on $\partial X\}$.
Clearly N is "real", and isotropic since we have seen K, $d\Gamma\mathcal{H}$ annihilate each other (in fact they are each other's annihilators in V).

N^0 consists of harmonic ω such that

$$\int_X df\,\omega = \int_{\partial X} f\,\omega = 0$$

for all f harmonic on locally constant on ∂X. Thus

$$N^0 = \{\overset{\in V}{\omega} \mid \int_{S_i} \omega = 0 \quad \text{all } i\}.$$

Thus $$W \cap N^0 = \{\omega \in \Gamma\Omega' \mid \int_{S_i} \omega = 0 \quad \text{all } i\},$$

and so we should by the previous discussion have a decomposition of the harmonic 1-forms

$$V = N \oplus (*N) \oplus (\Gamma\Omega')_0 \oplus (\Gamma\bar{\Omega}')_0 .$$

(Here J is the Hodge $*$ operator).

Now we know that

$$N \oplus (\Gamma\Omega')_0 \oplus (\Gamma\bar{\Omega}')_0 \subset d\Gamma\mathcal{H} + K$$

the latter being the space of harmonic forms ~~whose~~ whose classes lie in $H'(X, \partial X)$ — this is just N^0:

$$N^0 = d\Gamma\mathcal{H} + K$$

But then clearly we have

$$N^0 = N \oplus (\Gamma\Omega')_0 \oplus (\Gamma\bar{\Omega}')_0$$

and $*N \hookrightarrow H'(X)$ as a complement to $H'(X, \partial X)$.

But we can see this directly as follows. If $df \in N$ and $*df$ lands in $H^1(X, \partial X)$, then $*df \in N^0$ so

$$\int df \wedge *df = 0$$

showing that $df = 0$.

<u>Question:</u> Is $*N \hookrightarrow H^1(X)$ the complement described before ?

This means that we have to understand the pairing

$$H^1(X) \otimes H^1(X, \partial X) \longrightarrow H^2(X, \partial X) \xrightarrow{\int} \mathbb{C}$$

which is given by $\omega_1, \omega_2 \longmapsto \int_X \omega_1 \omega_2$. If $\omega_1 = df$ then

$$\int_X df\, \omega_2 = \int_{\partial X} f \omega_2 = 0$$

if $\omega | \partial X = 0$. Now there's something tricky here, if I try to use $N^0 = dH + K$ to represent elements of $H^1(X, \partial X)$, or more generally to replace forms supported in $X - \partial X$ with forms vanishing on ∂X. Thus if $\omega_2 = df$ we have

$$\int_X \omega_1\, df = - \int_{\partial X} \omega_1 f$$

which needn't vanish. Thus already $N \subset N^0$ pairs non-trivially with $*N$. ?

Let's return to the case $r=1$ where

$$d\Gamma\mathcal{H} \oplus K = \Gamma\Omega' \oplus \Gamma\overline{\Omega}'.$$

Set $V = C^\infty(S)/\mathbb{C} = \Gamma\mathcal{H}/\mathbb{C} \xrightarrow[\sim]{d} d\Gamma\mathcal{H}$
and $W = \Gamma\mathcal{O}/\mathbb{C}$. By Kronheimer's proof

$$W^\circ = \{f \in V \mid df \in \Gamma\Omega' + K\}$$

Now W is a partial polarization of V in the sense that $[W,W]=0$, $[W,\overline{W}]>0$, so the symplectic form on $W+\overline{W}$ is non-degenerate and we have a symplectic splitting

$$V = (W+\overline{W}) \oplus \underbrace{(W+\overline{W})^\circ}_{W^\circ \cap \overline{W}^\circ}$$

Now $W^\circ \cap \overline{W}^\circ$ is a complement to W in W°. (In effect $V = W^\circ \oplus \overline{W} = \overline{W}^\circ \oplus W$ so

$$W^\circ = W^\circ \cap (\overline{W}^\circ + W) = (W^\circ \cap \overline{W}^\circ) \oplus W.)$$

Thus we conclude that

$$W^\circ/W = W^\circ \cap \overline{W}^\circ$$

has a real structure.

Prop: On $W^\circ/W \xrightarrow{\sim} \Gamma\Omega'/d\Gamma\mathcal{O} \xrightarrow{\sim} H^1(X)$ there is a real structure. The real elements are represented by $\omega \in \Gamma\Omega'$ whose restriction to ∂X is real.

Proof: $W^\circ \cap \overline{W}^\circ$ is the space of $f \in V$ such that df belongs both to $\Gamma\Omega' + K$ and $\Gamma\overline{\Omega}$

A real element is thus an $f \in V_{real}$ such that $df \in \Gamma\Omega' + K$. This means

$$(*) \qquad df = \omega + k$$

and so ω is a holom. 1-form which is real on the boundary. Thus we have a map

$$(W^0 \cap \overline{W}^0)_{real} \xrightarrow{\sim} \{\omega \in \Gamma\Omega' \mid \omega|\partial X \text{ is real}\}$$

defined by taking f into its harmonic extension and then its ~~splitting~~ $(*)$ and taking the $\Gamma\Omega'$ component. The inverse map takes ω into its indefinite integral on ∂X.

So let $\omega \in \Gamma\Omega'$ be such that $\omega|\partial X$ is real, i.e. $\omega = df + k$ with f real. Then

$$\omega - \overline{\omega} = k - \overline{k} \in K$$

and $K \xrightarrow{\sim} H^1(X)$. It follows that $[\omega]$ is a real cohomology class only when $k = \overline{k}$ whence $\omega = \overline{\omega}$ which means $\omega = 0$. Therefore we conclude that this real structure on

$$W^0/W \xrightarrow{\sim} \Gamma\Omega^1/d\Gamma\mathcal{O} \simeq H^1(X)$$

is quite different from the obvious one, namely requiring a form to have real periods.

Let's summarize: We start with $W = \Gamma\mathcal{O}/\mathbb{C} \hookrightarrow V = C^\infty(\partial X)/\mathbb{C}$. W is a partial polarization so we have $W^0 \cap \overline{W}^0 \xrightarrow{\sim} W^0/W$ which defines a real structure on W^0/W. $W^0 = \{$indef. integrals of holom. 1-forms restricted to $\partial X\} \xrightarrow{\sim} \Gamma\Omega'$.

A real element of W^0, i.e. a real element of $W^0 \cap \overline{W}^0$ is ~~the~~ the indefinite integral of a holom. form ω such that $\omega|\partial X$ is real.

In general we can identify $W^0 \cap \overline{W}^0$ with the those forms on ∂X which are restrictions of both holomorphic and anti-holomorphic forms.

Let $R = \{\omega \in \Gamma\Omega^1 \mid \omega \in \Gamma\overline{\Omega}^1 + K\}$

ω has same restriction to ∂X as some elt. of $\Gamma\overline{\Omega}^1$.

Then let's check that $d\Gamma\mathcal{O} \cap R = 0$. In effect given $df = \overline{\omega}_1 + k$, $f \in \Gamma\mathcal{O}$, $\omega_1 \in \Gamma\Omega^1$ we have

$$\int_X df\, d\overline{f} = \int_{\partial X} f(\omega_1 + \overline{k}) = \int_{\partial X} \underbrace{f\omega_1}_{\substack{\text{holom} \\ \text{1-form}}} = 0$$

so $df = 0$.

Alternatively, go back to

$$0 \longrightarrow \underset{\dim 2g}{K} \longrightarrow \Gamma\Omega^1 \oplus \Gamma\overline{\Omega}^1 \longrightarrow \Omega^1(\partial X)_{\int=0} \longrightarrow 0$$

It follows that the image of $\Gamma\Omega^1$ in $\Omega^1(\partial X)_{\int=0}$ (which is dW^0) and the image of $\Gamma\overline{\Omega}^1$ have to overlap in a $2g$-diml space.

It would seem that $dW^0 \cap d\overline{W}^0$ is exactly the image of $*K$ in $\Omega^1(\partial X)_{\int=0}$. In effect suppose $\omega \in \Gamma\Omega^1$ such that $\omega - \overline{\omega} \in K$. Then

$$*(\omega - \overline{\omega}) \in *K$$
$$\parallel$$
$$-i(\omega + \overline{\omega})$$

and so $*(\omega - \overline{\omega})$ restricts the same as $-2i\omega$ to ∂X.

Better $*\left[\frac{i}{2}(\omega-\bar\omega)\right] = \frac{\omega+\bar\omega}{2}$ has the same restriction as ω to ∂X. Thus the real subspace in $\Omega^1(\partial X)_{\int=0}$ we want is exactly the image of $*(K_{real})$.

The conclusion seems then to be that if we take the harmonic forms vanishing on ∂X and look at their holomorphic components we get a natural complement to $d\Gamma\mathcal{O}$ in $\Gamma\Omega^1$. This complement contains a real subspace of $\omega \in \Gamma\Omega^1$ such that $\omega - \bar\omega$ vanishes on ∂X, and this defines a real structure on $\Gamma\Omega^1/d\Gamma\mathcal{O} = H^1(X)$ which is quite different from $H^1(X,\mathbb{R})$.

Two real structures on $H^1(X) = \Gamma\Omega^1/d\Gamma\mathcal{O}$ are as follows: One is $H^1(X,\mathbb{R})$, i.e. the subspace represented by holomorphic forms with real periods. The other is represented by holomorphic forms ω such that $\omega-\bar\omega$ vanishes on ∂X.

For tomorrow: Use the double $X \cup \bar X$, where we can identify $K + *K$ with the harmonic forms. The restriction of holomorphic forms on the double should give the complement of $d\Gamma\mathcal{O}$ in $\Gamma\Omega^1$.

June 20, 1987

We are going to be interested in the cohomology of the double $X \cup \overline{X}$ with its integral structure and Hodge decomposition. So it seems a good idea to learn about Riemann's conditions.

Up to now we have been thinking of

$$\underset{\text{self dual lattice}}{\Gamma} \subset V_{\mathbb{R}} \subset V = W \oplus \overline{W}$$

Then W determines a vacuum state $|W\rangle$ and Γ determines a (non normalizable) state $|\Gamma\rangle$ such that

$$F(v) = \langle \Gamma | e^{iv} | W \rangle$$

is a kind of Θ-function. (To be more specific we note that

$$F(v) = \langle \Gamma | e^{+i\gamma} e^{iv} | W \rangle = e^{-\frac{1}{2}[\gamma, v]} F(v+\gamma)$$

and

$$F(w+w^*) = \langle \Gamma | e^{iw^* + iw} | W \rangle$$

$$= e^{\frac{1}{2}[w, w^*]} \langle \Gamma | e^{iw^*} | W \rangle$$

$$= e^{\frac{1}{2}[w, w^*]} F(w^*).$$

This shows that F on $V_{\mathbb{R}}$ is essentially holomorphic relative to $V_{\mathbb{R}} \xrightarrow{\sim} \overline{W}$, and that it's quasi-periodic with respect to Γ.)

However if X is now a closed Riemann surface, then when we come to

$$H^1(X, \mathbb{Z}) \subset H^1(X, \mathbb{R}) \subset H^1(X, \mathbb{C}) = H^{1,0} \oplus H^{0,1}$$

the polarization is not arbitrary, but rather there are Riemann's conditions, NO which say that the complex torus

$$H^{1,0} / H^1(X, \mathbb{Z})$$

(these are automatic see below)

is an abelian variety, as well as other conditions (the Schottky problem is to find them) which specify that this abelian variety is a Jacobian.

In order to sort out the confusion, let's list various situations.

1) (Riemann case). Here one has a complex vector space W with hermitian inner product and a lattice Γ which is self-dual with respect to the symplectic form given by the imaginary part of the inner product: $\Gamma \subset V_{\mathbb{R}} \subset V = W \oplus \overline{W}$.

2) (p. 863) One has V complex symplectic with Γ self-dual in V and also a real structure $V_{\mathbb{R}}$. Note that $\operatorname{rank}(\Gamma) = \dim_{\mathbb{C}} V$, so we have "half" a lattice in V. Thus we have

$$\Gamma \underset{\text{half}}{\subset} V \supset V_{\mathbb{R}}$$

3) In case 2 if we take $\Lambda = \Gamma + \Gamma^*$, then we have

$$\Lambda \underset{\text{full}}{\subset} V \supset V_{\mathbb{R}}$$

where Λ is self-dual wrt $\operatorname{Im}[v, v']$.

So roughly it appears that there are two cases 1) + 3). In both we have a self-dual lattice in a real symplectic vector space V and a complex structure on V. And we are looking for an entire function satisfying some quasi-periodicity condition.

Let's begin with a real vector space V and lattice Γ inside it. We want a line bundle over the torus V/Γ and one way to obtain this is to start with suitable connection on the trivial line bundle over V. Take a skew form F on V and view it as a 2-form on V which is translation invariant. If we choose a bilinear form whose skew-symmetrization is F, then we get a connection on the trivial line bundle with curvature F. (Specifically if $F = \frac{1}{2} F_{\mu\nu}\, dx_\mu dx_\nu$, then $A = A_{\mu\nu} x_\mu dx_\nu$ satisfies $dA = A_{\mu\nu}\, dx_\mu dx_\nu = F$ when $A_{\mu\nu} - A_{\nu\mu} = F_{\mu\nu}$).

The connection is $d + A = dx^k \underbrace{(\partial_k + x_\mu A_{\mu k})}_{D_k}$

Let $$D'_\ell = \partial_\ell + A_{\ell\nu} x_\nu .$$ Then

$$[D_k, D'_\ell] = [\partial_k + x_\mu A_{\mu k}, \partial_\ell + A_{\ell\nu} x_\nu]$$
$$= A_{\ell k} - A_{\ell k} = 0.$$

Thus the translation operators

$$T_v = e^{v_\ell D'_\ell}$$

preserve the connection. Also

$$T_v T_w = e^{vD'} e^{wD'} = e^{\frac{1}{2}[v_\ell D'_\ell,\, w_m D'_m]} e^{(v+w)D'}$$

where $[v_\ell D'_\ell\ w_m D'_m] = v_\ell w_m [\partial_\ell + A_{\ell\mu}x_\mu, \partial_m + A_{m\nu}x_\nu]$

$$= v_\ell w_m (A_{m\ell} - A_{\ell m}) = -F_{\ell m} v_\ell w_m .$$

so

$$\boxed{T_v T_w = e^{-\frac{1}{2}F_{\ell m} v_\ell w_m}\, T_{v+w}}$$

Thus if $F_{\ell m} v_\ell w_m \in 2\pi i \mathbb{Z}$ for all $v, w \in \Gamma$ we will have an action of Γ on this line bundle with connection. We assume this is true whence F is purely imaginary.

At this point we have a line bundle with connection on the torus V/Γ.

Next we suppose V is given a complex structure, whence F being a 2-form in V will split into forms of types $(2,0)$, $(1,1)$, $(0,2)$. The connection on the line bundle determines a holomorphic structure when the curvature F is of type $(1,1)$.

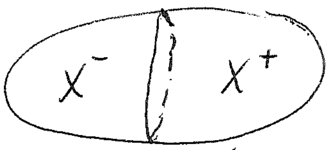
June 21, 1987

Conventions: $C^\infty(S)/\mathbb{C}$ is the space of linear functions on "phase space" so to each f belongs an operator $\rho(f)$ on the Heisenberg repn. CCR:

$$\begin{cases} \rho(f)^* = \rho(\bar{f}) \\ [\rho(f), \rho(g)] = \frac{1}{2\pi i}\int f\,dg \end{cases}$$

e.g. $[\rho(z^m), \rho(z^n)] = \frac{1}{2\pi i}\int z^m n z^n \frac{dz}{z} = n\,\delta_{-m,n}$

Given $\downarrow$ we think of X^- as a process in the past, hence it gives a ~~state~~ $|X^-\rangle$ on the circle. This state



is supposed to be fixed under $\Gamma(X^-, \mathcal{O}^\times)$. In particular

$$\rho(f)\,|X^-\rangle = 0 \qquad \text{if} \qquad f \in \Gamma(X^-, \mathcal{O}).$$

Thus in the plane where $X^- = \{|z| \le 1\}$ we want $|0\rangle$ to satisfy the physicists convention

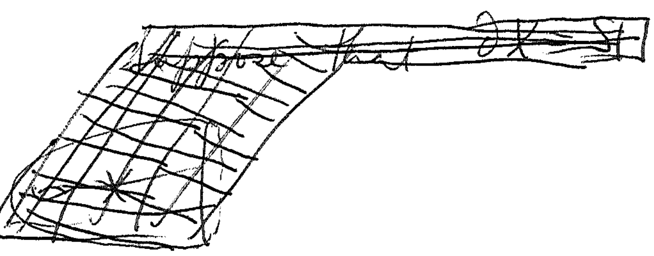
$$\rho(z^n)|0\rangle = 0 \qquad n > 0$$

This means we probably want to change the CCR to

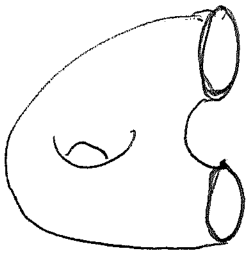
$$\boxed{[\rho(f), \rho(g)] = \frac{i}{2\pi}\int f\,dg}$$

Thus for $f \in \Gamma(X^-, \mathcal{O})$ $\quad f \neq 0$

$$[\rho(f), \rho(f)^*] = \frac{i}{2\pi}\int_{\partial X^-} f\,d\bar{f} = \frac{i}{2\pi}\int_{X^-} df\,d\bar{f} \quad > 0$$



Suppose that $\partial X = \bigcup_{i=1}^{r} S_i$. Given



$f \in \Gamma(X, \mathcal{O}^\times)$ such that

$$\frac{1}{2\pi i}\int_{S_i} \frac{df}{f} = \deg_{S_i}(f) = 0$$

for $i = 1, \dots, r$, the function $\log f$ on ∂X is well-defined modulo $H^0(\partial X)$. [redacted] For $\log f \in C^\infty(\partial X)/H^0(\partial X)$ we have an operator $\rho(\log f)$ on the Heisenberg representation $\mathcal{F}_{\partial X} = \bigotimes \mathcal{F}_{S_i}$

<u>Conjecture</u>: There is a unique line in $\mathcal{F}_{\partial X}$ which is fixed by the operators $e^{\rho(\log f)}$ for all $f \in \Gamma(X, \mathcal{O}^\times)$ with $\deg_{S_i}(f) = 0 \quad \forall i$.

Necessary condition:

$$\left(e^{\rho(\log f)}, e^{\rho(\log g)} \right) = e^{\frac{i}{2\pi}\int_{\partial X} (\log f)\frac{dg}{g}} = 1$$

for $f, g \in \Gamma(X, \mathcal{O}^\times)_0 = \left\{ f \in \Gamma(X, \mathcal{O}^\times) \;\middle|\; \int_{S_i} \frac{df}{f} = 0, \ \forall i \right\}$. However this follows from Deligne's [redacted] theory. The number is the total monodromy over ∂X of the flat line bundle defined by f, g over X.

Let's put $V = C^\infty(\partial X)/H^0(\partial X)$ and let $W = \Gamma(X, \mathcal{O})/\mathbb{C} \subset V$. We know W is a partial polarization:

$$[w, w'] = 0 \quad , \quad [w, w^*] > 0 \text{ for } w \neq 0.$$

The line we seek is contained in $(\mathcal{F}_{\partial X})^W$ the subspace of vectors killed by W. We know that $(\mathcal{F}_{\partial X})^W$ is the Heisenberg representation of

$$W^0 \cap \overline{W}^0 \simeq W^0/W$$

June 22, 1987 (47 years old)

Let V be a complex vector space. If $H(x,y)$ is a hermitian form on V:

$$H(ix,y) = iH(x,y)$$
$$H(x,iy) = -iH(x,y)$$
$$H(x,y) = \overline{H(y,x)}$$

then $\boxed{\Omega(x,y) = \operatorname{Im} H(x,y)}$ is real skew form on the underlying real vector space of V such that

$$(*) \qquad \Omega(ix,iy) = \Omega(x,y).$$

One can recover H from Ω because

$$H(x,y) = \operatorname{Re} H(x,y) + i \operatorname{Im} H(x,y)$$
$$= \operatorname{Im} iH(x,y) + i\,\Omega(x,y)$$

$$\boxed{H(x,y) = \begin{cases} \Omega(ix,y) + i\,\Omega(x,y) \\ \Omega(x,-iy) + i\,\Omega(x,y) \end{cases}}$$

Conversely given Ω satisfying $(*)$ the last two formulas define a hermitian form H. Thus we see hermitian forms are equivalent to skew forms Ω satisfying $(*)$.

Now suppose V is equipped with a skew ~~symplectic~~ form $S(x,y)$, that is, a $\mathbb{C}$-bilinear ~~non-degenerate~~ skew form. Let σ be a conjugation on V compatible with S:

$$\overline{S(x,y)} = S(\sigma x, \sigma y),$$

whence V, S is the complexification of (V^{σ}, S). Then

$$\boxed{H(x,y) = iS(x,\sigma y)}$$

is a hermitian form:

$$\overline{H(y,x)} = \overline{iS(y,\sigma x)} = -iS(\sigma y, x) = iS(x,\sigma y) = H(x,y).$$

It's clear that H, Ω, S are all non-deg. if one of them is.

Let's now consider V, S complex symplectic with real structure $v \mapsto v^*$ and a self-dual lattice $\Gamma \subset V$ satisfying the ~~positivity~~ condition

$$\oplus \qquad [v, v^*] > [v, \sigma v] \qquad v \neq 0$$

where σ is the conjugation fixing $\mathbb{R}\Gamma$. The goal is to understand properly why there is a unique state ψ in $\mathcal{F}_{V,*}$ fixed under $e^{i\Gamma}$. To simplify calculation I will replace $e^{i\Gamma}$ by e^{Γ}, but this probably changes the positivity condition $\oplus$ above ($v \mapsto -v$).

Put $\quad F(v) = \langle \psi | e^{v} \psi \rangle \quad$ so that F is entire such that

$$① \qquad F(v) = e^{\frac{1}{2}[v,\gamma]} F(v+\gamma) = e^{\frac{1}{2}[\gamma^*, v]} F(v+\gamma^*)$$

I propose to identify F with the Θ function on V associated to the lattice $\Gamma + \Gamma^*$ and the hermitian inner product $\oplus$.

The first step will be to determine the line bundle with connection over V whose translation operators give the quasi periodicity conditions ① on F. These translation operators are

$$e^{\frac{1}{2}[a^*, \gamma] + \gamma a}, \qquad e^{\frac{1}{2}[\gamma^*, a^*] + \gamma^* a^*}$$

here $\gamma a = \gamma_i \partial_{z_i}$
$[a^*, \gamma] = z_i [e_i, \gamma]$

e_i basis for V coords z_i

for $\gamma \in \Gamma$. These commute as

$$\left[\tfrac{1}{2}[a^*, \gamma_1] + \gamma_1 a, \ \tfrac{1}{2}[a^*, \gamma_2] + \gamma_2 a \right] = \tfrac{1}{2}[\gamma_1, \gamma_2] - \tfrac{1}{2}[\gamma_2, \gamma_1] = [\gamma_1, \gamma_2]$$

$$\left[\tfrac{1}{2}[\gamma_1^*, a^*] + \gamma_1^* a^*, \ \tfrac{1}{2}[\gamma_2^*, a^*] + \gamma_2^* a^* \right] = \tfrac{1}{2}[\gamma_2^*, \gamma_1^*] - \tfrac{1}{2}[\gamma_1^*, \gamma_2^*] = -[\gamma_1^*, \gamma_2^*]$$

$$= -[\gamma_2, \gamma_1]^* = [\gamma_1, \gamma_2]^* = -[\gamma_1, \gamma_2]$$

belong to $2\pi i \mathbb{Z}$, and

$$\left[\tfrac{1}{2}[a^*, \gamma_1] + \gamma_1 a \,,\, \tfrac{1}{2}[\gamma_2^*, a^*] + \gamma_2^* a \right] = \tfrac{1}{2}[\gamma_2^*, \gamma_1] - \tfrac{1}{2}[\gamma_2^*, \gamma_1] = 0$$

Thus we are dealing with the lattice $\Gamma \oplus \Gamma^*$ inside the real symplectic space $R\Gamma \oplus R\Gamma^*$ equipped with the symplectic form

$$\Omega(\gamma_1 + \gamma_2^*, \mu_1 + \mu_2^*) = \frac{1}{i}\left([\gamma_1, \mu_1] - [\gamma_2, \mu_2]\right)$$

What we want to see now is that the operator J of multiplication by i on $V = \mathbb{R}\Gamma + \mathbb{R}\Gamma^*$ is a polarization with respect to Ω i.e.

$$\Omega(Jv, Jv') = \Omega(v, v')$$

$$i\Omega(v, Jv) > 0 \qquad v \neq 0.$$

The first condition means that Ω is the imaginary part of a unique hermitian form on V, and I want to identify this form with

$$H(v, v') = [v, v^*] \pm [v, \sigma v].$$

Alternate approach. We are considering a complex symplectic space V with two real structures given by the conjugations σ and $*$. Then we have a complex symplectic transformation between the real symplectic spaces V_σ and $V_* = V_{\mathbb{R}}$. Assuming this transformation is implementable on the "quantum mech. level" we therefore have a transformation between the two Heisenberg representations. Put another way we have two inner products on the same representation of V.

Thus we should view σ as being the adjoint with respect to a different inner product

$$\langle \xi | A \eta \rangle$$

on $\mathcal{F}$. Here $A = A^* > 0$. This means $\sigma(v)$ satisfies

$$\langle \xi | A v \eta \rangle = \langle \sigma(v) \xi | A \eta \rangle$$

or

$$\langle \xi | A v \eta \rangle = \langle \xi | \sigma(v)^* A \eta \rangle$$

or

$$\boxed{A v = \sigma(v)^* A}$$

Thus $\sigma(v)^* = A v A^{-1}$ which means that A implements the transformation $v \longmapsto \sigma(v)^*$, which is a complex symplectic transformation:

$$[\sigma(v)^*, \sigma(v')^*] = -\overline{[\sigma(v), \sigma(v')]} = [v, v']$$

But I have seen that $v \mapsto T(v)$ is implementable when it is symplectic and positive

$$[Tv, Tv'] = [v, v']$$

$$[Tv, (Tv)^*] > [v, v^*] \qquad v \neq 0$$

Apply this to $Tv = \sigma(v)^* = (* \cdot \sigma)(v)$, and we get the condition

$$[*(\sigma v), \sigma v] \overset{?}{>} [v, v^*]$$

$$\cancel{[*\sigma(\sigma * v), \sigma(\sigma * v)] > [\sigma * v, *\sigma * v]}$$

$$\cancel{[v, *v] > [*\sigma\sigma * v, *\sigma * \sigma * v]}$$

$$\cancel{\Rightarrow [v, (*\sigma * \sigma *) v]}$$

Now ~~substitute~~ $\sigma \mapsto (\sigma *)\sigma$ and this becomes

$$[*\sigma\sigma*\upsilon,\ \sigma\sigma*\upsilon] \overset{?}{>} [\sigma*\upsilon,\ \cancel{*}\sigma*\upsilon]$$

$$\|$$

$$[\upsilon,\ *\upsilon] \overset{?}{>} [\upsilon,\ \underbrace{(*\sigma*\sigma*)}_{(*\sigma)*(*\sigma)^{-1} \,=\, T*T^{-1}}\sigma]$$

The involution $T*T^{-1}$ fixes $T(V_2) = *(\sigma V_2)$. ?

June 23, 1987

Let V be a complex symplectic vector space with skew form $S(v,v')$. If σ is a conjugation on V compatible with S:

$$\overline{S(v,v')} = S(\sigma v, \sigma v')$$

then one can associate to σ a hermitian form

$$H_\sigma(v,v') = iS(v,\sigma v').$$

Check:
$$\overline{H(v,v')} = -i\,\overline{S(v,\sigma v')} = -iS(\sigma v, v')$$
$$= i\,S(v',\sigma v) = H(v',v).$$

Moreover one can associate to σ a Hilbert space $\mathcal{F}_\sigma$ which is "the" irreducible representation of the CCR.

$$[\rho_\sigma(v), \rho_\sigma(v')] = i\,S(v,v')$$
$$\rho_\sigma(v)^* = \rho_\sigma(\sigma(v))$$

We can partially order the set of conjugations by declaring $\sigma \geq \tau$ if the associated hermitian forms satisfy $H_\sigma \geq H_\tau$, i.e. if

$$H_\sigma(v,v) \geq H_\tau(v,v).$$

It appears that when $\sigma \geq \tau$ there is an embedding of $\mathcal{F}_\sigma$ into $\mathcal{F}_\tau$ compatible with the operators associated to elements of V. This embedding is unique up to scalar factors.

My idea is that it might be possible to assemble all of these Hilbert spaces into a tower which we might think of as a kind of rigging.

Let's fix a basepoint $*$ with real subspace $V_{\mathbb{R}}$, and consider another conjugation σ. When $* \geq \sigma$ we can choose a complex symplectic T such that $TV_{\mathbb{R}} = V_\sigma$, i.e. such that

$$\sigma = T * T^{-1}$$

and we can implement T by an operator K on $\mathcal{F} = \mathcal{F}_*$ i.e.

$$T(v) = K v K^{-1}.$$

Then we consider the inner product

$$\text{(1)} \qquad \langle K^{-1}\xi \mid K^{-1}\eta \rangle$$

on $\mathcal{F}_*$, actually on the dense subspace $\operatorname{Im} K$. We have

$$\begin{aligned}
\langle K^{-1} \underbrace{\sigma(v)}_{T*T^{-1}v} \xi \mid K^{-1}\eta \rangle &= \langle (*T^{-1}v) K^{-1}\xi \mid K^{-1}\eta \rangle \\
&= \langle K^{-1}\xi \mid \underbrace{(T^{-1}v)}_{K^{-1}vK} K^{-1}\eta \rangle \\
&= \langle K^{-1}\xi \mid K^{-1} v \eta \rangle
\end{aligned}$$

Thus $\sigma(v)$ is the adjoint of v with respect to the inner product (1).

But we should be able to do this using σ alone. Note that T can be right multiplied by a real symplectic transformation.

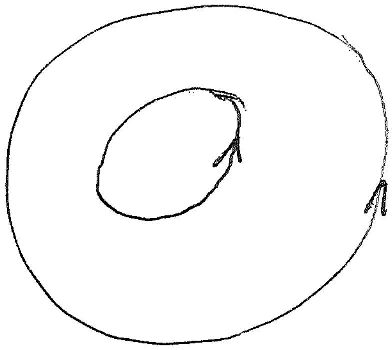
Consider the case where V is 2 dim'l. In this case a conjugation determines a circle on the Riemann sphere $\mathbb{P}V$, where circle means a real projective line. Thus σ is a Mobius transformation and the circle is the fixpoint set. If we remove this circle we are looking at the complex lines W in V such that $W \neq W^{\sigma}$. Then we can talk about the W such that $[w, \sigma w] > 0$ for $0 \neq w \in W$. These give an inside to the circle, which means that the circle is oriented.

The positivity condition

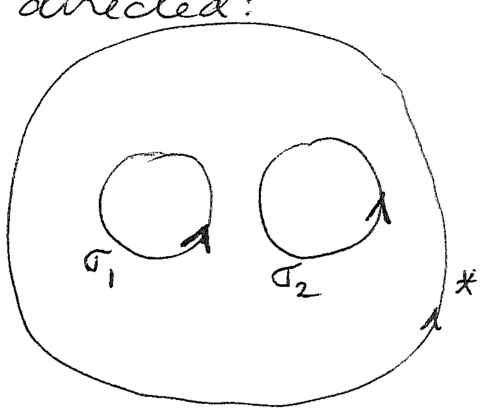
$$[v, \sigma v] > [v, v^*] \qquad v \neq 0$$

means that the σ-circle is contained in the inside of the $*$-circle.

So it is now clear that the partially ordered set in this case consists of all oriented (linear) circles on the Riemann sphere and the ordering is the same sort of thing encountered with Riemann surfaces



In particular we see that this partially-ordered set is not directed:



σ_1, σ_2 both $> *$ but no σ dominates both σ_1, σ_2.

It should be so that the Heisenberg representation associated to these (linear) circles actually sit inside the quantum field theory Fock spaces.

The next stage is to learn how to calculate in this situation.

Let V be a complex symplectic vector space of dim $2n$. Let's choose a basis so we can identify V with $\mathbb{C}^{2n}$ and the symplectic form becomes

$$S(x,y) = x^t J y$$

where J is invertible and $J^t = -J$. The symplectic group is

$$Sp(V) = \{ g \mid g^t J g = J \}$$

and its Lie algebra is

$$\mathfrak{sp}(V) = \{ X \in \mathfrak{gl}(V) \mid X^t J + J X = 0 \}$$

But $\quad X^t J + J X = J X - (J X)^t \quad$ so that

$$\mathfrak{sp}(V) = \{ J^{-1} A \mid A^t = A \}$$

Thus $\quad \dim_{\mathbb{C}} (Sp(V)) = \dfrac{2n(2n+1)}{2} = n(2n+1).$

We have two maps from the Lie algebra $\mathfrak{sp}(V)$ to $Sp(V)$, namely the exponential

$$X = J^{-1}A \longmapsto e^{J^{-1}A} \qquad \text{always defined}$$

and the Cayley transform

$$X = J^{-1}A \longmapsto \frac{1+X}{1-X}$$

which is defined as long as X doesn't have the eigenvalue ± 1. (Notice that as $X^t = -JXJ^{-1}$ the eigenvalues of X are stable under $\lambda \mapsto -\lambda$, and similarly the eigenvalues of g $(g^t = Jg^{-1}J^{-1})$ are stable under $\lambda \mapsto \lambda^{-1}$). Check:

$$J\left(\frac{1+X}{1-X}\right)^{-1} J^{-1} = \left(\frac{1-X^t}{1+X^t}\right)^{-1} = \frac{1+X^t}{1-X^t} = \left(\frac{1+X}{1-X}\right)^t$$

The Cayley transform is an analytic diffeom. from the open set of $\underline{sp}(V)$ consisting of all X not having the eigenvalues ± 1 to the open set of $Sp(V)$ consisting of all g not having the eigenvalue -1.

Take $V = \mathbb{C}^2$ with $J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. Then

$$X = J^{-1}A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} a & b \\ b & d \end{pmatrix} = \begin{pmatrix} -b & -d \\ a & b \end{pmatrix}$$

is a typical matrix of trace zero, and of course $\underline{Sp}(V) = SL_2(\mathbb{C})$.

The exponential map

$$sl_2(\mathbb{C}) \xrightarrow{e} SL_2(\mathbb{C})$$

is not onto. It is onto semi-simple elts, and onto unipotents, but it can't cover $-1+N$ where $N^2 = 0$, $N \neq 0$ since eigenvalues for any $X \in sl_2(\mathbb{C})$, $X \neq 0$ are distinct. The Cayley transform will be a diffeom. from the complement of the set of involutions to $\{g \in SL_2(\mathbb{C}) \mid g \neq -I\}$.

Let's next consider conjugations. To fix the ideas suppose $*$ = conjugation on $\mathbb{C}^{2n}$ and $J = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}$. The space of conjugations is

$$Sp(2n, \mathbb{C}) / Sp(2n, \mathbb{R}).$$

The tangent space to the basepoint can be identified with $X = J^{-1}A$ such that $*X = -X$, i.e. such that A is a purely imaginary symmetric matrix. The goal is now find a nice description of the space of conjugations.

Let σ be a conjugation $<$ $*$ and choose a symplectic T such that $\sigma = T * T^{-1}$, i.e. such that $V_\sigma = T(V_r)$. I believe there exists a contraction operator K implementing T:

$$K v K^{-1} = T(v) \qquad \text{or} \qquad K^{-1} v K = T^{-1}(v).$$

Then
$$\langle K^{-1}\sigma(v)\xi \mid K^{-1}\eta \rangle = \langle K^{-1}T(*T^{-1}(v))\xi \mid K^{-1}\eta\rangle$$
$$= \langle (*(T^{-1}v))K^{-1}\xi \mid K^{-1}\eta \rangle = \langle K^{-1}\xi \mid (T^{-1}v)K^{-1}\eta \rangle$$
$$= \langle K^{-1}\xi \mid K^{-1}v\eta \rangle$$

This shows that $\langle \xi \mid \sigma(v)^* K^{-1*}K^{-1}\eta \rangle = \langle \xi \mid K^{-1*}K^{-1}v\eta \rangle$

whence
$$(*\sigma(v))K^{-1*}K^{-1} = K^{-1*}K^{-1}v.$$

In other words the symplectic transformation
$$v \longmapsto *\sigma(v)$$
is implemented by a positive contraction operator $K^{-1*}K^{-1}$.

Let's analyze the 2 diml case geometrically using Mobius transformations in the Riemann sphere. Suppose we take our basepoint to be the unit circle $|\lambda| = 1$. This is convenient because it corresponds to the usual a, a^* operators. A point is PV is a line W in $\mathbb{C}a + \mathbb{C}a^*$. Let W be spanned by $a + \lambda a^*$. Then positivity means

$$[a+\lambda a^*, a^* + \bar{\lambda} a] = 1 - |\lambda|^2 > 0$$

Consider the contraction operator e^{-ta^*a}. This moves the line $\mathbb{C}(a + \lambda a^*)$ into the line killing

$$e^{-ta^*a} e^{-\lambda a^*} |0\rangle = e^{-\lambda e^{-t} a^*} |0\rangle$$

i.e. into the line $a + \lambda e^{-t} a^*$. Thus the circle gets shrunk.

Let's now note that all the coherent states $e^{\lambda a^*}|0\rangle$ belong to the subspace

$$\bigcap e^{-ta^*a} \mathcal{F}$$

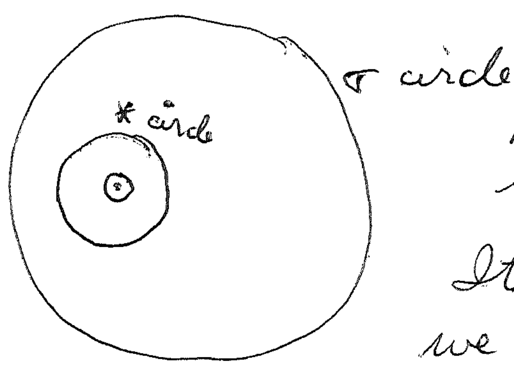
consisting of ξ such that $e^{ta^*a}\xi \in \mathcal{F}$ for all t.

In general given two circles one contained inside the other we can successively reflect, or better by applying a suitable Mobius transf. we can assume they are concentric with the larger one being the unit circle.

Suppose the larger circle is given by the conjugation σ and the smaller circle given by $*$ so that

$$[v, \sigma v] > [v, v^*] \quad \text{for} \quad v \neq 0.$$

Think of σ and $*$ as reflections.



Consider the symplectic transformation $*\sigma$. This carries the σ circle into its reflection thru the $*$ circle. It's clear that if we iterate we get a standard picture in which $*\sigma$ generates a translation in the radial direction from its two fixpoints.

If the $*$ circle is fixed, then the choice of σ depends on three parameters, two for the location of the outside fixpoint and 1 for the size of the translation. I can see that up to conjugacy we are dealing with a homothety $z \longrightarrow e^t z$ where $*$ and σ circles are concentric around 0.

It's clear that the sort of symplectic transformation we have is e^H where H is in the positive cone of the real symplectic Lie algebra.

Now suppose the circles intersect in two points. Then it's clear that the symplectic transformation $*\sigma$ will rotate around its fixpts, which are where the circles intersect. To see this put one at 0 and the other at ∞, whence $\sigma, *$ become reflections through lines passing thru the origin, and so $\sigma *$ is a rotation. Again one gets a 1-parameter group, but this time the eigenvalues of the generator are purely imaginary.

What this example shows is that it is not going to be possible to identify the space of conjugations with the exponentials of the X such that $\bar{X} = -X$. It still might be possible to use the C.T.

Review: Suppose $\sigma, *$ such that we can find a symplectic transf T implemented by K such that $T(V_r) = V_\sigma$ i.e.

$$T * T^{-1} = \sigma$$

$$T(v)\, K = K\, \sigma \qquad\qquad K^{-1} T(v) = v K^{-1}$$

Then from $\left(T(v) \otimes 1 - 1 \otimes v^t\right) K = 0$ we see that

$$[T(v), T(v)^*] > [v, v^*] \qquad \text{for } v \neq 0$$

Substituting $v \mapsto T^{-1}v$ gives the pos. condition

$$[v, v^*] > [T^{-1}v, (T^{-1}v)^*] = [v, \underbrace{(T * T^{-1})}_{\sigma}(v)]$$

which depends only on σ.

On the other hand we have

$$\langle K^{-1}\sigma(v)\, \xi \mid K^{-1}\eta \rangle = \langle K^{-1} T(*T^{-1}v)\, \xi \mid K^{-1}\eta \rangle$$

$$= \langle *(T^{-1}v)\, K^{-1}\xi \mid K^{-1}\eta \rangle = \langle K^{-1}\xi \mid (T^{-1}v)\, K^{-1}\eta \rangle$$

$$= \langle K^{-1}\xi \mid K^{-1} v\, \eta \rangle \qquad \text{or}$$

$$\langle (K^{-1})^* K^{-1} \sigma(v) \xi | \eta \rangle = \langle v^*(K^{-1})^* K^{-1} \xi | \eta \rangle$$

so that if $P = KK^*$, then we have

$$P^{-1}\sigma(v) = v^* P^{-1} \quad \text{or} \quad (\sigma *)(v) = PvP^{-1}$$

Thus the symplectic transformation $\sigma *$ is implementable. We should be able to check this by showing that $\sigma *$ satisfies

$$[(\sigma *)(v), *(\sigma *(v))] \overset{?}{>} [v, v^*]$$

starting from $[v, v^*] > [v, \sigma v]$. Indeed putting $*v$ in for v gives

$$[*v, v] > [*v, \sigma * v]$$

or

$$[v, *v] < [\sigma * v, *v].$$ On the other hand

$$[(\sigma *)(v), *(\sigma *)(v)] > [\sigma * v, \cancel{\sigma} * v],$$ so we win.

So at the moment starting from the positivity condition $[v, v^*] > [v, \sigma v]$ I get a positivity condition on the symplectic transformation $\sigma *$. What I would like to show is that $\sigma *$ has a "positive" square root. Thus the operator $(KK^*)^{1/2}$ hopefully comes from a "positive" complex symplectic transformation.

Let's shift to matrices and suppose T above is a symplectic matrix g so that

$$g^* J g = J$$

and $$\sigma(v) = g(\bar{g}^{-1}\bar{v}) = \underbrace{(g\bar{g}^{-1})}_{h}\bar{v}.$$ Thus h represents the transformation $\sigma *$. Special property of h is that $\bar{h} = h^{-1}$ as $g\bar{g}^{-1}\bar{g}g^{-1} = 1$.

Then

$$JhJ^{-1} = (h^t)^{-1} = \bar{h}^t = h^*$$

Now our positivity condition is that

$$[v, v^*] > [v, \sigma v]$$

or $$i\, v^t J \bar{v} > i\, v^t J h \bar{v}$$

or $$i\, v^* J v > i\, v^* J h v \qquad \text{for } v \neq 0.$$

Thus $$(iJ)(1-h) > 0$$

Note that any conjugation σ when composed with $*$ gives a linear operator, so $\sigma(\bar{v}) = hv$ for some h. Then $\sigma^2 = 1 \iff h\bar{h} = 1$, and σ preserves $S(x,y) \iff h$ symplectic. What I would like to see is that h has a natural square root, call it k, which is symplectic and satisfies $k\bar{k} = 1$.

June 24, 1987

The problem: We have a complex symplectic space V with conjugation $*$ and the assoc. Heisenberg repn. $\mathcal{F}$. We have another conjugation σ on V and choose T symplectic such that $T * T^{-1} = \sigma$. We suppose T implemented by $K \in \mathcal{F} \otimes \mathcal{F}^*$:

$$T(v) K = K v$$

We've seen this is equivalent to the positivity condition

$$[v, v^*] > [v, \sigma v] \qquad \text{for} \qquad v \neq 0.$$

Then $\quad K^* T(v^*)^* = v K^* \quad$ so K^* implements $(*T*)^{-1}$, hence KK^* implements $T * T^{-1} * = \sigma *$:

$$\sigma(v^*)(KK^*) = T(* T^{-1} * v) KK^*$$

$$= K (T^{-1}(v^*))^* K^* = (KK^*) v$$

Now we can extract a positive square root of KK^* and the problem is whether this square root corresponds to a symplectic transf.

In matrix notation T becomes g and $* T^{-1} *$ is $\bar{g}^{-1}$. Thus KK^* implements $g\bar{g}^{-1} = h$, and h satisfies $h = \bar{h}^{-1}$ which corresponds to KK^* being hermitian. So what corresponds to $KK^* > 0$?

Now one way to get transformations h satisfying $h = \bar{h}^{-1}$ is by

$$h = e^{iJ^{-1}A} \qquad A \text{ real symmetric}$$

What conditions on h guarantee that it is implemented by a positive operator?

I want to analyze the positivity condition on p. 918:

$$(iJ)(1-h) > 0$$

for $h \in Sp(2n, \mathbb{C})$ satisfying $\bar{h} = h^{-1}$. (Recall such h are in 1-1 correspondence with conjugations σ which are compatible with the symplectic structure – the corresp. is $\sigma(v) = h\bar{v}$.) I want to do this analysis in the 2×2 case.

$$h = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \qquad h^{-1} = \begin{pmatrix} \delta & -\beta \\ -\gamma & \alpha \end{pmatrix} = \begin{pmatrix} \bar\alpha & \bar\beta \\ \bar\gamma & \bar\delta \end{pmatrix}$$

$$\iff \quad \delta = \bar\alpha \qquad \bar\beta = -\beta, \quad \bar\gamma = -\gamma \qquad \text{and so}$$

$$h = \begin{pmatrix} \alpha & -ib \\ ic & \bar\alpha \end{pmatrix} \qquad \text{with } b, c \text{ real.}$$

$$\det(h) = |\alpha|^2 - bc = 1$$

Since $\mathrm{tr}(h)$ is real, the eigenvalues of h are mutually inverse real numbers, or conjugate points on the unit circle.

Positivity condition

$$(iJ)(1-h) = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}\begin{pmatrix} 1-\alpha & +ib \\ -ic & 1-\bar\alpha \end{pmatrix} = \begin{pmatrix} c & i(1-\bar\alpha) \\ -i(1-\alpha) & +b \end{pmatrix} > 0$$

Thus $\boxed{c > 0, \quad b > 0}$, and $\underbrace{+cb}_{|\alpha|^2-1} - |1-\alpha|^2 > 0$ or

$$|1-\alpha|^2 + 1 - |\alpha|^2 = 2 - \alpha - \bar\alpha < 0 \quad \text{or}$$

$$\boxed{\mathrm{Re}(\alpha) > 1}$$

Thus the eigenvalues of h are mutually inverse positive real numbers $\neq 1$.

Thus we know that there is a unique square root $h^{1/2}$ having positive eigenvalues; here I am taking $h^{1/2}$ in $GL(2n, \mathbb{C})$, but one can then see $h^{1/2}$ is symplectic because both $J h^{1/2} J^{-1}$ and $((h^{1/2})^{-1})^t$ will have the same square and positive eigenvalues. Similarly $\overline{h^{1/2}} = (h^{1/2})^{-1}$. And we can continue to get a whole 1-parameter group h^t, $t \in \mathbb{R}$.

Now $h^t = e^{tX}$ for [redacted] a unique $X \in sp(2n, \mathbb{C})$, and [redacted] X has to have the form

$$X = iJ^{-1}A \qquad A \text{ real + symmetric.}$$

So if $A = \begin{pmatrix} l & m \\ m & n \end{pmatrix}$, then

$$X = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\begin{pmatrix} l & m \\ m & n \end{pmatrix} = \begin{pmatrix} -im & -in \\ il & im \end{pmatrix}.$$

?(Assuming that h^t also satisfies the positivity condition for small $t > 0$, which seems reasonable) we see that $l, n > 0$ as $e^{tX} = 1 + tX$.
Furthermore the eigenvalues of X are necessarily real and of opposite sign so that

$$\det(X) = m^2 - ln < 0$$

hence we see $A > 0$.

One of the reasons I am having difficulties with these issues of positivity is that I don't have a feeling for the eigenvalues of $X = J^{-1}A$ where A is real symmetric. This is a typical element of the real symplectic Lie algebra $\underline{sp}(2n, \mathbb{R})$, and I am asking about the orbits in the adjoint representation.

Notice first of all that if g is symplectic then $\qquad g^t J g = J$

$$g^{-1}(J^{-1}A)g = \underbrace{(Jg)}^{-1}Ag = \left((g^t)^{-1}J\right)^{-1}Ag$$

$$= J^{-1}(g^t A g)$$

so that the adjoint action can be identified with the natural action of invertible matrices on quadratic forms, but restricted to symplectic matrices. The orbits of $GL(n,\mathbb{R})$ on real symmetric matrices are given by the signature, a certain number of $+$'s, $-$'s and 0's.

Now I might hope that the eigenvalues would specify the orbits of semi-simple elements. The eigenvalues of $X = J^{-1}A$ are stable under $\lambda \mapsto -\lambda$ and $\lambda \mapsto \bar{\lambda}$ and probably one can expect the $\underline{sl}(2,\mathbb{R})$ situation to explain the general case for eigenvalues and the semi-simple elements.

X is an arbitrary real 2x2 matrix of trace 0 being acted on by real matrices of determinant 1. The det of X determines the eigenvalues: $\lambda = \pm\sqrt{\det(X)}$

As long as $\det(X) \neq 0$ the eigenvalues are distinct, so X is ~~semi~~-simple. There are two cases: $\det(X) > 0$, whence the eigenvalues are purely imaginary and $\det(X) < 0$ whence the eigenvalues are real.

Instead of trying to classify orbits of the symplectic group on symmetric matrices, ~~say invertible symmetric matrices~~ one can consider orbits of the general linear group on pairs (J, A) where J is a non-degenerate skew form. These give the same orbit space.

So if we look at the A which are positive definite and hence form a single orbit under GL, the classification becomes that of invertible skew-symmetric operators under the orthogonal group, which means looking at the adjoint representation of the orthogonal group. Thus everything will break into little 2×2 blocks and the situation is completely semi-simple.

June 25, 1987

Conventions: Let V_r be a real symplectic space with skew form $S(x,y)$. Then we have the Heisenberg representation $\mathcal{F}$ where to each $x \in V_r$ there is an operator $\rho(x)$ ∋

$$\rho(x)^* = \rho(x) \qquad [\rho(x),\rho(y)] = iS(x,y)$$

Suppose we have a polarization of V_r. From the operator viewpoint this can be viewed as a splitting of the complexification $V_c = V_r \otimes \mathbb{C}$ into

$$V_c = W \oplus \overline{W}$$

such that
$$[w, w'] = 0$$
$$[w, w^*] > 0 \qquad w \neq 0$$

and we know there is a unique line $\mathbb{C}|0\rangle$ in $\mathcal{F}$ killed by W. Then

$$\langle 0 | e^{i\rho(x)} | 0 \rangle = e^{-Q(x)}$$

where Q is a positive definite quadratic form on V_r. Thus a polarization determines an inner product on V_r.

Set $J = -i$ on W and i on $\overline{W}$. Then J is real, i.e. preserves V_r, and it is also symplectic. $S(Jx, Jy) = S(x,y)$. We calculate $S(x, Jy)$ for $x = w_1 + w_1^*$, $y = w_2 + w_2^*$ in V_r

$$S(x, Jy) = \frac{1}{i}[w_1 + w_1^*, -iw_2 + iw_2^*]$$
$$= [w_1, w_2^*] + [w_2, w_1^*]$$

This is symmetric and positive definite. Moreover

$$e^{i\varphi(x)} = e^{i\omega^* + i\omega} = e^{-\frac{1}{2}[i\omega^*, i\omega]} e^{i\omega^*} e^{i\omega}$$

so
$$\langle 0 | e^{i\varphi(x)} | 0 \rangle = e^{-\frac{1}{2}[\omega, \omega^*]} = e^{-\frac{1}{4}S(x, Jx)}$$

Thus we can define a polarization of (V_r, S) to be a positive definite form $B(x, y)$ on V_r such that if we let J be defined by

$$B(Jx, y) = S(x, y)$$

then $J^2 = -1$. We then have

$$\langle 0 | e^{i\varphi(x)} | 0 \rangle = e^{-\frac{1}{4}B(x, x)}$$

Next I want to discuss a general quasi-free state on the Weyl algebra of V_r.

<u>June 26, 1987</u>

Let V be a real symplectic vector space with skew-form $S(x,y)$, let its Weyl algebra be defined so that $[x,y] = iS(x,y)$. By a <u>quasi-free state</u> one means a state on the Weyl algebra whose generating function is Gaussian:

$$\langle e^{ix}\rangle = e^{-\frac{1}{4}|x|^2}$$

superseded by July 1

where $|x|^2$ is an inner product on V. Here $\langle A\rangle$ is $\langle 0|A|0\rangle$ where $|0\rangle$ is the distinguished cyclic vector in the GNS representation.

Let (x,y) denote the inner product on V, and let K be the skew-symmetric operator on V such that

$$S(x,y) = (x, Ky)$$

I claim that the quadratic form has to satisfy the condition that K be a contraction

$$|Kx| \leq |x| \qquad \text{or} \qquad 1+K^2 \geq 0.$$

To see this consider the complex vector space L consisting of the vectors

$$(x+iy)|0\rangle \qquad\qquad x,y \in V.$$

One has from $\langle e^{ix}\rangle = e^{-\frac{1}{4}|x|^2}$ the formula

$$\langle x^2\rangle = \frac{1}{2}|x|^2.$$

See July 1

So

$$0 \leq \|(x+iy)|0\rangle\|^2 = \langle x^2\rangle + \langle y^2\rangle + \langle 0|x\, iy|0\rangle$$
$$+ \langle 0|(-iy)\,x|0\rangle$$

$$= \tfrac{1}{2}|x|^2 + \tfrac{1}{2}|y|^2 + i\langle xy - yx\rangle$$

$$= \tfrac{1}{2}|x|^2 + \tfrac{1}{2}|y|^2 - \underbrace{S(x,y)}_{(x,\,Ky)}$$

$$= \tfrac{1}{2}\left\{|x-Ky|^2 + |y|^2 - |Ky|^2\right\}$$

proving that $|Ky| \le |y|$ as claimed.

Conversely given $(\,,\,)$ on V such that [redacted] K is a [redacted] contraction we define [redacted] a (possibly degenerate) non-negative hermitian form on the complexification $V_c = V + iV$ by

$$\|x+iy\|^2 = |x|^2 + |y|^2 - 2(x, Ky)$$

$$= |x - Ky|^2 + |y|^2 - |Ky|^2.$$

[redacted] Note that this norm is compatible with mult. by i:

$$\|-y+ix\|^2 = |y|^2 + |x|^2 - 2\underbrace{(-y, Kx)}_{(Ky,\,x)}.$$

[redacted]

Hence it extends to a hermitian form ≥ 0. Let L [redacted] be the quotient by the subspace of elements of norm zero. [redacted]

Notice that $\|x\|^2 = |x|^2$, so that we have an isometric embedding [redacted] of $V \hookrightarrow L$, and that

$$\|x+iy\|^2 = |x|^2 + |y|^2 - 2(x, Ky)$$

$$\|$$

$$|x|^2 + |y|^2 + \underbrace{\langle x|iy\rangle + \langle iy|x\rangle}_{2\,\mathrm{Re}\,i\langle x|y\rangle} = |x|^2 + |y|^2 - 2\,\mathrm{Im}\,\langle x|y\rangle$$

so that

$$\operatorname{Im} \langle x | y \rangle = (x, Ky) = S(x,y)$$

Conclusion: When $(\ ,\ )$ and S are such that K is a contraction, then there is an essentially unique map from V to a complex space with inner product $\langle\ |\ \rangle$ such that

$$\langle x | y \rangle = (x,y) + i S(x,y) \qquad x,y \in V$$

Check: Here $\langle x | y \rangle = 2 \langle 0 | x^* y | 0 \rangle$ for $x, y \in V_c$, where the 2 comes from

$$\langle 0 | x^2 | 0 \rangle = \tfrac{1}{2} |x|^2$$

encountered before. Thus for x,y real

$$\operatorname{Im} \langle x | y \rangle = \frac{1}{2i} \left(\langle x | y \rangle - \langle y | x \rangle \right)$$

$$= \frac{1}{2i}\, 2 \langle 0 | [x,y] | 0 \rangle = \frac{1}{i} [x,y] = S(x,y)$$

June 27, 1987

V = real symplectic vector space with skew form $S(x,y)$; its Weyl algebra is defined so that

$$[x,y] = iS(x,y)$$

or

$$e^{ix}e^{iy} = e^{-\frac{1}{2}[x,y]}e^{i(x+y)}$$

$$= e^{-\frac{1}{2}iS(x,y)}e^{i(x+y)}$$

Definition: A state φ on the Weyl algebra is called <u>quasi-free</u> if

$$\varphi(e^{ix}) = e^{-\frac{1}{4}|x|^2}$$

where $|x|^2 = (x,x)$ and (x,y) is a positive-def. inner product on V.

~~The simplest examples are given by polarizations.~~

~~Def. A polarization on V is a complex structure i.e. an operator J satisfying $J^2 = -1$, such that~~

~~$S(Jx,x) > 0 \quad x \neq 0$.~~

By the GNS construction there is a unique (up to canonical isomorphism) unitary representation of the Weyl algebra with cyclic vector $|0\rangle$ such that

$$\varphi(e^{ix}) = \langle 0|e^{ix}|0\rangle.$$

One obtains a hermitian form on the complexification V_c of V by

$$H(v,v') = 2\langle 0|v^*v'|0\rangle.$$

This form is non-negative: $H(v,v) \geq 0$ and is [redacted] the obvious extension to V_c of the form on $\hat{V}$ given by

$$H(x,y) = 2\langle 0|xy|0\rangle.$$

Lemma: $\boxed{H(x,y) = (x,y) + i\,S(x,y)}$

Proof: $\operatorname{Im} H(x,y) = \frac{1}{i}\left(\langle 0|xy|0\rangle - \langle 0|yx|0\rangle\right)$

$$= \langle 0|\underbrace{\tfrac{1}{i}[x,y]}_{S(x,y)}|0\rangle = S(x,y).$$

Comparing second degree components of

$$\langle 0|e^{ix}|0\rangle = e^{-\frac{1}{4}|x|^2}$$

yields

$$\langle 0|\frac{(ix)^2}{2}|0\rangle = -\frac{1}{4}|x|^2 \quad \text{so}$$

$$H(x,x) = 2\langle 0|x^2|0\rangle = |x^2|$$

whence $\underline{\operatorname{Re} H(x,y) = (x,y).}$ done.

Now define K on V by

$$\boxed{S(x,y) = (Kx,y)}$$

Then 1) $(Kx,y) + (x,Ky) = 0 \implies K = -K^t$ relative to $(\,,\,)$

2) $S(Kx,y) = (KKx,y) = -(Kx,Ky) = -S(x,Ky)$

$\implies K$ infinitesimally symplectic

3) $S(x,Kx) = (Kx,Kx) > 0 \qquad x \neq 0.$

Lemma: $\boxed{(x,iy) = (x,Ky)}$ for $x,y \in V$

Proof: $(x,iy) = \operatorname{Re} H(x,iy) = -\operatorname{Im} H(x,y)$

$$= -S(x,y) = -(Kx,y) = (x,Ky).$$

<u>Lemma</u>: K is a contraction: $|Ky| \le |y|$.

Proof.

$$0 \le H(x+iy,\, x+iy) = |x|^2 + |y|^2 + 2\,\mathrm{Re}\,\underbrace{H(x,iy)}$$

$(x, iy) = (x, Ky)$

$$= |x + Ky|^2 + |y|^2 - |Ky|^2.$$

Now set $x = -Ky$

If $|Kx| = |x|$ for all y, then

$$(x,x) = (Kx, Kx) = (x, (-K^2)x)$$

and as $-K^2$ is symmetric this implies $K^2 = -I$. In this case K is a polarization of V in the following sense

<u>Def:</u> A <u>polarization</u> of a real symplectic space V, S is an operator J on V satisfying

1) $J^2 = -I$ so J defines a complex structure
2) $S(Jx, y) + S(x, Jy) = 0$, so J inf. symp.
(note that with 1) we have
$$S(Jx, Jy) = S(x, -J^2 y) = S(x,y)$$
so J is also symplectic.)
3) $S(x, Jx) > 0$ for $x \ne 0$.

Given such an operator we can define

$$H(x,y) = S(x, Jy) + i\,S(x,y)$$

Then
$$\begin{aligned} H(y,x) &= S(y, Jx) + i\,S(y,x) \\ &= S(-Jy, x) - i\,S(x,y) \\ &= S(x, Jy) - i\,S(x,y) = \overline{H(x,y)} \end{aligned}$$

$$H(x, Jy) = S(x,-y) + i\,S(x,Jy)$$
$$= i\left[S(x,Jy) + i\,S(x,y)\right] = i\,H(x,y)$$

so H is a hermitian form which is positive by 3).

When J is a polarization of V, the complexification splits into $V_c = W^{10} \oplus W^{01}$ where $J = -i$ on W^{10} and $+i$ on W^{01}. Also $\overline{W^{10}} = W^{01}$.

Lemma: Let S be extended $\mathbb{C}$ linearly to V_c. Then W^{10} and W^{01} are isotropic and

$$iS(w,\bar{w}) > 0 \qquad \text{for} \qquad 0 \neq w \in W^{10}.$$

Proof: If $w_1, w_2 \in W^{10}$, then

$$S(w_1,w_2) = S(Jw_1, Jw_2) = (-i)^2 S(w_1,w_2)$$

so this vanishes. If $w \in W^{10}$ and $w \neq 0$, then $x = w + \bar{w} \in V = V_R$ and

$$0 < S(x,Jx) = S(w+\bar{w}, -iw + i\bar{w})$$
$$= 2i\,S(w,\bar{w})$$

as claimed.

At this point we have to give an explicit construction of a representation of the Weyl algebra of V associated to a polarization. Thus there is a unique Hilbert space representation of the Weyl algebra having a cyclic vector $|0\rangle$ killed by the operators associated to elements of W^{10}. To each element v of V_c is an operator $\rho(v)$ depending $\mathbb{C}$-linearly on v such that

$$[\rho(v), \rho(v')] = i\, S(v, v')$$

$$\rho(v)^* = \rho(\bar{v})$$

$$\rho(w)|0\rangle = 0 \qquad \text{for } w \in W = W^{10}.$$

The Hilbert space is the symmetric tensor space in the Hilbert space sense of $W^{01} = \bar{W} \simeq W^*$ and

$$\rho(w) = \text{differentiation } \partial_w \qquad w \in W$$

$$\rho(\bar{w}) = \text{multiplication by } \langle w|$$

Denote these respectively by a_w, a_w^*. If $x = w \oplus \bar{w} \in V_r$, then

$$e^{ix} = e^{i(a_w^* + a_w)} = e^{-\frac{1}{2}[ia_w^*, ia_w]} e^{ia_w^*} e^{ia_w}$$

$$= e^{-\frac{1}{2}[a_w, a_w^*]} e^{ia_w^*} e^{ia_w}$$

Now

$$[a_w, a_w^*] = i\, S(w, \bar{w}) = \frac{1}{2} \underbrace{S(x, Jx)}_{= H(x,x) = |x|^2.}$$

by the previous page.

Thus

$$\langle 0| e^{ix} |0\rangle = e^{-\frac{1}{4}|x|^2}$$

This proves (See July)

<u>Prop</u>: If J is a polarization of V and $|x|^2 = S(x, Jx)$ is the associated norm, then there is a <u>state</u> on the Weyl algebra such that

$$\varphi(e^{ix}) = e^{-\frac{1}{4}|x|^2}$$

We can now present the following

Classification of quasi-free states on Weyl(V).

Theorem: If $|x|^2 = (x,x)$ with (x,y) an inner product on V, then there is a state on Weyl(V) with $\varphi(e^{ix}) = e^{-\frac{1}{4}|x|^2}$ iff the operator K defined by

$$S(x,y) = (Kx, y)$$

is a contraction.

<u>Proof</u>: Necessity has been proved on 931.

Sufficiency: Define a hermitian form on the complexification V_c by

$$H(x+iy, x+iy) = |x|^2 + |y|^2 + 2(x, Ky)$$

$$= |x+Ky|^2 + |y|^2 - |Ky|^2$$

since K is skew-symmetric $H(iv, iv) = H(v,v)$ and so this extends to a ~~skew-symmetric~~ hermitian form. Because K is a contraction $H \geq 0$, and so we may divide out by the space Z of elements $v \in V_c$ such that $H(v,v) = 0$ and obtain a complex vector space V_c/Z with inner product. By construction we have a map $V \longrightarrow V_c/Z$ such that

$$H(x,y) = (x,y) + i(Kx,y)$$

In effect if we extend this to be sesqui-linear we obtain

$$H(x, iy) = -(Kx, y) + i(x,y)$$
$$H(iy, x) = (Ky, x) - i(x,y)$$

so $H(x+iy, x+iy) = |x|^2 + |y|^2 + 2(x, Ky)$

as above.

But we know that the Weyl algebra of V_c/Z equipped with $\operatorname{Im} H$ has the quasifree state associated to the quadratic form $\operatorname{Re} H$, so by restriction the Weyl alg of V for $(Kx,y) = S(x,y)$ has the quasi-free state given by the [redacted] inner product $|x|^2$.

Now I would like to return to the case where I have two conjugations. Let's go back to [redacted] $V_\sigma \subset V_c \supset V_r$, where we have the positivity condition $[v, v^*] > [v, \sigma v]$. Choosing a symplectic basis for V_r we have

$$S(v, v') = v^t J v' \qquad J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

$$*(v) = \bar{v}$$

and $$\sigma(v) = h\bar{v}$$ where h satisfies

$$h\bar{h} = 1$$
$$h^t J h = J$$

Let's [redacted] consider the $\dim V_r = 2$ case, whence

$$h = \begin{pmatrix} a & -ic \\ ib & \bar{a} \end{pmatrix} \qquad \begin{array}{l} b, c \text{ real} \\ |a|^2 - bc = 1 \end{array}$$

and the positivity condition [redacted] says
$$b, c > 0 \quad \text{and} \quad \operatorname{Re}(a) > 1$$

which implies the eigenvalues are positive and <u>mutually</u> inverse.

General remark: Suppose h is a symplectic transformation. Then the λ-eigenspace pairs trivially with the μ-eigenspace for $\lambda\mu \neq 1$, since if $hv = \lambda v$ and $hv' = \mu v'$, then $S(v, v') = S(hv, hv') = \lambda\mu\, S(v, v')$.

Similarly if none of the eigenvalues of h lie on the unit circle, then we get a splitting $V = W' \oplus W''$ stable under h where W' is the sum of the generalized eigenspaces of h corresponding to eigenvalues $|\lambda| < 1$ and W'' is similar but for $|\lambda| > 1$. Then we claim that W' and W'' are isotropic.

In effect if

$$p' = \frac{1}{2\pi i} \int_{|\lambda|=1} \frac{1}{\lambda - h}\, d\lambda$$

is the projection on W', then clearly

$$h^n p' \longrightarrow 0 \qquad \text{as} \qquad n \longrightarrow +\infty$$

so that for $w_1, w_2 \in W'$ we have

$$S(w_1, w_2) = S(h^n w_1, h^n w_2) \longrightarrow 0.$$

and similarly for W''.

Next let $hv = \lambda v$ and apply $*$.

$$h^{-}\bar{v} = \overline{hv} = \bar{\lambda}\bar{v} \quad \Longrightarrow \quad h\bar{v} = (\bar{\lambda})^{-1}\bar{v}$$

so if $|\lambda| = 1$ one sees that the λ eigenspace is closed under $*$, i.e. defined over $\mathbb{R}$. Geometrically

This is the case where the σ and $*$ circles intersect.

June 28, 1987

Let's consider $V_{\mathbb{R}} \subset V_c$ as usual and σ as conjugation which is positive:

$$[v, \sigma *] > [v, \sigma v] \qquad \text{for } v \neq 0.$$

Put $h = \sigma *$; it is a symplectic ($\mathbb{C}$ linear) transf. of V_c which is inverted by $*$. We have

$$[*v, v] > [*v, \underbrace{\sigma *}_{h} v]$$

or

$$[v, *v] < [hv, *v]$$

Let v be an eigenvector for h:

$$hv = \lambda v$$

Then

$$[v, *v] < \lambda [v, *v]$$

This implies $[v, *v] \neq 0$ and that

$$\lambda > 1 \quad \text{if} \quad [v, *v] > 0$$

$$0 < \lambda < 1 \quad \text{if} \quad [v, *v] < 0.$$

From yesterday we know that we have a splitting

$$V_c = W \oplus \overline{W}$$

into isotropic subspaces where W is the generalized eigenspace where the eigenvalues are > 1, and $\overline{W}$ where the eigenvalues are < 1. This is a polarization since $[w, *w] = 0$ on W.

Since h is symplectic

$$[h\omega_1, \underbrace{h * \omega_2}_{* h^{-1}}] = [\omega_1, * \omega_2]$$

Thus with respect to the inner product $\underbrace{[\omega_1, * \omega_2]}_{\langle \omega_2 | \omega_1 \rangle}$ on W we have that

$$\langle \omega_2 | \omega_1 \rangle = \langle h^{-1}\omega_2 \,|\, h\omega_1 \rangle$$

or $\quad h = h^{*}$ adjoint. Thus h is a hermitian operator on W such that $h > 1$, and as $* h * = h^{-1}$, we have that h on $\overline{W}$ is the conjugate of h^{-1}.

At this point I have control over a conjugation satisfying the positivity condition. Namely, I know there is a polarization W and a hermitian $h > 1$ on W such that σ is an obvious shift of $*$ by h.

July 1, 1987

New conventions: If V, S is a real symplectic vector space define the Weyl algebra by
$$x^* = x$$
$$[x,y] = 2i\,S(x,y)$$
or in the Weyl form: e^{ix} unitary
$$e^{ix}e^{iy} = e^{-iS(x,y)}\,e^{i(x+y)}.$$

Define the quasi-free state associated to a inner product (x,y) on V by
$$\langle e^{ix}\rangle = e^{-\frac{1}{2}|x|^2}.$$

Thus
$$\boxed{\langle x^2\rangle = |x|^2}$$

and
$$\langle xy\rangle + \langle yx\rangle = |x+y|^2 - |x|^2 - |y|^2$$
$$= 2(x,y)$$
$$\langle x,y\rangle - \langle y,x\rangle = 2i\,S(x,y)$$

$$\cancel{\langle xy\rangle = (x,y) + iS(x,y) = H(x,y)}$$
$$\cancel{\langle \omega^*\omega\rangle = H(\omega,\omega) \qquad \omega,\omega \in V_c}$$

$$\boxed{\langle xy\rangle = (x,y) + i\,S(x,y)}$$

In the case of a polarization
$$\langle (\omega_1+\omega_1^*)(\omega_2+\omega_2^*)\rangle = \langle \omega_1\omega_2^*\rangle = \langle[\omega_1,\omega_2^*]\rangle$$
$$= [\omega_1, \omega_2^*]$$
Thus we see the hermitian inner product on V

is the one I am used to using on W^*.

The main disadvantage with these conventions is that when we come to $\mathbb{R}q + \mathbb{R}p$ we must define $S(q,p) = \frac{1}{2}$.

July 2, 1987

Conventions. Let V be a real vector space with skew form $S(x,y)$ and inner product (x,y). Suppose we have a representation of the CCR on a Hilbert space:

$$x^* = x \qquad \boxed{[x,y] = 2iS(x,y)}$$

or in Weyl form

$$e^{ix} \text{ unitary} \qquad e^{ix}e^{iy} = e^{-iS(x,y)}\,e^{i(x+y)}$$

and a vector $|0\rangle$ in this Hilbert space such that

$$\boxed{\langle 0|e^{ix}|0\rangle = e^{-\frac{1}{2}|x|^2}} \qquad |x|^2 = (x,x).$$

which implies $\langle 0|x^2|0\rangle = |x|^2$. Let

$$W' = \{(x+iy)|0\rangle \mid x,y \in V\}$$

It is a ~~quotient~~ complex vector space which is a quotient of $V_c = V \otimes \mathbb{C}$ and which has a hermitian inner product. There is an embedding

$$V \xrightarrow{j} W' \qquad x \longmapsto x|0\rangle$$

which is isometric, ~~Let K be the operator on V which is the compression of mult. by i on W':~~

$$\cancel{(x, Ky) = \langle 0|x\,iy|0\rangle = i\langle xy\rangle}$$

$$\cancel{= -}$$

as $\boxed{\langle x^2 \rangle = |x|^2}$ where $\langle \alpha \rangle = \langle 0|\alpha|0\rangle$

Then
$$\langle xy \rangle + \langle yx \rangle = \langle (x+y)^2 - x^2 - y^2 \rangle$$
$$= 2(x,y)$$
$$\langle xy \rangle - \langle yx \rangle = \langle [x,y] \rangle = 2iS(x,y)$$
so the hermitian form on W' restricted to V is
$$\boxed{\langle xy \rangle = (x,y) + iS(x,y)}$$

Now let K be the operator on V which is the compression of the operator of multiplication by i on W'. This means that we ~~use the~~ view W' as a real Hilbert space with inner product the real part of the hermitian inner products. Thus K is defined by
$$(x, Ky) = \operatorname{Re}\langle x iy \rangle = \operatorname{Re} i\langle xy \rangle$$
$$= -S(x,y)$$
or
$$\boxed{(Kx,y) = S(x,y)}$$
which shows that K is skew-symmetric.

Note that W' is obtained by dividing V_c by the kernel of the semi-norm
$$\langle (x+iy)^2 \rangle = |x|^2 + |y|^2 + 2\operatorname{Re}\langle x iy \rangle$$
$$= |x|^2 + |y|^2 + 2(x, Ky)$$

<u>Example</u>: Let W be complex with herm. inner product $(w|w')$, and consider the Hilbert space of holomorphic functions $f(w)$ on W such that
$$\|f\|^2 = \int e^{-|w|^2} |f(w)|^2 \frac{d^{2n}w}{\pi^n} < \infty$$
with the operators $a_w = \partial_{\bar{w}}$, $a_w^* =$ mult by $(w|$.
Let $V_{\mathbb{R}} = \{ a_w + a_w^* \mid w \in W \}$. Then we have a representation of $V_{\mathbb{R}}$ by hermitian ops satisfying
$$[a_w + a_w^*, a_{w'} + a_{w'}^*] = -2i \, \mathrm{Im}(w|w')$$
together with $|0\rangle =$ the fn. 1 satisfying
$$\langle 0| e^{i(a_w + a_w^*)} |0\rangle = e^{-\frac{1}{2}|w|^2}$$
In this case $(a_w + a_w^*)|0\rangle = a_w^*|0\rangle$ so W' is identified with the dual space W^* or conjugate space $\overline{W}$.

<u>July 3, 1987</u>

Yesterday I thought of trying to set things up in such a way as not to fix a real structure. So instead of looking for $\langle \psi | e^v | \psi \rangle$, where $\psi = |\Gamma\rangle$, the unique state fixed under the self-dual positive lattice Γ, I should look for

$$\langle \Gamma' | e^v | \Gamma \rangle$$

where Γ' is negative and Γ is positive relative to $V_\mathbb{R}$. This function on V_c should be independent of the real structure between Γ' and Γ. In the same vein I can replace Γ by a maximal isotropic subspace W.

Let's first look at two complementary isotropic subspaces: $V = W' \oplus W''$. Choose a real structure $*$ on V such that W' is positive and W'' is negative, and then define the states $|W'\rangle$ and $\langle W''|$ to be killed by the corresponding operators. Then given $v \in V$, write it $v = w' + w''$ and we have

$$\langle W'' | e^v | W' \rangle = \langle W'' | e^{-\frac{1}{2}[w'',w']} e^{w''} e^{w'} | W' \rangle$$

$$= e^{-\frac{1}{2}[w'',w']} \langle W'' | W' \rangle$$

This shows that this function on V is independent (up to normalization) of $*$.

The problem is still to pin down the state corresponding to a self-dual positive lattice. The idea is to use the Poincaré series

$$\sum_{\gamma \in \Gamma} e^{\gamma} \alpha$$

with a suitable ~~state~~ α to construct the ~~state~~ $|\Gamma\rangle$ fixed under e^{Γ}.

<u>Lemma</u>: Let f_i $i \in I$ be a family of elements in a Hilbert space such that

$$\sum_{(i,j) \in I \times I} |(f_i | f_j)| < \infty$$

Then as S runs over the finite subsets of I,

$$\lim_{S} \sum_{i \in S} f_i \qquad \text{exists.}$$

<u>Proof</u>: Here the limit is in the sense of directed systems. By Cauchy's criterion it suffices to show given $\varepsilon > 0$ there is a finite subset S_0 such that for all S disjoint from S_0 we have

$$\underbrace{\left\| \sum_{i \in S} f_i \right\|^2}_{} < \varepsilon$$

$$0 \leq \sum_{i,j \in S \times S} (f_i | f_j) \leq \sum_{i,j \in S \times S} |(f_i | f_j)|$$

But this is clear.

Now we can take α to be $|W\rangle$ for W positive isotropic whence

$$\left(e^{\gamma'}|W\rangle \,\middle|\, e^{\gamma}|W\rangle \right)$$

will be a Gaussian function of γ and γ'. Thus one might be able to verify the condition of the lemma and conclude that

$$\sum_{\gamma \in \Gamma} e^{\gamma}|W\rangle$$

converges.

However it seems harder to use self-duality to establish the uniqueness.

It seems to be useful to view a self-dual lattice as an analogue of a Lagrangian subspace. Evidently one thereby obtains a description of the Heisenberg representation as sections of a line bundle over the torus. This is formally analogous to describing the Heisenberg representation as functions on the maximal isotropic subspace, or really the quotient by the maximal isotropic subspace, (which is again the dual group).

As in Vergne's book with Lion there are various transformation formulas between the different representations.

Let's discuss some examples.

Let Γ be a self-dual lattice in $\mathbb{C}$, and put $T_\gamma = e^{\bar\gamma a^* - \gamma a} = e^{-\frac{1}{2}|\gamma|^2} e^{\bar\gamma a^*} e^{-\gamma a}$.

We consider the Poincaré operator:

$$f \longmapsto \sum_{\gamma\in\Gamma}{}' T_\gamma f$$

which produces a Γ-invariant when it converges. It doesn't converge in the actual Hilbert space but does in some distributional sense. Thus if we take $f = |0\rangle$ then we can pair with coherent states:

$$\left(\sum T_\gamma|0\rangle\right)(z) = \langle 0| e^{za} \sum_\gamma e^{\bar\gamma a^* - \gamma a}|0\rangle = \sum_\gamma e^{-\frac{1}{2}|\gamma|^2 + \bar\gamma z}$$

Similarly if we take $f = e^{\lambda a^*}|0\rangle$ ~~then~~ we get

$$\langle 0|e^{za} \sum_\gamma e^{\bar\gamma a^* - \gamma a} e^{\lambda a^*}|0\rangle$$

$$= \sum_\gamma e^{-\frac{1}{2}|\gamma|^2 + \bar\gamma z + \lambda(z-\gamma)}$$

However we know these two functions of z are the same except for a scalar factor depending on λ. But this is not obvious from this viewpoint.

It seems we have the following identity

$$\sum_\gamma e^{-\frac{1}{2}|\gamma|^2 + \bar\gamma z + \lambda(z-\gamma)} = \frac{\sum e^{-\frac{1}{2}|\gamma|^2 - \lambda\gamma}}{\sum e^{-\frac{1}{2}|\gamma|^2}} \sum e^{-\frac{1}{2}|\gamma|^2 + \bar\gamma z}$$

which isn't obvious.

July 4, 1987

Discuss different kinds of Θ functions. Basically a Θ function is associated to a lattice in a complex vector space equipped with inner product.

Let's begin with Riemann's Θ-function

$$\Theta(z|\tau) = \sum_{n\in\mathbb{Z}^g} e^{i\pi n^t\tau n + 2\pi i n^t z}$$

which is an entire function of $z \in \mathbb{C}^g$. Here τ is a symmetric complex matrix with pos definite imaginary part. $\Theta(z|\tau)$ is periodic with periods $\mathbb{Z}^g$ and quasi-periodic with quasi-periods $\tau\mathbb{Z}^g$, since

$$\Theta(z+\tau m|\tau) = \sum_n e^{i\pi[n^t\tau n + 2n^t z + 2n^t\tau m]}$$

$$= \sum_n e^{i\pi[\underbrace{(n-m)^t\tau(n-m)}_{n^t\tau n - \cancel{2n^t\tau m} + m^t\tau m} + 2(n-m)^t z + 2\cancel{(n-m)^t}\tau m]}$$

$$= e^{-i\pi(m^t\tau m + 2m^t z)} \underbrace{\sum_n e^{i\pi[n^t\tau n + 2n^t z]}}_{\Theta(z|\tau)}$$

In order to obtain this Θ function one must have expressed the lattice as a sum of complementary isotropic sublattices.

The Riemann Θ arises as follows. One is given a Hodge structure situation such as the following in the case of a Riemann surface X:

$$H^1(X,\mathbb{Z}) \subset H^1(X,\mathbb{R}) \subset H^1(X,\mathbb{C}) = H^{10} \oplus H^{01}$$

Thus one is given a real symplectic vector space $V_{\mathbb{R}}$ with a self dual lattice Γ and a polarization $V_{\mathbb{C}} = W \oplus \overline{W}$. Then one chooses a symplectic basis for Γ, whence the polarization will be given by a τ in the Siegel UHP.

Now I think there ought to be something more intrinsic. Ideally we would like a periodic entire function which is impossible. One way to obtain an intrinsic function is as follows. Working in the holomorphic function representation associated to the polarization, one forms the function

$$\sum_{\gamma \in \Gamma} T_\gamma |0\rangle$$

where say $\quad (T_\gamma f)(z) = e^{-\frac{1}{2}|\gamma|^2 + \bar{\gamma} z} f(z-\gamma)$

Thus the function in question is

$$\theta(z) = \sum_{\gamma \in \Gamma} e^{-\frac{1}{2}|\gamma|^2 + \bar{\gamma} z}$$

and it satisfies the quasi-periodicity condition

$$\theta(z) = e^{-\frac{1}{2}|\gamma|^2 + \bar{\gamma} z}\, \theta(z-\gamma)$$

A natural question or problem is ~~to~~ relate these different kinds of θ-functions. Now I feel that both result from a choice of trivialization on "the" line bundle over $V_{\mathbb{R}}$ whose curvature is the symplectic form. Such a bundle with connection ~~[illegible]~~ has a natural holomorphic structure, and the Heisenberg representation is just the space of holomorphic sections of the line bundle. The trivializations are such that 1 is a holomorphic section, hence the connection form is of type (1,0). We've [redacted] seen that the natural connection forms are equivalent to bilinear forms lifting the Kahler form.

July 5, 1987

I would like to understand the link between Riemann type θ functions

$$\theta(z|\tau) = \sum e^{i\pi\tau n^2 + 2i\pi n z}$$

and the [redacted] intrinsic θ functions

$$\theta(z) = \sum_{\gamma\in\Gamma} T_\gamma 1 = \sum_{\gamma\in\Gamma} e^{-\frac{1}{2}|\gamma|^2 + \bar{\gamma} z}$$

[redacted] which I would like to work with. [redacted] I propose to identify both of these with [redacted] holomorphic sections of a line bundle over a torus.

On the Riemann side one has fixed a symplectic basis for the lattice; the lattice is $\mathbb{Z}+\mathbb{Z} \subset \mathbb{R}^2$, so we can use the nice correspondence between functions on the line and sections of the line bundle on the torus:

$$f(y) \longmapsto \sum e^{2\pi i n x} f(y+n) = F(x,y)$$

$$2\pi i y \longleftrightarrow \nabla_x = \partial_x + 2\pi i y$$

$$\partial_y \longleftrightarrow \nabla_y = \partial_y$$

The holomorphic structure is given by $z = x + \tau y$. This is killed by $-\tau\nabla_x + \nabla_y = -\tau\left(\nabla_x + \left(-\frac{1}{\tau}\right)\nabla_y\right)$ which corresponds to the operator

$$\partial_y - 2\pi i \tau y \qquad \text{which kills} \qquad e^{i\pi\tau y^2}$$

Then
$$F(x,y) = \sum e^{2\pi i n x} e^{i\pi\tau(y+n)^2}$$
$$= e^{i\pi\tau y^2} \sum e^{i\pi\tau n^2 + 2\pi i n\overbrace{(x+\tau y)}^{z}}$$

$$= e^{i\pi\tau y^2}\,\theta(z|\tau)$$

Now

$$\theta(z+\tau|\tau) = \sum e^{i\pi\tau n^2 + 2i\pi n\tau + 2i\pi n z}$$

$$= \sum e^{i\pi\tau(n+1)^2 - i\pi\tau + 2i\pi(n+1)z - 2i\pi z}$$

$$= e^{-i\pi\tau - 2i\pi z}\,\theta(z|\tau)$$

so $\theta(z|\tau)$ is fixed under

$$e^{i\pi\tau + 2i\pi a^*}\, e^{\tau a} = e^{\frac{1}{2}[2i\pi a^*, \tau a]}\, e^{i\pi\tau + 2i\pi a^* + \tau a}$$

$$= e^{2i\pi a^* + \tau a}$$

Thus $\theta(z|\tau)$ is fixed under the operators

$$e^{a}, \quad e^{2i\pi a^* + \tau a}$$

~~And moreover it is clear~~

$\cancel{\theta(z|\tau) =}$ Call these operators T_1, T_2 respectively. Note that the constant function 1 is fixed under T_1 and that

$$\theta(z|\tau) = \sum_n \underbrace{e^{n(2i\pi a^* + \tau a)}}_{T_2^n} 1$$

Thus $\theta(z|\tau)$ is not in an obvious way a **Poincaré** series:

$$\sum_{m,n} T_1^m T_2^n f.$$

Let us now turn to

$$\theta(z) = \sum_{\gamma \in \Gamma} e^{-\frac{1}{2}|\gamma|^2 + \bar{\gamma} z}$$

where $\Gamma \subset \mathbb{C}$ is self-dual in the sense that $\Gamma = \check{\Gamma} = \{z \mid 2\,\mathrm{Im}(z,\gamma) \in 2\pi\mathbb{Z},\ \forall\gamma\}$

Let's choose the real axis so that $\Gamma \cap \mathbb{R} = \mathbb{Z}$ whence $\Gamma = \mathbb{Z} + \mathbb{Z}\tau$ with $\mathrm{Im}(\tau) = \pi$

Then the group of T_γ is generated by

$$T_{(-1)} = e^{-a^* + a}, \quad T_{(-\tau)} = e^{-\bar{\tau}a^* + \tau a}.$$

If we conjugate

$$e^{-\frac{1}{2}z^2}\Big(-\underbrace{a^*}_{z} + \underbrace{a}_{\partial z}\Big)e^{\frac{1}{2}z^2} = a$$

$$e^{-\frac{1}{2}z^2}\left(-\bar{\tau}a^* + \tau a\right)e^{\frac{1}{2}z^2} = \underbrace{(-\bar{\tau} + \tau)}_{2\pi i}a^* + \tau a$$

so the conjugated group of operators is again generated by

$$\underset{T_1}{\overset{\|}{e^{a}}}, \quad \underset{T_2}{\overset{\|}{e^{2\pi i a^* + \tau a}}}$$

So we have

$$\theta(z) = \sum_{m,n} \left(T_{(-\tau)}\right)^n \left(T_{(-1)}\right)^m 1$$

$$= \sum_{m,n} e^{\frac{1}{2}z^2}\left[T_2^{\,n}\, T_1^{\,m}\right] e^{-\frac{1}{2}z^2}$$

In other words we find

$$e^{-\frac{z^2}{2}}\theta(z) = \sum_{m,n} T_2^n T_1^m e^{-\frac{z^2}{2}}$$

Now by uniqueness, the fact that this line bundle over the torus has a unique (up to scalar factors) holomorphic section, we have

$$e^{-\frac{z^2}{2}}\theta(z) = c\,\Theta(z|\tau)$$

for some constant c. This constant can be evaluated by putting $z=0$ and gives

$$\sum_{m,n} e^{-\frac{1}{2}|m+n\tau|^2} = c\sum_n e^{i\pi n^2\tau}$$

<u>Lesson:</u> It seems necessary to use the Riemann approach precisely to handle the uniqueness. It will be possible to represent the unique holomorphic section in many ways as a Poincaré series

$$\sum_{m,n} T_1^m T_2^n f$$

and identities are hard to see from this viewpoint. However when we ask for an entire analytic function to be invariant under operators

$$e^{a}, \quad e^{2\pi i a^* + \tau a}$$

the uniqueness is clear.

The problem is now to find the proper formulation. I want to start with a complex vector space V and a lattice Γ in V such that $\Gamma \otimes_{\mathbb{Z}} \mathbb{R} \xrightarrow{\sim} V$. Then I have a complex torus V/Γ. I next want a holomorphic line bundle over this complex torus. The simplest way to obtain this is to take a connection on the trivial line bundle over V with suitable properties. First, one wants the curvature to be of type $(1,1)$, so that there is an induced holomorphic structure. Second, you want some condition relative to the lattice Γ enabling one to descend.

A connection on the trivial bundle is given by a 1-form θ. The simplest way to guarantee the curvature $d\theta$ is of type $(1,1)$ is to assume θ is of type $(1,0)$, whence 1 is a holomorphic section; more generally any holom. function on V is a holom. section of the line bundle and conversely.

I just noticed that $d\theta$ needs only to be of filtration 1, not necessarily of type $(1,1)$.

July 6, 1987

Let $A = \sum A_{ij}\, x_i\, dx_j$ be a connection form on $V = \mathbb{R}^n$. Then

$$D_j = \partial_j + x_i A_{ij} \qquad \text{covariant derivatives}$$

$$D'_k = \partial_k + A_{ke}\, x_e \qquad \text{generate translation operators preserving the connection since}$$

$$[D_j, D'_k] = A_{kj} - A_{kj} = 0.$$

The curvature is

$$[D_j, D_k] = A_{jk} - A_{kj}$$

Now suppose $V = \mathbb{C}^n$ and require A be of type $(1,0)$ and dA of type $(1,1)$.

$$A = A'_{ij}\, z_i\, dz_j + A''_{\bar{i}j}\, \bar{z}_i\, dz_j$$

$$dA = \underbrace{A'_{ij}\, dz\, dz_j}_{=0} + A''_{\bar{i}j}\, d\bar{z}_i\, dz_j$$

$$\Longrightarrow \quad A'_{ij} = A'_{ji}$$

But then $A'_{ij}\, z_i\, dz_j = d(\frac{1}{2} A'_{ij}\, z_i z_j)$ and so by a holom. gauge transformation ~~we~~ can suppose $A' = 0$.

Let's return now ~~to~~ the following problem: Let Λ be a (full) lattice in a complex V and let $F(v)$ be an entire function $\neq 0$ on V satisfying a quasi-periodicity condition

$$F(v) = e^{\frac{1}{2}q(\lambda) + \varphi(\lambda, v)}\, F(v+\lambda)$$

where $\varphi(v_1, v_2)$ is $\mathbb{R}$-bilinear on V and $\mathbb{C}$-linear in the variable v_2. Then we saw that

$$\tfrac{1}{2}g(\lambda+\lambda') - \tfrac{1}{2}g(\lambda) - \tfrac{1}{2}g(\lambda') \equiv \varphi(\lambda',\lambda) \quad \begin{pmatrix}\text{mod}\\ 2\pi i\mathbb{Z}\end{pmatrix}$$

In particular

$$\varphi(\lambda',\lambda) - \varphi(\lambda',\lambda) \in 2\pi i\mathbb{Z}$$

which implies

$$(*) \qquad \varphi(v',v) - \varphi(v,v') \in i\mathbb{R}$$

for all $v', v \in V$.

But we can write

$$\varphi(v_1,v_2) = \varphi'(v_1,v_2) + \varphi''(v_1,v_2)$$

where φ' is of type $(2,0)$, i.e. $\mathbb{C}$ linear in both v_1, v_2, and where φ'' is of type $(1,1)$, i.e. anti-linear in v_1 and linear in v_2. Then $(*)$ implies that φ' is symmetric and φ'' is hermitian symmetric.

I would ultimately like a clean statement that I can apply to pin down generating functions of the form

$$\langle \Gamma_1 | e^{v} | \Gamma_2 \rangle .$$

July 7, 1987

Let's introduce a category analogous to Graeme's but finite-dimensional. We consider real symplectic vector spaces for the objects, say of the same dimension to begin with. A morphism $V \longrightarrow V'$ is a complex symplectic transformation $T: V_c \simeq V'_c$ satisfying the positivity condition that it be implementable on the level of the Heisenberg representations. Thus we consider $V' \oplus V$ with the symplectic form

$$S\left(\begin{pmatrix} v_1' \\ v_1 \end{pmatrix}, \begin{pmatrix} v_2' \\ v_2 \end{pmatrix}\right) = S(v_1', v_2') - S(v_1, v_2)$$

and want a polarization in this direct sum. We've seen that the positivity condition on T is that

$$[Tv, (Tv)^*] > [v, v^*] \qquad \text{for } 0 \neq v \in V_c$$

Now this is not quite a category because one has to adjoin identities, and more generally one might weaken the positivity condition to include T with the above $>$ replaced by $\geq$.

Now this picture should be more general than what I considered before, namely, all real structures on a fixed complex symplectic vector space V_c. This set has a partial ordering. I would like to show it is a subcategory. This means that when I am given real structures $V_\sigma \subset V_c \supset V_\tau$ that I have (assuming positivity: $\sigma < \tau$)

a morphism from V_σ to V_τ in the category. But this is obvious.

Let V, W be real symplectic vector spaces of the same dimension. Then we can consider $\mathcal{F}_V \otimes \mathcal{F}_W^*$ as the Heisenberg representation of $V \oplus W$ equipped with the skew form $S_V \oplus (-S_W)$. Recall that

$$[\omega_1^t, \omega_2^t] = -[\omega_1, \omega_2]$$

A polarization of $V \oplus W$ determines a state in $\mathcal{F}_V \otimes \mathcal{F}_W^*$ which we can view as an operator $K : \mathcal{F}_W \longrightarrow \mathcal{F}_V$ determined up to a scalar factor. Such a polarization can be identified with a subspace of $V_c \oplus W_c$ which is the graph $\left\{ \begin{pmatrix} T\omega \\ -\omega \end{pmatrix} \,\middle|\, \omega \in W_c \right\}$ of a transf. $T : W_c \longrightarrow V_c$. T satisfies

$$[T\omega_1, T\omega_2] = [\omega_1, \omega_2] \qquad \omega_1, \omega_2 \in W_c \text{ i.e. } T \text{ is complex symplectic}$$

$$[T\omega, (T\omega)^*] > [\omega, \omega^*] \qquad 0 \neq \omega \in W_c$$

and K is killed by the operators $T(\omega) \otimes 1 - 1 \otimes \omega^t$

i.e.

$$T(\omega) K = K\omega$$

Now my goal is to pin down the function

$$\langle \Gamma | e^v | \Gamma' \rangle = F(v) \qquad v \in V_c$$

where Γ, Γ' are suitable self-dual lattices in a complex symplectic vector space.

I want to think of V as $R\Gamma$ and W as $R\Gamma'$. A better way might be to say that the complex symplectic vector space is on one hand V_c and on the other W_c and that T is the identification $V_c = W_c$. Then $|\Gamma'\rangle$ is really the image under K of the distributional state in $\mathcal{F}_W$ fixed under the translation operators belonging to Γ'.

My main problem is find a good assertion, specifying $F(v)$ up to a constant factor starting from $\Gamma \subset V_c \supset \Gamma'$. F is entire; it is quasi-periodic with respect to $\Gamma + \Gamma'$ with a certain $\varphi(v_1, v_2)$. I am missing a proof that

$$\varphi(v_1, v_2) - \varphi(v_2, v_1)$$

is a Kahler form, i.e. is a positive-definite hermitian form.

July 8, 1987

Calculation in the simplest case. The problem: Given a ~~[illegible]~~ complex symplectic vector space V_c with two real structures V, W the fixed spaces of the involutions $*$ and σ respectively. One assumes the positivity condition

$$[v, v^*] > [v, \sigma v] \qquad \text{for } 0 \neq v \in V_c$$

Then one has $V \oplus W = V_c$ and one can define a real symplectic form S on V_c by taking the direct sum

$$S = S_V \oplus (-S_W)$$

In other words

$$S(v_1, v_2) = \frac{1}{2i}[v_1, v_2] \qquad v_i \in V$$

$$S(v, w) = 0 \qquad v \in V, \ w \in W$$

$$S(w_1, w_2) = -\frac{1}{2i}[w_1, w_2] \qquad w_i \in W$$

The problem is to show that S is compatible with the complex structure, i.e.

$$S(v, iv') + S(iv, v') = 0 \qquad v, v' \in V_c$$

and that its positive

$$S(v, iv) > 0 \qquad 0 \neq v \in V_c.$$

Also we want to identify this hermitian inner product on V_c.

Let's set $h = \sigma *$. This is a complex symplectic transformation of V_c.

$$[v, v^*] > [v, \sigma v] \implies [*v, v] > [*v, hv]$$

$$\implies [v, *v] < [hv, *v]$$

We have seen this implies the eigenvalues of h are >0 and $\neq +1$ and that we have a decomposition

$$V_c = \underbrace{M}_{h>1} \oplus \underbrace{\overline{M}}_{h<1}$$

interchanged by both $*$ and σ. This is a polarization with respect to both real structures.

Now suppose $\dim M = 1$ and write $M = \mathbb{C}a$ where $[a, a^*] = 1$. We have

$$h = \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix} \quad \text{on} \quad \mathbb{C}a \oplus \mathbb{C}a^* \qquad \text{where } t>1.$$

$$\sigma a = h a^* = t^{-1} a^*$$
$$\sigma a^* = h a = t a$$

Then if $v = c_1 a + c_2 a^*$

$$[v, v^*] = [c_1 a + c_2 a^*, \bar{c}_1 a^* + \bar{c}_2 a] = |c_1|^2 - |c_2|^2$$

$$[v, \sigma v] = [c_1 a + c_2 a^*, \bar{c}_1 t^{-1} a^* + \bar{c}_2 t a] = t^{-1}|c_1|^2 - t|c_2|^2$$

and the signs check.

Now $V = V_r$ has the symplectic basis

$$\underset{(\sqrt{2}q)}{a + a^*}, \qquad \underset{(\sqrt{2}p)}{-ia + ia^*}$$

since $S_V(a+a^*, -ia+ia^*) = \frac{1}{2i}(i+i) = 1.$

Similarly $W = h^{1/2} V$ has the symplectic basis

$$t^{1/2} a + t^{-1/2} a^*, \qquad -it^{1/2} a + it^{(-1/2)} a^*$$

Then the real symplectic form S on $V \oplus W = V_c$ is determined by requiring

So

$$(1-t)\, S(a, ia^*) = \frac{t}{t-1} - \frac{t}{t-1} = 0$$

$$(1-t^{-1})\, S(a^*, ia^*) = \frac{t}{t-1} - \frac{1}{t-1} = 1$$

$$(1-t)\, S(a, ia) = \frac{1}{t-1} - \frac{t}{t-1} = -1$$

$$(1-t^{-1})\, S(a^*, ia) = \frac{1}{t-1} - \frac{1}{t-1} = 0$$

So

$$\boxed{\begin{aligned} S(a, ia) &= \frac{1}{t-1} \\ S(a^*, ia^*) &= \frac{t}{t-1} \\ S(a, ia^*) &= S(a^*, ia) = 0 \end{aligned}}$$

Now we can check the symplectic property

$S(a, ia) + S(ia, a) = 0$ as S is skew

$S(a, ia^*) + S(ia, a^*) = 0 + 0 = 0.$

$S(a^*, ia^*) + S(ia^*, a^*) = 0$ as S is skew.

What is the corresponding hermitian inner product?

$$\mathrm{Re}\{H(x,y)\} = \mathrm{Im}\{i H(x,y)\} = \mathrm{Im}\, H(x, iy)$$
$$= S(x, iy).$$

Thus

$$H(x,y) = S(x, iy) + i\, S(x,y)$$

and in our example we find

$$S(a+a^*,\ -ia+ia^*) = 1$$

$$S(t^{1/2}a+t^{-1/2}a^*,\ -it^{1/2}a+it^{-1/2}a^*) = -1$$

and all the other S values on the four basis elements are zero (except for the two in the opposite order from the above). In particular

$$S(a+a^*,\ t^{1/2}a+t^{-1/2}a^*) = 0$$

which means that the real subspace spanned by $a+a^*$, $t^{1/2}a+t^{-1/2}a^*$, i.e. $\mathbb{R}a+\mathbb{R}a^*$, is Lagrangian. Similarly for $\mathbb{R}ia+\mathbb{R}ia^*$.

Now we want to calculate the values of S on the basis a, a^*, ia, ia^*.

$$S(a+a^*,\ -ia+ia^*) = 1$$

$$\cdot t^{-1/2} \qquad S(a+a^*,\ -it^{1/2}a+it^{-1/2}a^*) = 0$$

$$(1-t^{-1})\, S(a+a^*,\ ia^*) = 1$$

$$\boxed{\begin{aligned} S(a+a^*, ia^*) &= \frac{1}{1-t^{-1}} = \frac{t}{t-1} \\ S(a+a^*, ia) &= \frac{-1}{1-t} = \frac{1}{t-1} \end{aligned}}$$

Similarly

$$S(t^{1/2}a+t^{-1/2}a^*,\ -it^{1/2}a+it^{-1/2}a^*) = -1$$

$$S(t^{1/2}a+t^{-1/2}a^*,\ -ia+ia^*) = 0$$

$$\boxed{\begin{aligned} S(t^{1/2}a+t^{-1/2}a^*,\ ia^*) &= \frac{-t^{-1/2}}{t^{-1}-1} = \frac{+t^{1/2}}{t-1} \\ S(t^{1/2}a+t^{-1/2}a^*,\ ia) &= \frac{+t^{1/2}}{t-1} \end{aligned}}$$

$$\boxed{\begin{aligned} H(a,a) &= \frac{1}{t-1} \\ H(a^*,a^*) &= \frac{t}{t-1} \\ H(a,a^*) &= 0 \end{aligned}} \quad \begin{aligned} &= S(a,ia) + \cancel{iS(a,a)} \\ &= S(a^*,ia^*) + i\,\cancel{S(a^*,a^*)} \\ &= S(a,ia^*) + i\,\cancel{S(a,a^*)} \end{aligned}$$

At this point we have found something reminiscent of KMS.

Here is a formula for H:

$$\boxed{H(v_1,v_2) = [(h-1)^{-1} v_2, v_1^*]}$$

In effect if $v_1 = c_1 a + c_2 a^*$, $v_2 = c_1' a + c_2' a^*$, then

$$H(v_1, v_2) = \bar{c}_1 c_1' \frac{1}{t-1} + \bar{c}_2 c_2' \frac{t}{t-1}$$

and

$$[(h-1)^{-1}(c_1' a + c_2' a^*), (c_1 a + c_2 a^*)^*]$$

$$= \left[\frac{c_1'}{t-1} a + \frac{c_2'}{t^{-1}-1} a^*, \bar{c}_1 a^* + \bar{c}_2 a\right]$$

$$= \frac{1}{t-1} c_1' \bar{c}_1 + \underbrace{\frac{-1}{t^{-1}-1}}_{\frac{t}{t-1}} c_2' \bar{c}_2$$

Actually it would have been enough to take $v_1 = v_2$ in the above. First of all we know that h is its adjoint with respect to the [redacted] non-degenerate hermitian symmetric form [redacted] $[v, v^*]$. In effect

$$[v_1, (hv_2)^*] = [v_1, h^{-1} v_2^*]$$

$$= [h v_1, v_2^*].$$

The same will hold for $h-1$ and $(h-1)^{-1}$. Thus $[(h-1)^{-1} v_1, * v_2]$ is hermitian symmetric.

~~New approach: We are trying to start with two real structures on a complex symplectic vector space~~

New approach: Let's go back to the idea that we interested in a polarization of $V \oplus W$ equipped with $S_V + (-S_W)$. More precisely we are interested in polarizations corresponding to complex symplectic transformations $T: W_c \simeq V_c$ satisfying a positivity condition. We have been thinking of T as being the identity and $W \subset V_c$, however it is more flexible to think in terms of triples (V, W, T).

We are trying to understand the inner product on V_c which results from this polarization. Thus if $V_c + W_c = M \oplus \overline{M}$ is the splitting we have

$$V_c \xrightarrow{\sim} V_c \oplus W_c / M \xleftarrow{\sim} W_c$$
$$\| \wr$$
$$\overline{M}$$

and V_c inherits a hermitian inner product from the symplectic form on $V \oplus W$.

However the important point is that the real symplectic space has been embedded in a complex vector space with hermitian inner product compatibly with the symplectic structures. This is exactly the situation we encountered with quasi-free states.

Therefore we should be able to understand everything from the viewpoint of quasi-free states on V. Thus we suppose given on V an inner product $(\ ,\ )$ such that if

$$S(x,y) = (Kx,y)$$

then K is a contraction. We then embed V isometrically and symplectically inside a complex vector space with hermitian inner product so that

$$\mathrm{Re}\langle x, iy\rangle = (x, Ky).$$

Assuming $\|K\| < 1$ we know that this complex vector space, if assumed to be generated by V, can be identified with V_c. Finally we can define W as the annihilator of V for the symplectic form $\mathrm{Im}\langle v, v'\rangle$ in V_c. As $\mathrm{Im}\langle x,y\rangle = S(x,y)$ for $x,y \in V$ is non-degenerate on V, it follows that $V_c = V \oplus W$.

So let's calculate everything in terms of S, K. As before identify V_c with $\{(x+iy)|0\rangle\}$. Then the annihilator of V consists of $(x+iy)|0\rangle$ such that for $x' \in V$ we have

$$0 = \mathrm{Im}\langle 0|x'(x+iy)|0\rangle = \mathrm{Im}\langle x', x\rangle + \mathrm{Im}\underbrace{\langle x', iy\rangle}_{i\langle x',y\rangle}$$

$$= S(x',x) + (x',y)$$
$$= -(x',Kx) + (x',y) \implies y = Kx$$

Thus $\quad W = \{(x+iKx)|0\rangle\}.$

July 9, 1987

Recall yesterday's idea. Given V, W real symplectic vector spaces and a complex symplectic transformation $T: W_c \xrightarrow{\sim} V_c$ satisfying the positivity condition $iS_V(T\omega, T\omega) > iS_W(\omega, \omega)$ for $0 \neq \omega \in W_c$, we obtain a hermitian inner product on V_c as follows. We let $M = \left\{\begin{pmatrix} T\omega \\ -\omega \end{pmatrix} \in V_c \oplus W_c\right\}$; this defines a polarization of $V \oplus W$ equipped with $S_V \oplus (-S_W)$. There is a vacuum state $|0\rangle$ in $\mathcal{F}_V \otimes \mathcal{F}_W^*$ killed by M, so we have isomorphisms

$$V_c \xrightarrow{\sim} V_c \oplus W_c/M \xrightarrow{\sim} \{(v+w)|0\rangle\}$$

and the latter has a canonical inner product.

Now it's clear we have produced a quasi-free state for V and that the hermitian inner product we seek is just the natural one in the following sense. To a quasi-free state one has an canonical embedding of V in a complex vector space which is a quotient of V_c in general, but which is isomorphic to V_c when the contraction K has no eigenvalues of abs. value 1.

The natural problem now is to set up an equivalence between quasi-free states with $\|K\| < 1$ and real structures W on V_c which are positive relative to V. We have just seen above how to go from W to K. Conversely given a quasi-free state with contraction K satisfying $\|K\| < 1$, we construct the complex inner product space associated to it; this turns out to be V_c and we let W be the annihilator of V for the symplectic form.

Now the problem is to make the whole business more explicit. Start with V, S and the inner product $(\ ,\ )$ on V. Let K be defined by

$$(Kx,y) = S(x,y) \qquad \text{or} \qquad (x,y) = S(K^{-1}x,y).$$

Then we have the representation of V with cyclic vector $|0\rangle$ such that

$$\langle 0|x^2|0\rangle = |x|^2 \quad , \quad [x,y] = 2iS(x,y).$$

We have an inner product on V_c given by

$$\|(x+iy)|0\rangle\|^2 = \langle 0|(x-iy)(x+iy)|0\rangle$$

$$= |x|^2 + |y|^2 - 2S(x,y)$$

$$= |x|^2 + |y|^2 + 2\,\mathrm{Re}\,(x, Ky)$$

$$= |x+Ky|^2 + |y|^2 - |Ky|^2$$

Let's review the calculation of the annihilator W of V in V_c for the symplectic form $=$ $\mathrm{Im}\langle\ |\ \rangle$. W consists of $(x+iy)$ such that for all $x' \in V$, we have

$$0 = \mathrm{Im}\,\langle 0|x'(x+iy)|0\rangle$$

$$= \underline{\mathrm{Im}\langle 0|x'x|0\rangle} + \mathrm{Re}\langle 0|x'y|0\rangle$$

$$\tfrac{1}{2i}\langle x'x\rangle - \langle xx'\rangle = S(x',x) = -(x',Kx)$$

$$= -(x',Kx) + (x',y). \qquad \therefore \quad \boxed{y = Kx}$$

So $W = \{(x+iKx) \in V_c \mid x \in V\}$.

As a check let's compute the complex linear symplectic form on V_c restricted to W and see if it's real.

$$\frac{1}{2i}[x+iKx,\ y+iKy] = S(x,y) + iS(Kx,y) + iS(x,Ky) - S(Kx,Ky)$$

But $S(Kx,y) + S(x,Ky) = (K^2x,y) + (Kx,Ky) \equiv 0$ as $K^t = -K$.

Next we want to find h, the complex symplectic transformation on V_c such that $h* = \sigma$ is the conjugation belonging to W. Thus we want

$$\underbrace{h*(x+iKx)}_{h(x-iKx)} = x+iKx$$

which means

$$\boxed{h = \frac{1+iK}{1-iK}}$$

July 10, 1987

Intrinsic Θ function. Let V be a complex vector space with hermitian inner product $\langle v | v' \rangle$. If $\lambda \in V$ define the translation operator T_λ acting on functions on V by

$$(T_\lambda f)(v) = e^{-\frac{1}{2}|\lambda|^2 - \langle \lambda | v \rangle} f(v+\lambda).$$

Note T_λ preserves analytic functions. We have

$$\boxed{T_\lambda T_\mu = e^{i\, \mathrm{Im}\langle \lambda | \mu \rangle} T_{\lambda+\mu}.}$$

[redacted] In effect,

$$(T_\lambda T_\mu f)(v) = e^{-\frac{1}{2}|\lambda|^2 - \langle \lambda | v \rangle - \frac{1}{2}|\mu|^2 - \langle \mu | v+\lambda \rangle} f(v+\lambda+\mu)$$

$$= e^{\underbrace{-\frac{1}{2}|\lambda|^2 - \langle \mu | \lambda \rangle - \frac{1}{2}|\mu|^2 + \frac{1}{2}|\lambda+\mu|^2}_{-\frac{1}{2}\langle \mu | \lambda \rangle + \frac{1}{2}\langle \lambda | \mu \rangle = i\, \mathrm{Im}\langle \lambda | \mu \rangle.}} (T_{\lambda+\mu} f)(v)$$

Thus

$$T_\lambda^{-1} = T_{-\lambda}$$

$$\boxed{(T_\lambda, T_\mu) = e^{2i\, \mathrm{Im}\langle \lambda | \mu \rangle}}$$

Let Γ be a lattice in V (a free abelian group [redacted] such that $\mathbb{R} \otimes \Gamma = V$.) Define the <u>dual lattice</u> to be

$$\Gamma^\vee = \{ v \mid \mathrm{Im}\langle \gamma | v \rangle \in \pi\mathbb{Z} \text{ for all } \gamma \in \Gamma \}$$

and call Γ <u>self-dual</u> when $\Gamma = \Gamma^\vee$.

But notice that even when $\Gamma = \Gamma^\vee$, the operators T_γ do not form a group. One has

$$T_\lambda T_\mu = e^{i \operatorname{Im}\langle\lambda|\mu\rangle} T_{\lambda+\mu}$$

and the exponent is in $i\pi\mathbb{Z}$, so we find only that $T_\lambda T_\mu = \pm T_{\lambda+\mu}$. Thus everything done earlier with the Poincaré series

$$\sum_{\lambda\in\Gamma} T_\lambda 1 = \sum_{\lambda\in\Gamma} e^{-\frac{1}{2}|\lambda|^2 - \langle\lambda|\sigma\rangle}$$

is apparently nonsense.

In fact the idea that there is a canonical state $|\Gamma\rangle$ is also nonsense. For example when $\Gamma = \mathbb{Z}p + \mathbb{Z}2\pi q$ with $p = \frac{1}{i}\partial_x$ and $q = x$, then

$$e^{i(mp + n2\pi q)} = e^{\overbrace{+\frac{1}{2}[2\pi i n q,\, i m p]}^{i\pi m n}} e^{2\pi i n x} e^{m\partial_x}$$

$$= (-1)^{mn} e^{2\pi i n x} e^{m\partial_x}$$

July 11, 1987

Let V be real symplectic and Γ a self-dual lattice in V. Then we don't get an action of Γ on the representation $\mathcal{F}$ of V, but rather we have a representation of an abelian extension of Γ

$$\tilde{\Gamma} = \{\pm e^{i\gamma} \mid \gamma \in \Gamma\}.$$

In general one can look at the full inverse image $\tilde{\Gamma}' = \{\zeta e^{i\gamma} \mid \gamma \in \Gamma,\ \zeta \in \mathbb{T}\}$ of Γ in the Heisenberg group. This is a maximal abelian subgroup and the Heisenberg representation decomposes with multiplicity one according to the characters of $\tilde{\Gamma}'$ which are the identity on the circle $\mathbb{T}$. ~~[illegible]~~ We would like to have a distinguished state in $\mathcal{F}$, that is, a distinguished character of $\tilde{\Gamma}'$, or equivalently a splitting of the extension of $\tilde{\Gamma}'$. We can obviously work with $\tilde{\Gamma}$ and ask for a ~~[illegible]~~ subgrp. complementary to $\{\pm 1\}$.

So to specify a state in $\mathcal{F}$ with the desired properties we must look for a ~~[illegible]~~ complement of $\{\pm 1\}$ in $\tilde{\Gamma}$. Such a complement necessarily contains $2\tilde{\Gamma} = \{e^{i2\gamma} \mid \gamma \in \Gamma\}$, so we wish to split the sequence

$$0 \longrightarrow \{\pm 1\} \longrightarrow \tilde{\Gamma}/2\tilde{\Gamma} \longrightarrow \Gamma/2\Gamma \longrightarrow 0$$

There doesn't seem to be any natural way to do this, so it seems that the best we can do is to choose a character $\chi : \tilde{\Gamma}/2\tilde{\Gamma} \longrightarrow \{\pm 1\}$ which is the identity on the subgroup $\{\pm 1\}$. Then we can look for the unique eigenvector for $\tilde{\Gamma}$ with the character χ.

Another possibility is to look at the space fixed by $2\tilde{\Gamma}$ which will have dimension = order of $\tilde{\Gamma}/2\tilde{\Gamma} = 2^{2g}$ where $2g = \text{rank}(\Gamma)$. Does this have a natural structure of some sort?

(Let's recall that extensions of an elementary abelian 2 group A by $\mathbb{Z}/2$ are classified by quadratic functions $Q : A \to \mathbb{Z}/2$ where quadratic functions are the same thing as elements of $S^2(A^*)$. Given Q because of the exact sequence

$$0 \longrightarrow \Lambda^2 A^* \longrightarrow A^* \otimes A^* \longrightarrow S^2(A^2) \longrightarrow 0$$

one can find $B : A \times A \longrightarrow \mathbb{Z}/2$ bilinear over $\mathbb{Z}/2$ such that $Q(x) = B(x,x)$. One uses B as cocycle to construct the extension.

Next we have

$$0 \longrightarrow A^* \longrightarrow S^2(A^*) \longrightarrow \Lambda^2(A^*) \longrightarrow 0$$
$$x \longmapsto x^2$$

where the second map associates to Q the bilinear form $Q(x+y) - Q(x) - Q(y)$ which is alternating.

Thus when $A = \Gamma/2\Gamma$ we can lift the

the symplectic form on $\Gamma/2\Gamma$ to a quadratic function and we obtain a non-commutative central extension of $\Gamma/2\Gamma$ by $\mathbb{Z}/2\mathbb{Z}$. My idea is that such a central extension will have a Clifford algebra whose dimension is 2^{2g} and maybe it can be identified with the space of states fixed under 2Γ. (Again this structure, i.e. the quadratic function is unique up to elements of $\mathrm{Hom}(\Gamma/2\Gamma, \mathbb{Z}/2) = A^*$.)

July 12, 1987

Let Γ be a self-dual lattice in V real symplectic, and $\tilde{\Gamma} = \{\pm e^{ix} \mid x \in \Gamma\}$ the extension of Γ by $\{\pm 1\}$ which acts on the Heisenberg representation $\mathcal{F} = \mathcal{F}_V$. For each character of $\tilde{\Gamma}$ which is the identity on $\{\pm\}$, there is an eigenvector of $\tilde{\Gamma}$ with this character, which is "distributional" and is unique up to scalar factors. Among these character $\chi : \tilde{\Gamma} \to \mathbb{T}$, there are those which are real, i.e. have values in $\{\pm 1\}$. These are splittings of the extension

$$0 \longrightarrow \{\pm 1\} \longrightarrow \tilde{\Gamma} \longrightarrow \Gamma \longrightarrow 0 .$$

The real characters are as close to being trivial as is possible.

Let's review some group cohomology. The group $\tilde{\Gamma}$ can be described as the set $\{\pm 1\} \times \Gamma$ with the group law defined by a 2-cocycle which is biadditive. Recall that a biadditive map $f : \Gamma \times \Gamma \longrightarrow$ some abelian gp. G is a 2-cocycle

$$(\delta f)(x,y,z) = f(y,z) - f(x+y,z) + f(x,y+z) - f(x,y)$$

$$= f(y,z) - f(x,z) - f(y,z) + f(x,y) + f(x,z) - f(x,y) = 0$$

and that a for a 1-cochain $g : \Gamma \longrightarrow G$ (normalized so that $g(0)=0$) to have coboundary f:

$$(\delta g)(x,y) = g(y) - g(x+y) + g(x)$$

means that g is quadratic with associated biadditive function $-f$.

~~Thus to split the extension $\tilde{\Gamma}$~~

The splittings of the extension $\tilde{\Gamma}$ are in one-one correspondence with ways of writing the 2 cocycle as a coboundary, i.e. with quadratic functions associated to the $\mathbb{Z}$-valued symplectic pairing on Γ mod 2.

But these quadratic functions occur already in the definition of quasi-periodic function

$$F(v) = e^{g(\lambda) + \varphi(\lambda, v)} F(v+\lambda).$$

We saw that to get consistent conditions with the addition in Γ, we wanted

$$\varphi(\lambda+\lambda', v) = \varphi(\lambda, v) + \varphi(\lambda', v)$$

$$g(\lambda+\lambda') \equiv g(\lambda) + g(\lambda') + \varphi(\lambda,\lambda') \qquad \text{mod } 2\pi i\mathbb{Z}$$

Now let's return to the problems of Θ functions. Suppose given a complex vector space V with herm. inner product $\langle\,|\,\rangle$. Define translation operators

$$(T_\lambda f)(v) = e^{-\frac{1}{2}|\lambda|^2 - \langle\lambda|v\rangle} f(v+\lambda) \qquad \lambda \in V$$

satisfying
$$T_\lambda T_\mu = e^{i\,\mathrm{Im}\langle\lambda|\mu\rangle}\, T_{\lambda+\mu} \qquad \lambda,\mu \in V$$

Let Γ be a lattice which is self-dual

$$\Gamma = \check{\Gamma} = \{ v \mid \mathrm{Im}\langle\lambda|v\rangle \in \pi\mathbb{Z},\ \forall \lambda \in \Gamma \}.$$

and let $g : \Gamma \longrightarrow \mathbb{Z}/2\mathbb{Z}$ satisfy

$$i\pi\left[g(\lambda+\mu) - g(\lambda) - g(\mu) \right] \equiv i\,\mathrm{Im}\langle\lambda|\mu\rangle \qquad \text{mod } 2\pi i\mathbb{Z}$$

Then the operators

$$\tilde{T}_\lambda = \underbrace{e^{i\pi g(\lambda)}}_{\pm 1} T_\lambda \qquad \lambda \in \Gamma$$

define a representation of Γ on the analytic functions on V.

Assertion: Up to scalar factors there is a unique analytic function on V which is fixed under the operators $\tilde{T}_\lambda$, $\lambda \in \Gamma$.

To prove this it is probably better to study the general quasi-periodic setup.

Lemma: Let V be a complex vector space and let $\varphi : V \times V \longrightarrow \mathbb{C}$ be an $\mathbb{R}$-bilinear map such that

$$\varphi(v, \lambda w) = \lambda (v, w) \qquad \lambda \in \mathbb{C}$$

$$\varphi(v, w) - \varphi(w, v) \in i\mathbb{R}$$

Then φ can be written uniquely

$$\varphi(v, w) = \varphi'(v, w) + \varphi''(v, w)$$

where φ' is $\mathbb{C}$-bilinear and symmetric and where φ'' is sesquilinear and hermitian symmetric.

Proof: Let $V = \mathbb{C}^n$. Then

$$\varphi(v, w) = \sum A_{k\ell} v_k w_\ell + \sum B_{k\ell} \bar{v}_k w_\ell$$

where the $A_{k\ell}$, $B_{k\ell}$ are easily found from the values of φ as v runs over the $\mathbb{R}$ basis e_j, ie_j of V and w runs over the $\mathbb{C}$ basis e_j. Put another way φ can be uniquely written as $\varphi' + \varphi''$, where φ' is $\mathbb{C}$ bilinear and φ'' is sesqui-linear. Now

$$[\varphi(v,w) - \varphi(w,v)] = \overbrace{[\varphi'(v,w) - \varphi'(w,v)]}^{\alpha} + \overbrace{[\varphi''(v,w) - \varphi''(w,v)]}^{\beta}$$

Consider the effect of $v \mapsto e^{i\theta}v$, $w \mapsto e^{i\theta}w$.
We then find

$$e^{2i\theta}\alpha + \beta \in i\mathbb{R}$$

for all θ, which implies $\alpha = 0$, showing φ' is symmetric. On the other hand, applying $v \mapsto e^{i\theta}v$, $w \mapsto w$ gives

$$e^{-i\theta}\varphi''(v,w) - e^{i\theta}\varphi''(w,v) \in i\mathbb{R}$$

for all θ, so

$$\mathrm{Re}\left(e^{i\theta}\varphi''(w,v)\right) = \mathrm{Re}\left(e^{-i\theta}\varphi''(v,w)\right)$$
$$= \mathrm{Re}\left(e^{i\theta}\overline{\varphi''(v,w)}\right)$$

showing $\varphi''(w,v) = \overline{\varphi''(v,w)}$. QED.

Now let us return to a general quasi-periodic function

$$F(v) = e^{q(v) + \varphi(\lambda,v)} F(v+\lambda) \qquad \lambda \in \Gamma$$

We have seen $\varphi = \varphi' + \varphi''$ where $\varphi'(v,w)$ is complex symmetric and $\varphi''(v,w)$ is hermitian symmetric.
Moreover

$$e^{q(\lambda+\lambda')} = e^{q(\lambda)} e^{q(\lambda')} e^{\varphi(\lambda',\lambda)}$$

which means that $e^{q(\lambda)}$ is a quadratic character with the associated pairing $e^{\varphi(\lambda',\lambda)} = e^{\varphi(\lambda,\lambda')}$.
Such a quadratic character is unique up to [illegible]

multiplying by an ordinary character $\chi : \Gamma \longrightarrow \mathbb{C}^\times$.

Suppose we consider a Gaussian gauge transf.

$$F(v) = e^{h(v)} F_1(v)$$

Then the automorphy factor becomes

$$e^{g(\gamma) + \varphi(\gamma, v) + h(v+\gamma) - h(v)}$$

so that $g(\gamma)$ changes to $(g+h)(\gamma)$ and $\varphi(\gamma, v)$ changes to $\varphi(\gamma, v) + h(v, \gamma)$

The problem under consideration is to relate the intrinsic Θ-function defined by the assertion on 977 say to the Riemann type Θ-function. Let's calculate for $\dim_{\mathbb{C}} V = 1$ from the other direction, ~~starting~~ with the symplectic side in standard form. Thus we suppose

$$V = \mathbb{R}q + \mathbb{R}p$$

$$\Gamma = \mathbb{R}\, 2\pi q + \mathbb{R}p \qquad [2\pi q, p] = 2\pi i$$

In view of the rule $[x,y] = 2i S(x,y)$, this means

$$2i\, S(q,p) = [q,p] = i \qquad \text{or}$$

$$\boxed{S(q,p) = \tfrac{1}{2}}$$

I also have to specify the lifting of Γ to operators, i.e. the quadratic functions. The simplest lifting seems to be

$$\tilde{T}_{m\, 2\pi q + np} = \left(e^{2\pi i q}\right)^m \left(e^{ip}\right)^n .$$

The ~~[illegible]~~ minimal possibilities corresponding to real characters are the four possible choices of sign

$$\tilde{T}_{2\pi q} = \pm e^{2\pi i q} \qquad \tilde{T}_p = \pm e^{ip}$$

Next let's consider the polarization. Simplest seems to give the annihilator subspace, say it's spanned by

$$ip + \omega q \qquad \text{with} \quad \mathrm{Re}(\omega) > 0$$

Put $\tau = i\omega$ so that $\mathrm{Im}(\tau) = \mathrm{Re}(\omega) > 0$. Then the annihilating space is spanned by

$$p - \tau q.$$

Thus if $|0\rangle$ is the ground state for this polarization we have

$$\boxed{p|0\rangle = \tau q|0\rangle}$$

Now we propose to look at the complex structure and inner product on $V = \mathbb{R}q + \mathbb{R}p$ obtained from the isomorphism

$$V \xrightarrow{\sim} \{ (xq + yp)|0\rangle \}$$

Define creation and annihilation operators by

$$a = c(p - \tau q) \qquad a^* = \bar{c}(p - \bar{\tau} q)$$

where c is to be determined so that

$$\begin{aligned}
[a, a^*] &= |c|^2 \, [p - \tau q, p - \bar{\tau} q] \\
&= |c|^2 (-\tau i + \bar{\tau} i) = |c|^2 \overbrace{(\tau - \bar{\tau})}^{2i\,\mathrm{Im}\,\tau}(-i) \\
&= |c|^2 (2\,\mathrm{Im}\,\tau).
\end{aligned}$$

Simplest choice $\quad c = \dfrac{1}{\sqrt{2\,\mathrm{Im}\,\tau}}$.

Now

$$\cancel{\langle 0 | a a^* | 0 \rangle = \frac{1}{2\,\mathrm{Im}\,\tau} \langle p - \tau q}$$

$$1 = \| a^* |0\rangle \|^2 = \frac{1}{2\,\mathrm{Im}\,\tau} \| (p - \bar{\tau} q)|0\rangle \|^2 =$$

$$\frac{1}{2\,\mathrm{Im}\,\tau}\;\underbrace{|\tau-\bar\tau|^2}_{2i\,\mathrm{Im}\,\tau}\;\underbrace{\|q|0\rangle\|^2}_{\langle q^2\rangle}$$

Thus
$$\langle q^2\rangle = \frac{1}{2\,\mathrm{Im}\,\tau}$$
$$\langle p^2\rangle = \|p|0\rangle\|^2 = \|\tau q|0\rangle\|^2 = \frac{|\tau|^2}{2\,\mathrm{Im}\,\tau}$$
$$\langle qp\rangle = \cancel{\langle q(p-\tau q)\rangle} + \tau\langle q^2\rangle$$
$$= \frac{\tau}{2\,\mathrm{Im}\,\tau}$$

Check: $\langle qp\rangle - \overline{\langle qp\rangle} = \langle qp-pq\rangle = i$

$$\frac{\tau-\bar\tau}{2\,\mathrm{Im}\,\tau} = i \qquad \checkmark$$

Let's recall that we are trying to show the existence and uniqueness of a certain kind of quasi-periodic analytic function. The proof method I wish to use is to reduce to the Riemann case. So whatever I'm doing should end up in the case when $\Gamma = \mathbb{Z}+\mathbb{Z}\tau \subset \mathbb{C}$ with
$$\theta(z|\tau) = \sum e^{i\pi n^2\tau + 2i\pi nz}.$$

This satisfies
$$\begin{cases}\theta(z+1|\tau) = \theta(z|\tau)\\ \theta(z+\tau|\tau) = e^{-i\pi\tau - 2i\pi z}\,\theta(z|\tau)\end{cases}$$

or
$$\boxed{\begin{aligned}\theta(z|\tau) &= \theta(z+1|\tau)\\ \theta(z+\tau|\tau) &= e^{i\pi\tau+2i\pi z}\,\theta(z+\tau|\tau)\end{aligned}}$$

For this particular analytic function we have

$$\varphi(1, z) = 0$$

$$\varphi(\tau, z) = 2\pi i z$$

Thus $\varphi(x + \tau y, \omega) = y\, 2\pi i \omega$

$$= \frac{(x+\tau y) - (x+\bar{\tau} y)}{\tau - \bar{\tau}}\, 2i\pi\omega$$

$$= (z - \bar{z}) \frac{\pi}{\operatorname{Im}\tau}\, \omega$$

or

$$\boxed{\varphi(z, \omega) = \frac{\pi}{\operatorname{Im}\tau} z\omega - \frac{\pi}{\operatorname{Im}\tau} \bar{z}\omega}$$

It would appear therefore that

$$\langle z | \omega \rangle = \frac{\pi}{\operatorname{Im}(\tau)} \bar{z}\omega$$

This certainly has the property that

$$\operatorname{Im} \langle 1 | \tau \rangle = \frac{\pi}{\operatorname{Im}\tau} \operatorname{Im}(\tau) = \pi$$

so that $\Gamma = \mathbb{Z} + \mathbb{Z}\tau$ is self-dual.

July 13, 1987

Our goal is to prepare an account of Θ-functions, starting from the equation

$$* \qquad F(v) = e^{Q(\lambda)+\varphi(\lambda,v)} F(v+\lambda) , \qquad \lambda \in \Gamma$$

Here F is a non-zero entire function on a complex vector space V, Γ is a lattice in V, and $\varphi(\lambda, v)$ is a $\mathbb{C}$-linear functional on V. Consistency requires

$$\varphi(\lambda+\lambda', v) = \varphi(\lambda, v) + \varphi(\lambda', v)$$

$$Q(\lambda+\lambda') \equiv Q(\lambda) + Q(\lambda') + \varphi(\lambda', \lambda) \qquad \text{mod } 2\pi i \mathbb{Z}$$

Note that $Q(\lambda)$ is determines only mod $2\pi i\mathbb{Z}$, and it might be preferable to say $e^{Q(\lambda)}$ is a quadratic character on Γ with associated pairing $e^{\varphi(\lambda',\lambda)}$

As $e^{\varphi(\lambda',\lambda)}$ is symmetric we have

$$** \qquad \varphi(\lambda',\lambda) - \varphi(\lambda',\lambda) \in 2\pi i \mathbb{Z}$$

As $\mathbb{R}\Gamma = V$, $\varphi(\lambda, v)$ extends to a pairing $\varphi(v, v')$ from $V \times V$ to $\mathbb{C}$ which is $\mathbb{R}$-linear in the first variable and $\mathbb{C}$ linear in the second. One can write

$$\varphi = \varphi' + \varphi''$$

where φ' is $\mathbb{C}$-bilinear and φ'' is sesquilinear. From ** one deduces that φ' is a symmetric $\mathbb{C}$-bilinear form, and φ'' is hermitian symmetric. One has

$$\mathrm{Im}\ \varphi''(\lambda,\mu) = \frac{1}{2i}\big(\varphi(\lambda,\mu)-\varphi(\mu,\lambda)\big) \in \pi\mathbb{Z}$$

for all $\lambda,\mu \in \Gamma$.

Next we analyze $Q(\lambda)$. The symmetric pairing $e^{\varphi(\lambda,\mu)}$ from $\Gamma\times\Gamma$ to $\mathbb{C}^\times$ defines an extension of Γ by $\mathbb{C}^\times$ which is abelian; the quadratic character $e^{Q(\lambda)}$ determines a splitting of this extension, and any two splittings differ by a character.

Set $q_1(\lambda) = Q(\lambda) - \frac{1}{2}\varphi(\lambda,\lambda)$. Then

$$e^{q_1(\lambda+\mu)} = e^{Q(\lambda+\mu)-\frac{1}{2}\varphi(\lambda+\mu,\lambda+\mu)}$$

$$= e^{Q(\lambda)+Q(\mu)+\varphi(\mu,\lambda)-\frac{1}{2}\varphi(\lambda,\lambda)-\frac{1}{2}\varphi(\lambda,\mu)-\frac{1}{2}\varphi(\lambda,\mu)-\frac{1}{2}\varphi(\mu,\mu)}$$

$$= e^{q_1(\lambda)+q_1(\mu)+\frac{1}{2}(\varphi(\mu,\lambda)-\varphi(\lambda,\mu))}$$

$$= e^{q_1(\lambda)+q_1(\mu)}\, e^{-i\,\mathrm{Im}\,\varphi''(\lambda,\mu)} \quad \text{values in } \pi\mathbb{Z}$$

Thus $e^{q_1(\lambda)}$ is a quadratic character associated to a pairing whose values are in $\{\pm 1\}$. The pairing defines an extension of Γ by $\{\pm 1\}$ which can be split. This means $e^{q_1(\lambda)}$ is the product of a quadratic character with values in $\{\pm 1\}$ associated to the pairing

$$e^{-i\,\mathrm{Im}\,\varphi''(\lambda,\mu)}$$

and an ordinary character of Γ.

Next I would like to show that $-\varphi''(v,w)$ is a positive definite inner product on V. Of course if one starts with F constant, $Q = \varphi = 0$, then one sees this isn't true. In general one can consider the hermitian form $-\varphi''(v|w)$ on V. For any complex subspace W of V such that $\Gamma \cap W$ is a lattice in W and such that this hermitian form is ≤ 0, one might hope to show that F is constant on W cosets. The argument would be to show the function is bounded.

Perhaps we can reformulate and view F as a section of a holomorphic line bundle over the torus V/Γ. Then the hermitian form $-\varphi''(v,w)$ is essentially the curvature form associated to a natural metric on this line bundle, everything is invariant under translation. But then F has to have a maximum somewhere which implies the curvature is ≥ 0. The rest is fairly clear.

Next we turn to the converse. We consider a complex vector space V, lattice Γ, herm. inner product $\langle v|w\rangle$ on V such that

$$\Gamma = \Gamma^\vee = \{v \mid \operatorname{Im}\langle \lambda|v\rangle \in \pi\mathbb{Z}, \ \forall \lambda \in \Gamma\}.$$

Set

$$\varphi(v,w) = \underbrace{\varphi'(v,w)}_{\substack{\text{symmetric}\\ \mathbb{C}\text{-linear}}} - \langle v|w\rangle \quad \text{and}$$

let $e^{Q(\lambda)}$ be a quadratic character on Γ associated to the pairing $e^{\varphi(\lambda,\mu)}$. Then we wish to demonstrate the existence of an entire function unique up to scalar factors satisfying $*$ on 983.

The first point is that the problem is independent of $Q(\lambda)$ and φ', it just depends on $V, \Gamma, \langle\,|\,\rangle$. This is because by making the gauge transformation

$$F(\sigma) = e^{\frac{1}{2}B(\sigma,\sigma) + C(\sigma)} F_1(\sigma)$$

we change $Q(\lambda) + \varphi(\lambda,\sigma)$ to

$$Q(\lambda) + \varphi(\lambda,\sigma) + \tfrac{1}{2}B(\sigma+\lambda,\sigma+\lambda) - \tfrac{1}{2}B(\sigma,\sigma) + C(\sigma+\lambda) - C(\sigma).$$

So if B is a symm. bilinear form and $C(\sigma)$ is a linear form, we change

$$\varphi(\lambda,\sigma) \longmapsto \varphi(\lambda,\sigma) + B(\lambda,\sigma)$$
$$Q(\lambda) \longmapsto Q(\lambda) + \tfrac{1}{2}B(\lambda,\lambda) + C(\lambda).$$

Thus we can alter the symmetric part φ' as we wish, and similarly we can alter $Q(\lambda)$ by any character of the form $e^{C(\lambda)}$, where C is a $\mathbb{C}$-linear functional. The other thing we can do is apply a translation, i.e. put $F(\sigma) = F_1(\sigma - \varepsilon)$. Then

$$F_1(\sigma) = e^{Q(\lambda) + \varphi(\lambda,\sigma+\varepsilon)} F_1(\sigma+\lambda),$$

which changes $Q(\lambda)$ to $Q(\lambda) + \varphi(\lambda,\varepsilon)$, and this latter contains anti-holomorphic linear functions of λ.

Review Graeme's proof of existence of these functions. Form the holom. line bundle over the torus V/Γ whose sections are the desired functions $F(v)$. The positivity condition & Kodaira vanishing implies the higher cohomology of the sheaf of holomorphic sections vanishes. Then apply the index theorem ~~to~~ calculate the dimension of the space of sections. Self-duality should imply the dimension is 1.

Let's try for a more direct proof. Let's choose a symplectic basis for Γ with respect to the form $\frac{1}{\pi}\mathrm{Im}\langle\lambda|\mu\rangle$. This means a $\mathbb{Z}$-basis $\alpha_1,\dots,\alpha_g,\beta_1,\dots,\beta_g$ such that the nonzero pairings are between α_j and β_j and

$$\frac{1}{\pi}\mathrm{Im}\langle\alpha_j|\beta_j\rangle = 1.$$

Then consider the real subspaces $\mathbb{R}\alpha_1+\dots+\mathbb{R}\alpha_g$, $\mathbb{R}\beta_1+\dots+\mathbb{R}\beta_g$. These are complementary and isotropic for the symplectic form. It follows that the complexification of $W_0=\mathbb{R}\alpha_1+\dots+\mathbb{R}\alpha_g$ must be V. In effect consider $W_0\cap iW_0$. This is stable under multiplication by i, and hence is a complex subspace of V. Then the restriction of $\mathrm{Im}\langle\,|\,\rangle$ ~~to~~ $W_0\cap iW_0$ is non-degenerate, which is possible only if $W_0\cap iW_0=0$.

Now we will show we can choose $q(\lambda)$ and $\varphi(\lambda,v)$ to be zero when $\lambda\in W_0$. We take the inner product $-\varphi''(v,w)=\langle v|w\rangle$ restrict it to V_0 where it is a positive definite real quadratic form. We can then extend this real form $\mathbb{C}$ linearly to obtain a symmetric $\mathbb{C}$-bilinear form $\varphi'(v,w)$ on V

which agrees with $\langle\ |\ \rangle$ on V_0.

Thus $\varphi(v,w) = \varphi'(v,w) - \langle v|w\rangle$ vanishes when $v \in V_0$ since it is $\mathbb{C}$-linear in W.

Now that we've chosen φ we have to choose $Q(\lambda)$. The simplest thing to do will be to choose the lifting of Γ into the central extension defined by the pairing $e^{\varphi(\lambda,\lambda')}$; and we can do this by saying what to do on the generators of Γ.

Let's next discuss an example. Suppose V is 1-dimensional, and $\Gamma = \mathbb{Z}\alpha + \mathbb{Z}\beta$, where $\frac{1}{\pi}\operatorname{Im}\langle\alpha|\beta\rangle = 1$. Write $\beta = \tau\alpha$, so that

$$1 = \frac{1}{\pi}\operatorname{Im}\langle\alpha|\tau\alpha\rangle = \frac{|\alpha|^2}{\pi}\operatorname{Im}(\tau).$$

We have

$$\begin{aligned}\varphi(z\alpha, w\alpha) &= \varphi'(z\alpha, w\alpha) - \langle z\alpha|w\alpha\rangle \\ &= zw\,\varphi'(\alpha,\alpha) - \bar{z}w\frac{\pi}{\operatorname{Im}\tau} \\ &= \frac{\pi}{\operatorname{Im}\tau}(z-\bar{z})w\end{aligned}$$

since we want $\varphi(z\alpha, w\alpha)$ to vanish when $z \in \mathbb{R}$.

Thus taking the simplest choice for Q on the generators gives

$$\begin{aligned}(T_\alpha f)(z) &= f(z+1) \\ (T_\beta f)(z) &= e^{\varphi(\tau\alpha, z\alpha)} f(z+\tau) \\ &= e^{2i\pi z} f(z+\tau)\end{aligned}$$

Let's solve for f assuming it is fixed under T_α, T_β.

$$f(z) = \sum c_n e^{2i\pi n z}$$

$$= \sum c_n \, e^{2i\pi z} \, e^{2i\pi n(z+\tau)}$$

$$= \sum \left(c_{n-1} \, e^{2i\pi(n-1)\tau}\right) e^{2i\pi n z}$$

$$\therefore \quad c_n = c_{n-1} \, e^{2i\pi(n-1)\tau} \implies c_n = e^{i\pi \frac{n(n-1)}{2}\tau} c_0$$

So the unique solution up to scalars is

$$f(z) = \sum_{n \in \mathbb{Z}} e^{i\pi \frac{n(n-1)}{2}\tau + 2i\pi n z}$$

Next try to carry this out in general.
First let's use

$$e^{2\pi i a^* + \tau a} = e^{\underbrace{-\frac{1}{2}[2\pi i a^*, \tau a]}_{i\pi\tau}} e^{2\pi i a^*} e^{\tau a}$$

and use the following modified formulas

$$F(z) = F(z+1) = e^{i\pi\tau + 2i\pi z} F(z+\tau)$$

The recursion relation is then

$$c_n = c_{n-1} \, e^{i\pi\tau + 2i\pi(n-1)\tau} = c_{n-1} \, e^{i\pi(2n-1)\tau}$$

and leads to

$$F(z) = \sum e^{i\pi n^2 \tau + 2i\pi n z}$$

which is the Riemann Θ function.

July 14, 1987

General case: Let V be a complex vector space with hermitian inner product $\langle \mid \rangle$ and let Γ be a lattice in V which is self-dual with respect to $\frac{1}{\pi}\,\mathrm{Im}\langle \mid \rangle$.

We choose a $\mathbb{Z}$ basis $\alpha_1, \dots, \alpha_g, \beta_1, \dots, \beta_g$ which is symplectic:

$$* \quad \begin{cases} \mathrm{Im}\langle \alpha_j \mid \alpha_k \rangle = \mathrm{Im}\langle \beta_j \mid \beta_k \rangle = 0 \\ \mathrm{Im}\langle \alpha_j \mid \beta_k \rangle = \pi\, \delta_{jk} \end{cases}$$

We saw that $\alpha_1, \dots, \alpha_g$ is a $\mathbb{C}$-basis for V. In effect, if $V_0 = \mathbb{R}\alpha_1 + \dots + \mathbb{R}\alpha_g$, then $V_0 \cap iV_0$ is a complex subspace of V, so $\mathrm{Im}\langle \mid \rangle$ is non-degenerate on it; as V_0 is isotropic this implies $V_0 \cap iV_0 = 0$, whence $V_0 + iV_0 = V$.

Let τ be the matrix defined by

$$\beta_j = \sum \alpha_k \tau_{kj}$$

In other words, using the basis $\alpha_1, \dots, \alpha_g$, we can identify V with $\mathbb{C}^n$ so that $\alpha_j = e_j$ (the standard basis vector) and $\beta_j = \tau e_j$. Thus

$$\Gamma = \mathbb{Z}^n + \tau \mathbb{Z}^n$$

The above conditions $*$ imply

$$\mathrm{Im}\langle e_\ell \mid \tau e_j \rangle = \mathrm{Im}\langle e_\ell \mid e_k \tau_{kj} \rangle$$

$$= \mathrm{Im}\big(\langle e_\ell \mid e_k \rangle \tau_{kj}\big) = \pi\, \delta_{\ell j}$$

Let $N_{\ell k} = \langle e_\ell \mid e_k \rangle$; this is real symmetric and positive-definite. We have

$$N(\operatorname{Im}\tau) = \pi$$

or

$$\boxed{N = \frac{\pi}{\operatorname{Im}\tau}}$$

Next I want to see that τ is symmetric. Start with

$$\begin{aligned}\langle\beta_j|\beta_k\rangle &= \langle \tau e_j|\tau e_k\rangle \\ &= \langle e_p\tau_{pj}\,|\,e_q\tau_{qk}\rangle \\ &= \bar\tau_{pj}\langle e_p|e_q\rangle\tau_{qk} = (\tau^*N\tau)_{jk}\end{aligned}$$

being a real symmetric matrix. Then

$$\begin{aligned}\tau^*N\tau &= (N\bar\tau)^t\tau \\ &= (\underbrace{N\bar\tau - N\tau}_{N(-2i\operatorname{Im}\tau) = -2i\pi} + N\tau)^t\tau\end{aligned}$$

$$\underbrace{\tau^*N\tau}_{\text{symm.}} = -2i\pi\tau + \underbrace{\tau^t N\tau}_{\text{symm.}}$$

showing $\boxed{\tau \text{ is symmetric}}$.

Now we consider $\varphi(\lambda, z)$. Recall this differs from $-\langle\lambda|z\rangle$ by a symmetric form, and it vanishes where $\lambda \in \mathbb{R}^n \longleftrightarrow \sum \mathbb{R}\alpha_j$. Since $\langle\lambda|z\rangle = \bar\lambda^{*} N z$ it is clear that

$$\varphi(\lambda, z) = (\lambda^t - \bar\lambda^t)\,N z$$

Then with $n \in \mathbb{Z}^g$ we have

$$\varphi(n, z) = 0$$

$$\varphi(\tau n, z) = n^t(\tau - \bar{\tau}) \frac{\pi}{\operatorname{Im} \tau} z$$

$$= 2i\pi\, n^t z$$

We seek an entire function $F(z)$ satisfying

$$F(z) = F(z+n) = e^{i\pi n^t \cdot \tau n + 2i\pi n^t z} F(z + \tau n)$$

for all $n \in \mathbb{Z}^g$. The solution is unique up to a scalar factor and is Reimann's θ.

$$F(z) = \sum_{n \in \mathbb{Z}^g} e^{i\pi n^t \tau n + 2i\pi n^t z}$$