March 27, 1986

Problem: Consider the family of Dirac operators modulo gauge transformations on the trivial line bundle over the circle. The parameter space is $U(1)$. There are many spaces of operators representing K^1, for example $\mathcal{F}_1$, $U(\mathcal{K})$, $\mathcal{J}(2)$, graded restricted unitary group. The index of this family is an element of $K^1(U(1))$. The problem is to explicitly construct a map from $U(1)$ to a suitable space representing K^1 which is the index of the family.

This morning I tried the approach using Kuiper's theorem to trivialize the Hilbert bundle.

$$H = L^2(S^1) \qquad S^1 = \mathbb{R}/\mathbb{Z}$$

$$\tilde{H} = \mathbb{R} \times^{\mathbb{Z}} H \qquad \text{with } 1 \in \mathbb{Z} \text{ acting as } +2\pi \text{ on } \mathbb{Z} \text{ and } \cdot e^{2\pi i x} \text{ on } H.$$

$$(\theta, e^{2\pi i x} f) \sim (\theta + 2\pi, f)$$

$$\tilde{H}_{\bar\theta} = \text{fibre over } \bar\theta \in \mathbb{R}/2\pi\mathbb{Z}$$

$$\begin{aligned} \iota_\theta : H &\longrightarrow \tilde{H}_{\bar\theta} \\ f &\longmapsto \text{class of } (\theta, f) \end{aligned}$$

$$\boxed{\begin{aligned} \iota_\theta \circ e^{2\pi i x} &= \iota_{\theta + 2\pi} \\ \iota_\theta^{-1} \circ D_{\bar\theta} \circ \iota_\theta &= \tfrac{1}{i}\partial_x + \theta \end{aligned}}$$

To trivialize the Hilbert bundle we have to deform $e^{2\pi i x} \in \mathrm{Map}(S^1, U(1))$ to the identity. This is not possible in continuous function but it's true in measurable functions for example

$$e^{i\theta\beta(x)} \qquad\qquad \beta(x) = \begin{cases} x & 0 \leq x < 1 \\ x - [x] \end{cases}$$

where the deformation is $0 \leq \theta \leq 2\pi$. The trivialization is

$$j_{\bar\theta} : H \xrightarrow{\sim} \tilde{H}_{\bar\theta}$$

where

$$\boxed{j_{\bar\theta} = \iota_\theta \circ e^{-i\theta\beta(x)}}$$

whence $D_{\bar\theta}$ becomes

$$j_{\bar\theta}^{-1} \circ D_{\bar\theta} \circ j_{\bar\theta} = e^{i\theta\beta}\left(\frac{1}{i}\partial_x + \theta\right) e^{-i\theta\beta}$$

$$= \boxed{\frac{1}{i}\partial_x + \theta \sum_{n\in\mathbb{Z}} \delta(x+n)}$$

There is a way to interpret this as a ~~[illegible]~~ self-adjoint operator on H, and upon performing the transformation $D \mapsto D/\sqrt{1+D^2}$ one ~~[illegible]~~ gets a map from $R/2\pi\mathbb{Z}$ to $\mathcal{F}_1(H)$. This is not obvious from the above formula, but seems to follow from its derivation. ? In any case this use of discontinuous functions is very confusing.

Review the <u>easy part of Kuiper's theorem</u>:

The inclusion $U(H) \longrightarrow U(H \oplus H')$, $g \mapsto g \oplus 1$ is null-homotopic. Here $H' \simeq \underbrace{H \oplus H \oplus \cdots\cdots}_{\text{infinitely many times}}$

The proof is to note first that

$$g \longmapsto \begin{pmatrix} g & 0 \\ 0 & g^{-1} \end{pmatrix} \qquad \text{from } U(H) \text{ to } U(H \oplus H)$$

is null-homotopic. In effect

$$\begin{pmatrix} g & 0 \\ 0 & g^{-1} \end{pmatrix} \sim \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} g & 0 \\ 0 & g^{-1} \end{pmatrix} = \begin{pmatrix} 0 & g^{-1} \\ g & 0 \end{pmatrix}$$

using a path joining $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ to the identity. Then we have the path $0 \leq \theta \leq \frac{\pi}{2}$

$$(\cos\theta)\varepsilon + (\sin\theta)F = \begin{pmatrix} \cos\theta & (\sin\theta)g^{-1} \\ (\sin\theta)g & -\cos\theta \end{pmatrix}$$

joining $\begin{pmatrix} 0 & g^{-1} \\ g & 0 \end{pmatrix}$ to $\varepsilon = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, which can be joined to the identity.

Then observe that

$$\begin{pmatrix} g & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \\ & & & & \ddots \end{pmatrix} = \begin{pmatrix} g & & & & \\ & g^{-1} & & & \\ & & g & & \\ & & & g^{-1} & \\ & & & & \ddots \end{pmatrix}\begin{pmatrix} 1 & & & & \\ & g & & & \\ & & g^{-1} & & \\ & & & g & \\ & & & & g^{-1} \\ & & & & & \ddots \end{pmatrix}$$

so the map $U(H) \longrightarrow U(H \oplus H')$ is the product of two null-homotopic maps. QED.

March 28, 1986

Let's review the Milnor classifying space.

If X is a G-space, then the join $X * G$ is a G-space with the following property: A G-map $F: X * G \longrightarrow Y$ is the same thing as a G-map $f: X \longrightarrow Y$ and a null-homotopy h_t of f through not necessarily G-equivariant maps. In effect

$$P * G = P \times I \times G \Big/ \begin{array}{l}(p,0,g) = p \\ (p,1,g) = g\end{array}$$

so that $F: X * G \longrightarrow Y$ equivariant has the form

$$F(p,t,g) = h_t(pg^{-1})g$$

$$\begin{array}{lll} F(p,0,g) \text{ ind. of } g & \Longrightarrow & h_0 \text{ } G\text{-equivariant} \\ F(p,1,g) \text{ ind. of } p & \Longrightarrow & h_1(P) = pt \end{array}$$

Given a principal G-bundle $P \to X$ and a G-bundle $E \to B$ with E contractible, suppose we have a bundle map
$$\begin{tikzcd} P \arrow[r, "\varphi"] \arrow[d] & E \arrow[d] \\ X \arrow[r] & B \end{tikzcd}$$
. Then because E is contractible we can successively extends φ to

$$P \subset P * G \subset P * G * G \subset \cdots\cdots$$

The union of this sequence is contractible, because each space contracts to a point in its successor. Notice that

$$\bigcup_n P * \overbrace{G * \cdots * G}^{n} \quad = \quad P \times^G \underbrace{\bigcup_n \overbrace{G * \cdots * G}^{n+1}}_{\text{Milnor's } E_G}.$$

In the easy part of Kuiper's thm. we prove that $U(H) \longrightarrow U(H \oplus H')$ $\quad H' = \bigoplus_{\mathbb{N}} H$ is null-homotopic. This is not the same as proving the map on classifying spaces is null-homotopic. However we should be able to prove ~~that~~ this stronger assertion by infinite repetition.

Let's first look at the situation from the viewpoint of obstruction theory. Suppose that I have $X = Y \cup e$ and a principal G-bundle P over X which has been trivialized over Y. Over a disk mapping onto e P ~~becomes~~ trivial and we can compare over the boundary of the disk the two trivializations, thereby obtaining a map $\partial e \longrightarrow G$ which is the obstruction to extending the triv. of P over Y to X. If we have a map $G \to G'$ which is null-homotopic, then the obstruction vanishes for the induced G'-bundle.

This implies that if we have a sequence of groups

$$G = G_0 \longrightarrow G_1 \longrightarrow G_2 \longrightarrow G_3 \longrightarrow \cdots$$

each null-homotopic in the succeeding one, and a principal G_0-bundle $P_0 \to X$, then P_0 is trivial over the 0-skeleton, $P_0 \times^{G_0} G_1$ is trivial over the 1-skeleton, and .. $P_0 \times^{G_0} G_n$ is trivial over the n-skeleton.

Suppose we take $P \to X$ to be the Milnor model for $EG \to BG$, and ~~by~~ take the n-skeleton to be $\overbrace{G * \cdots * G}^{n+1}/G$. Call this X_n so that

$$X_n - X_{n-1} = C(G^{*n}) - \text{base } G^{*n}$$

To be more precise we have

$$\begin{tikzcd} G^{*n}\times\{0\} \arrow[r, phantom, "\subset"] \arrow[d] & G^{*n}\times I \arrow[r, phantom, "\supset"] \arrow[d] & G^{*n}\times\{1\} \arrow[d] \\ X_{n-1} \arrow[r, phantom, "\subset"] & X_n \arrow[r, phantom, "\supset"] & pt \end{tikzcd}$$

better we have a pushout

$$\begin{tikzcd} G^{*n}\times\{0,1\} \arrow[r, phantom, "\subset"] \arrow[d] & G^{*n}\times I \arrow[d] \\ X_{n-1}\amalg pt \arrow[r, phantom, "\subset"] & X_n . \end{tikzcd}$$

The point is that the principal bundle becomes trivial over $X_n - X_{n-1}$ because it is contractible. Thus the same argument as before work and we see that the principal G_n-bundle

$$\begin{tikzcd} (G^*)^{n+1}\times^G G_n \arrow[d] \\ X_n \end{tikzcd}$$

is trivial.

So we have learned how to think constructively about Kuiper trivializations. ∎ In order to trivialize over the n skeleton, we will need ~~to~~ use n successive Hilbert space embeddings.

Question: Let H be a unitary representation of G, and form the induced vector bundle over BG. Is it possible to embed this vector bundle into a trivial Hilbert space bundle?

For example consider the standard repn. of $U(n)$ on $\mathbb{C}^n$. Can you find an embedding of the associated vector bundle over $BU(n)$ in a trivial Hilbert bundle? This is tautologically obvious if we take $BU(n)$ to be the Grassmannian of n-diml subspaces in Hilbert space.

Now in the case of $U(1)$ we know that the Milnor model gives $G^{*n} = S^{2n-1}$ over $\mathbb{C}P^{n-1}$. In general the Milnor model for $BU(n)$ sits inside the Grassmannian of n planes; this generalizes the graph construction which embeds the suspension of $U(n)$ inside $\mathrm{Grass}_n(\mathbb{C}^{2n})$.

In general then, given a representation H of G we let $H' = \bigoplus_{n \geq 0} H$. Consider the frame bundle over $U(H)$ consisting of all embeddings of H into H'. We a canonical embedding i_n for each n. We can map $G^{*(n+1)}$ into this frame bundle by sending $\sum_{j=0}^{n} t_j g_j$ $\quad (t_j \geq 0, \sum t_j = 1)$ into the embedding

$$\sum \sqrt{t_j}\, i_j\, g_j : H \longrightarrow H'$$

Thus we have a canonical map

$$EG \times H \longrightarrow H'$$

which passes to the quotient to yield a

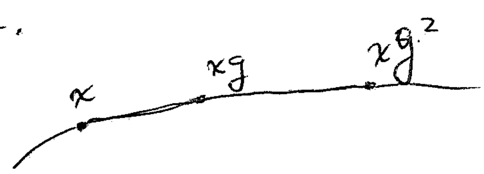
map $$EG \times^G H \longrightarrow H'$$
which is the desired embedding of $EG \times^G H$ over BG into the trivial Hilbert bundle $BG \times H' / BG$.

Now you can play the infinite repetition game: If $P \oplus Q \simeq F$, then

$$P \oplus F \oplus F \oplus \cdots\cdots = P \oplus (Q \oplus P) \oplus (Q \oplus P) \oplus \cdots$$
$$= (P \oplus Q) \oplus (P \oplus Q) \oplus \cdots\cdots$$
$$= F \oplus F \oplus \cdots\cdots$$

Thus if I add a trivial Hilbert bundle to $EG \times^G H$, then I get a trivial Hilbert bundle.

Suppose that the group $G = \mathbb{Z}$; here you want to use $EG = \mathbb{R}$. Consider the space X of embeddings $\mathrm{Inj}(H, H')$, where we recall H is the repn. of G and H' some Hilbert space. G acts to the right on X and we seek an equivariant map $EG \longrightarrow X$. So pick an $x \in X$ and choose a path from x to xg, g being the generator of $\mathbb{Z}$.



Then equivariance specifies the rest.

In other words an equiv. map $EG \to X$ is the same as a point x and a path joining x to gx. It is necessary to suppose H' is big enough for this to exist.

Problem: In the case of the Hilbert bundle over $U(n)$ associated to the representation on $\Omega U(n)$ on $L^2(S^1)^n$ you want to represent the index as a map of $U(n)$ to a space of operators in a nice way.

This is already clear for $n=1$, where I can suppose $G = \mathbb{Z}$ with 1 acting as $z = e^{ix}$ on $L^2(S^1)$. Suppose I try to construct an equivariant map from $EG = \mathbb{R}$ to $\mathrm{Inj}(H,H')$. This is the same as given a family of embeddings $j_t : H \longrightarrow H'$ such that $j_1 = j_0 z$. To do this we need an orthonormal sequence $j_t(z^n)$ $n \in \mathbb{Z}$. The only way I can see to do this is to have $H' = H \oplus H$, and consider imbeddings compatible with multiplication by z.

We can then construct j_t as follows. First rotate in the plane spanned by $\binom{1}{0}$ and $\binom{0}{1}$ so as to carry $\binom{1}{0}$ to $\binom{0}{1}$. Then rotate back in the plane spanned by $\binom{z}{0}$, $\binom{0}{1}$. These planes generate outgoing subspaces and correspond to loops in $U(2)$. I ~~think~~ we are explicitly contracting the loop $\begin{pmatrix} z & 0 \\ 0 & z^{-1} \end{pmatrix}$ to the identity.

General Ideas: I am trying to find a clean way to identify BU_{res} with $\mathcal{I}(2)$. This relies on Kuiper's thm. which I want to replace with something a bit more constructive.

In particular I feel that the natural thing to look for is a contractible G-space mapping equivariantly to the space of embeddings of H into some H'.

In the case of U_{res} there is a natural contractible G-space, namely the $A \in \mathcal{F}_1(H)$ lying over the involution. So the question is how to associate naturally an embedding to an A.

In the loop group case we have the space of connections which is a natural "building". Connections have eigenvalues and there's so much structure that what I am looking for should be obvious. (If not I am using the wrong approach.)

March 29, 1986:

The problem is as follows: Consider the family of Dirac operators on the circle parametrized by $U(n)$. The index of this family is a class in $K^1(U(n))$. The problem is to represent this class concretely by a map from $U(n)$ to a suitable space of operators which represents K^1.

Rmk 1. Over $U(n)$ is a Hilbert bundle with self-adjoint Fredholm family. According to Kasparov theory such a gadget represents an element of $K^1(U(n))$. To realize it by a map $U(n) \longrightarrow \mathcal{F}_1$ one must resort to a Kuiper type theorem.

Rmk 2. I saw yesterday that the easy type of Kuiper result $(E \oplus H_X \simeq H_X)$ results from being able to imbed the Hilbert bundle in a trivial one. Thus arises the problem of ~~assigning to~~ constructing an equivariant map from a contractible $\mathcal{G}$-space to the space of embeddings $\mathrm{Inj}(H, H')$, where $\mathcal{G}$ acts trivially on H'. A simpler question would be ~~the maps~~ to construct sections of $E\mathcal{G} \times^{\mathcal{G}} H$, i.e. to map a contractible $\mathcal{G}$-space equivariantly into H. Thus

Question: Is there a natural (i.e. $\mathcal{G}$-equivariant) way to assign to a connection $A \in \mathcal{A}$ a vector $v_A \in H = L^2(S^1)^n$?

Yes. Take a vector in the fibre over the basepoint and start parallel transporting it along S^1, then multiply by a fn. . Notice what we have done is to construct a map from the trivial bundle over S^1 to any bundle with connection over S^1 of the same rank. By means of a partition of unity over S^1 we can produce 2 such maps which span the fibres and are compatible with the metrics.

So what I see now is that I should have been thinking in terms of ~~the~~ vector bundle E over $U(n) \times S^1$ obtained by taking the trivial rank n bundle over $\mathcal{A} \times S^1$ and dividing out by the action of $\mathcal{G} = \Omega U(n)$. The problem of embedding the family $L^2(S^1, E_y) = H_y$ for $y \in U(n)$ into a trivial Hilbert family becomes the same (at least if we stick to smooth sections and ask for $C^\infty(S^1)$-

module isoms.) as embedding E in a trivial bundle over $U(n)\times S^1$.

What bundle E over $U(n)\times S^1$ is obtained? It is clear from the partition of 1 argument that E embeds in a trivial bundle of rank $2n$, hence comes from a map

$$U(n)\times S^1 \longrightarrow \mathrm{Grass}_n(\mathbb{C}^{2n}).$$

which suggests that we have the periodicity map.

This is of course clear if you recall how you used to construct the family of Dirac operators on the indexed trivial bundle over S^1. You take the trivial bundle over $U(n)\times\mathbb{R}$ and define the action of $\mathbb{Z}$ to translate on $\mathbb{R}$ and upstairs to use the element of $U(n)$ as the monodromy. Thus we are using the obvious clutching function to glue the ends of $U(n)\times[0,1]$ together.

The bundle E over $U(n)\times S^1$ has a canonical connection in the S^1-direction. One can see this either by the construction

$$E = \mathcal{G}\backslash \mathcal{A}\times S^1\times\mathbb{C}^n$$

where there is a tautological connection in the S^1 direction, or also from the construction

$$E = U(n)\times\mathbb{R}\times\mathbb{C}^n/\mathbb{Z}.$$

Thus on sections of E over $\{g\}\times S^1$ we have the Dirac operator.

The problem now is to determine the index of this family as concretely as possible.

What we are doing is to mix, i.e. take the Kasparov cup product of, the Dirac operator on S^1 with the Bott element in $K^1(U(n)\times S^1)$.

March 30, 1986

We consider a family E_y of vector bdles (with inner product) over S^1 param. by Y, i.e. a v.b. E over $Y\times S^1$. Suppose partial connections given in the S^1 direction, whence we have a family of Dirac operators on S^1 param. by Y. The problem is to identify the index of this family with the map $Y \to U(n)$ defined by associating to a connection its monodromy.

In order for this monodromy to be well-defined it is necessary to suppose that over the basepoint 1 of S^1 the family E is trivialized. Otherwise the monodromy will be an automorphism of the bundle $i_1^*(E)$ where $i_1 : Y \to Y\times\{1\} \subset Y\times S^1$

(Question: Do bundle automorphisms naturally link up with the graded restricted unitary group?)

The difficulty with the problem seems to be in suitably defining the index of the family. The natural definition is to take the Hilbert bundle over Y with $H_y = L^2(S, E_y)$ together with the self-adjoint Fredholm $F_y = D_y/\sqrt{1+D_y^2}$, where D_y

is the Dirac operator. In order to convert this to a map from Y to $\mathcal{F}_1$, we embed the Hilbert bundle in a trivial bundle and extend F_y by the identity on the complement.

Remark: The specific F_y on H_y is not relevant; it suffices to consider any self-adjoint contraction operator whose image mod compacts is the involution given by the symbol of the Hilbert transform. Thus what connections are chosen in the bundle E_y don't matter up to homotopy.

Now in order to embed the family H_y in a trivial family, it suffices to embed E as a trivial bundle over $Y \times S^1$. Since S^1 has an open covering by two open intervals, we can do the embedding in the trivial bdle of rank $2n$. In other words we can suppose E comes a map $Y \times S^1 \longrightarrow \mathrm{Grass}_n(\mathbb{C}^{2n})$. Thus to each $y \in Y$ we get a projector over $C^\infty(S^1)$, a $2n \times 2n$ idempotent self-adjoint matrix, call it e_y. $H_y = L^2(S^1, E_y)$ is the image of e_y on $H' = L^2(S^1)^{2n}$.

As said above the actual connection on E_y is not important up to homotopy, so we can equip E_y with the Grassmannian connection associated to e_y. (For that matter, any connection on E over $Y \times S^1$ can be assumed to be Grassmannian relative to an embedding of E in a trivial bundle.)

So we assume then that the families E_y, H_y are given by the images of e_y, where e_y is a

family of ~~[illegible]~~ projectors in $C^\infty(S^1)$.
Now we can describe the family index as follows. Let $F \in \mathcal{F}_1(H')$ coincide with the Hilbert involution mod compacts. Then set

$$\tilde{F}_y = e_y F e_y + (1-e_y)$$

Alternatively we could use instead of $e_y F e_y$, the operator $D_y/\sqrt{1+D_y^2}$, where $D_y = e_y \circ \frac{1}{i}\partial_x \circ e_y$ is the Dirac operator on $H_y = e_y H'$ associated to the Grassmannian connection.

<u>Summary</u>: We have constructed a map $Y \longrightarrow \mathcal{F}_1(H')$ representing the ~~image~~ index of the family. To do this we take E and embed it in a trivial bundle over $Y \times S^1$; this gives ~~[illegible]~~ $H_y \simeq e_y H'$, $H' = L^2(S^1)^N$ where e_y is a projector over $C^\infty(S^1)$. Then we extend $F_y \in \mathcal{F}_1(H_y)$ to $\tilde{F}_y \in \mathcal{F}_1(H')$.

Now the problem is to prove the following square commutes up to homotopy

$$\begin{tikzcd}
\Omega\, \mathrm{Grass}_n(\mathbb{C}^N) \arrow[r, "e \longmapsto eFe + (1-e)"] \arrow[d, "\text{monodromy}"'] & \mathcal{F}_1(H') \arrow[d, "\text{AS exponential map}"] \\
U(n) \arrow[r, leftrightarrow, "\text{natural inclusion}", "\text{(unique up to homotopy as frame bundle is cont.)}"'] & U(\mathcal{K})
\end{tikzcd}$$

Apparently I am trying to prove a version of

$$\det(\partial_t + A) \sim \det(1 - \text{monodromy})$$

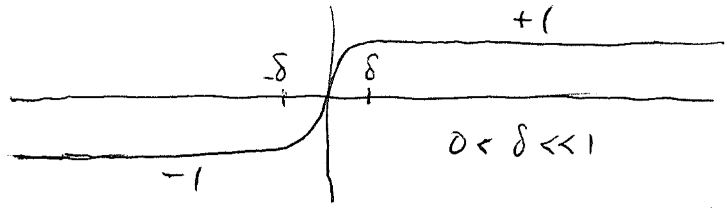
<u>Question</u>: What is the link between a Dirac operator over S^1 and the monodromy of its connection?

The spectrum of the Dirac operator is completely determined by the monodromy. In particular the number of zero modes equals the number of $+1$ eigenvalues of the monodromy. Recall the formula:

$$\det(\partial_t + A) \text{ "="} \det(1 - M)$$

March 31, 1986

Let $f(x)$:

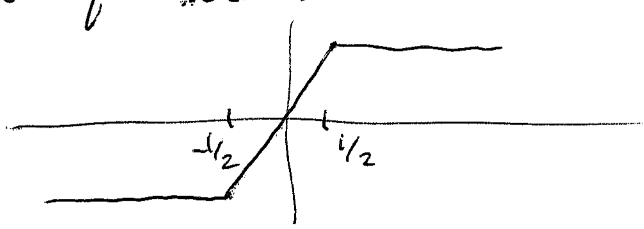


be a smooth approximation to $\operatorname{sgn}(x)$. Given a Dirac op D on $L^2(S^1, E) = H$ we form $f(D) \in \mathcal{F}_1(H)$ and $-\exp(i\pi f(D)) \in +\mathcal{U}(\mathcal{H})$. In fact $-\exp(i\pi f(D))$ differs from 1 by a finite rank operator. To see this take $D_a = \frac{1}{i}\partial_x + a$ on $L^2(S^1)$, where $D_a e^{inx} = (n+a)e^{inx}$, $S^1 = \mathbb{R}/2\pi\mathbb{Z}$. Then $-\exp(i\pi f(n+a)) \neq 1$ only if a is very close to $-n$, which can happen for at most 1 value of $n \in \mathbb{Z}$.

As a varies over $\mathbb{R}$, $a \mapsto U_a = -\exp(i\pi f(D_a))$ maps $\mathbb{R}$ onto a wedge of loops in $\mathcal{U}(\mathcal{H})$. As a moves from $n-\delta$ to $n+\delta$, U_a runs around the group $U(1) \subset \mathcal{U}(\mathcal{H})$ viewed as unitaries in the 1 dim'l space $\mathbb{C}e^{inx} \subset L^2(S^1)$.

In general D can have at most n eigenvalues in the range $(-\delta, \delta)$, so that $-\exp(i\pi f(D)) = U_D$ is supported in a space of dimension $\leq n$.

It's clear that U_D is roughly "equivalent" to the monodromy of D. This "equivalence" would be better if we used instead of f the function



Now the problem is the following. If we have a family of Dirac operators ~~on~~ the circle parametrized by Y, say D_y on $H_y = L^2(S^1, E_y)$, then $U_{D_y} = -\exp(i\pi f(D_y))$ is a bundle automorphism of the Hilbert bundle $\{H_y\}$, which is congruent to 1 mod $\mathcal{K}$. On the other hand using the basepoint $*$ of S^1, and the monodromy for the connections, over Y I get a vector bundle $y \longmapsto E_y|*$ and a bundle automorphism. The problem is to show that these two bundles with automorphism over Y represent the same element of $K^1(Y)$.

Let's try to be explicit as possible in the case of the family over $U(n)$.

There are two ways to construct the canonical vector bundle E over $U(n)\times S^1$, depending on ~~whether we use the~~ which of the principal bundles

$$\mathcal{G}\times S^1 \xrightarrow[\mathcal{G}]{} U(n)\times S^1 \xleftarrow[\mathbb{Z}]{} U(n)\times \mathbb{R}$$

we use. This may be useful later, but for the moment we will use the latter. We make $\mathbb{Z}$ act on $U(n)\times\mathbb{R}\times\mathbb{C}^n$ by letting the generator acting on (g, x, v) be $(g, x+1, gv)$.

Maybe a better description for our purposes is to give the sections of E_g/S^1 for $g\in U(n)$. This is the space of smooth functions $f(x)$ on $\mathbb{R}$

with values in $\mathbb{C}^n$ such that

$$\star \qquad f(x+1) = g^{-1} f(x)$$

(In other words at the point $g \in U(n)$ we are taking the vector bundle over $S^1 = \mathbb{R}/\mathbb{Z}$ associated to the representation $n \mapsto g^n$ of $\mathbb{Z}$ on $\mathbb{C}^n$). Thus

$$\Gamma(S^1, E_g) = \left\{ f \in C^\infty(\mathbb{R}, \mathbb{C}^n) \,\middle|\, f(x+1) = g^{-1} f(x) \right\}$$

(If we use the F.T.

$$f(x) = \int \frac{d\xi}{2\pi} e^{i\xi x} \hat{f}(\xi)$$

then the condition $\star$ becomes

$$e^{i\xi} \hat{f}(\xi) = g^{-1} \hat{f}(\xi)$$

which means that $\hat{f}$ is supported on those ξ such $e^{i\xi}$ is an eigenvalue of g^{-1}.)

Having described $\Gamma(S^1, E_g)$, we have the Hilbert space bundle

$$H_g = L^2(S^1, E_g)$$

consisting of measurable f (mod null equivalence) satisfying $\star$ with norm

$$\int_0^1 |f(x)|^2 dx .$$

The Dirac operator on $\Gamma(S^1, E_g)$ and H_g is given by $\frac{1}{i} \partial_x$.

Let us now consider the problem of identifying the index of this family param. by $U(n)$ with the canonical elt of $K^1(U(n))$. I've seen above that the index is represented by ~~unitary operator~~ family of unitary operators $\equiv 1 \mod \mathcal{K}$ in the family of Hilbert space H_g; that is, by a bundle automorphism of this Hilbert bundle over $U(n)$. I still have to somehow see the K-elt represented this way is the canonical element of $K^1(U(n))$.

I have not been able to figure out a way to proceed other than by embedding the Hilbert bundle in a trivial one. So I want to embed the vector bundle E over $U(n) \times S^1$ into a trivial bundle. I know how to do this using the trivial bundle of rank $2n$, but the construction involves choices, and it would be nice if there were a more canonical embedding possibly in infinite dims.

Embedding in a trivial bundle is roughly equivalent to producing sections of the dual bundle.

Recall that in the case $n=1$ we can identify sections of the line bundle E over $S^1 \times U(1)$ with $\mathcal{S}(\mathbb{R})$. With a slight change in notation we can identify sections of E with smooth fns. $F(x,y)$ on $\mathbb{R}^2$ satisfying

$$\begin{cases} F(x+1,y) = e^{2\pi i y} F(x,y) \\ F(x,y+1) = F(x,y) \end{cases}$$

To $f(x) \in \mathcal{S}(\mathbb{R})$ we assign

$$F_f(x,y) = \sum_{n\in\mathbb{Z}}{}' e^{2\pi i n y} f(x-n)$$

and this gives the isomorphism of $\mathcal{S}(\mathbb{R})$ with $C^\infty(\mathbb{T}^2, E)$.

Sections of the dual bundle $\check{E}$ can be identified with smooth $G(x,y)$ on $\mathbb{R}^2$ $\ni$

$$G(x+1,y) = e^{-2\pi i y} G(x,y)$$
$$G(x,y+1) = G(x,y)$$

The pairing of $E, \check{E}$ sends F, G to $F(x,y)G(x,y)$ which is doubly-periodic.

To embed E in a trivial bundle means we find sections F_i of E and sections G_i of $\check{E}$ such that

$$F \longmapsto \sum_i F_i \cdot G_i F$$

is the identity on $C^\infty(\mathbb{T}^2, E)$. Thus we want

$$\sum_i F_i(x,y)\, G_i(x,y) = 1$$

Now put in the representations

$$F_i(x,y) = \sum e^{2\pi i n y} f_i(x-n)$$

$$G_i(x,y) = \sum e^{2\pi i m y} g_i(x+m)$$

and we want

$$1 = \sum_i \sum_{n,m} e^{2\pi i (n)y} f_i(x-n+m)\, g_i(x+m)$$

i.e. we want $K(x,y) = \sum_i f_i(x)\, g_i(y)$

to satisfy

$\bigstar$ $$\sum_m K(x-n+m,\, x+m) = \begin{cases} 1 & n=0 \\ 0 & n\neq 0 \end{cases}$$

Conversely if we have a smooth Schwartz function K with this property, we can apply

$$\mathcal{S}(\mathbb{R}) \hat{\otimes} \mathcal{S}(\mathbb{R}) = \mathcal{S}(\mathbb{R}^2)$$

to obtain f_i, g_i (possibly infinitely many, but still one will have some sort of embedding in a trivial bundle).

We analyze $\bigstar$ as follows (in the hope of obtaining some sort of nice candidate for K.) Change the axes so that the diagonal $x=y$ becomes the x-axis. We then want

$$\sum_m K(x+m,\, n) = \begin{cases} 1 & n=0 \\ 0 & n\neq 0 \end{cases}$$

Recall the P.S. formula

$$\sum_{m\in\mathbb{Z}} f(x+m) = \sum_m e^{2\pi i m x} \hat{f}(2\pi m)$$

so that is constant $\iff \hat{f}(2\pi m) = 0$ for $m \neq 0$, $m \in \mathbb{Z}$. Thus if we transform $K(x,y)$ to $K(\xi, y)$ we want

$$K(m,n) = \begin{cases} 1 & (m,n) = (0,0) \\ 0 & (m,n) \in \mathbb{Z}^2 - (0,0) \end{cases}$$

Thus arises the question whether there are nice elements of $\mathcal{S}(\mathbb{R}^2)$ whose restriction to $\mathbb{Z}^2$ is $\delta_{(0,0)}$.

A simpler question is whether there exists a nice $f(x) \in \mathcal{S}(\mathbb{R})$ whose restriction to $\mathbb{Z}$ is δ_0. This is the same as a section of E

$$F(x,y) = \sum_n{}' e^{2\pi i n y} f(x-n)$$

such that $F(0,y) = 1$.

I don't seem to be able to find really nice canonical formulas. The simplest way I know to construct a function $\varphi(x) \in \mathcal{S}(\mathbb{R})$ such that $\sum' \varphi(x+n) = 1$ is to let

$$\varphi(x) = f(x+1) - f(x)$$

where $f(\pm\infty)$ exist and $f(+\infty) - f(-\infty) = 1$. Actually we need $f(x) \to f(\pm\infty)$ as $x \to \pm\infty$ to be very rapid. This is roughly like saying the Fourier transform $\hat{\varphi}(\xi)$ is a multiple of $\frac{e^{i\xi}-1}{\xi}$.

Anyway we don't seem to be able to find simple elegant ways of embedding E in a trivial bundle.

April 1, 1986

Let's consider the canonical family of Dirac operators on S^1 parametrized by $U(n)$. The index of this family is self adjoint Fredholm bundle map on the bundle of Hilbert spaces associated to this family. The problem is to identify this index with the canonical element of $K^1(U(n))$.

The best I seem to be able to do is to use stratification. This is based on the fact that if D is a Dirac associated to a connection with monodromy g, then

$$\mathrm{Ker}\, D \xrightarrow{\sim} \mathrm{Ker}(g-1)$$

where the isomorphism is evaluation at the basepoint.

April 2, 1986

Let D be a Dirac on S^1 with monodromy g. I claim that evaluation at the basepoint sets up an isomorphism

$$\mathrm{Ker}\, D \xrightarrow{\sim} \mathrm{Ker}(g-1)$$

which is unitary up to a factor depending on the length of the circle. Hence it will be unitary if we make the length (volume) of the circle one. More generally for λ real

$$\mathrm{Ker}(D-\lambda) \xrightarrow{\sim} \mathrm{Ker}\left(e^{-2\pi i\lambda}g-1\right)$$

Formulas: $D = \frac{1}{i}(\partial_x + A)$, $g = T\left\{e^{-\int_0^{2\pi} A\,dx}\right\}$

Now use the fact that eigenspaces belonging to different eigenvalues are orthogonal. This will tell us that if we choose an interval $I \subset \mathbb{R}$ such that I injects into $\mathbb{R}/\mathbb{Z}$, the evaluation at the basepoint gives a unitary isomorphism between the subspace ~~with~~ where D has values in I and the subspace where g has values in $\exp(2\pi i I)$.

$$L^2(S^1, E) \supset V_I \xrightarrow{\sim} W_I \subset E(*)$$

April 4, 1986

Recall old viewpoint: If I want to study the cohomology of $B\mathcal{G}$, $\mathcal{G}$ a group of gauge transfs, ~~then I am forced to~~ using differential forms, then I am forced to use connections. The reason: In order to assign forms on Y to a ~~bundle~~ vector bundle E over $Y \times M$, I need a connection on E, and this is a connection in the associated $\mathcal{G}$-bundle P over Y together with an equivariant map $P \longrightarrow \mathcal{A} =$ the space of connections. Thus the cohomology of $B\mathcal{G}$ is to be handled by means of equivariant forms on $\mathcal{G}$, $\mathcal{A}$.

However <u>the whole idea of connection seems to become different in non-commutative differential geometry</u>. We have seen that it is relatively easy to attach left invariant forms on $\mathcal{G}$ starting from an $A \in \mathcal{F}_1(H)_\varepsilon$, whereas it seemed trickier starting from a connection. (The two are eventually to be linked.)

The point is that there should be some kind of non-commutative differential forms on $B\mathcal{G}$ that fits nicely with ~~all~~ the cyclic cocycle constructions.

I want to understand Connes connections

$$\nabla : \mathcal{E} \longrightarrow \mathcal{E} \otimes_{\mathcal{A}} \hat{\Omega}^1$$

in the case where $\mathcal{A} = C^\infty(M)$, $\mathcal{E} = \Gamma(M, E)$. It seems to me such a ∇ ought to be roughly the same as ~~an isomorphism~~ homomorphism

$$\varphi : pr_2^* E \longrightarrow pr_1^* E \quad ; \ \varphi(x,y) : E_y \to E_x$$

such that $\varphi / \Delta M$ is the identity.

Notice that

$$0 \longrightarrow \hat{\Omega}^1 \longrightarrow a \otimes a \longrightarrow a \longrightarrow 0$$

splits as left (or right) a-modules so

$$\mathcal{E} \otimes_a \hat{\Omega}^1 = \mathrm{Ker}\left(\underbrace{\mathcal{E} \otimes a}_{\Gamma(pr_1^* E)} \xrightarrow{\Delta^*} \underbrace{\mathcal{E}}_{\Gamma(E)}\right)$$

Given φ define ∇ by

$$\Gamma(E) \longrightarrow \Gamma(pr_1^* E)$$

$$s \longmapsto \big((x,y) \longmapsto \varphi(x,y)s(y) - s(x)\big)$$

As $\varphi(x,x) = id$, this ∇ lands in $\mathcal{E} \otimes_a \hat{\Omega}^1$. Thus

$$\nabla(s)f \quad (x,y) = [\varphi(x,y)s(y) - s(\cancel{x})] f(y)$$

$$+$$

$$s \cdot df \quad (x,y) = s(x)[1\otimes f - f \otimes 1](x,y) = s(x)[f(y) - f(x)]$$

$$\nabla(sf)(x,y) = \varphi(x,y)s(y)f(y) - s(x)f(x)$$

and so ∇ is a connection.

Conversely given a connection $\nabla : \mathcal{E} \to \mathcal{E} \otimes_a \hat{\Omega}^1$ define φ by

$$\varphi(x,y)s(y) = (\nabla s)(x,y) + s(x)$$

i.e.

$$\varphi : \mathcal{E} \longrightarrow \mathcal{E} \otimes a$$

$$\varphi(s) = \nabla s + s \otimes 1$$

Then

$$\varphi(sf) = \nabla(sf) + (sf)\otimes 1$$

$$= (\nabla s)(1\otimes f) + (s\otimes 1)(1 \otimes f - \cancel{f\otimes 1}) + \cancel{(sf)\otimes 1}$$

$$= \varphi(s)(1 \otimes f)$$

so

$$\varphi(x,y) : E_y \longrightarrow (pr_1^* E)_{(x,y)} = E_x$$

so φ is bundle homomorphism $pr_2^* E \to pr_1^* E$.

An important example of a connection in Connes sense is a Grassmann connection. Here $E_x = \operatorname{Im} e_x$ where $e = \{e_x\}$ is a projector over $\mathcal{A}$. Then

$$\varphi(x,y) = e_x : E_y \longrightarrow E_x$$

This suggests the analogue of preserving the metric is that

$$\varphi(x,y)^* = \varphi(y,x)$$

(*) $$\|\varphi(x,y)\| \leq 1.$$

A natural question is to find conditions on a Connes connection which guarantees that it is Grassmannian. Take M to a finite set for example. We are given the family of Hilbert spaces V_x and $\varphi(x,y) : V_y \to V_x$. What we want to do is find an embedding into a larger family $V_x \overset{j}{\subset} H_x$ where $\tilde{\varphi}(x,y) : H_y \xrightarrow{\sim} H_x$ is unitary, where $j^* \tilde{\varphi} j = \varphi$, and where $\tilde{\varphi}$ satisfies the <u>cocycle</u> <u>condition</u>.

<u>Ex:</u> When there are two points this can be done assuming (*). This is just the result that any graded $A = \begin{pmatrix} 0 & T^* \\ T & 0 \end{pmatrix}$ can be expanded to a graded involution. Here the cocycle condition is empty.

This problem is related to <u>Stinespring's thm.</u>, that any completely positive map $\varphi : A \to \mathcal{L}(V)$, where A is a $*$ algebra is the contraction of a $*$-representation on a Hilbert space.

<u>Complete positivity</u> seems to mean that given any finite sequence $a_1, \ldots, a_n \in A$, the operator on $V^{\oplus n}$

given by $\left(\varphi(a_i^* a_j)\right)$ is $\geqslant 0$.

The proof. Assuming H found, one looks at the A-invariant subspace spanned by V. This is the closure of [redacted] the image of

$$A \otimes V \longrightarrow H \qquad (a, v) \longmapsto a\, j(v).$$

It can be recovered from the (degenerate) inner product on $A \otimes V$:

$$\left\langle \sum_i a_i\, j(v_i),\ \sum_j a_j'\, j(v_j') \right\rangle$$

$$\|$$

$$\sum_{ij} \left\langle v_i,\ \underbrace{j^* a_i^* a_j j}_{\varphi(a_i a_j)}\ v_j' \right\rangle$$

Conversely [redacted] the complete positivity of φ guarantees this is [redacted] a degenerate inner product on $A \otimes V$, and by completing one obtains the desired expansion.

Problem: We have even forms on U_{res} and odd forms on $\mathcal{F}_1$. The problem is to link them.

Recall that U_{res} $\sim \Omega\mathcal{F}_1$ because of the fibration $U_{res} \longrightarrow U(H) \longrightarrow \mathcal{J}(2)$ and the h.eq $\mathcal{F}_1 \longrightarrow \mathcal{J}(2)$. Thus over $\mathcal{F}_1$ is a principal U_{res} bundle with contractible total space.

However this isn't completely canonical because the map $U(H) \rightarrow \mathcal{J}(2)$ depends on the choice of point of $\mathcal{J}(2)$. What is canonical is the map $\mathcal{J}(H) \rightarrow \mathcal{J}(2)$. So we define W to be a pull-back

$$\begin{tikzcd}
 & & \mathcal{J}_{res}(H,\varepsilon) \arrow[dl] \arrow[d] \\
W \arrow[r] \arrow[d] & \mathcal{J}(H) \arrow[d] & \{\varepsilon\} \arrow[dl] \\
\mathcal{F}_1 \arrow[r] & \mathcal{J}(2) &
\end{tikzcd}$$

W is contractible; its fibre over A is $\mathcal{J}_{res}(H,\varepsilon)$ where $\varepsilon = A$ in 2.

Let's now take up the map $\mathcal{F}_1 \longrightarrow -U(\mathcal{K})$ defined either by $A \longmapsto \exp(i\pi A)$ or $A \mapsto (B+iA)^2$. The canonical odd biinvariant forms on $-U(\mathcal{K})$ pull-back to give the odd forms on $\mathcal{F}_1$. We want to lift them to W, write them as d of something, then restrict these even forms to the fibre to get even closed forms on $\mathcal{J}_{res}(H,\varepsilon)$. We have

$$\begin{tikzcd}
 & & \{-1\} \arrow[d, hook] \\
\mathcal{J}(H) \subset \mathcal{F}_1 \arrow[urr, bend left] \arrow[r, "\exp(itA)"] \arrow[d] & \Omega(U(a);1,-U(\mathcal{K})) \arrow[r, "\text{endpt}"'] \arrow[d, "\sim"] & -U(\mathcal{K}) \\
\mathcal{J}(2) \arrow[r, "\text{Bott}"', "\text{map}"] & \Omega(U(a);1,-1) &
\end{tikzcd}$$

The point is that we have two maps from W to $-\mathcal{U}(\mathcal{K})$:

$$\begin{tikzcd} & \mathcal{I}(H) \arrow[d] \arrow[rr, "-1"] & & -\mathcal{U}(\mathcal{K}) \\ W \arrow[ur] \arrow[dr] & \mathcal{I}(2) \arrow[rr, dotted] & & -\mathcal{U}(\mathcal{K}) \\ & \mathcal{F}_1 \arrow[u] \arrow[urr, "A \mapsto \exp(i\pi A)"'] & & \end{tikzcd}$$

and these two maps are homotopic because they are different representatives of the map in the homotopy category

$$W \longrightarrow \mathcal{I}(2) \longrightarrow \Omega(\mathcal{U}(\mathcal{Q}); 1, -1) \xrightarrow[\sim]{} -\mathcal{U}(\mathcal{K}).$$

Put another way a point of W is a pair (A, F) with $A \equiv F \pmod{\mathcal{K}}$ and one has the two maps $W \longrightarrow -\mathcal{U}(\mathcal{K})$ given by $(A,F) \longmapsto \exp i\pi A$, $\exp i\pi F = -1$. The claim is these are homotopic, and in fact an obvious homotopy is

$$-(\exp itA)(\exp itF)^{-1} \qquad 0 \le t \le \pi$$

We could also use a convex combination $\xrightarrow{h_t} (1-t)A + tF$.

Suppose we choose such a homotopy h_t from $(A,F) \longmapsto \exp(i\pi A)$ to -1. Now restrict this homotopy to the fibre $\{F \mid F \in \mathcal{I}_{res}(H)\}$ over $A \in \mathcal{F}_{1,\varepsilon}$. Then what we have is a map

$$\begin{array}{ccc} \mathrm{Susp}\ \mathcal{I}_{res}(H) & \longrightarrow & -\mathcal{U}(\mathcal{K}) \\ (t, F) & \longmapsto & h_t(A,F) \overset{e.g.}{=} -\exp(itA)\exp(itF)^{-1} \end{array}$$

What has to be done now is to set things up so that we get invariant forms on $\mathcal{I}_{res}(H)$.

We would like to get a Bott map from $\mathrm{Susp}(\mathcal{I}_{res}(H))$ to $\mathcal{U}(\mathcal{K})$, but there's an obvious problem in that $\exp(itF)$ is not in $\mathcal{U}(\mathcal{K})$. The method to handle this perhaps is to use the graded restricted unitary group

$$U^{\varepsilon}_{res} = \left\{ \begin{pmatrix} g_1 & 0 \\ 0 & g_2 \end{pmatrix} \,\middle|\, g_1 \equiv g_2 \mod \mathcal{K} \right\} \subset U(H)$$

$H = H^+ \oplus H^-$ where $H^+ = H^-$

This is the group of unitaries g satisfying

$$g \varepsilon g^{-1} = \varepsilon$$

$$g F g^{-1} \equiv F \pmod{\mathcal{K}} \qquad F = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

We can make U^{ε}_{res} act on $U(\mathcal{K})$ by

$$\begin{pmatrix} g_1 & 0 \\ 0 & g_2 \end{pmatrix} \cdot u = g_1 u g_2^{-1}.$$

The action is transitive, and the stabilizer of 1 is $\Delta U(H^+)$, hence we have fibration

$$\begin{tikzcd} 1 \arrow[r] & U(H^+) \arrow[r, "\Delta"] & U^{\varepsilon}_{res} \arrow[r] & U(\mathcal{K}) \arrow[r] \arrow[l, bend right, dashed] & * . \end{tikzcd}$$

with an obvious section. Hence $U^{\varepsilon}_{res} \sim U(\mathcal{K})$.

Recall also that we have the factorization of the Cayley map

$$\mathcal{F}_1 \longrightarrow U^{\varepsilon}_{res} \xrightarrow{\sim} U(\mathcal{K})$$

$$A \longmapsto \begin{pmatrix} -iB+A & \\ & iB+A \end{pmatrix} \longmapsto (-iB+A)(iB+A)^{-1} = -(B+iA)^2$$

April 7, 1986

Yesterday I concluded that trying to desuspend the odd forms, defined on $\mathcal{F}_1$ via the h.eq. $\mathcal{F}_1 \to U(\mathcal{K})$, using the fibration

$$\begin{tikzcd}
\mathcal{I}_{res} \arrow[r, equal] \arrow[d] & \mathcal{I}_{res} \arrow[d] \\
W \arrow[r] \arrow[d] \arrow[dr, phantom, "\text{cart}"] & \mathcal{I}(H) \arrow[d] \\
\mathcal{F}_1 \arrow[r] & \mathcal{I}(Q)
\end{tikzcd}$$

doesn't seem to work. We do get a map

$$\operatorname{Susp} \mathcal{I}_{res}(H) \longrightarrow +\, U(\mathcal{K})$$

but it doesn't appear to give invariant forms on the restricted Grassmannian.* This suggests the

<u>Problem:</u> Find a "Bott map" $\mathcal{I}_{res}(H) \longrightarrow \Omega\, U(\mathcal{K})$, hich is suitably equivariant for the U_{res} action.

After some thought it appears that I'm up gainst the difficulty encountered previously, namely , construct the index of the family over BG iven by the Hilbert bundle $EG \times^G H$ over BG nd the family of Fredholms given by the canonical <u>section of</u> $EG \times^G \mathcal{I}(Q)$ over BG.

* It's possible that the forms on $\mathcal{I}_{res}(H)$ obtained from this map could be deformed into invariant forms.

From working with families index thm. I know that in order to obtain forms on the parameter space I need a connection in the associated G-bundle. So to define an index map $BG \longrightarrow \mathcal{F}_1$ I [redacted] probably have to have a connection in EG [redacted] over BG; better the model I use for $EG \to BG$ ought to come with a canonical connection.

For example the Milnor model has a canonical connection. To see this, think of $G \subset U(H)$, [redacted] let $H' = \bigoplus_{n \geq 0} H$, & let $i_n : H \hookrightarrow H'$ be the canonical inclusions. We identify EG with the set of embeddings of H into H' [redacted] of the form

$$s = \sum_{n \geq 0} \sqrt{t_n}\, i_n g_n \quad : H \longrightarrow H'$$

The connection form for the Grassmannian connection is

$$s^* ds = \sum \sqrt{t_n}\, g_n^* \, d(\sqrt{t_n}\, g_n)$$

$$= \sum t_n g_n^* dg_n + \underbrace{\sum_n \sqrt{t_n} \frac{1}{2\sqrt{t_n}} dt_n}_{\frac{1}{2}\sum_{n \geq 0} dt_n \;=\; 0}$$

$$\boxed{\theta = \sum_{n \geq 0} t_n g_n^* dg_n}$$

This obviously makes sense in general.

Now because this model for BG comes equipped with a canonical embedding of $EG \times^G H$ into $BG \times H'$, we therefore get an index map

$$\begin{tikzcd}
E\mathcal{G}\times^{\mathcal{G}}\mathcal{F}_{1,\varepsilon} \arrow[d, "\pi"', "\sim"] \arrow[r, phantom, "\subset"] & E\mathcal{G}\times^{\mathcal{G}}\mathcal{F}_1(H)=\mathcal{F}_1(E\mathcal{G}\times^{\mathcal{G}}H/B\mathcal{G}) \arrow[d, hook] & \\
 & \mathcal{F}_1(B\mathcal{G}\times H'/B\mathcal{G}) \arrow[d, equal] & \\
B\mathcal{G} & B\mathcal{G}\times\mathcal{F}_1(H') \arrow[l] \arrow[r] & \mathcal{F}_1(H')
\end{tikzcd}$$

The index map goes from $BG \longrightarrow \mathcal{F}_1(H')$, and depends upon choosing a section of the map π, say using convexity of $\mathcal{F}_{1,\varepsilon}$.

The next step is to carry this construction out over the "1-skeleton" of $B\mathcal{G}$ which is $\Sigma\mathcal{G}$, because in this way I hope to go from the odd forms on $\mathcal{F}_1(H')$ to even forms on $\mathcal{G}$.

For the 1-skeleton we have $H' = H\oplus H$ and $\mathcal{G}*\mathcal{G}$ maps to the space of embeddings of H in H' by

$$t_0g_0 + t_1g_1 \longmapsto \sqrt{t_0}\, i_0g_0 + \sqrt{t_1}\, i_1g_1$$

his last embedding gives the subspace

$$\operatorname{Im}\left(i_0 + i_1\sqrt{\frac{t_1}{t_0}}\, g_1g_0^{-1}\right) = \text{graph of } \sqrt{\frac{t_1}{t_0}}\, g_1g_0^{-1}.$$

Thus we are thinking of the Bott or graph map of ΣU into the Grassmannian. This is given by the formulas

$$(\theta,g) \longmapsto F_{\theta,g} = (\cos\theta)\varepsilon + (\sin\theta)\begin{pmatrix}0 & g^*\\ g & 0\end{pmatrix} \qquad 0\le\theta\le\pi$$

$$F_{\theta,g} = U_{\theta,g}\,\varepsilon\, U_{\theta,g}^{-1} = U_{2\theta,g} \qquad \text{where}$$

$$U_{\theta,g} = \begin{pmatrix} \cos\frac{\theta}{2} & (-\sin\frac{\theta}{2})g^* \\ (\sin\frac{\theta}{2})g & \cos\frac{\theta}{2} \end{pmatrix}$$

Now that we have embedded $(\mathcal{G}*\mathcal{G})\times^{\mathcal{G}} H$ into the trivial bundle with fibre H' over $\Sigma\mathcal{G}$, we must construct a section of $(\mathcal{G}*\mathcal{G})\times^{\mathcal{G}}\mathcal{F}_{1,\varepsilon}$ over $\Sigma\mathcal{G}$. This is done by picking A_0, A_1 at the ends and filling in by convexity. Set

$$A_{\theta,g} = U_{\theta,g}\begin{pmatrix} (\cos\frac{\theta}{2})^2 A_0 + (\sin\frac{\theta}{2})^2 g^* A_1 g & 0 \\ 0 & 1 \end{pmatrix} U_{\theta,g}^{-1}$$

Clearly this commutes with $F_{\theta,g}$; it is an appropriate convex combination of A_0, A_1 in H identified with the $+1$ eigenspace of $F_{\theta,g}$, and is 1 on the complement. (Instead of 1 I should be able to put any invertible operator.)

Note that at $\theta = 0, \pi$ this is independent of g:

$$\begin{pmatrix} 0 & -g^* \\ g & 0 \end{pmatrix}\begin{pmatrix} g^*A_1 g & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & g^* \\ -g & 0 \end{pmatrix} = \begin{pmatrix} 0 & -g^* \\ A_1 g & 0 \end{pmatrix}\begin{pmatrix} 0 & g^* \\ -g & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 \\ 0 & A_1 \end{pmatrix}$$

Thus $A_{0,g} = \begin{pmatrix} A_0 & 0 \\ 0 & 1 \end{pmatrix}$, $A_{\pi,g} = \begin{pmatrix} 1 & 0 \\ 0 & A_1 \end{pmatrix}$.

2

Here is a better way to proceed. Think in terms of the embedding into a trivial bundle. For each $(\theta, g) \in \Sigma\mathcal{G}$ we have the involution

$$F_{\theta g} = (\cos\theta)\varepsilon + (\sin\theta)\begin{pmatrix} 0 & g^* \\ g & 0 \end{pmatrix}$$

whose $+1$ eigenspace is the fibre of the vector bundle $(I \times \mathcal{G}) \times^{\mathcal{G}} H$ at (θ, g). Call this fibre $H_{\theta g}$. We can use

$$j_{\theta g} : H \longrightarrow H \oplus H$$

$$j_{\theta g} = u_{\theta g}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} (\cos\frac{\theta}{2}) \\ (\sin\frac{\theta}{2})g \end{pmatrix}$$

to identify $H_{\theta g}$ with H. ~~...~~ ~~...~~. Then

$$j_{\theta g}^* \begin{pmatrix} A_0 & 0 \\ 0 & A_1 \end{pmatrix} j_{\theta g} = \begin{pmatrix} \cos\frac{\theta}{2} & (\sin\frac{\theta}{2})g^* \end{pmatrix}\begin{pmatrix} A_0 & 0 \\ 0 & A_1 \end{pmatrix}\begin{pmatrix} \cos\frac{\theta}{2} \\ (\sin\frac{\theta}{2})g \end{pmatrix}$$

$$= \left(\cos\frac{\theta}{2}\right)^2 A_0 + \left(\sin\frac{\theta}{2}\right)^2 g^* A_1 g$$

which is the appropriate convex combination of A_0, A_1.

So we end up with the following setup. We have a fixed operator $A = \begin{pmatrix} A_0 & 0 \\ 0 & A_1 \end{pmatrix}$ on H' and a mapping $(\theta, g) \longmapsto F_{\theta g}$ from $\Sigma\mathcal{G}$ into the Grassmannian; maybe I should think of this as a mapping to projectors. Now to fix the ideas suppose A is an involution F (one can achieve this by expanding). Now we have a fixed operator F on H' which is being reduced by a family of

3 idempotents. From this setup I want to produce odd forms on the parameter space.

To fix the ideas we consider a map from $Y \times S^1$ to $\mathrm{Grass}_n(\mathbb{C}^{2n})$, i.e. a family of projectors e_y over $C^\infty(S^1)$ indexed by y. Fix F to be the Hilbert involution on $L^2(S^1)^{2n}$. That's essentially the Dirac operator. So we have this **odd** operator being reduced with respect to a family of idempotents.

April 8, 1986

Conclusions from yesterday. Given a Hilbert space H with an involution F_0 we have seen how o define even forms on the group U_{res} of unitaries commuting with F_0 mod $\mathcal{K}$. In order to arry out the transgression it seems that I ant to consider the space of involutions F ommuting with F_0 mod $\mathcal{K}$, i.e. the elements of der 2 in U_{res}. I want to define odd ~~classes~~ forms this space of involutions.

What we are looking at is the family of perators obtained by reducing F_0 with respect a family of projectors $\{e_y\}$, and we want assign odd forms on the parameter space to this mily.

Change notation $F_0 \mapsto \varepsilon$. We can map our rassmannian to unitaries as follows. Given F ve let A ~~be~~ the contraction of ε to the $+1$

eigenspace of F, then convert A to a unitary via Cayley: $(A+iB)^2$ and then extend by 1 on the -1 eigenspace of F.

Recall the eigenvalue concept for F relative to ε. The involutions F, ε generate a dihedral group; one sees the spectrum of $g = F\varepsilon$ is invariant under conjugation, so eigenvalues occur in pairs on S^1.

Notation for the plane:

$$\varepsilon = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \qquad F = \begin{pmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{pmatrix} \qquad \begin{cases} F = 1 \text{ on } \begin{pmatrix} \cos\theta/2 \\ \sin\theta/2 \end{pmatrix} \\ F = -1 \text{ on } \begin{pmatrix} -\sin\theta/2 \\ \cos\theta/2 \end{pmatrix} \end{cases}$$

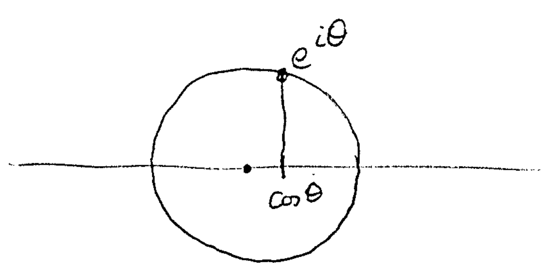
$$g = F\varepsilon = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \quad \text{has eigenvalues } e^{\pm i\theta}$$

A = contraction of ε to $+1$ eigenspace of F

$$= \begin{pmatrix} \cos\theta/2 & \sin\theta/2 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} \cos\theta/2 \\ \sin\theta/2 \end{pmatrix} = \cos^2\theta/2 - \sin^2\theta/2$$

$$= \cos\theta.$$

Picture of the spectrum of g



What I want to do is to convert A to a unitary which means taking $A + iB = \cos\theta + i\sin\theta$, $0 \le \theta \le \pi$ and squaring to get

$$(A+iB)^2 = e^{i2\theta}$$

This is what we want where $F = +1$, and we want

-1 where $F=-1$. So we want

$$\frac{e^{2i\theta}+1}{2}\,Id + \frac{e^{2i\theta}-1}{2}\,F$$

$$= e^{i\theta}\left(\cos\theta + i\sin\theta\cdot F\right)$$

So we have some sort of Bott map with
a subtle parametrization.

April 9, 1986

Try to understand the 1-form. Recall that we
re trying to construct odd forms on the Grassmannian
hich depend on a fixed involution ε. Actually
e do have a construction as follows by means of
map to the unitary group. Given a subspace V
 involution contracts to an $A \in \mathcal{F}_1(V)$, which we can
n convert to a unitary, say by either $-\exp(i\pi A)$ or
$(A+iB)^2$, and then we extend this unitary to H by 1
 $V^\perp$.

Let's look at the map to the circle we get by
king the determinant of this unitary, say

$$\det(-\exp i\pi A) = \det\left(e^{i\pi(A+1)}\right)$$
$$= e^{i\pi\, tr(A+1)} = e^{iN\pi}\, e^{i\pi\, tr(A)}$$

or $\det\,(A+iB)^2$

In terms of the example yesterday where $N=2$

$$\varepsilon = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \qquad F = \begin{pmatrix} \cos\theta & (\sin\theta)\bar{j} \\ (\sin\theta)j & -\cos\theta \end{pmatrix}$$

$F=1$ on line $\begin{pmatrix} \cos\frac{\theta}{2} \\ (\sin\frac{\theta}{2})j \end{pmatrix}$

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so $A = \left(\cos\frac{\theta}{2} \quad (\sin\frac{\theta}{2})\bar{\zeta}\right)\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\begin{pmatrix}\cos\frac{\theta}{2}\\ (\sin\frac{\theta}{2})\zeta\end{pmatrix}$

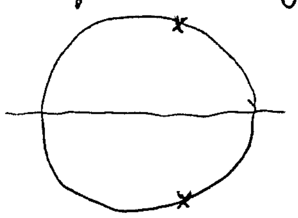
$= \cos\theta$

and we get either the

$$\det(-\exp i\pi A) = -e^{i\pi\cos\theta}$$

or $$\det(A+iB)^2 = e^{2i\theta}$$

The second formula suggests the following. Given F, [redacted] form the unitary $g = F\varepsilon$, which satisfies $\varepsilon g \varepsilon = g^{-1}$. Then take



$$\prod_{\mathrm{Im}\,\lambda > 0} \lambda^2 \qquad \lambda \text{ eigenvalue of } g$$

Actually a square root $\prod_{0 < \arg\lambda \le \pi} \lambda$ can be taken because when [redacted] eigenvalues approach -1, they do so in pairs.)

Fiddling with the formulas shows that one can construct lots of closed forms out of ε, F, dF e.g.

$$\mathrm{tr}(\varepsilon\, dF) = d\,\mathrm{tr}(\varepsilon F)$$
$$\mathrm{tr}(\varepsilon F \varepsilon\, dF) = d\,\tfrac{1}{2}\mathrm{tr}(\varepsilon F)^2$$

Check: $d\,\mathrm{tr}(\varepsilon F)^2 = \underset{\mathrm{tr}(\varepsilon F\cdot\varepsilon dF)}{\underset{\Vert}{\mathrm{tr}(\varepsilon dF\cdot\varepsilon F)}} + \mathrm{tr}(\varepsilon F\cdot\varepsilon dF)$)

So one can take d of any invariant function of εF. [redacted] Such a function will give a 1-form invariant under the centralizer of ε.

7 It is clear that [redacted] invariant functions such as $\operatorname{tr}(F\varepsilon)^k$ are not what we want since they don't involve selecting out the [redacted] eigenvalues in the UHP.

Set $g = F\varepsilon$. Form $\frac{g+g^{-1}}{2}$ which is in $\mathcal{F}_1$ and then construct a unitary by wrapping $[-1,1]$ around S^1. Since

$$\left(\frac{g+g^{-1}}{2}\right)^2 + \left(\frac{g-g^{-1}}{2i}\right)^2 = 1$$

the unitary we want is

$$\left(\frac{g+g^{-1}}{2} + i\left|\frac{g-g^{-1}}{2i}\right|\right)^2.$$

Notice that $\frac{g+g^{-1}}{2}$ commutes with ε, $\frac{g-g^{-1}}{2i}$ anti-commutes with ε, but that $\left|\frac{g-g^{-1}}{2i}\right| = \sqrt{1-\left(\frac{g+g^{-1}}{2}\right)^2}$ commutes with

Suppose $F = \begin{pmatrix} \alpha & \beta^* \\ \beta & \gamma \end{pmatrix}$ relative to $\varepsilon = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

Then

$$\frac{g+g^{-1}}{2} = \frac{F\varepsilon + \varepsilon F}{2} = \begin{pmatrix} \alpha & 0 \\ 0 & -\gamma \end{pmatrix}$$

$$\frac{g-g^{-1}}{2} = \frac{F\varepsilon - \varepsilon F}{2} = \begin{pmatrix} 0 & -\beta^* \\ \beta & 0 \end{pmatrix}$$

$$\left|\frac{g-g^{-1}}{2i}\right| = \begin{pmatrix} \beta^*\beta & 0 \\ 0 & \beta\beta^* \end{pmatrix}^{1/2} = \begin{pmatrix} \sqrt{\beta^*\beta} & 0 \\ 0 & \sqrt{\beta\beta^*} \end{pmatrix}$$

$$\frac{g+g^{-1}}{2} + i\left|\frac{g-g^{-1}}{2i}\right| = \begin{pmatrix} \alpha + i\sqrt{\beta^*\beta} & 0 \\ 0 & -\gamma + i\sqrt{\beta\beta^*} \end{pmatrix}$$

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Notice that the element

$$\left(\frac{g+g^{-1}}{2}\right) + i\left|\frac{g-g^{-1}}{2i}\right| \qquad \text{where} \quad g = F\varepsilon$$

commutes with both F and ε. Hence it should come from the center of the group ring of the dihedral group. The dihedral group I have in mind is $(\mathbb{Z}/2) * (\mathbb{Z}/2) = \mathbb{Z}/2 \ltimes \mathbb{Z}$, and the group ring is some sort of C^*-algebra, whose repns. are Hilbert spaces equipped with pairs of involutions.

One question would be to relate the different unitary transformations produced by this element on the different eigenspaces, e.g. on $F = +1$ and $\varepsilon = +1$.

April 12, 1986

I want to write down ideas acquired on the plane, etc. The problem is the following. Let's suppose given $\varepsilon \in \mathcal{J}(\mathcal{Z})$. We have already introduced the restricted unitary group

$$U_{res}(H,\varepsilon) = \{ g \in U(H) \mid g\varepsilon g^{-1} \equiv \varepsilon \}$$

and discussed even forms on it. Now we look at a corresponding Grassmannian

$$\mathcal{J}' = \{ F \in \mathcal{J}(H) \mid F\varepsilon F \equiv \varepsilon \}$$

and we want to produce odd forms on it. I think I should add the condition that the involutions induced on the $\varepsilon = +1$, $\varepsilon = -1$ pieces be non-trivial. In other words we have

$$\begin{tikzcd} & U_{res}(H,\varepsilon) \arrow[r, hook] \arrow[d] \arrow[dr, phantom, "\text{cart.}"] & U(H) \arrow[d] \\ U(\mathcal{Z}^+) \times U(\mathcal{Z}^-) \arrow[r, phantom, "="] & U(\mathcal{Z})^\varepsilon \arrow[r, hook] & U(\mathcal{Z}) \end{tikzcd}$$

and

$$\begin{tikzcd} \mathcal{J}' \arrow[rr] \arrow[d] \arrow[drr, phantom, "\text{cart.}"] & & \mathcal{J}(H) \arrow[d] \\ \mathcal{J}(\mathcal{Z}^+) \times \mathcal{J}(\mathcal{Z}^-) \arrow[r, phantom, "\subset"] & \mathcal{J}(\mathcal{Z})^\varepsilon \arrow[r, hook] & \mathcal{J}(\mathcal{Z}) \end{tikzcd}$$

The point is that

$$\{\pm 1\} \cup \mathcal{J}(\mathcal{Z})^\varepsilon = \left(\{\pm 1\} \cup \mathcal{J}(\mathcal{Z}^+)\right) \times \left(\{\pm 1\} \cup \mathcal{J}(\mathcal{Z}^-)\right)$$

It's clear from the above squares that

$$U_{res} \sim U(2) \sim \mathbb{Z}\times BU$$
$$\mathcal{J}' \sim \mathcal{J}(2) \sim U$$

Another approach is to introduce the algebra of restricted bounded operators
$$\mathcal{L}_{res} = \{ T \in \mathcal{L}(H) \mid [T,\varepsilon] \equiv 0 \}$$
so that U_{res} is the group of unitaries in $\mathcal{L}_{res}$ and $\mathcal{J}'$ is a compenent of $\mathcal{J}(\mathcal{L}_{res})$.

The process I wish to understand is that of reducing ε, or a self-adjoint Fredholm operator in the class of ε, by a family $\{F_y\}$ in $\mathcal{J}'$. Let's try to relate this to the Kasparov product.

Recall the product goes
$$KK(A,B) \otimes KK(B,C) \longrightarrow KK(A,C)$$
hence $KK(A,B)$ is like $\mathrm{Hom}(A,B)$, contravariant in A and covariant in B. This means that we look at Hilbert right B-modules (analogues of Hilbert bundles over B) left acted on by A, and such that the F commutes with B, and quasi-commutes with A. ~~In our situation a family of $\{F_y\}$ will be a single F on a Hilbert bundles over Y. and so will determine an element~~

$$\cancel{\varphi \in KK^0(\mathbb{C}, C(Y))}.$$

~~But more is true because we have $\mathcal{L}_{res}$ acting on H, quasi~~

Now consider H with $\mathcal{L}_{res}$ acting to the left and with an F in the class of ε. This defines an element

$$[\varepsilon] \in KK^1(\mathcal{L}_{res}, \mathbb{C}).$$

On the other hand a family of projectors $y \to e_y$ (or involutions $y \mapsto F_y$) with values in $\mathcal{L}_{res}$ parametrized by Y represents an element

$$[\{F_y\}] \in KK^0(\mathbb{C}, C(Y) \otimes \mathcal{L}_{res})$$

(Think of a map $y \longmapsto e_y$ over $C(X)$ as a family of vector bundles on X param. by Y, i.e. a v.b. over $Y \times X$ i.e. $[e] \in KK^0(\mathbb{C}, C(Y) \otimes C(X))$. Now then we take

$$\begin{tikzcd}
{[\{F_y\}] \otimes (1 \otimes [\varepsilon])} \arrow[d, "\cup"] & \in KK^0(\mathbb{C}, C(Y)\otimes\mathcal{L}_{res}) \otimes KK^1(C(Y)\otimes\mathcal{L}_{res}, C(Y)) \arrow[d, "\cup"] \\
{[\{F_y\}] \cup (1 \otimes [\varepsilon])} & \in KK^1(\mathbb{C}, C(Y))
\end{tikzcd}$$

April 13, 1986

Recall the problem is to produce odd forms on the Grassmannian [redacted] of $F \in \mathcal{I}(H)$ satisfying $[F,A] \in \mathcal{K}$, where $\bar{\varepsilon}$ is a fixed involution in $\mathcal{I}(\mathcal{Q})$ and [redacted] $A \in \mathcal{F}_{1,\bar{\varepsilon}}$. For example we take $H = L^2(S^1)^N$ with $A = \frac{D}{\sqrt{1+D^2}}$, $D = \frac{1}{i}\partial_x$, and we want odd forms on the space of projectors in $M_N \otimes C^\infty(S^1)$.

First look at the case where A is an involution ε. Then we have two involutions F, ε on H and they generate a dihedral group. Set $g = F\varepsilon$ so that g is reversed by both F, ε. Suppose

$$F = \begin{pmatrix} \alpha & \beta^* \\ \beta & \gamma \end{pmatrix} \qquad \varepsilon = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Then

$$g = F\varepsilon = \begin{pmatrix} \alpha & -\beta^* \\ \beta & \gamma \end{pmatrix} \qquad g^{-1} = \varepsilon F = \begin{pmatrix} \alpha & \beta^* \\ -\beta & -\gamma \end{pmatrix}$$

so

$$\frac{g+g^{-1}}{2} = \begin{pmatrix} \alpha & 0 \\ 0 & -\gamma \end{pmatrix}. \qquad \left|\frac{g-g^{-1}}{2i}\right| = \begin{pmatrix} \sqrt{\beta^*\beta} & 0 \\ 0 & \sqrt{\beta\beta^*} \end{pmatrix}$$

Observe that $\frac{g+g^{-1}}{2}$ commutes with both F, ε and that its restriction to the $+1$ eigenspace of ε is the contraction of [redacted] F to that eigenspace. Similarly its restriction to the $+1$ eigenspace of F is the contraction of ε to that eigenspace.

Because $\frac{g+g^{-1}}{2}$ commutes with both F, ε and hence with the associated projection operators, we see that in the ~~generic~~ case where β is non-singular, there are intertwining operators [redacted] for $\frac{g+g^{-1}}{2}$ acting on $\varepsilon = 1$, $\varepsilon = -1$, $F = 1$, $F = -1$. For example $F^2 = 1$ says

$$\alpha^2 + \beta^*\beta = 1, \qquad \gamma^2 + \beta\beta^* = 1, \qquad \beta\alpha + \gamma\beta = 0$$

so β intertwines α and $-\gamma$, which are the effect of $\frac{g+g^{-1}}{2}$ on $\varepsilon = 1$, $\varepsilon = -1$.

Here's the problem to concentrate upon: Take $H = L^2(S^1)^N$ with $D = \frac{1}{i}\partial_x$, and take a family $\{e_y, y \in Y\}$ of projectors in $C^\infty(S^1, M_N)$. Then I want to assign odd degree differential forms on Y [redacted] which represent the Chern character of the index of the family.

The idea originally was to choose $A \in \mathcal{F}_1(H)_\varepsilon$, e.g. $A = \frac{D}{\sqrt{1+D^2}}$, and then reduce with respect to e_y. This gives a family $e_y A e_y$ on $e_y H$. Then we extend from $e_y H$ to H by using an invertible family, e.g. $F = 1$ on $(1-e_y)H$. This then gives a family in $\mathcal{F}_1(H)$ representing the index, which we can convert via Cayley transform to a family in $U(\mathcal{K})$.

Another approach: Notice that the family in $\mathcal{F}_1(H)$

$$e_y A e_y + (1-e_y) A (1-e_y)$$

modulo $\mathcal{K}$ is $e_y A + (1-e_y) A = A$, which is constant. Thus $e_y A e_y$, $(1-e_y)A(1-e_y)$ on $e_y H$ and $(1-e_y)H$ respectively are additive inverses in $K^1(Y)$. [redacted]

On the other hand the map the map $\alpha \mapsto -\alpha$ on $\mathcal{F}_1(H)$ represents -1 on K^1, since adding α to $-\alpha$ in the sense of Whitney sum lifts to $\mathcal{I}(H^{\oplus 2}) \sim pt$:

$$\begin{pmatrix} \alpha & 0 \\ 0 & -\alpha \end{pmatrix} \equiv \begin{pmatrix} \alpha & \beta \\ \beta & -\alpha \end{pmatrix} \in \mathcal{I}(H^{\oplus 2})$$

Thus the family

$$\begin{aligned} & e_y A e_y - (1-e_y)A(1-e_y) \\ = \; & e_y A + A e_y - A \\ = \; & \tfrac{1}{2}(F_y A + A F_y) \qquad\qquad \left(\tfrac{1}{2}F_y = e_y - \tfrac{1}{2}\right) \end{aligned}$$

represents twice the index of the family $e_y A e_y$ in $K^1(Y)$.

Note that $\frac{1}{2}(FA+AF)$ commutes with F:

$$F(FA+AF) = A + FAF = (FA+AF)F$$

or
$$[F,[F,A]] = [\underbrace{[F,F]}_{\substack{\| \\ [2,\,A]=0}},A] - [F,[F,A]] \qquad \text{graded comm.}$$

Further version: If we convert A to $\varepsilon = \begin{pmatrix} A & B \\ B & -A \end{pmatrix}$ on $H^{\oplus 2}$, then we want to reduce A via $\begin{pmatrix} e & 0 \\ 0 & 0 \end{pmatrix}$. There are four reductions differing in sign in K-theory, and

$$\begin{pmatrix} eAe - (1-e)A(1-e) & 0 \\ 0 & \text{same} \end{pmatrix}$$

represents 4-times the index. Also we could use the family

$$\frac{1}{2}\left\{\begin{pmatrix} F & 0 \\ 0 & -F \end{pmatrix}\begin{pmatrix} A & B \\ B & -A \end{pmatrix} + \begin{pmatrix} A & B \\ B & -A \end{pmatrix}\begin{pmatrix} F & 0 \\ 0 & -F \end{pmatrix}\right\} = \frac{1}{2}\begin{pmatrix} FA+AF & FB-BF \\ -FB+BF & FA+AF \end{pmatrix}$$

which represents 4 times the index.

Note $\frac{1}{2}(FA+AF)$ over S^1 is Dirac on $\operatorname{Im} e$ and $-$ Dirac on $\operatorname{Im}(1-e)$.

April 16, 1986

Here's a promising approach to solve the transgression problem. Let H be a Hilbert space with an involution mod $\mathcal{K}$: $\varepsilon \in \mathcal{I}(\mathcal{Q}(H))$ given. [redacted] We consider the operator algebra

$$\mathcal{A} = \mathcal{L}_{res}(H,\varepsilon) = \{ a \in \mathcal{L}(H) \mid [a,\varepsilon] = 0 \}.$$

Clearly

$$U(\mathcal{A}) = U_{res}(H,\varepsilon) \sim \mathbb{Z} \times BU.$$

For the transgression problem we [redacted] want to understand $BU(\mathcal{A})$ which should be essentially $\mathcal{I}(\mathcal{A})$. More precisely, in order to realize the Grassmannian over $\mathcal{A}$ equipped with a Bott map from Susp $U(\mathcal{A})$, one forms a direct sum H' of copies of H, usually two. H' will come equipped with an involution ε' mod $\mathcal{K}$. We then get a principal $U(\mathcal{A})$-bundle by considering embeddings $s: H \longrightarrow H'$ compatible with $\varepsilon, \varepsilon'$; call this principal bundle P. It's clear we then have something like a fibration

$$\begin{tikzcd} U(\mathcal{A}) \arrow[r] & P \arrow[d] \\ & \mathcal{I}(\mathcal{A}') \end{tikzcd}$$

where $\mathcal{A}' = \mathcal{L}_{res}(H',\varepsilon')$.

Now the problem becomes the following. We pick [redacted] an F on [redacted] H' in the class of ε'. We've seen how this induces even forms on U_{res}. The problem is to transgress these forms to $\mathcal{I}(\mathcal{A}')$.

Notice we have by definition

$$\begin{tikzcd} 0 \arrow[r] & \mathcal{K} \arrow[r] \arrow[d, equal] & \mathcal{A} \arrow[r] \arrow[d, phantom, "\cap"] & \mathcal{Q}^+ \times \mathcal{Q}^- \arrow[r] \arrow[d, phantom, "\cap"] & 0 \\ 0 \arrow[r] & \mathcal{K} \arrow[r] & \mathcal{L} \arrow[r] & \mathcal{Q} \arrow[r] & 0 \end{tikzcd}$$

which suggests that $\mathcal{A}$ is K-equivalent to $\mathcal{Q}^+$.
In fact this is clear since we know that
$\mathcal{U}(\mathcal{A}) = \mathcal{U}_{res} \sim$ [redacted] $\mathcal{U}(\mathcal{Q}^+)$

April 17, 1986

Let $H = H^+ \oplus H^-$ and consider the map

$$\mathrm{Grass}(H) \longrightarrow \mathcal{F}_1(H^+)$$

which assign to an involution F the contraction of F to H^+.

Question: To what extent is this map the quotient by the action of $U(H^-)$ on the Grassmannian?

Suppose given $A \in \mathcal{F}_1(H^+)$. Then A has a minimal expansion $H^+ \subset W$ to an involution F. The codimension of H^+ in W is the number of eigenvalues of A not equal to ± 1. Assuming this number is $\leq \dim H^-$, one can extend the embedding $H^+ \subset H$ to an embedding of $W \subset H$. This last embedding is unique up to the action of $U(H^-)$. Now the F on W can be extended to H by picking an involution in $H \ominus W$. These fall into orbits under $U(H \ominus W)$ depending on the number of ± 1 eigenvalues.

Thus we see that the space of F on H which contract to A on H^+ falls into orbits under $U(H^-)$ determined by the number of ± 1 eigenvalues of F on H^-. Now suppose we restrict the number of $+1$ eigenvalues of F on H, i.e. we look at $\mathrm{Grass}_p(H)$. How do we count the number of $+1$ eigenvalues of F when it contracts to $A \in \mathcal{F}_1(H^+)$. It is the number of $+1$ eigenvalues of A plus the number of eigenvalues in $(-1,1)$ ~~plus~~ the number of $+1$ eigenvalues of F on H^-. In other words the number of $+1$ eigenvalues of F is the number of eigenvalues of A not equal to -1 plus the number of $+1$ eigenvalues of F on H^-. So if p is fixed there can be only one orbit of F under $U(H^-)$.

So let us see what we have. Given $A \in \mathcal{F}_1(H^+)$ we want to know when it is the contraction of an $F \in Gr_p(H)$. It is necessary that the number of eigenvalues of A in $(-1, 1]$ be $\leqslant p$ and that the number of eigenvalues of A in $(-1, 1)$ be $\leqslant \dim(H^-)$. When this is the case the possible F form an orbit under $U(H^-)$. Special case:

<u>Prop:</u> $Gr_p(V) \longrightarrow \mathcal{F}_1(H^+)$ is the quotient by the action of $U(H^-)$ provided $\dim H^+ \leq p \leq \dim(H^-)$. (In general $U(H^-)$ acts transitively on the fibres which may be empty)

<u>Question:</u> We have a map

$$Gr_p(H) \longrightarrow \mathcal{F}_1(H^+)$$

which is equivariant for $U(H^+) \times U(H^-)$. To what extent is this like a moment map? To what extent can one describe

$$\left(Gr_p(H) \times Gr_q(H)\right) / U(H) \qquad q = \dim H^+$$

as a simplex?

The last question is clear, for the quotient space where $q \leq p \leq \dim H - q$ is the quotient of $\mathcal{F}_1(H^+)$ by $U(H^+)$ which is given by the eigenvalue $-1 \leq \lambda_1 \leq \cdots \leq \lambda_q \leq 1$.

There is something wrong with the counting, so let's do it more carefully.

Given $A \in \mathcal{F}_1(H^+)$ let us denote the number of eigenvalues in the sets $\{-1\}$, $(-1,1)$, $\{1\}$ by ℓ^-, k, ℓ^+ respectively, and q^+, q^- be the dimensions of H^+, H^-, resp. Let W be as before so that

$$\dim W = q^+ + k \qquad q^+ = \ell^- + k + \ell^+$$

$$\underline{\text{dim of}}\ F=1 \text{ on } W = \ell^+ + k$$

$$\text{———}\quad F=-1 \quad \text{—} \quad = \ell^- + k$$

In order to extend this F on W to H so that the extension has $\dim(F=1) = p$, we need

$$\ell^+ + k \leq p$$

$$\ell^- + k \leq q^+ + q^- - p$$

$$= \ell^- + k + \ell^+ + q^- - p$$

i.e.

$$\boxed{\ell^+ + k \leq p \leq \ell^+ + q^-}$$

In the generic case $\ell^+ = \ell^- = 0$ and $k = q^+$, so we have

$$q^+ \leq p \leq q^-$$

or

$$\boxed{\dim H^+ \leq p \leq \dim H^-}$$

<u>Prop</u>: $\left(\mathrm{Grass}_p(H) \times \mathrm{Grass}_q(H)\right)/U(H)$ is always a simplex. If we arrange $p, q \leq \frac{1}{2}(\dim H)$, then the dimension of the simplex is $\min\{p,q\}$.

Given $\bar{\varepsilon} \in \mathcal{J}(\mathcal{Q})$ we have defined the restricted operator algebra
$$\mathcal{A} = \{ a \in \mathcal{L}(H) \mid [a, \bar{\varepsilon}] = 0 \}$$
and this gives rise to a restricted unitary group
$$U(\mathcal{A}) = U_{res}$$
and corresponding Grassmannian $\mathcal{J}(\mathcal{A})$. (one of nine components?)

Lift $\bar{\varepsilon}$ to $\varepsilon \in \mathcal{J}(H)$, let $H = H^+ \oplus H^-$ be the corresponding splitting. We have maps

$$\begin{tikzcd} \mathcal{F}_0(H^+) & U_{res} \arrow[l] \arrow[r, "g \mapsto g\varepsilon g^{-1}"] & \mathcal{J}_{res} = \{ F \in \mathcal{J}(H) \mid F \equiv \varepsilon \ (\mathcal{K}) \} \arrow[from=1-1, to=1-2, dashed, bend left, "s"] \end{tikzcd}$$

$$s(T) = \begin{pmatrix} T & -\sqrt{1-TT^*} \\ \sqrt{1-T^*T} & T^* \end{pmatrix}$$

The section s uses an isom. of H^+ with H^-.

In a similar way I would like to construct arrows

$$\begin{tikzcd} \mathcal{F}_1(H^+) & \mathcal{J}(\mathcal{A}) \arrow[l] \arrow[r] & U(\mathcal{K}^+) \arrow[from=1-1, to=1-2, dashed, bend left, "s"] \end{tikzcd}$$

$\mathcal{J}(\mathcal{A})$ consists of F on H such that $[F, \varepsilon] \equiv 0 \ (\mathcal{K})$ and such that F gives rise to non-trivial involutions in $\mathcal{Q}^+, \mathcal{Q}^-$. In other words
$$F = \begin{pmatrix} \alpha & \beta^* \\ \beta & \gamma \end{pmatrix}$$
where $F^2 = 1$, β is compact, and $\alpha \in \mathcal{F}_1(H^+)$, $\gamma \in \mathcal{F}_1(H^-)$.

The section s is
$$s(A) = \begin{pmatrix} A & B \\ B & -A \end{pmatrix}$$
based on an isom. of $H^+ = H^-$.

Now define $\mathcal{I}(\mathfrak{a}) \longrightarrow \mathcal{U}(\mathcal{K}^+)$ as follows. Given $F = \begin{pmatrix} \alpha & \beta^* \\ \beta & \gamma \end{pmatrix}$ in $\mathcal{I}(\mathfrak{a})$, set $g = F\varepsilon$ and form

$$\left(\frac{g+g^{-1}}{2} + i \left| \frac{g-g^{-1}}{2i} \right| \right)^2 = \begin{pmatrix} \alpha + i\sqrt{\beta^*\beta} & 0 \\ 0 & -\gamma + i\sqrt{\beta\beta^*} \end{pmatrix}^2$$

and take the top block. Thus

$$\mathcal{I}(\mathfrak{a}) \longrightarrow \mathcal{U}(\mathcal{K}^+)$$

seems to take $F = \begin{pmatrix} \alpha & \beta^* \\ \beta & \gamma \end{pmatrix}$ into $\left(\alpha + i\sqrt{1-\alpha^2} \right)^2$

Obvious question: Is it obvious that the above map is a homotopy equivalence?

Not without a stratification argument, because this map is not a fibre bundle.

April 19, 1986

Yesterday I reviewed the calculation showing that if the odd character forms on $U(2n)$ are pulled back under the Bott map

$$\Sigma\, Gr_n(\mathbb{C}^{2n}) \longrightarrow U(2n)$$

then one obtains the even character forms on the Grassmannian. An important point is that the above map is equivariant for the action of $U(2n)$, [redacted] where $U(2n)$ acts on itself via conjugation, so that a priori the biinvariant forms on $U(2n)$ give invariant forms on the Grassmannian. In this calculation the factor $2\pi i$ appears.

A basic problem seems to be to find a suitable Bott map for the restricted Grassmannian and the unitary group $U(\mathcal{K})$. [redacted] The usual Bott map assigns to a point in the Grassmannian a geodesic going from 1 to -1; [redacted] but $-1 \notin U(\mathcal{K})$. The Bott map is

$$(\theta, F) \longmapsto \cos\theta + (i\sin\theta)F \qquad 0 \leq \theta \leq \pi$$

and if $F \equiv \varepsilon \mod \mathcal{K}$, then for a given θ this map lies in the coset of $\cos\theta + (i\sin\theta)\varepsilon \mod U(\mathcal{K})$.

So we have a Bott map from $\mathcal{I}_{res}$ to the space of paths in $U(H)$ going from 1 to -1 which lie over the path $\cos\theta + (i\sin\theta)\bar{\varepsilon}$ in $U(\mathcal{Q})$. It might be possible to make sense of the odd character forms on this space.

<u>Problem:</u> Bott maps relating $U(\mathcal{K})$ + $\mathcal{I}_{res}$.

What seems to be poor for differential form

So we now want to find even differential forms on the space of pairs of involutions (F^+, F^-) which are congruent mod $\mathcal{K}$.

April 20, 1986

Let's return to Grasmé's idea of constructing an extension of $U_{res,(0)}$ by $U(\mathcal{K})$.

$$\begin{tikzcd} U(\mathcal{K}^+) \arrow[r, equal] \arrow[d] & U(\mathcal{K}^+) \arrow[d] \\ \mathcal{E} \arrow[r] \arrow[d] & U(H^+) \arrow[d] \\ U_{res,(0)} \arrow[r] & U(\mathcal{Q}^+) \end{tikzcd}$$

In other words ~~...~~ $\mathcal{E}$ consists of pairs (g, u) where

$$g = \begin{pmatrix} \alpha & \gamma \\ \beta & \delta \end{pmatrix} \in U(H) \qquad \beta, \gamma \equiv 0 \quad (\mathcal{K})$$

$$u \in U(H^+)$$

$$u \equiv \alpha \qquad (\mathcal{K}).$$

Let $\mathcal{G} = U_{res,(0)}$, and $\mathcal{G}' = U(H^+) \times_{U(\mathcal{Q}^+)} U(H^+)$.

Notice that $\mathcal{G}$ = unitaries in the restricted operator alg of H relative to ε, $\mathcal{G}'$ = unitaries in the graded restricted operator algebra.

We have an embedding

$$\mathcal{G}' \subset \mathcal{E}$$

as the subgrp of pairs (g, u) with $g \in U(H^+)$. As $\mathcal{E}$ is contractible, we see $\mathcal{E}/\mathcal{G}'$ is a model for $B\mathcal{G}'$. This may be useful.

But I want to divide out by the group $U(H^+) \times U(H^-)$, because this gives a fibring with base $I_{res,(0)}$ and group $U(\mathcal{K}^+)$. Note that $U(H^+) \times U(H^-) \subset U_{res,(0)}$ lifts to $\mathcal{E}$ by

$$(u^+, u^-) \longmapsto \left(\begin{pmatrix} u^+ & 0 \\ 0 & u^- \end{pmatrix}, u^+ \right).$$

Then left multiplication by this subgroup on $\mathcal{E}$ will commute with right multiplication. So we can take the quotient

$$\begin{tikzcd} \mathcal{E} \arrow[r] \arrow[d] & U(H^+)\times U(H^-) \backslash \mathcal{E} \arrow[d] \\ U_{res,(0)} \arrow[r] & U(H^+)\times U(H^-) \backslash U_{res,(0)} \simeq I_{res,(0)} \end{tikzcd}$$

Let's now identify this principal bundle.

Recall U_{res} consists of $\begin{pmatrix} \alpha & \gamma \\ \beta & \delta \end{pmatrix} \in U(H) \ni \beta, \gamma \equiv 0$.

We can identify $U_{res}/U(H^-)$ with the set of

$$\begin{pmatrix} \alpha \\ \beta \end{pmatrix} : H^+ \longrightarrow H^+ \oplus H^- \quad \ni \quad \alpha^*\alpha + \beta^*\beta = 1 \quad \text{(isom emb)}$$
$$\cancel{\beta \equiv 0 \ (\mathcal{K})} \quad \alpha^*\alpha \equiv \alpha^*\alpha \equiv 1 \mod \mathcal{K}$$

In effect the orthogonal complement of such an embedding is infinite diml, hence we can extend it to a unitary $\begin{pmatrix} \alpha & \gamma \\ \beta & \delta \end{pmatrix}$ and necessarily γ is compact since α is essentially unitary.

Thus $\mathcal{E}/U(H^-)$ can be identified with pairs $\left(\begin{pmatrix} \alpha \\ \beta \end{pmatrix}, u \right)$ with $\begin{pmatrix} \alpha \\ \beta \end{pmatrix}$ as above and $u \in U(H^+)$ such that $\alpha \equiv u$. $U(H^+)$ acts diagonally

$$\left(\begin{pmatrix} \alpha \\ \beta \end{pmatrix}, u \right) u' = \left(\begin{pmatrix} \alpha u' \\ \beta u' \end{pmatrix}, u u' \right)$$

essentially unitary. Let $\mathcal{A}$ be the subalgebra of $a \in \mathcal{L}(H)^{\varepsilon}$ commuting with A mod compacts. $\mathcal{G} = \mathcal{U}(\mathcal{A})$ consists of pairs g_+ on H^+ g_- on H^- of unitaries intertwined by T mod $\mathcal{K}$.

To simplify suppose the index of the involution in $\mathcal{I}(\mathcal{Q})$ is zero whence we can choose T unitary and identify H^+ with H^-. One has then

$$\Delta\, \mathcal{U}(H^+) \longrightarrow \mathcal{G} \longrightarrow \mathcal{U}(\mathcal{K})$$
$$(g_+, g_-) \longmapsto g_-(g_+)^{-1}$$

In complete analogy with
$$\mathcal{U}(H)^{\varepsilon} \longrightarrow \mathcal{U}(H) \longrightarrow \mathcal{I}_{res}.$$

Observe that $\mathcal{A}$ is defined by fibre product:

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{K}^{\varepsilon} \arrow[r] \arrow[d, equal] & \mathcal{A} \arrow[r] \arrow[d] & \mathcal{Q} \arrow[r] \arrow[d, "\Delta"] & 0 \\
0 \arrow[r] & \mathcal{K}^{\varepsilon} \arrow[r] & \mathcal{L}(H)^{\varepsilon} \arrow[r] & \mathcal{Q}^{\varepsilon} \arrow[r] & 0
\end{tikzcd}$$

so that one has a natural embedding

$$\mathcal{K}^+ \hookrightarrow \mathcal{A}$$

which ought to be a K-equivalence. (In the ungraded case we have $\mathcal{A} \to \mathcal{Q}^+$.)

Next we look at $\mathcal{I}(\mathcal{A})$. This consists of pairs of involutions⊛ $F^{\pm}$ of $H^{\pm}$ which are compatible with T mod $\mathcal{K}$. Thus when $T = 1$, $\mathcal{I}(\mathcal{A})$ is pairs of involutions congruent mod $\mathcal{K}$.

$$\mathcal{I}(\mathcal{A}) = \mathcal{I}(H) \times_{\mathcal{I}(\mathcal{Q})} \mathcal{I}(H)$$

$$\sim \text{Fibre of } \mathcal{I}(H) \to \mathcal{I}(\mathcal{Q}) = \mathcal{I}_{res}$$

⊛ $F^{\pm}$ are to be in $\mathcal{I}(H^{\pm})$; ∞ many eigenvalues $+1$ and -1.

purposes is to use the Toeplitz maps

$$U_{res} = \mathcal{U}(\mathfrak{a}) \longrightarrow \mathcal{F}_0(H^+)$$

$$\mathcal{I}(\mathfrak{a}) \longrightarrow \mathcal{F}_1(H^+)$$

In the case of $\mathcal{U}(\mathfrak{a})$ we produce nice diff'l forms on U_{res} using the map $U_{res} \to I_{res}$ given by ε. Note we have

$$\begin{tikzcd} \mathcal{F}_0(H^+) \arrow[r, dashed, bend left] & U_{res} \arrow[l] \arrow[r] & I_{res} \end{tikzcd}$$

and to produce the section requires an isomorphism of H^+ with H^-.

This is strange because if we were to use instead of $\mathcal{F}_0(H^+) = \left\{ T: H^+ \to H^+ \;\middle|\; \begin{matrix} T \text{ contraction} \\ T \text{ essentially unitary} \end{matrix} \right\}$, the $\mathcal{F}_0$ consisting of degree 1 operators $\begin{pmatrix} 0 & T^* \\ T & 0 \end{pmatrix}$ on $H^+ \oplus H^-$, then we have a canonical map

$$\mathcal{F}_0(H^+) \longrightarrow I_{res}(H, -\varepsilon)$$

Now when I come to $\mathcal{I}(\mathfrak{a})$ I have canonical arrows.

$$\begin{tikzcd} \mathcal{F}_1(H^+) \arrow[r, dashed, bend left] & \mathcal{I}(\mathfrak{a}) \arrow[l] \arrow[r] & \mathcal{U}(\mathcal{K}^+) \end{tikzcd}$$

$$\alpha \longleftarrow\!\shortmid \begin{pmatrix} \alpha & \beta \\ \beta^* & \gamma \end{pmatrix} \longmapsto \left(\alpha + i\sqrt{1-\alpha^2}\right)^2.$$

The problem is to relate the odd forms thus defined on $\mathcal{I}(\mathfrak{a})$ to the even forms on U_{res}.

Graded case: In the graded case we have a graded Hilbert space $H = H^+ \oplus H^-$ and we fix a degree 1 involution in $\mathcal{I}(2)$. This can be lifted to an $A = \begin{pmatrix} 0 & T^* \\ T & 0 \end{pmatrix} \in \mathcal{F}_0$; here $T: H^+ \to H^-$ is so there is an obvious ~~section~~ slice to the action given by pairs with $u = 1$. So we have

$$\mathcal{E}/\,\mathcal{U}(H^+)\times\mathcal{U}(H^-) = \left\{ \begin{pmatrix}\alpha\\ \beta\end{pmatrix} : H^+ \longrightarrow H^+\oplus H^- \;\middle|\; \begin{matrix}\alpha^*\alpha+\beta^*\beta=1\\ \alpha\equiv 1\end{matrix} \right\}$$

We have both a left and right action of $\mathcal{U}(\mathcal{K}^+)$:

$$\begin{pmatrix}\alpha\\ \beta\end{pmatrix} u = \begin{pmatrix}\alpha u\\ \beta u\end{pmatrix} \qquad u\begin{pmatrix}\alpha\\ \beta\end{pmatrix} = \begin{pmatrix}u\alpha\\ \beta\end{pmatrix} \quad ?$$

[This is not surprising as it happens with Stiefel manifolds and the Milnor principal bundle.)

Something is wrong: I haven't calculated the left action correctly as the action written isn't free: [redacted] The whole group $\mathcal{E}^{(\text{left})}$ acts on $\mathcal{E}/\,\mathcal{U}(H^+)\times\mathcal{U}(H^-)$. In particular $u \in \mathcal{U}(\mathcal{K}^+)$, which becomes $\left(\begin{pmatrix}1&0\\0&1\end{pmatrix}, u\right) \in \mathcal{E}$ acts as

$$u\cdot\left(\begin{pmatrix}\alpha\\ \beta\end{pmatrix}, 1\right) = \left(\begin{pmatrix}\alpha\\ \beta\end{pmatrix}, u\right) \sim \left(\begin{pmatrix}\alpha u^{-1}\\ \beta u^{-1}\end{pmatrix}, 1\right)$$

and so we see that [redacted] the left $\mathcal{U}(\mathcal{K}^+)$ action on $\mathcal{E}/\,\mathcal{U}(H^+)\times\mathcal{U}(H^-)$ is the same as the standard right action on the "restricted" frame bundle over the restricted Grassmannian.

The upshot of the above is that there is a natural frame bundle over the Grassmannian $\mathcal{J}_{res,(0)}$ for the group $\mathcal{U}(\mathcal{K}^+)$. Namely it consists of all isometric embeddings $\begin{pmatrix}\alpha\\ \beta\end{pmatrix} : H^+ \longrightarrow H^+\oplus H^-$ such that $\alpha \equiv 1$. $\mathcal{U}(\mathcal{K}^+)$ acts by composition on the right.

Next we consider $\mathcal{G}' = U(H^+) \times_{U(\mathcal{Q}^+)} U(H^+)$, which sits inside $\mathcal{E}$ as the subgp of pairs (g,u) with $g \in U(H^+)$. $\mathcal{U}(\mathcal{K}^+)$ sits inside $\mathcal{G}'$ as the subgp of pairs $(1,u)$, and $\mathcal{G}'$ is naturally the semi-direct product of $\Delta U(H^+)$ with $\mathcal{U}(\mathcal{K}^+)$; $U(H^+)$ acts on $\mathcal{U}(\mathcal{K}^+)$ by conjugation.

Now [redacted] notice that $\mathcal{G}' \times U(H^-)$ sits inside $\mathcal{E}$ as the subgp. of pairs (g,u) with $g \in U(H^+) \times U(H)^-$. In other words if we let $\mathcal{E}$ act on $\mathcal{I}_{res,(0)}$ thru the homomorphism $\mathcal{E} \longrightarrow U_{res,(0)}$, then $\mathcal{G}' \times U(H^-)$ is the stabilizer of F. Thus

$$\mathcal{E} / \mathcal{G}' \times U(H^-) = \mathcal{I}_{res,(0)}$$

and so $\mathcal{E}/U(H^-)$ is a principal $\mathcal{G}'$-bundle with base $\mathcal{I}_{res,(0)}$. Since $\mathcal{E}/U(H^-)$ is contractible we see that we have a model for $B\mathcal{G}'$ which has canonical even forms.

Probably $\mathcal{E}/U(H^-) \longrightarrow \mathcal{I}_{res,(0)}$ is induced from the principal $U(\mathcal{K}^+)$ bundle P with base $\mathcal{I}_{res,(0)}$ by the homomorphism $U(\mathcal{K}^+) \hookrightarrow \mathcal{G}'$. To check this let us identify $\mathcal{E}/U(H^-)$ with pairs $\left(\begin{pmatrix}\alpha\\ \beta\end{pmatrix}, u\right)$ where $\begin{pmatrix}\alpha\\ \beta\end{pmatrix} : H^+ \hookrightarrow H$ is an isometric embedding [redacted] $u \in U(H^+)$ and $u \equiv \alpha \bmod \mathcal{K}^+$. We clearly have a map

$$P \longrightarrow \mathcal{E}/U(H^-) \qquad \begin{pmatrix}\alpha\\ \beta\end{pmatrix} \longmapsto \left(\begin{pmatrix}\alpha\\ \beta\end{pmatrix}, 1\right)$$

compatible with the action of $U(\mathcal{K}^+)$ provided we use the embedding $u \longmapsto (u,1)$ of $U(\mathcal{K}^+)$ in $\mathcal{G}'$. Put another way we have the map

$$\mathcal{E}/U(H^-) \longrightarrow U(H^+)$$
$$\left(\begin{pmatrix}\alpha\\ \beta\end{pmatrix}, u\right) \longmapsto u$$

which is $\mathcal{G}'$-equivariant, hence $\mathcal{E}/U(H^-)$ is the $\mathcal{G}'$-space induced by the fibre over $u=1$, which is $\mathcal{P}$, for the action of the isotropy gp of $u=1$, which is $U(\mathcal{K}^+)$ embedded as $\{(u,1)\}$ in $\mathcal{G}'$.

Next one might want to understand what it might mean to use this model for $B\mathcal{G}'$. So for example, let's take a graded Hilbert space $\tilde{H} = \tilde{H}^+ \oplus \tilde{H}^-$ with $F = \begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}$ for some identification $\tilde{H}^+ \simeq \tilde{H}^-$. Now we consider degree zero involutions or projectors on $\tilde{H}$ commuting with F mod compacts. The space of such involutions is supposed to be a classifying space for $\mathcal{G}'$.