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March 12, 1986

Some ideas: Go back over the Atiyah-Singer proof of Bott periodicity using spaces of Fredholm operators. In this proof Kuiper's thm. is the key; it is used in two ways, the first being to show the total spaces of certain fibrations are contractible (e.g. $U(k) \longrightarrow U(B) \longrightarrow U(B/k)$). The second is to prove certain exponential maps are homotopy equivalences because the fibres are contractible e.g.

$$\begin{array}{ccc} A & \longmapsto & \exp(i\pi A) \\ \cap & & \cap \\ \mathcal{F}_1 & & -U(k) \end{array}$$

Now the even version of the second map is an exponential map for the Grassmannian. I should understand this map a lot better.

For example the map itself goes from

$$F = \begin{pmatrix} 0 & P^* \\ P & 0 \end{pmatrix} \in \mathcal{F}_0 \qquad PP^* \equiv P^*P \equiv 1 \mod k$$
$$P^*P,\ PP^* \leq 1$$

to involutions congruent to $-\varepsilon$ mod k.
On the other hand we have a map the other way maybe ~~which~~ assign to ε' the projection from the $\varepsilon = +1$ subspace to the $\varepsilon' = +1$ subspace.

March 14, 1986

Let $H = H^+ \oplus H^-$, $\varepsilon = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, where both $H^\pm$ are infinite diml Hilbert spaces. Introduce the restricted unitary group consisting of unitary operators u on H such that $[u, \varepsilon]$ is compact. Another description is as a pull-back

$$\begin{tikzcd}
U(\mathcal{K}) \arrow[r, phantom, "="] \arrow[d] & U(\mathcal{K}) \arrow[d] \\
U_{res}(H, \varepsilon) \arrow[r, hook] \arrow[d] & U(\mathcal{B}(H)) \arrow[d] \\
U(\mathcal{B}/\mathcal{K})^\varepsilon \arrow[r, phantom, "\subset"] & U(\mathcal{B}/\mathcal{K})
\end{tikzcd}$$

Here $\mathcal{B}$ = bounded operators on $\mathcal{H}$, $\mathcal{B}/\mathcal{K}$ is the Calkin algebra. Note

$$(\mathcal{B}/\mathcal{K})^\varepsilon = \text{two copies of the Calkin algebra.}$$
$$= \mathcal{B}^+/\mathcal{K}^+ \times \mathcal{B}^-/\mathcal{K}^-$$

<u>Grame's Assertion:</u> $U_{res}(H) \longrightarrow U(\mathcal{B}^+/\mathcal{K}^+)$ <u>is a homotopy equivalence.</u>

This is a homomorphism of groups, so it suffices to prove the map is surjective, and show the kernel is contractible.

Let's describe things more clearly. The grading of H defines a block decomposition for operators on H. $U_{res}(H)$ consists of unitary matrices g whose block picture

$$g = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$$

is such that γ, β are compact. Then the map

$$g \longmapsto ege = \alpha \qquad \text{from } U_{res}(H) \longrightarrow \mathcal{F}_1(H^+)$$

is the Toeplitz operator map. ~~Since~~ Now

$$\mathcal{F}_1(H^+) \longrightarrow U(\mathcal{B}^+/\mathcal{K}^+)$$

is a homotopy equivalence, as its surjective and the fibres are the self-adjoint elements of $\mathcal{K}^+$. Thus Graeme's assertion implies <u>the Toeplitz map</u>

$$U_{res}(\mathcal{H}) \longrightarrow \mathcal{F}_1(\mathcal{H}^+)$$

<u>is a homotopy equivalence</u>, and is equivalent to this.

Another consequence is that we can pull back the basic extension

$$\begin{tikzcd}
U(\mathcal{K}^+) \arrow[r, equal] \arrow[d] & U(\mathcal{K}^+) \arrow[d] \\
\mathcal{E} \arrow[r] \arrow[d] & U(\mathcal{B}^+) \arrow[d] \\
U_{res}(H) \arrow[r] & U(\mathcal{B}^+/\mathcal{K}^+).
\end{tikzcd}$$

Then $\mathcal{E}$ is an extension of the identity component of $U_{res}(H)$ by $U(\mathcal{K}^+)$. Moreover Graeme's assertion implies that $\mathcal{E}$ is heq to $U(\mathcal{B}^+)$ and hence it is contractible.

One interest in this is that $U_{res}(H)$ has a natural sequence of subgroups defined by Schatten conditions on $[g, \varepsilon]$. There should be some relation with the work of Dan Freed.

We have a homotopy equivalence between the restriction Grassmannian and the restricted unitary group

$$U_{res}(H) \longrightarrow U_{res}(H)/U(H^+) \times U(H^-)$$

because of Kuiper's theorem. Thus we see the equivalence in the loop group ~~situation~~ between the maps from $\mathcal{L}(G)$ to the Grassmannian or to the Fredholm operators.

$$\begin{tikzcd}
\mathcal{L}(G) \arrow[r] & U_{res}(H) \arrow[dl] \arrow[dr] & \\
U(\mathcal{B}^+/\mathcal{K}^+) & & U_{res}(H)/U^+ \times U^-
\end{tikzcd}$$

Further questions:

1) How to prove Graeme's assertion?

2) Can we understand the homotopy equivalence

$$\mathcal{F}_1(H^+) \longrightarrow Grass_{res}(H)$$

well enough to explain Connes "doubling process" for defining cyclic cocycles?

March 15, 1986

Spectral thm. for subspaces $W \subset V^+ \oplus V^-$

Let's first look at the case where $V^\pm$ are one dimensional. Cases:

1) $W = 0$, $W^\perp = V^+ \oplus V^-$

2) $\dim W = 1$. Three subcases

a) $W = V^+$, $W^\perp = V^-$

b) $W = \mathrm{Im}\begin{pmatrix} 1 \\ \lambda \end{pmatrix}$, $W^\perp = \mathrm{Im}\begin{pmatrix} -\lambda^* \\ 1 \end{pmatrix}$ where $\lambda : V^+ \xrightarrow{\sim} V^-$

c) $W = V^-$, $W^\perp = V^+$

3) $W = V^+ \oplus V^-$, $W^\perp = 0$.

Let us next consider the case when the involution ε defining the splitting $V = V^+ \oplus V^-$ commutes with the involution F defining the splitting $V = W \oplus W^\perp$. In this case we have

$$\begin{array}{lcccc} W & = & (W \cap V^+) & \oplus & (W \cap V^-) \\ W^\perp & = & (W^\perp \cap V^+) & \oplus & (W^\perp \cap V^-) \\ & & \underbrace{\qquad\qquad}_{V^+} & & \underbrace{\qquad\qquad}_{V^-} \end{array}$$

I would like to label things with a parameter $0 \le \lambda \le \infty$, which I think of roughly as the slope of W relative to the splitting $\begin{array}{c} V^- \\ \big| \end{array}\!\!\!-\!\!\!-\, V^+$. Thus we have

$$\begin{array}{ccccccccc} V^+ & & V^+_0 & \oplus & V^+_\infty & & W \cap V^+ & & W^\perp \cap V^+ \\ \oplus & = & & \oplus & & = & & & \\ V^- & & V^-_0 & \oplus & V^-_\infty & & W^\perp \cap V^- & & W \cap V^- \end{array}$$

~~In the general case we~~ The next case we consider is where all these intersections $W \cap V^{\pm}$, $W^{\perp} \cap V^{\pm}$ are zero. Since

$$W / W \cap V^{+} \hookrightarrow V / V^{+} \cong V^{-}$$

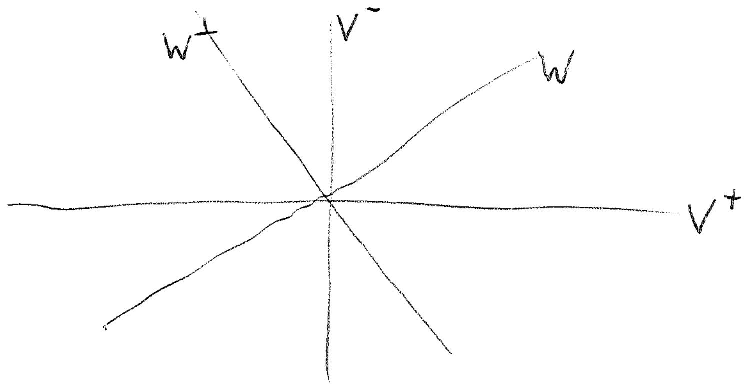
we have $\dim(W) \leq \dim(V^{-})$. Similarly

$$\dim(W^{\perp}) \leq \dim(V^{+})$$

and adding these equalities gives an equality, hence

$$\dim(W) = \dim(V^{-})$$
$$\dim(W^{\perp}) = \dim(V^{+}).$$

Upon reversing $V^{\pm}$ we see $\dim(W) = \dim(W^{+}) = \dim(V^{+}) = \dim(V^{-})$. Moreover we have W is complementary to $V^{\pm}$



hence $W = \Gamma_T = \operatorname{Im}\begin{pmatrix} 1 \\ T \end{pmatrix}$ where $T : V^{+} \xrightarrow{\sim} T^{-}$

$$W^{\perp} = (\Gamma_T)^{\perp} = \operatorname{Im}\begin{pmatrix} -T^{*} \\ 1 \end{pmatrix}.$$

In this case using the eigenvalues of $T^{*}T : V^{+} \circlearrowleft$ and $TT^{*} : V^{-} \circlearrowleft$ we can decompose

$$V^{+} = \bigoplus_{0 < \lambda < \infty} V^{+}_{\lambda}$$

$$V^{-} = \bigoplus_{0 < \lambda < \infty} V^{-}_{\lambda}$$

where $V^{+}_{\lambda} \underset{T^{*}}{\overset{T}{\rightleftarrows}} V^{-}_{\lambda}$ satisfy $T^{*}T = \lambda^{2}$, $TT^{*} = \lambda^{2}$

Let's now consider the general case.
We project W onto V^+. This is

$$(W + V^-) \cap V^+$$

and its orthogonal complement in V^+ is $W^\perp \cap V^+$.
Thus we have

$$\begin{matrix} V^+ & = & (W+V^-)\cap V^+ & \oplus\ (W^\perp \cap V^+) \\ \oplus & & \oplus & \\ V^- & & \underbrace{\quad V^- \quad}_{\text{W lives in here}} & \end{matrix}$$

So what this means is that we can split off $W^\perp \cap V^+ = V^+_\infty$ from both $W^\perp$ and V^+ without changing W, V^-. We then obtain the direct sum of a situation where W projects onto V^+ together with a situation where $V^+ = W^\perp$, $W = V^- = 0$.

This step we have gone thru picks up the failure of the correspondence defined by W to be defined on all of V^+. We split off V^+_∞.

Next step will be to remove the indeterminacy which is the subspace $W \cap V^-$. But this will be essentially the same as

$$\begin{matrix} V^- & = & (W^\perp+V^+)\cap V^- & \oplus\ (W \cap V^-) \\ \oplus & & \oplus & \\ V^+ & & \underbrace{\quad V^+ \quad}_{W^\perp \text{ inside here}} & \end{matrix}$$

Thus we split off $V^-_\infty = W \cap V^-$.

Now that we are reduced to the case where $V^+_\infty = (W + V^-) \cap V^+ = 0$ (so W projects onto V^+) and $V^-_\infty = W \cap V^- = 0$, we have that W is the graph of a map $T : V^+ \to V^-$. Then $\operatorname{Ker} T = W \cap V^+ = V^+_0$ and $(\operatorname{Im} T)^\perp = W^\perp \cap V^- = V^-_0$. I think the picture is now clear and also that we have established the following spectral theorem.

<u>Thm:</u> Let $W \subset V^+ \oplus V^-$. There is a canonical decomposition

$$V^\pm = \bigoplus_{0 \le \lambda \le \infty} V^\pm_\lambda$$

together with unitary maps $\varphi_\lambda : V^+_\lambda \xrightarrow{\sim} V^-_\lambda$ for $0 < \lambda < \infty$, such that

$$W = \begin{matrix} V^+_0 \\ \oplus \\ 0 \end{matrix} \oplus \bigoplus_{0<\lambda<\infty} \overbrace{\Gamma_{\lambda\varphi_\lambda}}^{\binom{1}{\lambda\varphi_\lambda} V^+_\lambda} \oplus \begin{matrix} 0 \\ \oplus \\ V^-_\infty \end{matrix}$$

$$W^\perp = \begin{matrix} 0 \\ \oplus \\ V^-_0 \end{matrix} \oplus \bigoplus_{0<\lambda<\infty} \begin{pmatrix} -\lambda\varphi^*_\lambda \\ 1 \end{pmatrix} V^-_\lambda \oplus \begin{matrix} V^+_\infty \\ \oplus \\ 0 \end{matrix}$$

Questions: 1) Can one deduce this spectral theorem from the usual one in an easy way?

2) Discuss the restricted Grassmannian and its homotopy type.

3) Relation of this spectral thm. with graph maps and exponential type maps from the tangent space at ε.

The program: Recall how the Grassmannian can be identified with unitaries g such that $\varepsilon g \varepsilon = g^{-1}$. Given an involution ε', put $g = \varepsilon' \varepsilon$. Then $g^{-1} = \varepsilon\varepsilon' = \varepsilon g \varepsilon$. Conversely $\varepsilon g \varepsilon = g^{-1} \implies g\varepsilon g \varepsilon = 1$ so $g\varepsilon$ is an involution.

The first thing I want to do is to discuss the Cayley transform in the "anticommuting with ε" situation. The second is to explore the idea that the group generated by two involutions is dihedral. Thus it should be the case that the spectral thm above is just a special case of representations of the dihedral group. Thirdly I would like to go back to the Toeplitz operator.

Consider then a self-adjoint anti-commuting with ε

$$A = \begin{pmatrix} 0 & T^* \\ T & 0 \end{pmatrix}$$

The Cayley transform of A is the unitary operator

$$U = \frac{1+iA}{1-iA}$$

and it satisfies $\varepsilon U \varepsilon^{-1} = U^{-1}$. Now we can conjugate

$$\begin{pmatrix} i & 0 \\ 0 & 1 \end{pmatrix}^{-1} iA \begin{pmatrix} i & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} i & 0 \\ 0 & 1 \end{pmatrix}^{-1} \begin{pmatrix} 0 & iT^* \\ iT & 0 \end{pmatrix} \begin{pmatrix} i & 0 \\ 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & T^* \\ -T & 0 \end{pmatrix} = \varepsilon A$$

Let's use instead of the Cayley transform

$$\begin{pmatrix} i & 0 \\ 0 & 1 \end{pmatrix}^{-1} \left(\frac{1+iA}{1-iA} \right) \begin{pmatrix} i & 0 \\ 0 & 1 \end{pmatrix} = \frac{1+\varepsilon A}{1-\varepsilon A}.$$ Call this U.

Then U is unitary and $\varepsilon U \varepsilon^{-1} = U^{-1}$!
We can also write

$$\begin{aligned} U &= (1-\varepsilon A)^{-1}(1+\varepsilon A) \\ &= (\varepsilon(\varepsilon - A))^{-1}(1+\varepsilon A) = (\varepsilon - A)^{-1}\varepsilon(1+\varepsilon A) \\ &= (\varepsilon - A)^{-1}(\varepsilon + A) \end{aligned}$$

except that we must be careful because $\varepsilon + A$, $\varepsilon - A$ don't commute.

Note that

$$(1-\varepsilon A)(1+\varepsilon A) = 1 - (\varepsilon A)^2 = 1 + A^2 = \begin{pmatrix} 1+T^*T & 0 \\ 0 & 1+TT^* \end{pmatrix}$$

so

$$\begin{aligned} U &= (1-\varepsilon A)^{-1}(1+\varepsilon A) = (1+A^2)^{-1}(1+\varepsilon A)^2 \\ &= \begin{pmatrix} 1+T^*T & 0 \\ 0 & 1+TT^* \end{pmatrix}^{-1} \begin{pmatrix} 1-T^*T & 2T^* \\ -2T & 1-TT^* \end{pmatrix} \\ &= \begin{pmatrix} \frac{1-T^*T}{1+T^*T} & \frac{2}{1+T^*T}T^* \\ -\frac{2}{1+TT^*}T & \frac{1-TT^*}{1+TT^*} \end{pmatrix} \end{aligned}$$

The corresponding involution is either $U\varepsilon$ or εU, say

$$\varepsilon U = \begin{pmatrix} \frac{1-T^*T}{1+T^*T} & \frac{2}{1+T^*T}T^* \\ \frac{2}{1+TT^*}T & -\frac{1-TT^*}{1+TT^*} \end{pmatrix}$$

This is the involution ε' with $\varepsilon' = 1$ on $\Gamma_T = \mathrm{Im}\binom{1}{T}$ and $\varepsilon' = -1$ on $(\Gamma_T)^\perp$

Now we know the Cayley transform sets up a bijection between skew-adjoint operators and unitaries not having the eigenvalue -1.

At this point we have completed step 1 of our program, namely we have identified the Cayley transform going from self adjoints anti-commuting with ε to unitaries carried into their inverses by ε. The map is

$$A = \begin{pmatrix} 0 & T^* \\ T & 0 \end{pmatrix} \longmapsto U = \frac{1+A\varepsilon}{1-A\varepsilon} = (\varepsilon + A)(\varepsilon - A)^{-1}$$

and the involution is

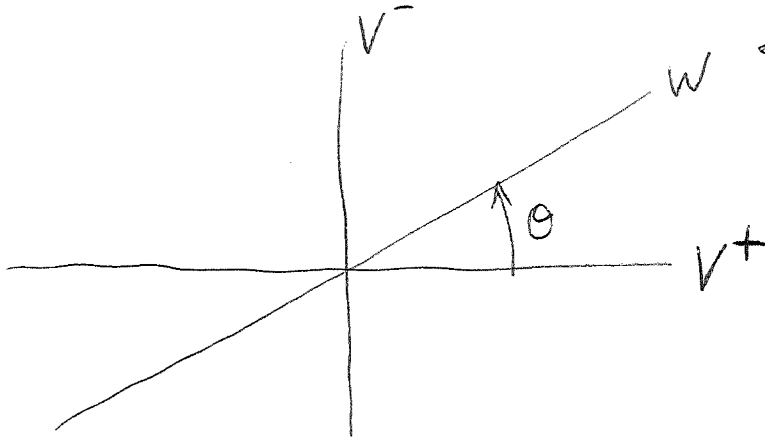
$$\varepsilon' = U\varepsilon = \begin{pmatrix} \frac{1-T^*T}{1+T^*T} & \frac{2}{1+T^*T}T^* \\ \frac{2}{1+TT^*}T & \frac{-1+TT^*}{1+TT^*} \end{pmatrix} \qquad \begin{array}{l} \varepsilon' = 1 \text{ on } \Gamma_T = \operatorname{Im}\begin{pmatrix} 1 \\ T \end{pmatrix} \\ \varepsilon' = -1 \text{ on } (\Gamma_T)^\perp = \operatorname{Im}\begin{pmatrix} -T^* \\ 1 \end{pmatrix} \end{array}$$

so we really do have the graph map.

Next we turn to our second project which is to associate to an involution ε' the [redacted] group generated by ε' and ε. This is a dihedral group the normal cyclic subgroup being generated by the unitary $g = \varepsilon'\varepsilon$ which satisfies $\varepsilon g \varepsilon = \varepsilon\varepsilon' = g^{-1}$. Apply the spectral theorem to g. ε carries the eigenspace V_λ on which $g = \lambda$ to $V_{\bar{\lambda}}$. This means that [redacted] $V_1 = V_1^+ \oplus V_1^-$ where $\varepsilon = \varepsilon' = +1$ on V_1^+ and $\varepsilon = \varepsilon' = -1$ on V_1^-. Also $V_{-1} = V_{-1}^+ \oplus V_{-1}^-$ where $\varepsilon = -\varepsilon' = \pm 1$ on $V_{-1}^\pm$.

We see that there are two different irreducibles for $\lambda = 1$, and for $\lambda = -1$, and one irreducible with $\lambda \in \mathbb{T} - \{1, -1\}$.

We can visualize things by means of the following picture:



← ε' reflects around W

← ε reflects around V^+

Then $\varepsilon, \varepsilon'$ generate the dihedral group. Note that in the 2 dimensional real plane drawn $\varepsilon'\varepsilon$ is rotation through the angle 2θ.

In general one has

$$\varepsilon' = \begin{pmatrix} \cos 2\theta & \sin 2\theta \\ \sin 2\theta & -\cos 2\theta \end{pmatrix} \qquad \text{corresp to} \qquad W = \mathbb{C}\begin{pmatrix} \sin\theta \\ \cos\theta \end{pmatrix}$$

$$U = \varepsilon'\varepsilon = \begin{pmatrix} \cos 2\theta & -\sin 2\theta \\ \sin 2\theta & \cos 2\theta \end{pmatrix} \qquad \text{has eigenvalues } e^{\pm 2i\theta}$$

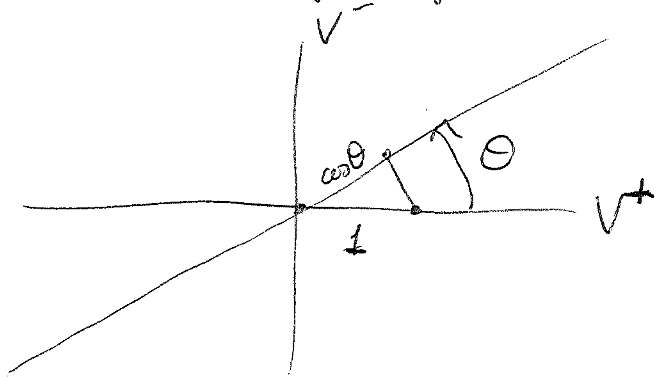
Now I want to study the map which associates to a unitary g the Toeplitz operator ege from V^+ to itself. In practice we are interested in this Toeplitz operator as a Fredholm operator from one space to another because we want its index. This means that we might as well consider the operators

$$W = gV^+ \underset{geg^{-1}e = geg^{-1}}{\overset{e = egeg^{-1}}{\rightleftarrows}} V^+$$

i.e. the projections $\quad W \underset{e' = e'e}{\overset{e = ee'}{\rightleftarrows}} V^+$. Thus I am looking at

$$\begin{pmatrix} 0 & ee' \\ e'e & 0 \end{pmatrix} \text{ on } V^+ \oplus W$$

In the case where W is a line in the plane $V^+ \oplus V^-$ of angle θ with $0 < \theta < \frac{\pi}{2}$



it is clear that the characteristic value of $T = e'e : V^+ \to W$ is just $\cos\theta$, and it lies between $0, 1$ strictly:

$$1 > \cos\theta > 0 \quad \text{for} \quad 0 < \theta < \frac{\pi}{2}$$

The remaining 4 irreducible cases are

$$\begin{array}{lll}
W = V^+, & V^- = 0 & T = 1 : \mathbb{C} \to \mathbb{C} \\
W = V^-, & V^+ = 0 & T = 0 : 0 \to \mathbb{C} \\
W = 0 = V^-, & W^\perp = V^+ & T = 0 : \mathbb{C} \to 0 \\
W = 0 = V^+, & W^\perp = V^- & T = 0 : 0 \to 0
\end{array}$$

The last case doesn't register with the Toeplitz operator apparently, i.e. if $\varepsilon' = \varepsilon = -1$, then $V^+ = W = 0$ and so nothing appears.

March 16, 1986:

I would like to understand why the restricted unitary group is homotopy equivalent to the space of Fredholm operators by the Toeplitz operator map $g \longmapsto ege : V^+ \longrightarrow V^+$. I propose to do this by means of eigenvalues and the resulting stratifications.

Let's begin on the Grassmannian side. Given $\varepsilon' \equiv \varepsilon \mod \mathcal{K}$ we consider $g = \varepsilon'\varepsilon \equiv 1 \mod \mathcal{K}$ and its spectral decomposition. The eigenvalues $\lambda \neq 1$ of g are discrete of finite multiplicity, and they cluster to $\lambda = 1$. The $\lambda = -1$ eigenspace splits into 2 pieces

$$\varepsilon = 1, \varepsilon' = -1 \qquad W^\perp \cap V^+$$
$$\varepsilon = -1, \varepsilon' = 1 \qquad W \cap V^-$$

We define the p,q stratum of Gr to consist of all W where

$$\dim(W^\perp \cap V^+) = q, \quad \dim(W \cap V^-) = p.$$

Denoting this stratum by $Gr_{(p,q)}$ we have a map

$$Gr_{(p,q)} \longrightarrow \mathrm{Grass}_p(V^-) \times \mathrm{Grass}_q(V^+).$$

This map is a homotopy equivalence because we can use the spectral thm. to move all $\lambda \neq -1$ into $\lambda = 1$. This retracts $Gr_{(p,q)}$ to a subspace isomorphic to $\mathrm{Grass}_p(V^-) \times \mathrm{Grass}_q(V^+)$.

Next consider the operators $A = \begin{pmatrix} 0 & T^* \\ T & 0 \end{pmatrix}$ on $V^+ \oplus V^+$ such that $A^2 \leq 1$ and $A^2 \equiv 1 \mod \mathcal{K}$. This is our space $\mathcal{F}_1$. We can stratify it by letting

$$\mathcal{F}_{1\,(p,q)} = \{ A \mid \dim(\operatorname{Ker} T) = p, \dim(\operatorname{Ker} T^*) = q \}$$

We have an obvious map

$$\mathcal{F}_{1(p,q)} \longrightarrow \mathrm{Grass}_p(V^+) \times \mathrm{Grass}_q(V^+)$$

which we claim is a homotopy equivalence. To see this we can deform the non zero eigenvalues of A to ± 1; this retracts $\mathcal{F}_{1(p,q)}$ to the subspace of A such that $A^2 = 1$ on the orthogonal complement of $\mathrm{Ker}\,A$. This subspace is a fibre bundle over $\mathrm{Grass}_p(V^+) \times \mathrm{Grass}_q(V^-)$ whose fibre at (K^+, K^-) is the unitary isos. of V^+/K^+ with V^-/K^-, and this fibre is contractible by Kuiper's theorem.

Next we consider over $\mathcal{G}r = \{W \mid W \equiv V^+ \bmod \mathcal{K}\}$ the principal bundle P for $U(V^+)$ consisting of isometric embeddings $V^+ \xrightarrow{h} V$ such that $hV^+ \in \mathcal{G}r$. Kuiper's thm. implies $P \longrightarrow \mathcal{G}r$ is a h.eq. We can map P to $\mathcal{F}_1$ by assigning to h the operator

$$T : \; V^+ \xrightarrow[\sim]{h} hV^+ \xrightarrow{e} V^+.$$

(Motivation comes from the fact that if $g \in U_{res}$, and $h = g\big|_{V^+} : V^+ \xrightarrow{\sim} gV^+$, then

$$T = ege : \; V^+ \xrightarrow{h = g|V^+} gV^+ \xrightarrow{e} V^+.)$$

Note that

$$\mathrm{Ker}(T) = h^{-1}(W \cap V^-)$$
$$\mathrm{Ker}(T^*) = W^\perp \cap V^+.$$

Set $P_{(p,q)}$ = inverse image of $\mathcal{G}r_{(p,q)}$. So now we have the diagram

$$\begin{tikzcd}
\mathcal{P}_{(p,q)} \arrow[r, "\sim"] \arrow[d] & \mathcal{G}r_{(p,q)} \arrow[dd, "\sim"] \\
\mathcal{F}_{1\,(p,q)} \arrow[d, "\sim"] & \\
\mathrm{Grass}_p(V^+) \times \mathrm{Grass}_q(V^+) & \mathrm{Grass}_p(V^-) \times \mathrm{Grass}_q(V^+)
\end{tikzcd}$$

$$\begin{tikzcd}
& h \arrow[dl] \arrow[dr, bend left] & \\
\left(h^{-1}(W \cap V^-),\ W^\perp \cap V^-\right) & & \left(W \cap V^-,\ W^\perp \cap V^+\right)
\end{tikzcd}$$

$$W = hV^+$$

If we choose any isom $V^+ \simeq V^-$, then we will get a homotopy commutative diagram, because $\mathrm{Grass}_p(V^+) = BU_p$ and the induced bundles are isomorphic back over $\mathcal{P}_{(p,q)}$.

So we conclude that

$$\mathcal{P}_{(p,q)} \sim \mathcal{F}_{1\,(p,q)}.$$

∎

March 17, 1986

Let's go over Gaussian or quasi-free states on a Weyl or Clifford algebra. I recall that such a state is the restriction of an irreducible Gaussian state on a larger Weyl algebra. So we begin by describing the irreducible states.

Let H be a complex Hilbert space. Then we have the boson Fock space $\hat{S}(H)$ with the skew-adjoint operators $a_h^* - a_h$ satisfying

$$[a_h^* - a_h, a_{h'}^* - a_{h'}] = -\langle h|h'\rangle + \langle h'|h\rangle$$

$$= -2i \operatorname{Im} \langle h|h'\rangle$$

These relations can be written in Weyl form:

$$W(h) = e^{a_h^* - a_h}$$

Then

$$W(h)W(h') = e^{-i \operatorname{Im}\langle h|h'\rangle} W(h+h').$$

The irreducible state $|0\rangle$ gives the generating function

$$\langle 0|W(h)|0\rangle = \langle 0|e^{a_h^* - a_h}|0\rangle = e^{-\frac{1}{2}\|h\|^2}.$$

Fock space $\hat{S}(H)$ is a cyclic repn. of ~~the~~ Weyl algebra $W(H)$ with the cyclic vector $|0\rangle$. The corresponding state is Gaussian with variance

$$-\langle 0|(a_h^* - a_h)^2|0\rangle = \langle 0|a_h a_h^*|0\rangle = \|h\|^2$$

Now let V be a real subspace of H. Restricting $\langle\,|\,\rangle$ to V we get an inner product and a skew-symmetric form. The Weyl algebra $W(V)$ for V with the skew-symmetric form is a subalgebra of $W(H)$, and the restriction of $\langle 0|?|0\rangle$ is a Gaussian state on $W(V)$ for the inner product.

Conversely suppose given in the real space V an inner product and skew form. Then we can form a complex Hilbert space H generated by V such that in H one has

$$\langle v|v'\rangle = \underbrace{(v,v')}_{\text{inner prod}} + i\,\underbrace{\omega(v,v')}_{\text{skew-form}}$$

provided one assumes that

$$|\omega(v,v')| \leq \|v\|\cdot\|v'\|$$

(Positivity.

$$\|v_1+iv_2\|^2 = \|v_1\|^2+\|v_2\|^2+i\big(\langle v_1|v_2\rangle - \langle v_2|v_1\rangle\big)$$

$$= \|v_1\|^2+\|v_2\|^2 - 2\,\underbrace{\operatorname{Im}\langle v_1|v_2\rangle}_{\omega(v_1,v_2)} \geq \|v_1\|^2+\|v_2\|^2-2\|v_1\|\|v_2\| \geq 0.\Big)$$

If we write $\omega(v,v') = (v, Jv')$ with J skew-symmetric relative to the inner product, then

$$\|v_1+iv_2\|^2 = \|v_1\|^2+\|v_2\|^2-2(v_1,Jv_2) = \|v_1\|^2+\|v_2\|^2+2(Jv_1,v_2)$$

$$= \|v_1-Jv_2\|^2 + \|v_2\|^2-\|Jv_2\|^2$$

$$= \|v_1\|^2-\|Jv_1\|^2 + \|Jv_1+v_2\|^2.$$

So $\|v_1+iv_2\|^2=0 \implies v_1 = Jv_2,\ Jv_1=-v_2$. So as long as $-J^2\leq 1$ we have positivity and if J^2 doesn't have the eigenvalue -1, then $H=V\oplus iV$.

The next step will be to suppose ω is symplectic, i.e. that J is non-singular. In this case we have a unique complex structure on V such that the eigenvalues of J are of the form $i\lambda$ with $0 < \lambda \leq 1$. I should have said that i preserves $(\ ,\ )$ and $\omega(\ ,\ )$, and so i commutes with J.

So far what I have done is to start with V, $(\ ,\ )$, ω and construct H.

March 18, 1986

Let's review the theory of Gaussian (quasi-free ?) ~~states~~ on Clifford and Weyl algebras. Everything finite dimensional in spite of language

In this theory a central role is played by the following structure: a real vector space V equipped with a quadratic form $(\ ,\ )$ and a skew-symmetric form $\omega(\ ,\ )$. One supposes that the quadratic form is an inner product, and that if K is the skew-symmetric operator representing ω;

$$\omega(v, v') = (v, Kv')$$

then $-K^2 \leq 1$; equivalently

$$|\omega(v,v')| \leq \|v\|\cdot\|v'\|.$$

The critical case occurs when $K^2 = -1$ in which case there is a complex structure on V with multiplication by i given by K. Lets write (H, J) instead of (V, K). H is a complex Hilbert space with hermitian inner product

$$\langle h | h' \rangle = (h, h') - i(h, Jh')$$

$$\boxed{\begin{array}{l}\omega(h,h') \\ = -\mathrm{Im}\langle h|h'\rangle\end{array}}$$

There are two quantizations one can attach to a complex Hilbert space H:

Bosonic Fock space $\hat{S}(H)$. The skew form $\omega = -\mathrm{Im}\langle\,|\,\rangle$ describes the commutation relations among the creation and annihilation operators. This is a representation of the Weyl algebra $W(H)$ constructed from the underlying real vector space of H with the symplectic form ω. This representation has the cyclic vector $|0\rangle$ which gives a Gaussian state on $W(H)$ with variance given by the inner product $(\ ,\ )$ on H. Formulas:

$$\tfrac{1}{2}\left[a_h^* - a_h \,,\, a_{h'}^* - a_{h'}\right] = \tfrac{1}{2}\left(-\langle h|h'\rangle + \langle h'|h\rangle\right)$$

$$= -i\,\mathrm{Im}\langle h|h'\rangle = i\,\omega(h,h')$$

$$-\langle 0|\,(a_h^* - a_h)^2\,|0\rangle = \langle 0|\,a_h a_h^*\,|0\rangle = \|h\|^2$$

Fermionic Fock space $\hat{\Lambda}(H)$. ~~The quadratic form~~ This is a representation of the Clifford algebra $C(H)$ constructed from the ~~real~~ underlying real vector space of H with the quadratic form $(\ ,\ )$. The cyclic vector $|0\rangle$ gives rise to a Gaussian state on $C(H)$ with variance determined by the skew form ω.

Formulas:

$$(a_h^* + a_h)^2 = \|h\|^2$$

$$\tfrac{1}{2}\langle 0|\,[a_h^* + a_h \,,\, a_{h'}^* + a_{h'}]\,|0\rangle = \tfrac{1}{2}\langle 0|a_h a_{h'}^*|0\rangle - \tfrac{1}{2}\langle 0|a_{h'} a_h^*|0\rangle$$

$$= \tfrac{1}{2}\left(\langle h|h'\rangle - \langle h'|h\rangle\right) = i\,\mathrm{Im}\langle h|h'\rangle$$

Now let's return to V, $(\ ,\ )$, $\omega_{\wedge}(v,v') = (v, Kv')$ where $-K^2 \le 1$.

<u>Prop:</u> Up to canon isomorphism there is a unique isometric embedding $j : V \longrightarrow H$ where H is a Hilbert space with complex structure J such that

i) $j^* J j = K$ i.e. $\omega(v,v') = -\mathrm{Im}\langle jv|jv'\rangle$

ii) $H = V + JV$.

Idea of proof: Given (H, j, J) as in the proposition the inner product on H is given by

$$\|jv_1 + Jjv_2\|^2 = \|v_1\|^2 + \|v_2\|^2 + 2\underbrace{(jv_1, Jjv_2)}_{(v_1,\, j^*Jj v_2) \,=\, (v_1,\, Kv_2)}$$

The basic assumption on K:
$$|(v_1, Kv_2)| \leq \|v_1\| \|v_2\|$$
implies that we get a non-negative norm² on the set of pairs (v_1, v_2) by this formula. Moreover if we define $J(v_1, v_2) = (-v_2, v_1)$ this is an isometry:
$$\|-v_2\|^2 + \|v_1\|^2 + 2(-v_2, Kv_1) = \|v_1\|^2 + \|v_2\|^2 + 2(v_1, Kv_2)$$
by skew-symmetry of K. So completing gives H.

<u>Applications</u>: Given $(V, (\ ,\ ), \omega)$ we can construct a Gaussian state on $W(V, \omega)$ having the variance determined by $(\ ,\ )$. Namely we take the bosonic Fock space $\hat{S}(H)$ which gives a Gaussian state on $W(H)$ with variance $\|h\|^2$ and restrict this representation to the subalgebra $W(V, \omega) \subset W(H)$.

Similarly we construct a Gaussian state on the Clifford algebra $C(V, (\ ,\ ))$ with variance determined by ω, by taking the cyclic representation of $C(H)$ given by the <u>fermionic</u> Fock space $\hat{\Lambda}(H)$.

Let us now consider the complex case of the above. By this we mean that there is given in addition a complex structure on V compatible with $(\ ,\ )$ and ω, i.e. $v \mapsto iv$ preserves these two forms. Then V becomes a complex Hilbert space via $(\ ,\ )$ and i, and K is a skew-adjoint operator on V considered as a complex Hilbert space. We can then replace K by the self-adjoint operator $A = iK$ which satisfies $A^2 \leq 1$, or $-1 \leq A \leq 1$.

If we use the above proposition to construct (H, j, J), then by uniqueness the complex structure on V induces one on H, which we denote by $h \mapsto ih$. Then j is an isometric embedding of complex Hilbert spaces such that $j^*Jj = K$ where J is a completely different complex structure on H. The following is clearly the complex version of the above proposition.

<u>Prop</u>: Let V be a complex Hilbert space equipped with a self-adjoint operator A such that $-1 \leq A \leq 1$. Then up to isomorphism there is a unique triple (H, j, F) where $j : V \longrightarrow H$ is an isometric embedding of complex Hilbert spaces and F is an involution on H satisfying

i) $\quad j^* F j = A$

ii) $\quad H = jV + FjV$

Remarks: 1) This result is essentially a variation of the idea that any positive definite function on $\{\pm 1\}$ comes from a cyclic representation of this group.

2) If A doesn't have the eigenvalue ± 1, then H will be twice the dimension of V. One can see this by splitting V into the eigenspaces for A. So one should look first at the case where A is a scalar λ with $-1 < \lambda < 1$.

Take $H = V \oplus V$ and let $j : V \hookrightarrow H$ be the embedding in_1. Take

$$F_\theta = \begin{pmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{pmatrix}$$

then $$j^* F_\theta j = \cos\theta.$$

which given $\lambda = \cos\theta$ for any θ. [redacted] If $\sin\theta \neq 0$, then $H = jV + F_\theta jV$. [redacted]
Notice that changing θ to $\theta + 2\pi$ doesn't affect $j^* F_\theta j$, but changes the sign of F_θ. This is taken care of by the automorphism $\varepsilon = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ of H.

[redacted] Let's now try to describe the picture that emerges. Let's consider the generic case where A doesn't have the eigenvalues ± 1. I start with $H = V \oplus V$, $j = in_1$ and the flipping involution
$$F_{\pi/2} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$
which gives rise to $A = 0$. The assertion then is that the map which assigns to an involution F on $H = V \oplus V$ the contraction $A = j^* F j$ is close to being bijective. Of course this is crazy — I haven't allowed for isomorphisms of the second factor of H.

There seems to be a simple proof that the restricted unitary group ~~[illegible]~~ is homotopy equivalent to the unitary group of the Calkin alg., the equivalence being given by the Toeplitz map.

As usual let U_{res} = the group of unitaries on $H = H^+ \oplus H^-$ which commute with ε mod $\mathcal{K}$. We know that any non-trivial projector in the Calkin algebra

$\mathcal{A} = \mathcal{B}/\mathcal{K}$ lifts to a projector in $\mathcal{B}$ with infinite rank and nullity and that any two such projectors are conjugate under the unitary group $U(H) = U(\mathcal{B}(H))$. This gives us a fibration

$$1 \longrightarrow U_{res} \longrightarrow U(H) \longrightarrow \mathcal{I}(\mathcal{A}) \longrightarrow *$$

where $\mathcal{I}(\mathcal{A})$ is the space of non-trivial involutions in $\mathcal{A}$. This shows immediately that

$$U_{res} \sim \Omega\, \mathcal{I}(\mathcal{A})$$

and since we know that $\mathcal{I}(\mathcal{A}) \sim U(\mathcal{K}) = U$ we see U_{res} has the homotopy type of $\Omega U = \mathbb{Z} \times BU$.

But we can be a bit more precise because we can make the above fibration to the principal fibration.

$$\begin{array}{ccccccccc} 1 & \longrightarrow & U(\mathcal{A}_+) & \longrightarrow & U(\mathcal{A})/U(\mathcal{A}_-) & \longrightarrow & U(\mathcal{A})/U(\mathcal{A}_+)\times U(\mathcal{A}_-) & \longrightarrow & * \\ &&&&&& \| && \\ &&&&&& \mathcal{I}(\mathcal{A}) && \end{array}$$

Here $\mathcal{A}_+ \times \mathcal{A}_-$ is the centralizer of ε in the Calkin algebra $\mathcal{A}$. We know that

$$U(\mathcal{A}_-) \hookrightarrow U(\mathcal{A})$$

is a homotopy equivalence; I can reduce this to the Kuiper result, but it should be simpler, actually elementary since one knows $U(\mathcal{K}_-) \hookrightarrow U(\mathcal{K})$ is a h.eq., etc.. Thus we conclude easily that the Toeplitz map

$$U_{res} \longrightarrow U(\mathcal{A}_+) \qquad g \longmapsto \varepsilon g \varepsilon$$

is a homotopy equivalence.

March 19, 1986

~~[illegible]~~ Given a self-adjoint operator A on V with $|A| \leq 1$ we have seen that up to canonical isomorphism there is a unique triple (V_A, j, F) where $j : V \longrightarrow V_A$ is an isometric embedding, & F is an involution on V_A such that $j^* F j = A$, $V_A = jV + FjV$.

Suppose now that V is graded $V = V^+ \oplus V^-$ and that $A = \begin{pmatrix} 0 & T^* \\ T & 0 \end{pmatrix}$ anti-commutes with $\varepsilon = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. Then there is a unique ~~grading ε of V_A [illegible]~~ involution ε on V_A compatible with this ε on V and anti-commuting with F. To see this we have

$$\begin{array}{ccc} (V, A) & \xrightarrow[\sim]{\varepsilon} & (V, -A) \\ \downarrow j & & \downarrow j \\ (V_A, F) & \dashrightarrow & (V_{+A}, -F) \end{array}$$

whence the dotted arrow gives ε on V_A commuting with j and anti-commuting with F.

Thus we have $j^{\pm} : V^{\pm} \longrightarrow V_A^{\pm}$ and $F = \begin{pmatrix} 0 & u^* \\ u & 0 \end{pmatrix}$ where $u : V_A^+ \longrightarrow V_A^-$ is unitary & such that $(j^-)^* u j^+ = T$.

Now let us fix an embedding $j^{\pm} : V^{\pm} \hookrightarrow H^{\pm}$ i.e. a graded embedding $j : V \hookrightarrow H$. We fix also $A = \begin{pmatrix} 0 & T^* \\ T & 0 \end{pmatrix}$ on $V^+ \oplus V^-$. Let's consider the space of all $F = \begin{pmatrix} 0 & u^* \\ u & 0 \end{pmatrix}$ on H such that $j^* F j = A$. Given such an F we have a diagram

$$\begin{tikzcd} V \arrow[r, hook, "j"] \arrow[d, hook, "j"'] & H \\ V_A \arrow[ur, hook, dotted, "\varphi"'] & \end{tikzcd}$$

where φ is an (isometric) embedding commuting with F. Conversely given an embedding φ of V_A into H extending j ~~we can choose a~~ such that for $[(\varphi V_A)^{\perp}]^{\pm} \subset H^{\pm}$; these two space $[\varphi(V_A)^{\perp}]^{\pm}$ have the same dimension, we can choose a unitary isomorphism between them and we then construct an F on H compatible with the F on V_A and φ.

So we see that an F on H contracting to A on V is the same as an extension of $j : V \to H$ to an embedding φ of V_A in H together with an isomorphism between the two pieces of the orthogonal complement of $\varphi_A(V)$ in H.

It's clear that for dimensional reasons such an F needn't exist. So we will assume that A has only finitely many eigenvalues $\neq \pm 1$ and that $((jV)^{\perp})^{\pm}$ are inf. diml subspaces of $H^{\pm}$. In this case V is of finite codimension in V_A, and the space of embeddings φ of $V_A \ominus jV$ in $H \ominus jV$ is contractible. Furthermore the orthogonal complement of φ has 2 infinite dimensional pieces and the space of F's on the orthogonal complement is contractible. Thus we have proved the following.

<u>Proposition</u>: Let V be a graded subspace of the ~~graded~~ Hilbert space $H = H^+ \oplus H^-$ and let $A = \begin{pmatrix} 0 & T^* \\ T & 0 \end{pmatrix}$ be a degree 1 self-adjoint operator on V with $\|A\| \leq 1$. Assume that

1) A has only finitely many eigenvalues $\neq \pm 1$
2) $H^\pm \ominus V^\pm$ is countably inf. diml.

Then the space of odd degree involutions $F = \begin{pmatrix} 0 & u^* \\ u & 0 \end{pmatrix}$ of H such that $j^* F j = A$ ($j : V \longrightarrow H$ is the embedding) is contractible.

It remains only to identify the space of F on H whose contraction to V has only finitely many eigenvalues ~~$\neq \pm 1$~~. Thus we want to consider unitaries $u : H^+ \longrightarrow H^-$ whose block decomposition

$$u = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$$

is such that $\alpha = T$ is unitary up to finite rank operators. Thus $\alpha^* \alpha$ and $\alpha \alpha^*$ differ from the identity by finite rank. If we then use that

$$u^* u = \begin{pmatrix} \alpha^* & \gamma^* \\ \beta^* & \delta^* \end{pmatrix} \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

we see that $\alpha^* \alpha + \gamma^* \gamma = 1$, so $\gamma^* \gamma$ has finite rank which implies γ has finite rank. Similarly β has finite rank.

Thus it appears that the space of $F = \begin{pmatrix} 0 & u^* \\ u & 0 \end{pmatrix}$ contracting to the space of such A is the group of $u = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} : H^+ \to H^-$ such that γ, β have finite rank.

Let us now assume that $H^+ = H^-$, $V^+ = V^-$ and $j^+ = j^-$, and in this case write $H = H^\pm$, $V = V^\pm$. Then we have the group of unitaries of H which preserve the decomposition $H = V \oplus V^\perp$ up to finite rank operators. We have a Toeplitz map from this group to the space of $T : V \circlearrowleft$ with $\|T\| \leq 1$ and T unitary modulo finite rank. We have seen the fibres of this Toeplitz [redacted] map are contractible.

Let us discuss next the ungraded case. Here I again have an A on V, A selfadjoint, $-1 \leq A \leq 1$ with only finitely many eigenvalues different from ± 1. Suppose $H = V \oplus V^\perp$ with $V^\perp$ countably-infinite. [redacted] We want to study F involutions on H such that $j^* F j = A$. Such an F leads to a dotted arrow

$$\begin{tikzcd} V \arrow[r, hook, "j"] \arrow[d] & H \\ V_A \arrow[ur, hook, dashed, "\varphi"'] & \end{tikzcd}$$

together with an F on H compatible with F_A on V_A.

Conversely if we choose a φ, and then extend $\varphi F_A \varphi^{-1}$ on φV_A to an F on H, which is possible as V_A / V is finite-dimensional, and H/V is countable dimensional, then we obtain an F contracting to A.

So we see that the space of F's contracting to A is contractible, even if we require $V^\perp \cap H_F^+ + V^\perp \cap H_F^-$ to be finite codimensional in $V^\perp$, better, I want the contraction of F to $V^\perp$ to be of the same type as A, I mean essential spectrum $\{\pm 1\}$ and finitely many eigenvalues $\neq \pm 1$.

We have sort of proved the following. Consider $H = H^+ \oplus H^-$ where both $H^\pm$ are countable diml. Consider all involutions F on H which commute with ε mod $\mathcal{K}$, and which mod $\mathcal{K}^\pm$ induce involutions in $\mathcal{a}^\pm$ which are non-trivial. Call this set $\mathcal{I}$. Then we have a map $\mathcal{I} \longrightarrow \mathcal{I}(\mathcal{a}^+)$, and we claim this map is a homotopy equivalence.

What this means is that given a family in $\mathcal{I}(\mathcal{a}^+)$ there is a family in $\mathcal{I}(\mathcal{a}^-)$ such that their direct sum lifts to $\mathcal{I}(H)$, and hence is homotopic to a constant family in $\mathcal{I}(\mathcal{a})$.

Next I would like to compare maps between the restricted Grassmannian and the space $\mathcal{F}_0$ of special Fredholms $A = \begin{pmatrix} 0 & T^* \\ T & 0 \end{pmatrix}$. I have in mind the Toeplitz map which goes from U_{res} or $U_{res}/U(H^-)$ to $\mathcal{F}_0(H^+)$ on one hand. On the other hand we have the exponential map

$$A \longmapsto \exp(\pi A \varepsilon) \cdot \varepsilon$$

from $\mathcal{F}_0$ to the restricted Grassmannian consisting of involutions $\equiv -\varepsilon$ mod $\mathcal{K}$.

So we have these maps

$$\begin{tikzcd} \mathcal{F}_0 \arrow[r, "A \mapsto \exp(\pi A\varepsilon)\cdot\varepsilon"] & \mathrm{Grass}_{res}(H, -\varepsilon) & g(-\varepsilon)g^{-1} \\ & U_{res} \arrow[u] \arrow[dl, "\text{Toeplitz}"] & g \arrow[u, mapsto] \\ \mathcal{F}_0 & & \end{tikzcd}$$

which we want to be consistent.
It is not clear how to lift to U_{res}.

Recall my old understanding of the AS periodicity proof using Fredholm ops. The following fibrations can be constructed "algebraically" in some sense. In any case one has the fibrations

$$1 \longrightarrow U \longrightarrow U(H) \longrightarrow U(a) \longrightarrow \mathbb{Z} \longrightarrow 1$$

$$1 \longrightarrow U(a^+) \longrightarrow U(a)/U(a^-) \longrightarrow \mathcal{J}(a) \longrightarrow *$$

which give

$$U \sim \Omega U(a) \qquad\qquad U(a) \sim \Omega \mathcal{J}(a)$$

and so what is needed is a h.eq.

$$U \sim \mathcal{J}(a).$$

This is provided by $\mathcal{F}_1 \xrightarrow{\sim} \mathcal{J}(a)$ and
$A \mapsto \exp(i\pi A) : \mathcal{F}_1 \xrightarrow{\sim} U$

One has instead of the second fibration above

$$\begin{array}{l} 1 \longrightarrow U_{res} \longrightarrow U(H) \longrightarrow \mathcal{J}(a) \longrightarrow * \\ \qquad\quad \wr \\ \quad Grass_{res} \sim \mathbb{Z} \times BU \end{array}$$

and instead of the first

$$1 \longrightarrow U \longrightarrow EU \longrightarrow BU \longrightarrow *$$

which give

$$U \sim \Omega(\mathbb{Z} \times BU) \quad , \quad \mathbb{Z} \times BU \sim \Omega \mathcal{J}(a).$$

But again one still needs the connection between $\mathcal{U}$ and $\mathcal{I}(\mathcal{Q})$.

Now it is time to go back to the loop group and Dirac operators on the circle.

March 20, 1986

Suppose we have a self-adjoint operator D on V, and let $A = \frac{D}{\sqrt{1+D^2}}$, and then construct the canonical expansion of A to an involution F on a larger Hilbert space H. Since A has no eigenvalues $= \pm 1$, H is twice as large as V. We can identify, or rather construct this expansion as follows.

Set $H = V \oplus V$, let $j : V \longrightarrow H$ be the inclusion of the first factor $j = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, and let

$$F = \frac{1}{\sqrt{1+D^2}} \begin{pmatrix} D & 1 \\ 1 & -D \end{pmatrix} = \frac{1}{\sqrt{1+D^2}} (\gamma^1 + \varepsilon D)$$

It clear that F is an involution such that $j^* F j = A$.

Note that F anti-commutes with γ^2, so that F is equivalent to a unitary operator constructed from D. To see which one let's replace ε, γ^1 by γ^1, γ^2 whence F becomes

$$F = \frac{1}{\sqrt{1+D^2}} (\gamma^1 D + \gamma^2) = \frac{1}{\sqrt{1+D^2}} \begin{pmatrix} 0 & D-i \\ D+i & 0 \end{pmatrix}$$

while j becomes the inclusion of the $+1$ eigenspace of γ^1, i.e.

$$j = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

Check: $$j^* F j = \frac{1}{2} \begin{pmatrix} 1 & 1 \end{pmatrix} \frac{1}{\sqrt{1+D^2}} \begin{pmatrix} 0 & D-i \\ D+i & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$= \frac{D}{\sqrt{1+D^2}}$$

Thus F is equivalent to the unitary operator

$$\frac{D+i}{\sqrt{1+D^2}}$$

This is not the Cayley transform as the map $x \longrightarrow \frac{x+i}{\sqrt{1+x^2}}$ maps $\mathbb{R}$ onto the upper half of the unit circle. On the other hand if we act by the unitary F on the grading whose $+1$ eigenspace is $j(V)$ we get

$$F\gamma^1 F = \frac{1}{1+D^2}\begin{pmatrix} 0 & D-i \\ D+i & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & D-i \\ D+i & 0 \end{pmatrix}$$

$$= \frac{1}{1+D^2}\begin{pmatrix} 0 & D-i \\ D+i & 0 \end{pmatrix}\begin{pmatrix} D+i & 0 \\ 0 & D-i \end{pmatrix}$$

$$= \frac{1}{1+D^2}\begin{pmatrix} 0 & (D-i)^2 \\ (D+i)^2 & 0 \end{pmatrix}$$

which does give the Cayley transform as

$$\frac{D+i}{D-i} = \frac{(D+i)^2}{D^2+1}.$$

In the above we went from an ungraded self adjoint operator to a graded involution. Now we want to consider a graded self-adjoint operator $D = \begin{pmatrix} 0 & T^* \\ T & 0 \end{pmatrix}$ and get an ungraded involution.

Let's think of D as a self-adjoint operator on $V = V^+ \oplus V^-$ anti commuting with the grading η. As in the ungraded case we form

$$H = S \otimes V$$

where S is the 2 diml spinors and consider

$$F = \frac{1}{1\otimes\sqrt{1+D^2}}\left(\varepsilon\otimes D + \gamma^1\otimes 1\right)$$

which is obviously an involution. Now notice that on $S \otimes V$ we have two involutions: the total grading $\varepsilon \otimes \eta$ and $\gamma^2 \otimes 1$ which anti-commute with themselves and F. This means that F is an odd involution of a graded C_1-module, the grading given by $\varepsilon \otimes \eta$, the C_1-structure given by $\sigma = \gamma^2 \otimes 1$. But if we recall that graded C_1-modules $\simeq$ ungraded $\mathbb{C}$-modules

$$S \otimes W = W \oplus W \longleftarrow W$$

$$\varepsilon = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \qquad \sigma = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

then any odd endo, anti-comm. with σ is of the form $\begin{pmatrix} 0 & \alpha \\ \alpha & 0 \end{pmatrix}$ with α an endo of W. Any $\begin{pmatrix} 0 & \alpha \\ \alpha & 0 \end{pmatrix}^2 = \begin{pmatrix} \alpha^2 & 0 \\ 0 & \alpha^2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ iff α is an involution of W.

Thus $$F = \frac{1}{1 \otimes \sqrt{1 + D^2}} \left(\varepsilon \otimes D + \gamma' \otimes 1 \right) \quad \text{on } S \otimes V$$

is equivalent to an involution on the even part of $S \otimes V$ for the ~~total~~ grading $\varepsilon \otimes \eta$.

Now let's use the canonical isomorphism

$$S \otimes V \xrightarrow{\sim} V \otimes S \qquad x \otimes y \mapsto (-1)^{d^\circ x \, d^\circ y} y \otimes x.$$

to transform the above F to

$$F = \frac{1}{\sqrt{1 + D^2} \otimes 1} \left(D \otimes 1 + \eta \otimes \gamma' \right)$$

which anti-commutes now with the two involutions

$$\varepsilon_{total} = \eta \otimes \varepsilon \quad , \quad \sigma = \eta \otimes \gamma^2$$

We write $V \otimes S$ as a direct sum of the four pieces $V^\pm \otimes S^\pm$ in the following order

$$V \otimes S = \begin{matrix} V^+ \otimes S^+ \\ V^- \otimes S^- \\ V^+ \otimes S^- \\ V^- \otimes S^+ \end{matrix} = \begin{matrix} V^+ \\ V^- \\ V^+ \\ V^- \end{matrix}$$

so that we have

$$\varepsilon_{\text{total}} = \eta \otimes \varepsilon = \left(\begin{array}{cc|cc} 1 & & & \\ & 1 & & \\ \hline & & -1 & \\ & & & -1 \end{array}\right), \quad \sigma = \eta \otimes \gamma^2 = \left(\begin{array}{cc|cc} & & -i & \\ & & & -i \\ \hline i & & & \\ & i & & \end{array}\right)$$

and

$$D \otimes 1 + \eta \otimes \gamma^1 = \left(\begin{array}{cc|cc} & & 1 & T^* \\ & & T & -1 \\ \hline 1 & T^* & & \\ T & -1 & & \end{array}\right)$$

Hence we obtain the ungraded involution

$$\dot{F} = \begin{pmatrix} (1+T^*T)^{-1/2} & (1+T^*T)^{-1/2}T^* \\ (1+TT^*)^{-1/2}T & -(1+TT^*)^{-1/2} \end{pmatrix}$$

As usual we find this involution mysterious, however if we define the unitary

$$g = \dot{F}\varepsilon = \begin{pmatrix} (1+T^*T)^{-1/2} & -(1+T^*T)^{-1/2}T^* \\ (1+TT^*)^{-1/2}T & (1+TT^*)^{-1/2} \end{pmatrix}$$

Then

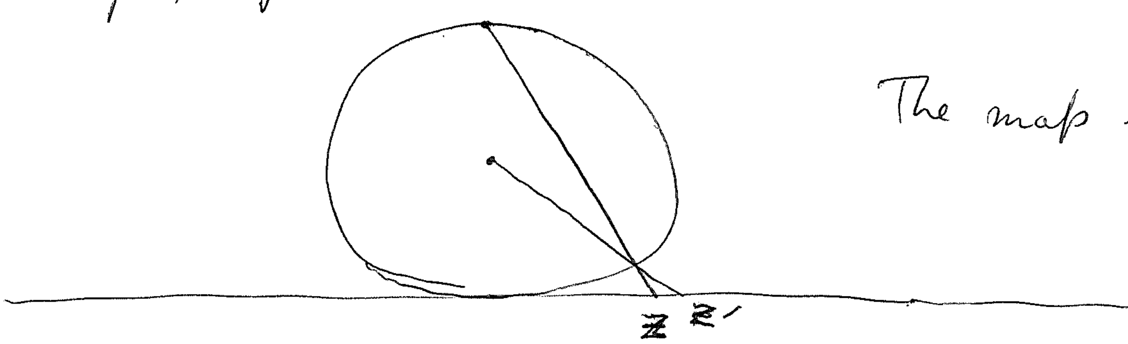
$$g^2\varepsilon = \begin{pmatrix} \dfrac{1-T^*T}{1+T^*T} & \dfrac{2}{1+T^*T}T^* \\ \dfrac{2}{1+TT^*}T & \dfrac{-1+TT^*}{1+TT^*} \end{pmatrix}$$

is the involution corresponding to the graph of T.

I should have seen instantly that $\mathring{F}$ is the "phase" for the operator $\begin{pmatrix} 1 & T^* \\ T & -1 \end{pmatrix}$ and so therefore corresponds to the positive subspace for this **massive** "Dirac" operator $D + \eta$.

Something is curious here. We have managed by means of this massive Dirac business to take a square <u>root</u> of the Cayley transform.

Recall how to map the disk $|z| < 1$ onto the complex plane.



The map is $z \mapsto z'$.

In the same vein we can map $-1 \leq x \leq 1$ onto the circle so that $-1, 1$ go to -1, and this can be done by an alg. map.

Given $A \in \mathcal{F}_1$; this means A self-adjoint $\|A\| \leq 1$, essential spectrum of $A = \{\pm 1\}$. Let $B = \sqrt{1 - A^2}$ and put

$$F = \begin{pmatrix} A & B \\ B & -A \end{pmatrix} = \cancel{} = A \otimes \varepsilon + B \otimes \gamma^1 \quad \text{on } V \otimes S$$

Clearly $F^2 = 1$, and as $A^2 - 1$ is compact, so is B.

In the graded case

$$A = \begin{pmatrix} 0 & T^* \\ T & 0 \end{pmatrix} \qquad B = \begin{pmatrix} (1-T^*T)^{1/2} & 0 \\ 0 & (1-TT^*)^{1/2} \end{pmatrix}$$

and the involution is

$$F = A \otimes 1 + B\eta \otimes \gamma^1 \qquad \text{on} \qquad V \otimes S$$

A, B commute and η anti-commutes with A so that the two [redacted] terms in F anti-commute. Thus

$$F^2 = A^2 \otimes 1 + B^2 \otimes 1 = 1.$$

The other point is that F anti commutes with $\eta \otimes \varepsilon$, and $\eta \otimes \gamma^2$ and so is equivalent to an ungraded involution. [redacted] Relative to

$$V \otimes S = \begin{matrix} V^+ \otimes S^+ \\ V^- \otimes S^- \\ V^+ \otimes S^- \\ V^- \otimes S^+ \end{matrix} \quad \text{one has} \quad F = \left(\begin{array}{cc|cc} & & (1-T^*T)^{1/2} & T^* \\ & & T & -(1-TT^*)^{1/2} \\ \hline (1-T^*T)^{1/2} & T^* & & \\ T & -(1-TT^*)^{1/2} & & \end{array}\right)$$

so the ungraded involution is

$$\begin{pmatrix} (1-T^*T)^{1/2} & T^* \\ T & -(1-TT^*)^{1/2} \end{pmatrix}$$

~~I seem to be missing something still. So far what I have done is to~~

March 21, 1986

Consider the following diagram

(1) $$\begin{tikzcd} \mathcal{F}_0 \arrow[r, dashed, bend left] & U_{res} \arrow[l, "\text{Toeplitz}"] \arrow[r] & Gr_{res} \end{tikzcd}$$

where U_{res} is the restricted unitary group of $H = H^+ \oplus H^-$; it consists of unitary ops on H such that in the block decomposition

$$u = \begin{pmatrix} \alpha & \gamma \\ \beta & \delta \end{pmatrix} \quad \text{one has} \quad \beta \equiv \gamma \equiv \ \text{mod } \mathcal{K}$$

The two solid arrows are

$$\alpha \longleftarrow\!\!\shortmid\ u \longmapsto u \varepsilon u^{-1}$$

We know that the second arrow is a h.eq. because

$$U_{res}/U(H^+)\times U(H^-) \simeq Gr_{res}$$

and $U(H^{\pm})$ is contractible by Kuiper's thm.

We have a section of the Toeplitz map given by

$$\alpha \longmapsto u = \begin{pmatrix} \alpha & -\sqrt{1-\alpha\alpha^*} \\ \sqrt{1-\alpha^*\alpha} & \alpha^* \end{pmatrix}$$

because

$$uu^* = \begin{pmatrix} \alpha & -\sqrt{1-\alpha\alpha^*} \\ \sqrt{1-\alpha^*\alpha} & \alpha^* \end{pmatrix}\begin{pmatrix} \alpha^* & \sqrt{1-\alpha^*\alpha} \\ -\sqrt{1-\alpha\alpha^*} & \alpha \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

(One can alter this section by left and right multiplying by elements of $U(H^-)$.)

Next let's map u to the res. Grassm.

$$u \mapsto u\varepsilon u^* = \begin{pmatrix} \alpha & -\sqrt{1-\alpha\alpha^*} \\ \sqrt{1-\alpha^*\alpha} & \alpha^* \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} \alpha^* & \sqrt{1-\alpha^*\alpha} \\ -\sqrt{1-\alpha\alpha^*} & \alpha \end{pmatrix}$$

$$= \begin{pmatrix} \alpha & -\sqrt{1-\alpha\alpha^*} \\ \sqrt{1-\alpha^*\alpha} & \alpha^* \end{pmatrix}\begin{pmatrix} \alpha^* & \sqrt{1-\alpha^*\alpha} \\ \sqrt{1-\alpha\alpha^*} & -\alpha \end{pmatrix}$$

$$= \begin{pmatrix} 2\alpha\alpha^*-1 & 2\alpha\sqrt{1-\alpha^*\alpha} \\ 2\alpha^*\sqrt{1-\alpha\alpha^*} & 1-2\alpha\alpha^* \end{pmatrix}$$

This is very close to the Cayley transform (graph map). Notice that $u\varepsilon u^{-1}$ is $\equiv \varepsilon \pmod{\mathcal{K}}$, whereas for the graph map we get something $\equiv -\varepsilon \pmod{\mathcal{K}}$.

(Other formulas: Assign to α the involution

$$F = \begin{pmatrix} \sqrt{1-\alpha^*\alpha} & \alpha^* \\ \alpha & -\sqrt{1-\alpha\alpha^*} \end{pmatrix}$$

write $F = u\varepsilon$ or $u = F\varepsilon = \begin{pmatrix} \sqrt{1-\alpha^*\alpha} & -\alpha^* \\ \alpha & \sqrt{1-\alpha\alpha^*} \end{pmatrix}$

and then $\varepsilon u \varepsilon = u^{-1}$ and so

$$u\varepsilon u^{-1} = u^2\varepsilon = F\varepsilon F = \begin{pmatrix} \sqrt{1-\alpha^*\alpha} & -\alpha^* \\ \alpha & +\sqrt{1-\alpha\alpha^*} \end{pmatrix}\begin{pmatrix} \sqrt{1-\alpha^*\alpha} & \alpha^* \\ \alpha & -\sqrt{1-\alpha\alpha^*} \end{pmatrix}$$

$$= \begin{pmatrix} 1-2\alpha^*\alpha & 2\alpha^*\sqrt{1-\alpha\alpha^*} \\ 2\alpha\sqrt{1-\alpha^*\alpha} & -(1-2\alpha\alpha^*) \end{pmatrix}$$

is the familiar graph involution. Notice $\alpha \in \mathcal{F}_0$ $\Longrightarrow$ $u\varepsilon u^{-1} \equiv -\varepsilon \pmod{\mathcal{K}}$.)

Let's go back to the diagram ①. We see for trivial reasons that is at least a factor of $U_{res} \sim Gr_{res}$ up to homotopy. It's probable that by expanding on the idea that inverses are unique (the fact that $\begin{pmatrix} \alpha & 0 \\ 0 & \alpha^* \end{pmatrix}$ lifts to $U(H)$ mod $\mathcal{K}$, means $\alpha \mapsto \alpha^*$ is the homotopy inverse in some sense for the Whitney sum), one can show the Toeplitz map is a h.eq.

Then we would have a proof without eigenvalues of the easy homotopy equivalence

$$\mathcal{F}_0 \sim U_{res} \sim \Omega \underbrace{\mathcal{J}(a)}_{U(H)/U_{res}}$$

in the periodicity thm.

I tried without success to find an ungraded analogue of the space U_{res} over $\mathcal{F}_0$. In other words I wanted to find a space $\mathcal{J}$ mapping both to $\mathcal{F}_1$ and $U(\mathcal{K})$. The obvious ~~candidate~~ for $\mathcal{J}$ is to take the space of involutions on $H = H^+ \oplus H^-$ which mod $\mathcal{K}$ ~~are congruent to~~ become non-trivial involutions in H^+ and H^-. I could not find a map of this space to $U(\mathcal{K})$, which would be consistent with the map

$$A \longmapsto \left(A + i\sqrt{1-A^2}\right)^2$$

March 22, 1986

It seems that one can give the Atiyah-Singer periodicity proof using Cayley maps instead of the exponential map. This might be advantageous when it comes to Dirac operators.

Let's begin with the ungraded case. We wish to prove that the Bott map

$$\mathcal{I}(a) \longrightarrow \Omega(\mathcal{U}(a); 1, -1)$$

is a homotopy equivalence. We have a principal fibration

$$\mathcal{U}(\mathcal{K}) \longrightarrow \mathcal{U}(\mathcal{B}(H)) \longrightarrow \mathcal{U}(a)_{(0)}$$

with contractible total space. From homotopy theory this gives us a homotopy equivalence

$$\Omega(\mathcal{U}(a); 1, -1) \sim \text{fibre over } -1 = -\mathcal{U}(\mathcal{K}).$$

This is obtained by lifting paths in $\mathcal{U}(a)$ starting at 1 to paths in $\mathcal{U}(\mathcal{B}(H))$ ~~starting~~ at 1, and then taking the endpoint. Specifically one has h.eq.'s

$$\begin{tikzcd}
\Omega(\mathcal{U}(\mathcal{B}); 1, -\mathcal{U}(\mathcal{K})) \arrow[r, "\sim", "\text{contractibility of } \mathcal{U}(\mathcal{B})"'] \arrow[d, "\sim \leftarrow \text{path lifting property}"] & -\mathcal{U}(\mathcal{K}) \\
\Omega(\mathcal{U}(a); 1, -1) &
\end{tikzcd}$$

Next we need the h.eq. $\mathcal{F}_1 \longrightarrow \mathcal{I}(a)$ which results from the fibres being convex and non-empty. We now construct an explicit path lifting Φ of the Bott map composed with this map obtaining the diagram

$$\begin{tikzcd} \mathcal{F}_1 \arrow[r, dashed, "\Phi"] \arrow[d, "\sim"'] & \Omega(\mathcal{U}(\mathcal{B}); 1, -\mathcal{U}(\mathcal{K})) \arrow[r, "\sim"'] \arrow[d, "\sim"] & -\mathcal{U}(\mathcal{K}) \\ \partial(a) \arrow[r] & \Omega(\mathcal{U}(a); 1, -1) & \end{tikzcd}$$

$$\Phi_t(A) = \left(\sqrt{1-t^2A^2} + itA\right)^2 \qquad 0 \le t \le 1.$$

If we project this into the Calkin algebra, then A^2 becomes 1 and this becomes

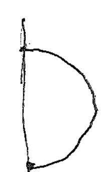
$$\left(\sqrt{1-t^2} + itA\right)^2 = (\cos\theta + i\sin\theta\, A)^2$$
$$= \cos(2\theta) + i(\sin 2\theta)A$$

where $0 \le \theta \le \frac{\pi}{2}$ and $t = \sin\theta$, $\sqrt{1-t^2} = \cos\theta$; thus we have a lifting of the Bott map.

What we therefore have to prove is that the endpoint map

$$\Phi_1 : \mathcal{F}_1 \longrightarrow -\mathcal{U}(\mathcal{K})$$
$$A \longmapsto \left(\sqrt{1-A^2} + iA\right)^2$$

is a homotopy equivalence. Note that



$$x \longmapsto \left(\sqrt{1-x^2} + ix\right)$$

maps $[-1,1]$ onto $\{e^{i\theta} \mid -\frac{\pi}{2} \le \theta \le \frac{\pi}{2}\}$, so that it square maps the unit interval onto the circle with $-1, 1$ going to -1. This map has the same properties as the map

$$x \longmapsto \exp(i\pi x)$$

used by Atiyah and Singer.

Next we consider the graded case. The Bott map to be proved a homotopy equivalence is

$$\mathcal{U}(a) \longrightarrow \Omega(\mathcal{J}(a);\, \varepsilon, -\varepsilon)$$

$$g \longmapsto (\cos 2\theta)\varepsilon + (\sin 2\theta)\begin{pmatrix} 0 & g^{-1} \\ g & 0 \end{pmatrix}$$

$$0 \leq \theta \leq \frac{\pi}{2}$$

Let's recall the version of the Cayley transform in this case. ~~We want to think of~~ $\cancel{U(a)}$ ~~as involutions~~ $\cancel{\begin{pmatrix} 0 & g^* \\ g & 0 \end{pmatrix}}$ ~~anti-commuting with~~ $\cancel{\varepsilon}$.

The Cayley transform in the graded case goes from $A = \begin{pmatrix} 0 & \alpha^* \\ \alpha & 0 \end{pmatrix}$ anti-commuting with ε, to unitaries g carried into their inverses by ε, and these are identified with involutions via $g \mapsto g\varepsilon$. The Cayley transform is usually

$$A \longmapsto \frac{1+iA}{1-iA}$$

however conjugation by $\begin{pmatrix} i & 0 \\ 0 & 1 \end{pmatrix}$ which centralize ε carries this into

$$A \longmapsto \frac{1+A\varepsilon}{1-A\varepsilon}$$

(see p. 240). Now we are also not in the unbounded operator setup, so we really want to work with

$$A \longmapsto \frac{B+A\varepsilon}{B-A\varepsilon} \qquad B = \sqrt{1-A^2} = \begin{pmatrix} \sqrt{1-\alpha^*\alpha} & 0 \\ 0 & \sqrt{1-\alpha\alpha^*} \end{pmatrix}$$

and so with the map

$$\Phi : \mathcal{F}_0 \longrightarrow \mathrm{Grass}_{res}(H, -\varepsilon)$$

$$A \longmapsto \frac{B + A\varepsilon}{B - A\varepsilon}\,\varepsilon = (B + A\varepsilon)^2 \varepsilon$$

$$\|$$

$$B + A\varepsilon = \begin{pmatrix} \sqrt{1-\alpha^*\alpha} & -\alpha^* \\ \alpha & \sqrt{1-\alpha\alpha^*} \end{pmatrix}, \qquad \begin{pmatrix} 1 - 2\alpha^*\alpha & 2\alpha^*\sqrt{1-\alpha\alpha^*} \\ 2\alpha\sqrt{1-\alpha^*\alpha} & -1 + 2\alpha\alpha^* \end{pmatrix}$$

The rest of the proof of the periodicity equivalence proceeds as in the ungraded case. We have the fibration

$$\mathcal{J}(\mathcal{B}) \longrightarrow \mathcal{J}(\mathcal{A})$$

$$\mathrm{Grass}_{res}(H, \pm\varepsilon)$$
$$\|$$

with contractible total space with fibre $\mathcal{J}_{res}(\mathcal{B}; \varepsilon')$ over ε'. Thus we have a diagram

$$\begin{tikzcd}
\mathcal{F}_0 \arrow[r, dashed, "\Phi_t"] \arrow[d, "\sim"] & \Omega(\mathcal{J}(\mathcal{B}); \varepsilon, \mathcal{J}_{res}(\mathcal{B}; \varepsilon)) \arrow[r, "\sim"] \arrow[d, "\sim"] & \mathcal{J}_{res}(\mathcal{B}, -\varepsilon) \\
\mathcal{U}(\mathcal{A}) \arrow[r] & \Omega(\mathcal{J}(\mathcal{A}); \varepsilon, -\varepsilon).
\end{tikzcd}$$

The dotted arrow is

$$\Phi_t(A) = \left(\sqrt{1 - t^2 A^2} + tA\varepsilon\right)^2 \varepsilon \qquad 0 \leq t \leq 1.$$

March 23, 1986

Let η be an involution on V. We have seen how to identify the space of involutions $\mathcal{I}(V)$ with the set of $g \in \mathcal{U}(V)$ $\ni$ $\eta g \eta = g^{-1}$; the map is $g \longrightarrow g\eta$.

I want to describe another way to see this. The idea is that $\mathcal{U}(V)$ should be identified with involutions F on $V \oplus V$ which anti-comm. with ε; the map being $g \longmapsto \begin{pmatrix} 0 & g^{-1} \\ g & 0 \end{pmatrix}$. Then the condition that η transforms g to g^{-1} should be interpreted as saying F anti-commutes or commutes with an involution. Finally a Clifford algebra periodicity should say F is equivalent to an involution on V.

In practice this works as follows

$$\gamma^2 F \gamma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\begin{pmatrix} 0 & g^{-1} \\ g & 0 \end{pmatrix}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} = \begin{pmatrix} 0 & -g \\ -g^{-1} & 0 \end{pmatrix} = -\eta F \eta$$

$$\gamma^1 F \gamma^1 = \eta F \eta$$

so that F anti-commutes with $\eta\gamma^2, \varepsilon$ or F commutes with $\eta\gamma^1$ and anti-comm. with ε.

One can compute all these endos of $V \oplus V$ and find they are of the form

$$\begin{pmatrix} 0 & \eta\alpha\eta \\ \alpha & 0 \end{pmatrix} \qquad \text{with } \alpha \in \mathrm{End}(V).$$

Finally this is an involution $\Longleftrightarrow$ $\alpha\eta\alpha\eta = 1$.

I'm still missing the good way to think about this. Ultimately many one should be thinking in terms of modules over Clifford algs. So that to give an F anti-commuting with $\eta\gamma^2, \varepsilon$

which generate C_2 is the same as giving a grading on a C_2-module, and by periodicity this is the same as a grading on the Morita equivalent C_0-module.

IDEA: "Higher K-groups" should be defined as lower K-groups (even K_0) of certain categories not as homotopy groups. This what happens in the Kasparov machine: Instead of maps from X to $\mathcal{U}(\mathcal{K})$, $\mathcal{U}(a)$ etc. one forms the free Hilbert module over $A = C(X)$ and the corresponding unitary groups etc. for this Hilbert modules. Then one is just talking about ordinary K_0, K_1 for $C(X) \otimes \mathcal{K}$, etc.

This point of view would perhaps allow one to localize, possibly even complete, algebraic K-gps. Also the odd case might work at the same time as the even because A can be non-commutative. (Question: Could the fact that the naive K-theory of graded C_1-modules gives the wrong answer contradict this last assertion?)

Idea: Atiyah + Singer have given a proof of periodicity based mostly on Kuiper's thm. Now the Kuiper thm. is so well-incorporated in Kasparov's machine that there might be a simple proof of periodicity along the A·S lines. (Maybe one has to use the K theory exact sequence (6 terms) relative to

$$A \otimes \mathcal{K} \longrightarrow \mathcal{L}(H_A) \longrightarrow Q(A) = \mathcal{L}(H_A)/A \otimes \mathcal{K}$$

March 24, 1986

Let us consider a compact odd dim'l Riemannian spin manifold M, and a vector bundle E with inner product over it. Let $H = L^2(M, S\otimes E)$, + let $\mathcal{G}$ be the group of gauge transformations of E; $\mathcal{G}$ acts on H. For each connection in E we have a Dirac operator D which is an unbounded self-adjoint operator on H. If $a \in \mathbb{R}$ is not an eigenvalue of D, then $\frac{D-a}{|D-a|}$ is an involution on H. Clearly the class modulo ~~compacts~~ finite rank ops of this involution is independent of the choice of a, since the spectrum of D is discrete of finite multiplicity. So we have associated to D a non-trivial involution in the Calkin algebra $\mathcal{A}$.

Question: Is this involution in $\mathcal{A}$ independent of the connection in E? (Yes, see below)

This is certainly the case for $M = S^1$ and probably in general because changing the connection alters the Dirac operator by a zeroth order operator.

A consequence is that one has a canonical homomorphism

$$\mathcal{G} \longrightarrow U_{res}$$

where U_{res} is the restricted unitary group of H relative to this involution in the Calkin algebra.

If I pick an involution in the restricted Grassmannian $\mathcal{I}_{res}$, then I get a map

$$U_{res} \longrightarrow \mathcal{I}_{res}$$

and the canonical invariant character forms (defined on Schatten class subsets) will pull back to given left-invariant forms on $\mathcal{G}$.

However these forms depend on the choice of involution. (Involutions are like extreme points – maybe one could define forms for arbitrary $A \in \mathcal{F}_1$. Say by the usual trick of expanding A to an involution?)

Graded versions of U_{res}: Let us consider the space of Dirac operators with coefficients in E over an even diml compact manifold M. Put $\mathcal{G}$ for the group of gauge transformations and $H = L^2(M, S\otimes E)$. Pick any Dirac operator D and set $A = \frac{D}{\sqrt{1+D^2}}$. Then A anti-commutes with the grading ε on H, and $A \in \mathcal{F}_0(H)$, i.e. it becomes an involution in the Calkin algebra $\mathcal{Q}$. I claim this involution in $\mathcal{Q}$ ~~which I should have said is essentially the sa~~ does not depend on the choice of D. In effect what we see in the Calkin algebra depends only on the symbol of A and this is just the phase $\frac{\sigma(D)}{|\sigma(D)|}$, which is given by Clifford multiplication and is independent of the connection chosen in E to define to Dirac operator. $\therefore$ we have proved the

Proposition: If D is the Dirac operator on $S\otimes E$ associated to a connection in E, then $D/\sqrt{1+D^2}$ projects to an involution in the Calkin algebra which is independent of the connection and m.

What we have then is a representation of $\mathcal{G}$ on $H = H^+ \oplus H^-$ which preserves the grading, and an operator $A = \begin{pmatrix} 0 & T^* \\ T & 0 \end{pmatrix} \in \mathcal{F}_0(H)$ such that $\mathcal{G}$ preserves A mod $\mathcal{K}$. This suggests defining a restricted in the graded setup as follows. Put $F =$ image of A

Set $$U^{\varepsilon}_{res}(F) = \{ g \in U(H)^{\varepsilon} \mid gFg^{-1} \equiv F \}$$

The difference in the ungraded case is that F can be lifted to involution. Here this can only be done when T has index zero.

Anyway we can describe this graded restricted unitary group as follows. Let $U(\mathcal{Q})_{(n)}$ be the component of $U(\mathcal{Q})$ consisting of unitaries of index n. Then we have an exact sequence

$$1 \longrightarrow U(\mathcal{Q}^+)_{(0)} \longrightarrow U(\mathcal{Q}^-)_{(0)} \times U(\mathcal{Q}^+)_{(0)} \longrightarrow U(\mathcal{Q})_{(n)} \longrightarrow *$$

$$(g^-, g^+) \longmapsto g^- \bar{T} (g^+)^{-1}$$

$$g^+ \longmapsto (\bar{T} g^+ \bar{T}^{-1}, g^+)$$

Actually $U(\mathcal{Q})_{(n)}$ stands for graded involutions $F = \begin{pmatrix} 0 & \bar{T}^{-1} \\ \bar{T} & 0 \end{pmatrix}$ of index n.

Then $U^{\varepsilon}_{res}(F)$ ~~is a pullback~~ fits in a diagram

$$\begin{tikzcd}
U(\mathcal{K}^-)\times U(\mathcal{K}^+) \arrow[d] \arrow[r, phantom, "="] & U(\mathcal{K}^-)\times U(\mathcal{K}^+) \arrow[d] & \\
1 \to U^{\varepsilon}_{res}(F) \arrow[r, hook] \arrow[d] \arrow[dr, phantom, "\text{cart}"] & U(H^-)\times U(H^+) \arrow[r] \arrow[d] & U(\mathcal{Q})_{(n)} \to * \arrow[d, equal] \\
1 \to U(\mathcal{Q}^+)_{(0)} \arrow[r, hook] & U(\mathcal{Q}^-)_{(0)} \times U(\mathcal{Q}^+)_{(0)} \arrow[r] & U(\mathcal{Q})_{(n)} \to *
\end{tikzcd}$$

and by Kuiper's thm., one has

$$U^{\varepsilon}_{res}(F) \sim \Omega\, U(\mathcal{Q})_{(n)} \sim U(\mathcal{K}).$$

We can summarize the construction as follows:

Given $A = \begin{pmatrix} 0 & T^* \\ T & 0 \end{pmatrix} \in \mathcal{F}_0$, define $\mathcal{U}^\varepsilon_{res}$ to be the subgroup of $\mathcal{U}^\varepsilon(H) = \mathcal{U}(H^+) \times \mathcal{U}(H^-)$ which leaves A invariant modulo compacts. In the case where A has index 0, we can replace T by a unitary mod $\mathcal{K}$. In this case we use T to identify $H^+ \xrightarrow{\sim} H^-$ and $\mathcal{U}^\varepsilon_{res}$ consists of pairs $(g_1, g_2) \in \mathcal{U}(H^+) \times \mathcal{U}(H^+)$ ~~such that~~ which agree mod $\mathcal{K}$.

March 25, 1986

Question: To begin let's work in finite dimensions, let $A \in \mathcal{F}_1(V)$, $G = U(V)$. Expand A to an ~~contraction~~ involution $j: V \subset H$, $F \in \mathcal{I}(H)$, $j^* F_j = A$. Then we have an embedding $G \subset U(H)$ and a map $U(H) \to \mathcal{I}(H)$ determined by F, so we can pull back the $U(H)$-invariant forms on $\mathcal{I}(H)$ given by $\mathrm{tr}(F(dF)^{2k})$ to obtain closed left-invariant forms on G. The first question is whether these are independent of the choice of expansion, and if so the second question is whether there's a simple formula. Thus we have linear functionals on $\Lambda^{ev}\mathfrak{g}^*$, $\mathfrak{g} = Lie(G)$ determined by $A \in \mathcal{F}_1(V)$.

Let's first check independence of the expansion. We know there is a minimal expansion which embeds uniquely into any other. Thus we have

$$\begin{array}{ccccc} V & \subset & H & \subset & H' \\ A & & F & & F' \end{array}$$

$$G = U(V) \hookrightarrow U(H) \hookrightarrow U(H')$$

We want to show that the forms $\mathrm{tr}_{H'}(F'[F',\theta]^{2k})$ on $U(H')$ and $\mathrm{tr}_H(F[F,\theta]^{2k})$ on $U(H)$ pull-back to the same form on $U(V)$. So we can suppose $V = H$. Let's work relative to the decomposition $H' = H \oplus H^\perp$. Because F' contracts to F one has

$$F' = \begin{pmatrix} F & 0 \\ 0 & F_1 \end{pmatrix}$$

Also $$\theta_{U(H')} \restriction U(H) = \begin{pmatrix} \theta_{U(H)} & 0 \\ 0 & 0 \end{pmatrix}$$

Thus $$[F',\theta]\big|_{u(H)} = \begin{pmatrix} [F,\theta] & 0 \\ 0 & 0 \end{pmatrix}$$

and so
$$\mathrm{tr}_{H'}\left(F'[F',\theta]^{2k}\right)\Big|_{u(H)} = \mathrm{tr}_{H'}\begin{pmatrix} F[F,\theta]^{2k} & 0 \\ 0 & F_1\cdot(0)^{2k} \end{pmatrix}$$
$$= \mathrm{tr}_H\left(F[F,\theta]^{2k}\right) \qquad k \geq 1.$$

Let's see now if we can come up with a formula. The idea here is that the forms we are interested in are characteristic forms associated to an invariant connection in the subbundle over the Grassmannian, all for the action of the unitary group. (In fact we know there are equivariant forms defined, because the subbundle has an invariant connection. This maybe be useful.)

Now I want to pull back via a map $G \longrightarrow \mathrm{Grass}$, $g \longmapsto gFg^{-1}$. Now G-equivariant bundles over G are canonically trivial. The subbundle, quotient, and total bundles over the Grassm. become trivial bundles with fibres H^+, H^-, and H, resp., where $H = H^+ \oplus H^-$ is the decomposition defined by F.

We have invariant connections in these trivial bundles and the connection forms will be matrix valued forms on $\Lambda\mathfrak{g}^*$. What we seek therefore are canonical elements of
$$\mathfrak{g}^* \otimes \mathrm{End}^0(H) \qquad\qquad \Lambda^2\mathfrak{g}^* \otimes \mathrm{End}^0(H)$$
giving the connection and curvature forms. These should depend on A.

March 26, 1986

Problem: How to define the index in K-theory for the family of Dirac operators parametrized by $\mathcal{A}/\mathcal{G}$ consistent with the canonical map $\mathcal{G} \to U_{res}$.

Here is the original idea. Consider the ungraded case. To each connection assign the Dirac operator D and form $\frac{D}{\sqrt{1+D^2}}$ which belongs to $\mathcal{F}_1(H)$, $H = L^2(M, S\otimes E)$. Over $\mathcal{A}/\mathcal{G}$ one has the Hilbert bundle $\mathcal{A} \times^{\mathcal{G}} H$. I can consider bundle over $\mathcal{A}/\mathcal{G}$ whose fibre at ξ is the space $\mathcal{F}_1(H_\xi)$ where H_ξ = fibre of $\mathcal{A}\times^{\mathcal{G}} H$ at ξ. This bundle is

$$\mathcal{A} \times^{\mathcal{G}} \mathcal{F}_1(H)$$

and it is clear that by assigning to each point of A the corresp. $D/\sqrt{1+D^2} \in \mathcal{F}_1(H)$, we get a $\mathcal{G}$-equivariant map $\mathcal{A} \longrightarrow \mathcal{F}_1(H)$, hence a canonical section

$$\begin{tikzcd} \mathcal{A}\times^{\mathcal{G}}\mathcal{F}_1(H) \arrow[d] \\ \mathcal{A}/\mathcal{G} \arrow[u, bend left, dashed] \end{tikzcd}$$

Thus we get a Hilbert bundle and a family of self adjoint Fredholms in the fibres

The idea is use Kuiper's thm. to say that Hilbert bundles are trivial which implies that

$$\mathcal{A}\times^{\mathcal{G}}\mathcal{F}_1(H) = \mathcal{A}/\mathcal{G} \times \mathcal{F}_1(H)$$

Put another way

$$\mathcal{A}\times^{\mathcal{G}}\mathcal{F}_1(H) = \left(\mathcal{A}\times^{\mathcal{G}} U(H)\right)\times^{U(H)}\mathcal{F}_1(H)$$

and the principal $\mathcal{U}(H)$-bundle $\mathcal{A} \times^{\mathcal{G}} \mathcal{U}(H)$ is trivial by the contractibility of $\mathcal{U}(H)$.

Using the trivialization we obtain a

① map $\qquad \mathcal{A}/\mathcal{G} \longrightarrow \mathcal{F}_1(H)$

which ought to represent the index of the family.

On the other hand we have seen that we have a canonical map

② $\qquad \mathcal{G} \longrightarrow \mathcal{U}_{res}$

where $\mathcal{U}_{res}$ is the subgroup of $\mathcal{U}(H)$ preserving the involution in the Calkin algebra defined by $D/\sqrt{1+D^2}$ for any of the Dirac operators.

<u>Problem</u>: Show that ① and ② are consistent with periodicity.

~~Here is a better version of ① which avoids a choice based on Kuiper. In fact its canonical We first~~

Here is a fancier version of ①. We have $\mathcal{G}$ acting on $\mathcal{I}(2)$ with the involution defined by any of the Diracs as fixpoint. Hence we have a section

$$B\mathcal{G} \dashrightarrow E\mathcal{G} \times^{\mathcal{G}} \mathcal{I}(2)$$

and then as before

$$\begin{aligned} &\| \\ &\left(E\mathcal{G} \times^{\mathcal{G}} \mathcal{U}(H)\right) \times^{\mathcal{U}(H)} \mathcal{I}(2) \\ &\| \\ &B\mathcal{G} \times \mathcal{I}(2) \xrightarrow{pr_2} \mathcal{I}(2) \end{aligned}$$

defines the index of the family.

But there is a nicer way to say this. We have an explicit universal bundle for U_{res}, namely

$$(*) \qquad U_{res} \longrightarrow U(H) \longrightarrow \mathcal{J}(2)$$

and so corresponding to the map

$$\mathcal{G} \longrightarrow U_{res}$$

is a unique map up to homotopy

$$B\mathcal{G} \longrightarrow \mathcal{J}(2).$$

This map is constructed as follows [redacted]

[redacted]

$$\begin{tikzcd} E\mathcal{G} \arrow[d, "\mathcal{G}"'] & E\mathcal{G} \times U(H) \arrow[l, "pr_1"'] \arrow[r, "pr_2"] \arrow[d, "\mathcal{G}"'] & U(H) \arrow[d, "U_{res}"] \\ B\mathcal{G} & E\mathcal{G} \times^{\mathcal{G}} U(H) \arrow[l, "p_1"'] \arrow[r, "p_2"] & \mathcal{J}(2). \end{tikzcd}$$

Because $U(H)$ is contractible p_1 has a section, whence a map $B\mathcal{G} \longrightarrow \mathcal{J}(2)$. Choosing the section [redacted] of p_1 is [redacted] the same as trivializing the Hilbert bundle over $B\mathcal{G}$.

At this point we have succeeded in explaining the index of the family of operators over $\mathcal{A}/\mathcal{G} \sim B\mathcal{G}$. It is just the induced map on classifying spaces assoc. to $\mathcal{G} \longrightarrow U_{res}$ together with the identification $BU_{res} = \mathcal{J}(2)$ provided by $(*)$ and Kuiper's theorem.

Problem: Now that we have partially identified the index of the family over $\mathcal{A}/\mathcal{G}$, let's consider the case of the circle. Here one has an explicit situation. I take the trivial bundle of rank n over S^1, $\mathcal{A}$ space of connection (preserving the inner product), $\mathcal{G} = \Omega U(n)$. Then the monodromy gives a isom.

$$\mathcal{A}/\mathcal{G} \xrightarrow{\sim} U(n).$$

We have the map $A \longmapsto D_A = \frac{1}{i}(\partial_x + A)$ from connections to operators on $H = L^2(S^1)^{\oplus n}$, which is equivariant for the action of the free loop group. So we have:

$$\begin{tikzcd}
\Omega U(n) = \mathcal{G} \arrow[rr] \arrow[d] & & U_{res} \arrow[d] \\
\mathcal{A} \arrow[rr, dotted] \arrow[d] & & U(H) \arrow[d] \\
U(n) = \mathcal{A}/\mathcal{G} \arrow[r, dotted] & \mathcal{F}_1 \arrow[r] & \mathcal{I}(2)
\end{tikzcd}$$

The problem is one of the dotted arrows being obtained only through Kuiper's theorem.

Question: We know there are homotopy equivalences

$$\mathcal{I}(2) \twoheadleftarrow \mathcal{F}_1 \longrightarrow -U(\mathcal{H})$$

where the second map is either $A \longrightarrow \exp(i\pi A)$ or the Cayley map $A \longrightarrow \dfrac{B+iA}{B-iA} = \left(\sqrt{1-A^2} + iA\right)^2$.

So in homotopy we have a map $U(n) \longrightarrow U(\mathcal{H})$ which ought to be the inclusion, and the question is how to see this.

What kind of approaches should be tried?

It appears that we ought to apply some version of the index theorem for families to the situation. I am thinking about on the level of differential forms. One has an explicit example of a family of Dirac operators over a nice manifold $U(n)$. One should pick a connection in the Hilbert bundle $\mathcal{A} \times^{\mathcal{G}} H$ and see what kind of forms result on $U(n)$. I think I should know what the classical limit of the superconnection character forms should be. These are certain equivariant forms for the $\mathcal{G}$-action on $\mathcal{A}$.

But before losing myself in computation let us see if there is a nice avoid some of the problems.

I recall that if I ignore the $\mathcal{G}$ action, then the superconnection forms are roughly

$$tr_\sigma\left(e^{L^2+dL}\right)$$

on the space of operators L, and that upon taking a suitable Laplace transform these forms become essentially the forms

$$tr\left(\frac{2}{1-L^2}\,dL\right)^{odd}$$

which one can also obtain by using the Cayley map from skew-adjoint L to unitary operators

$$L \longmapsto \frac{1+L}{1-L}$$

and pulling back the biinvariant forms on the unitary group.

So we have to somehow bring in the group G, i.e. we have to find differential forms on a/G.

Another approach: Given a connection we have associated D, ~~[illegible]~~ $D/\sqrt{m^2+D^2}$, $F = \frac{1}{\sqrt{m^2+D^2}}\begin{pmatrix} D & m \\ m & -D \end{pmatrix}$

and this gives us left invariant closed forms on G associated to a connection. Find these, especially the 2 form. One should be able to easily describe them in the $m \to +\infty$ limit.