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January 31, 1986

I want to work out the quantum mechanics of a forced harmonic oscillator. I have the idea that the discontinuity problems of the "oscillator coupled to string" might also occur in a simpler form with the forced oscillator.

It is first necessary to describe the situation classically, and to this end we review the classical - quantum correspondence. For an oscillator this correspondence is very close. The idea is that a quantum state determines a classical state by taking the average values of position and momentum.

Suppose the quantum mechanics is described as usual by a Hilbert space with operators q, p. Then states are given by unit vectors ψ in the Hilbert space. The time evolution of ψ is given by the path ψ_t with $\psi_0 = \psi$ and satisfying

$$\partial_t \psi_t = -i H(t) \psi_t$$

where $H(t)$ is the Hamiltonian operator. Thus

$$\psi_t = U(t) \psi$$

where $U(t)$ is the path of unitary operators $\ni$

$$\partial_t U(t) = -i H(t) U(t)$$

$$U(0) = I$$

The average value of an operator A in the state ψ is

$$\langle A \rangle = \langle \psi | A | \psi \rangle$$

and the average value of this operator in the ψ_t is

$$\langle A \rangle_t = \langle \psi_t | A | \psi_t \rangle$$
$$= \langle \psi | \underbrace{\mathcal{U}(t)^{-1} A\, \mathcal{U}(t)}_{A_t} | \psi \rangle = \langle A_t \rangle$$

As a rule the subscript t denotes time evolution. Note that

$$\partial_t A_t = -\mathcal{U}(t)^{-1} \overbrace{\partial_t \mathcal{U}(t)}^{-iH(t)\mathcal{U}(t)}\, \mathcal{U}(t)^{-1} A\ \mathcal{U}(t)$$
$$+ \ \mathcal{U}(t)^{-1} A \underbrace{\partial_t \mathcal{U}(t)}_{-iH(t)\mathcal{U}(t)}$$
$$= i\ \mathcal{U}(t)^{-1} [H(t), A]\, \mathcal{U}(t)$$
$$= i\ [H(t), A]_t$$

For example if $H(t) = \frac{p^2}{2} + V(q,t)$, then

$$\partial_t q_t = i \left[\frac{p^2}{2} + V(q,t),\ q \right]_t$$
$$= p_t$$
$$\partial_t p_t = -\left[\partial_q V(q,t) \right]_t = -\partial_q V(q_t, t)$$

and so

$$\partial_t \langle q_t \rangle = \langle p_t \rangle$$
$$\partial_t \langle p_t \rangle = -\langle \partial_q V(q_t, t) \rangle$$
$$\sim -\partial_q V(\langle q_t \rangle, t).$$

This approximation is exact when V is quadratic in q. So for the harmonic oscillator we see that a quantum trajectory ψ_t determines a classical trajectory by taking the average values of q, p.

The next stage will be to consider a general harmonic oscillator. Let's not start with a configuration space, but rather with the quantum mechanical description in terms of a boson Fock space. Thus we suppose V is a complex Hilbert space and our state space is $\hat{S}(V)$; instead of p's and q's we have the space of self-adjoint operators $a_v^* + a_v$.

Recall that a quantum state Ψ determines a real linear functional on this space of operators

$$a_v^* + a_v \longmapsto \langle \Psi | a_v^* + a_v | \Psi \rangle.$$

$$= \langle \Psi | a_v^* | \Psi \rangle + \text{c.c.}$$

and that this linear functional is the classical state assoc. to Ψ. Thus we can identify phase space with V as follows. A point of phase space is a linear functional λ on $\{a_v^* + a_v\}$, and any such λ can be represented as

$$\lambda(a_v^* + a_v) = \langle w | v \rangle + \langle v | w \rangle$$

for a unique $w \in V$. Thus given a quantum state Ψ the associated classical state w is defined by

$$\langle \Psi | a_v^* + a_v | \Psi \rangle = \langle w | v \rangle + \langle v | w \rangle \qquad \forall v$$

i.e.
$$\langle \Psi | a_v | \Psi \rangle = \langle v | w \rangle \qquad \forall v.$$

For example if we take Ψ to be the coherent state $\Psi_w = e^{-\frac{1}{2}|w|^2} e^{w}$, then

$$\langle \Psi_w | a_v | \Psi_w \rangle = \langle v | w \rangle$$

whence w is the classical state belonging to the coherent state Ψ_w.

<u>Summary</u>: Given a complex ~~vector~~ Hilbert space V we form the boson Fock space $\hat{S}(V)$ and consider it as an irreducible repn. of the Weyl algebra generated by the operators $\{a_v^* + a_v\}$. The corresponding classical state space can be identified with V; given ~~[illegible]~~ a unit vector $\Psi \in \hat{S}(V)$, the ~~[illegible]~~ classical state which is the mean of Ψ is the $w \in W$ such that

$$\langle \Psi | a_v | \Psi \rangle = \langle v | w \rangle \qquad \forall w.$$

Also the coherent state $\Psi_w = e^{-\frac{1}{2}|w|^2} e^{a_w^*} |0\rangle$ has mean w.

For the Hamiltonian $H = \sum_i \omega_i a_i^* a_i$ one has

$$\langle \Psi_w | H | \Psi_w \rangle = \sum_i \omega_i |\langle i | w \rangle|^2$$

and so the energy for Ψ_w is $\langle w | H | w \rangle$ where $H = \sum \omega_i |i\rangle\langle i|$. For time evolution

$$e^{-iHt}\Psi_w = \Psi_{\sum_i |i\rangle e^{-i\omega_i t}\langle i | w\rangle} = \Psi_{e^{-itH}w}.$$

This implies that the time evolution on phase space (identified with V) is given by e^{-itH}, and hence it has positive frequencies.

Let's finish by characterizing the coherent states for the simple harmonic oscillator in term of the energy. If as usual

$$a = \frac{1}{\sqrt{2\omega}}(\omega q + ip), \quad a^* = \frac{1}{\sqrt{2\omega}}(\omega q - ip)$$

then $\quad H = \frac{1}{2}(p^2 + \omega^2 q^2) = \omega\left(a^* a + \frac{1}{2}\right)$ and so the energy in the state Ψ is

$$\langle H \rangle = \omega\left(\langle a^* a\rangle + \frac{1}{2}\right)$$
$$\geq \omega\left(|\langle a\rangle|^2 + \frac{1}{2}\right)$$

where we have used

$$0 \leq \langle (a^* - \langle a^*\rangle)(a - \langle a\rangle)\rangle = \langle a^* a\rangle - \langle a^*\rangle\langle a\rangle$$
$$= \langle a^* a\rangle - |\langle a\rangle|^2$$

Equality holds $\iff$ $(a - \langle a\rangle)\Psi = 0$ which means $\Psi = \Psi_w$, where $w = \langle a\rangle$.

Thus coherent states are characterized by having energy = ground energy + classical energy.

(For tomorrow take a forced oscillator Hamiltonian and compute its time evolution both quantum + classical)

February 1, 1986

I want to make precise the relation between the quantum and classical pictures of a harmonic oscillator. I will start with the quantum picture. In the quantum picture one has a Heisenberg group acting on a Hilbert space $\mathcal{H}$ and the action is irreducible. The time evolution is given by a family $U(t)$ of unitaries on $\mathcal{H}$ such that conjugation by $U(t)$ ~~carries the Heisenberg group operators into themselves~~ induces an automorphism of the Heisenberg group. The Heisenberg group is the central extension by S^1 of a real vector space $\tilde{V}$ associated ~~given~~ to a cocycle given ~~~~ by a ~~skew~~ symplectic form on $\tilde{V}$. $\tilde{V}$ is the space of real linear combinations of position and momentum operators.

In the classical picture $\tilde{V}$ becomes the space of linear functions on phase space.

Now suppose we start with a specific model. Let V be a complex Hilbert space, and take $\mathcal{H}$ to be the boson Fock space $\hat{S}(V)$. The Heisenberg group will consist of S^1 and the unitary operators

$$e^{a_v^* - a_v}$$

Suppose time evolution is $U(t) = e^{-itH}$ where $H = \sum_k \omega_k a_k^* a_k$ is the extension to Fock space of a positive self-adjoint operator $H = \sum \omega_k |k\rangle\langle k|$ on V. Then the induced action on the Heisenberg group is

$$e^{itH} e^{a_v^* - a_v} e^{-itH} = e^{a^*_{(e^{itH}v)} - a_{(e^{itH}v)}}$$

In other words we can identify the generators of the Heisenberg group with elements of V:

$$e^{a_v^* - a_v} \longleftrightarrow v$$

and then we have that the time evolution is $v_t = e^{+itH} v$.

The points of phase space are real linear functions on V, which can be represented by elements of V. So we can identify phase space with V, however, the time evolution in phase space is then $v_t = e^{-itH} v$. To see this a bit better, let's assign to $w \in V$ the coherent state $\Psi_w = e^{a_w^* - a_w}|0\rangle = e^{-\frac{1}{2}|w|^2} e^{w}$. The associated classical state of Ψ_w is w. Time evolution is

$$e^{-itH}\Psi_w = \Psi_{e^{-itH}w}$$

Now that I have found the classical picture I can inquire about the symplectic form on phase space and the energy function whose Hamiltonian flow gives the time evolution.

Let's consider next the forced harmonic oscillator. Let's begin with a simple h. osc. so the Hamiltonian is

$$H(t) = \omega a^* a + J a^* + \bar{J} a$$

where J is a function of t. ~~has compact support~~ We work out the time evolution in the interaction picture.

$$e^{itH_0} U(t) = T\left\{ e^{-i\int_0^t (J(t')a^*_{t'} + \bar{J}(t')a_{t'})dt'} \right\}$$

$$= \begin{matrix}\text{scalar of}\\ \text{mod } 1\end{matrix} \cdot e^{\left(-i\int_0^t e^{i\omega t'} J(t')dt'\right)a^* + \left(-i\int_0^t e^{-i\omega t'}\bar{J}(t')dt'\right)a}$$

$$= \quad '' \quad e^{\gamma a^* - \bar{\gamma} a}$$

where $\boxed{\gamma = -i\int_0^t e^{i\omega t'} J(t')\,dt'.}$ Note that

$$e^{\gamma a^* - \bar{\gamma} a}\, \Psi_w = e^{\gamma a^* - \bar{\gamma} a} e^{w a^* - \bar{w} a} |0\rangle$$

$$= \begin{matrix}\text{scalar}\\ \text{mod } 1\end{matrix} \cdot \Psi_{\gamma + w}.$$

Therefore on the classical level the scattering $\overbrace{\text{}}^{\text{over } [0,t]}$ is $w \longmapsto w + \gamma$.

Another way to see this is to assume a solution of the Schroedinger equ.

$$\partial_t \psi_t = -i(\omega a^* a + J a^* + \bar{J} a)\psi_t$$

of the form $\psi_t = \xi(t)\Psi_{w_t}$ $\quad |\xi(t)| = 1.$

~~Substituting~~

$$\xi'(t) - \tfrac{1}{2}(|w_t|^2)^{\circ} + \mathring{w}_t z = -i\left(\omega z w_t + J(t) z + \bar{J}(t) w_t\right)$$

from which we get

$$\mathring{w}_t = -i(\omega w_t + J(t)).$$

This is our classical equation of motion and it has solutions

$$w_t = e^{-i\omega t}\underbrace{\left(w_0 - i\int_0^t e^{i\omega t'} J(t')\,dt'\right)}_{w_0 + \gamma}$$

as expected.

In the case of a general oscillator

$$H(t) = H_0 + a_J^* + a_J$$

the classical equation should be

$$\partial_t v_t = -iH_0 v_t - iJ(t)$$

and its solution is

$$v_t = e^{-iH_0 t}\left(v_0 - i\int_0^t e^{iH_0 t'} J(t')\,dt'\right)$$

Suppose we consider the oscillator which arises from the vector space of real $f(x)$ with symplectic form $\int f g'$ and time evolution given by $H = \frac{1}{i}\partial_x$. Thus

$$e^{-itH} f = e^{-t\partial_x} f = f(x-t).$$

The classical equation of motion is

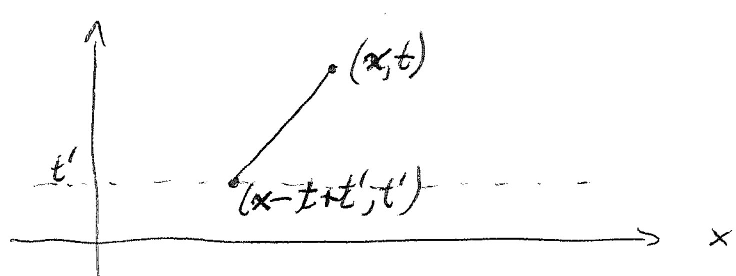
$$\partial_t u = -\partial_x u + f(t,x)$$

in the forced case. It's solution satisfying $u=0$ for $t \ll 0$ is

$$u = \int_{-\infty}^{t} e^{-(t-t')\partial_x} f(t',x)\,dt'$$

$$u(t,x) = \int_{-\infty}^{t} f(t', x-t+t')\,dt'$$

Picture



so u is obtained by adding the propagation of $f(t',\cdot)$ for each t'.

The question is whether there are periodic forcing terms $f(t,x)$ which have negative frequency relative to the complex structure on the space of real $u(x)$ such that $\frac{1}{i}\partial_x > 0$. (This isn't very clear and may not be meaningful. But it seems as if one could write down $J(t) = e^{-i\omega_0 t} g$ with $\omega_0 < 0$, and then this $J(t)$ would be periodic and would not feed into the oscillator.)

Let's due the frequency analysis

$$f(x) = \int_{-\infty}^{\infty} \frac{dk}{2\pi}\, e^{ikx}\, \hat{f}(k) \qquad \hat{f}(-k) = \overline{\hat{f}(k)}$$

$$e^{-itH} f(x) = e^{-t\partial_x} f(x) = \int_{-\infty}^{\infty} \frac{dk}{2\pi}\, e^{ikx} e^{-ikt}\, \hat{f}(k).$$

To construct

$$f(t,x) = \int_{+0}^{\infty} \frac{dk}{2\pi}\, e^{ikx}\, e^{-i\omega_0 t}\, a(k) + \int_{-\infty}^{0} \frac{dk}{2\pi}\, e^{ikx}\, e^{i\omega_0 t}\, \overline{a(-k)}$$

$$= \int_{-\infty}^{\infty} \frac{dk}{2\pi}\, e^{ikx}\, e^{-i\omega_0 t\, \mathrm{sgn}(k)}\, a(k)$$

$$\overline{a(-k)} = a(k).$$

In this way we can construct periodic forcing terms of negative energy.

What does this mean? The first point to keep in mind is that it is all classical, what has just been done. To be more specific we have been studying the hyperbolic equation

$$\partial_t u = -\partial_x u + f(t)$$

whose solution is

$$u(t) = e^{-t\partial_x} v_- + \int_a^t e^{-(t-t')\partial_x} f(t')\,dt'.$$

The scattering is then

$$v_- \longmapsto v_- + \int_{-\infty}^{\infty} e^{t'\partial_x} f(t')\,dt' = v_+$$

Next we can analyze this in terms of frequencies appearing in $f(t)$. Write

$$f(t) = \int \frac{d\omega}{2\pi} e^{-i\omega t} \hat{f}(\omega)$$

whence

$$\int_{-\infty}^{\infty} e^{t\partial_x} f(t)\,dt = \int \frac{d\omega}{2\pi} \int e^{t\partial_x} e^{-i\omega t} \hat{f}(\omega)\,dt$$

$$= \int \frac{d\omega}{2\pi} \int \frac{dk}{2\pi} \int dt\; e^{ikt} e^{ikx} e^{-i\omega t} \hat{f}(\omega, k)$$

$$= \int \frac{d\omega}{2\pi} \int \frac{dk}{2\pi} e^{ikx}\; 2\pi\delta(k-\omega)\, \hat{f}(\omega, k)$$

$$= \int \frac{dk}{2\pi} e^{ikx} \hat{f}(k,k) \qquad ?$$

Try

$$\int_{-\infty}^{\infty} e^{t\partial_x} f(t,x)\,dt = \int \frac{dk}{2\pi} \int dt\, e^{ikx} e^{ikt} \hat{f}(t,k)$$

So if [redacted] we fix attention on a given k we get the component

$$\int dt\, e^{ikt} \hat{f}(t,k) = \hat{f}(k,k) \qquad ?$$

Try again to compute

$$\gamma = \int_{-\infty}^{\infty} dt \; e^{t\partial_x} f(t) \qquad \text{when} \quad f(t) = e^{-i\omega_0 t} g$$

in the sense of the complex structure i.e.

$$f(t) = e^{-J\omega_0 t} g$$

$$Jg = \int \frac{dk}{2\pi} \, e^{ikx} (i \operatorname{sgn} k) \, \hat{g}(k).$$

Thus

$$\gamma = \int_{-\infty}^{\infty} dt \; e^{t\partial_x} e^{-J\omega_0 t} g$$

$$= \int_{-\infty}^{\infty} dt \; e^{t\partial_x} e^{-J\omega_0 t} \int \frac{dk}{2\pi} e^{ikx} \, \hat{g}(k)$$

$$= \int_{-\infty}^{\infty} dt \int \frac{dk}{2\pi} \; e^{ikt - i \operatorname{sgn}(k)\omega_0 t} \; e^{ikx} \; \hat{g}(k)$$

$$= \int \frac{dk}{\cancel{2\pi}} \; \cancel{2\pi}\, \delta\big(k - \operatorname{sgn}(k)\omega_0\big) \, e^{ikx} \, \hat{g}(k)$$

$$= \begin{cases} \overbrace{e^{i\omega_0 x} \, \hat{g}(\omega_0)}^{+\, e^{-i\omega_0 x} \hat{g}(-\omega_0)} & \text{if} \quad \omega_0 > 0 \\ 0 & \text{if} \quad \omega_0 < 0 \end{cases}$$

February 3, 1986

So far I have been considering a forced harmonic oscillator. I have learned that the quantum setup is easily obtained from the classical. Once the classical response to the forcing term is understood, the quantum response is easily understood in terms of it, up to a phase factor of no importance for probabilities. Moreover, the Fermi "Golden Rule" is to be found on the classical level.

To be more specific, let's consider an oscillator quantized in the standard way. The Hilbert space is $\hat{S}(V)$, where V is a complex Hilbert space, and the Hamiltonian is the extension as derivation on $\hat{S}(V)$ of a positive self-adjoint operator H on V. We now consider adding a forcing term to the Hamiltonian:

$$H + a_J^* + a_J$$

where $J = J(t)$ is a function of compact support with values in V. The scattering operator S_J associated to this perturbation is up to a phase factor the unitary operator

$$e^{a_\gamma^* - a_\gamma}$$

where $$\gamma = -i \int e^{itH_0} J(t)\,dt$$

If the system is initially in the ground state $|0\rangle$, then the effect of the forcing is to put it in the state

$$e^{a_\gamma^* - a_\gamma} |0\rangle = e^{-\frac{1}{2}|\gamma|^2} e^{a_\gamma^*} |0\rangle$$

If we resolve γ into eigenvectors for H

$$\gamma = \sum_i \gamma_k \qquad \text{where} \qquad H = \sum \omega_k a_k^* a_k$$

then the probability distribution of particles of type k in the above state is a Poisson distribution

$$\text{prob.}(n_k = n) = e^{-|\gamma_k|^2} \frac{|\gamma_k|^{2n}}{n!}$$

Now suppose we take $J(t) = e^{-i\omega_0 t} v$ on $0 \leq t \leq T$ and 0 outside. Then

$$\gamma_T = -i \int_0^T e^{i(H-\omega_0)t} v \, dt$$

$$= \frac{1 - e^{i(H-\omega_0)T}}{H - \omega_0} v$$

$$= \sum_k \frac{1 - e^{i(\omega_k - \omega_0)T}}{\omega_k - \omega_0} v_k$$

$$\|\gamma_T\|^2 = \sum_k \frac{4\sin^2(\omega_k - \omega_0)T}{(\omega_k - \omega_0)^2} \|v_k\|^2$$

Recall that

$$\int \chi_{(-a,a)}(x) e^{-i\xi x} dx = \frac{e^{ia\xi} - e^{-ia\xi}}{i\xi} = \frac{2\sin a\xi}{\xi}$$

$$\int \frac{d\xi}{2\pi} \frac{2\sin a\xi}{\xi} e^{+i\xi x} = \chi_{(-a,a)}(x)$$

$$\therefore \int d\xi \frac{1}{\pi} \frac{\sin\xi}{\xi} = 1.$$

$$\int \frac{d\xi}{2\pi} \; 4 \, \frac{\sin^2 a\xi}{\xi^2} = 2a$$

$$\int d\xi \, \frac{1}{\pi} \frac{\sin^2 a\xi}{\xi^2} = a$$

Assuming that the spectrum of H is sufficiently smooth and the v_k is smooth in k, the Riemann-Lebesgue lemma implies that as $T \longrightarrow \infty$, γ_T tends to peak around $T \sum\limits_{\substack{k \\ \omega_k = \omega_0}} v_k$. On the other hand one has

$$|\gamma_T|^2 \sim \text{const} \; T \cdot \sum_k \delta(\omega_k - \omega_0) |v_k|^2$$

February 5, 1986

The problem is how to handle Gaussian integrals with complex exponents in infinite dimensions. The approach is to find a suitable space of test functions such that the integral can be defined as a linear functional on test functions. One attempt to find a good defn. of test function is to consider the family of Gaussian measures; each of these defines a Hilbert space completion of the polynomials, and one can hope to intersect these completions in some sense to get the space of test functions.

A better approach seems to be the following. First note that the set of Gaussian which have complex exponents with positive real part forms a kind of Siegel UHP. A test function will, if the theory is constructed, give rise to a function on this UHP which should be analytic. Thus it might be possible to construct the space of test functions as a space of analytic fns. on the UHP.

Let's see how this works in 1 dimension. The kind of Gaussian integrals being considered are

$$f(x) \longmapsto \int f(x)\, e^{-\frac{1}{2}\tau x^2} \frac{\sqrt{\tau}\, dx}{\sqrt{2\pi}} \qquad Re(\tau) > 0.$$

We can think of this integral as the L^2 inner product of $f(x)$ and $e^{-\frac{1}{2}\bar{\tau}x^2}\sqrt{\frac{\bar{\tau}}{2\pi}}$. The latter is, up to a normalization factor, a Gaussian state (quasi-free is the accepted terminology perhaps?). So if we change to the holomorphic representation we get the linear fnl.

$$f \longmapsto \langle \Phi_\lambda | f \rangle.$$

But now fix f and think of this as the conjugate of an analytic function on the unit disk $|\lambda| < 1$. In other words we get [redacted] a space of analytic functions on the disk which is isomorphic to the Hilbert space $\hat{S}(V)$ of the holomorphic representation. Not quite because if f is odd, then $\langle \Phi_\lambda | f \rangle = 0$.

Suppose we have a holomorphic map $\lambda \mapsto \varphi_\lambda$ from an open subset U of $\mathbb{C}$ to a Hilbert space $\mathcal{H}$. Then each $h \in \mathcal{H}$ determines a holom. function on $\bar{U}$:

$$\hat{h}(\lambda) = \langle \varphi_{\bar{\lambda}} | h \rangle$$

Assuming the φ_λ span $\mathcal{H}$ we get an isomorphism of $\mathcal{H}$ with a space of holomorphic functions on $\bar{U}$, hence a Hilbert space of analytic fns. on $\bar{U}$. One can represent evaluation at a point as an inner product

$$\hat{h}(\mu) = \langle e_{\bar{\mu}} | \hat{h} \rangle$$

$$\langle \varphi_{\bar{\mu}} | h \rangle = \langle \hat{\varphi}_{\bar{\mu}} | \hat{h} \rangle$$

whence [redacted] $e_{\bar{\mu}} = \hat{\varphi}_{\bar{\mu}}$ or $\hat{\varphi}_\mu = e_\mu$.

In the case of $\varphi_\lambda = e^{\frac{1}{2}\lambda z^2}$ we compute the function $\hat{\varphi}_\mu$ on the unit disk

$$\hat{\varphi}_\mu(\lambda) = \langle \varphi_{\bar{\lambda}} | \varphi_\mu \rangle = \langle e^{\frac{1}{2}\bar{\lambda} z^2} | e^{\frac{1}{2}\mu z^2} \rangle$$

$$= \sum_n \frac{1}{n! \cdot n! \cdot 2^n \cdot 2^n} (\lambda\mu)^n \, 2n!$$

$$= \sum_n \frac{\frac{1}{2} \frac{3}{2} \cdots \frac{2n-1}{2}}{n!} (\lambda\mu)^n = \sum \binom{-\frac{1}{2}}{n} (-\lambda\mu)^n$$

$$= (1-\lambda\mu)^{-1/2}$$

We now want to determine the inner product on the analytic functions on the disk. Thus we want $\rho(\lambda)$ such that

$$\langle \hat{\varphi}_\mu | \hat{\varphi}_\nu \rangle \overset{\text{def}}{=} \int \overline{\varphi_\mu(\lambda)}\, \varphi_\nu(\lambda)\, \rho(\lambda)\, d^2\lambda$$

$$\langle \varphi_\mu | \varphi_\nu \rangle = (1-\bar{\mu}\nu)^{-1/2}$$

Thus we want

$$\int \underbrace{(1-\bar{\mu}\bar{\lambda})^{-1/2}(1-\nu\lambda)^{-1/2}}_{\sum_n \binom{-\frac12}{n}(-\bar{\mu}\bar{\lambda})^n \sum_m \binom{-\frac12}{m}(-\nu\lambda)^m} \rho(\lambda)\, d^2\lambda = (1-\bar{\mu}\nu)^{-1/2} = \sum_n \binom{-1/2}{n}(-\bar{\mu}\nu)^n.$$

This forces $\rho(\lambda)$ to be a function of $r = |\lambda|$ with

$$\binom{-\frac12}{n}(-1)^n \binom{-\frac12}{n}(-1)^n \int (\bar{\lambda}\lambda)^n \rho(\lambda)\, d^2\lambda = \binom{-\frac12}{n}(-1)^n$$

$$\int (\bar{\lambda}\lambda)^n \rho(\lambda)\, d^2\lambda = \frac{1}{\binom{-\frac12}{n}(-1)^n} = \frac{n!}{\frac12 \frac32 \cdots \frac{2n-1}{2}}$$

$$2\pi \int_0^1 r^{2n} \rho(r)\, r\, dr = \frac{\Gamma(n+1)\, \cancel{\Gamma(\tfrac12)}\ \sqrt{\pi}}{\Gamma(n+\frac12)}$$

Recall $$\int_0^1 t^{a-1}(1-t)^{b-1}\, dt = \beta(a,b) = \frac{\Gamma(a)\,\Gamma(b)}{\Gamma(a+b)}$$

$$\text{so} \quad \pi \int_0^1 (r^2)^n \rho(r)\, d(r^2) = \frac{\Gamma(n+1)\ \Gamma(-\frac12)}{\Gamma(n+\frac12)} \cdot \frac{\overbrace{\Gamma(\tfrac12)}^{(-\frac12)\Gamma(-\frac12)}}{\Gamma(-\frac12)}$$

$$\therefore \quad \pi \int_0^1 (r^2)^n \rho(r)\, dr^2 = \int_0^1 t^{(n+1)-1}(1-t)^{-\frac12-1}\, dt \cdot \left(-\tfrac12\right) \qquad ?$$

February 7, 1986

Previously I tried to realize the even subspace of the holomorphic representation, considered as a representation of the metaplectic group $\widetilde{SL(2,\mathbb{R})}$, as holomorphic functions on the UHP, but I ran into difficulties. From Lang's book on $SL(2,\mathbb{R})$ I learn to look at tensor products of this representation in order to construct discrete series representations.

First I want to get the generators for the Lie algebra. I will always work with a single pair a^*, a and the fundamental repn. given by holom. fns. $f(z)$ with $\|f\|^2 = \int e^{-|z|^2}|f(z)|^2 \frac{d^2z}{\pi}$ $a = \partial_z$, $a^* = z$.

The complexified Lie algebra has the basis $\frac{a^{*2}}{2}, \frac{a^*a + aa^*}{2}, \frac{a^2}{2}$. Work out the action on the linear combinations of a^*, a.

$$\left[-\frac{a^{*2}}{2}, ca^* + c'a\right] = \left[-\frac{a^{*2}}{2}, (a^* \;\; a)\right]\begin{pmatrix} c \\ c' \end{pmatrix}$$

$$= (0 \;\; a^*)\begin{pmatrix} c \\ c' \end{pmatrix} = (a^* \;\; a)\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} c \\ c' \end{pmatrix}$$

$$\left[\frac{a^*a + aa^*}{2}, (a^* \;\; a)\right]\begin{pmatrix} c \\ c' \end{pmatrix} = (a^* \;\; -a)\begin{pmatrix} c \\ c' \end{pmatrix} = (a^* \;\; a)\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} c \\ c' \end{pmatrix}$$

$$\left[\frac{a^2}{2}, (a^* \;\; a)\right] = (a \;\; 0) = (a^* \;\; a)\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

Thus

$$\boxed{X = -\frac{a^{*2}}{2} \to \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \qquad H = \frac{a^*a + aa^*}{2} \to \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \qquad Y = \frac{a^2}{2} \to \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}}$$

Standard sl_2 relations

$$[H, X] = 2X, \quad [H, Y] = -2Y, \quad [X, Y] = H$$

Casimir operator relative to $tr(\alpha\beta)$ pairing is

$$\Delta = XY + YX + \tfrac{1}{2}H^2$$

check: $$[X, \Delta] = XH + HX + \tfrac{1}{2}\Big(\underbrace{[X,H]}_{-2X}H + H\underbrace{[X,H]}_{-2X}\Big) = 0$$

Now suppose we have an irreducible module M over sl_2 which as a representation of the Cartan subalgebra generated by H is semi-simple. This is the kind of module obtained from a top. repn. of $SL(2,\mathbb{R})$, or better $SU(1,1)$, where the max. torus is $\left\{\begin{pmatrix} \zeta & 0 \\ 0 & \zeta^{-1}\end{pmatrix}, \zeta \in \mathbb{T}\right\}$.
Then M is a direct sum of 1-diml eigenspaces for H with X, Y acting as shifting operators: As

$$HX = XH + 2X = X(H+2)$$

$$Hv = \lambda v \implies H(Xv) = (\lambda + 2)Xv$$

so X shifts the H values $+2$, and Y shifts -2. On any H-eigenspace M_λ one has the commuting operators H, XY, $YX = XY - H$, so if v_λ is a common eigenvector it follows M is spanned by the $X^n v_\lambda$, $Y^n v_\lambda$.

The obvious invariants of M are the scalar Δ acts as, and the coset in $\mathbb{C}/2\mathbb{Z}$ containing the eigenvalues λ of H. We can instead of this coset look at $e^{i\pi\lambda}$ which is constant as λ ranges over the coset. In other words we want the scalar $e^{i\pi H}$ acts as. Note

$$e^{i\pi H} \longrightarrow \begin{pmatrix} e^{i\pi} & 0 \\ 0 & e^{-i\pi} \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$$

generates the center of $SU(1,1)$, and it is of order 2. If we want a representation of $SU(1,1)$ and not a covering group, then $e^{i\pi H}$ must be of order$\mid$2 i.e. we must have $\lambda \in \mathbb{Z}$.

Next we wish to relate the scalars $e^{i\pi H}$ and Δ. If M has a lowest weight λ, then $Yv_\lambda = 0$ and so

$$\Delta = 2XY + \tfrac{1}{2}H^2 - H$$

will have the value $\frac{1}{2}\lambda^2 - \lambda$. Similarly if there is a highest weight μ, then

$$\Delta = 2YX + \tfrac{1}{2}H^2 + H$$

will have the value $\frac{1}{2}\mu^2 + \mu$. Generically $\Delta \notin \{ \frac{1}{2}\lambda^2 - \lambda \mid \lambda \text{ weight of } M\}$; this gives the "principal series" type representation. Call this Case 1 + put $Wt(M)$ for the set of weights

<u>Case 1</u>: $\Delta = \frac{1}{2}\lambda^2 - \lambda$ has no solutions in $Wt(M)$. In this case there is only one irreducible M with the given Δ, $Wt(M)$.

<u>Case 2</u>: $\Delta = \frac{1}{2}\lambda^2 - \lambda$ has exactly one solution in $Wt(M)$. In this case there ~~are~~ two irreducible modules are with lowest weight ~~the~~ solution λ and the other with the highest weight $\lambda - 2$.

Case 3. $\Delta = \frac{1}{2}\lambda^2 - \lambda$ has two distinct solutions in $Wt(M)$. In this case there will be three irreducible M with the invariants Δ, $Wt(M)$. One will be finite diml with weights

$$-n, -n+2, \ldots\ldots, n \quad , \quad \Delta = \tfrac{1}{2}n^2 + n \quad , \quad n \in \mathbb{N}$$

and the other two will be infinite dimensional, one having lowest weight $n+2$, the other with highest weight $-n-2$.

Question: Lang describes the representations in terms of a continuous parameter s. What is s? It must essentially be the value of Δ, since for $SL(2,\mathbb{R})$ the representations have only two choices for weight coset, namely $2\mathbb{Z}$ and $1+2\mathbb{Z}$.

Now let's look at the Weyl, or holom. fn. representation, where

$$X = -\frac{a^{*2}}{2} \quad , \quad H = \frac{a^*a + aa^*}{2} = a^*a + \frac{1}{2} \quad , \quad Y = \frac{a^2}{2}$$

~~This has lowest weight $\lambda = \frac{1}{2}$ so $\Delta = \frac{1}{8} - \frac{1}{2} = -\frac{3}{8}$~~
~~Here $e^{i\pi H} = e^{i\pi a^*a} e^{i\frac{\pi}{2}}$~~

Here
$$e^{i\pi H} = e^{i\frac{\pi}{2}} e^{i\pi a^*a} = i\begin{cases} +1 & \text{on even } f(z) \\ -1 & \text{on odd } f(z) \end{cases}$$

Thus the repn is reducible; the even and odd parts have lowest weights $\frac{1}{2}$, $\frac{3}{2}$ respectively. Since $\lambda \notin \mathbb{Z}$ the representation is not a representation of $SU(1,1)$ but of a double covering.

Next we form the r fold tensor product of the Weyl representation. This means we use polys in $z_1, \ldots, z_r$ and our operators are

$$X = -\frac{1}{2}\sum_1^r a_i^{*2} \qquad Y = \frac{1}{2}\sum_1^r a_i^2$$

$$H = \left(\sum_1^r a_i^* a_i\right) + \frac{r}{2}$$

We see that for r even we do get a representation of $SU(1,1)$. Let's take the cyclic representation generated by $|0\rangle = 1$. It has the orthogonal basis

$$X^n|0\rangle = \left(-\tfrac{1}{2}\right)^n \left(\sum_1^r z_i^2\right)^n$$

and has the lowest weight $\frac{r}{2}$.

How does this fit with Case 3, assuming r is even? Case 3 occurs when the lowest weight is $n+2$, where $n = 0, 1, 2, \cdots$. So we see that if $\frac{r}{2} = 2, 3, 4, \cdots$ then we are in case 3; we have the upper part complementary to a finite diml. repn. But if $r = 2$, we are in case 2. So there is something special about $r=2$ even among the repns. of $SU(1,1)$; $r=1$ should be even stranger still.

Now we will try to identify the cyclic representation of sl_2 generated by $|0\rangle$ in the r-fold Weyl representation. If we act on this vector by unitary representations coming from $\widetilde{SU(1,1)}$, we get the vectors

$$\varphi_\lambda = e^{\lambda \frac{z^2}{2}} = e^{\frac{\lambda}{2}\sum_1^r z_i^2}$$

In this situation we have

$$\langle \varphi_\mu | \varphi_\nu \rangle = (1-\bar{\mu}\nu)^{-\frac{r}{2}} = \sum c_n^d (\bar{\mu}\nu)^n$$

where $$c_n^d = \binom{-\frac{r}{2}}{n}(-1)^n = \frac{\left(\frac{r}{2}\right)\left(\frac{r}{2}+1\right)\cdots\left(\frac{r}{2}+n-1\right)}{n!}$$

So we can do the computation done before and try to find $\rho_r(|\lambda|^2)$ such that

$$\int \overline{\hat{\varphi}_\mu(\lambda)}\,\hat{\varphi}_\nu(\lambda)\,\rho_r(|\lambda|^2)\,d^2\lambda = \langle \varphi_\mu | \varphi_\nu \rangle$$

As on p. 179 this leads to

$$\int |\lambda|^{2n} \rho_r(|\lambda|^2)\,d^2\lambda = \frac{1}{c_n^d}$$

$$\pi \int_0^1 u^n \rho_r(u)\,du = \frac{n!\ \Gamma\left(\frac{r}{2}\right)}{\Gamma\left(\frac{r}{2}\right)\left(\frac{r}{2}\right)\left(\frac{r}{2}+1\right)\cdots\left(\frac{r}{2}+n-1\right)}$$

$$= \frac{\Gamma(n+1)\ \Gamma\left(\frac{r}{2}\right)}{\Gamma\left(n+\frac{r}{2}\right)}$$

Now if $r=2$ this is 1 for all n, whence

$$\rho_2(u) = \frac{1}{\pi}\delta(1-u).$$

This probably means that we are dealing with a subspace of L^2 of the circle which is a Hardy space, i.e. consists of functions (or $\frac{1}{2}$densities?) which extend analytically over the disk.

Suppose now that $r > 2$.

$$\pi \int_0^1 u^n \rho_r(u)\,du = \underbrace{\frac{\Gamma(n+1)\,\Gamma\left(\frac{r}{2}-1\right)}{\Gamma\left(n+\frac{r}{2}\right)}}_{\beta\left(n+1,\frac{r}{2}-1\right) = \int_0^1 t^n (1-t)^{\frac{r}{2}-2}\,dt} \ \overbrace{\frac{\Gamma(r)}{\Gamma\left(\frac{r}{2}-1\right)}}^{\frac{r}{2}-1}$$

whence

$$\rho_r(u) = \frac{1}{\pi}\left(\frac{r}{2}-1\right)(1-u)^{\frac{r}{2}-2}$$

A natural question is whether these representations which I have constructed for integral $r>2$, can be defined for all real r. So I want to see if the representation can be constructed on the level of the Lie algebra; it should act on something like polynomials in λ.

We use the transform

$$\hat{h}(\lambda) = \langle \varphi_{\bar\lambda} | h \rangle \qquad \varphi_\lambda = e^{\frac{\lambda}{2}\sum z_i^2}$$

and want the operators on holomorphic functions corresponding to X, Y, H.

$$\widehat{Xh}(\lambda) = \langle \varphi_{\bar\lambda} | (-\tfrac{1}{2})\sum a_i^{*2} | h \rangle$$

$$= \langle (-\tfrac{1}{2})\sum a_i^2 \varphi_{\bar\lambda} | h \rangle$$

$$\sum a_i^2 \varphi_{\bar\lambda} = \sum \partial_{z_i}^2 e^{\bar\lambda \frac{1}{2}\sum z_i^2}$$

$$= \sum \partial_{z_i}\left(\bar\lambda z_i e^{\frac{\bar\lambda}{2}\sum z_i^2}\right)$$

$$= \left[\underbrace{\sum_i \left(\bar\lambda + \bar\lambda^2 z_i^2\right)}\right] e^{\frac{\bar\lambda}{2}\sum z_i^2}$$

$$= (r\bar\lambda + \bar\lambda^2 z^2)\, e^{\frac{\bar\lambda}{2}z^2}$$

$$= (r\bar\lambda + 2\bar\lambda^2 \partial_{\bar\lambda})\, e^{\frac{\bar\lambda}{2}z^2}$$

$$\therefore \widehat{Xh}(\lambda) = (-\tfrac{1}{2})(r\lambda + 2\lambda^2\partial_\lambda)\langle \varphi_{\bar\lambda} | h \rangle$$

$$= (-\tfrac{1}{2})(r\lambda + 2\lambda^2\partial_\lambda)\,\hat{h}(\lambda)$$

$$\widehat{Yh}(\lambda) = \langle \varphi_{\bar\lambda} | \tfrac{1}{2}\textstyle\sum a_i^2 | h \rangle$$

$$= \langle \underbrace{\tfrac{1}{2}\textstyle\sum a_i^{*2}\varphi_{\bar\lambda}}_{} | h \rangle = \partial_\lambda \langle \varphi_{\bar\lambda} | h \rangle$$

$$\tfrac{1}{2} z^2 e^{\frac{\bar\lambda}{2}z^2} = \partial_{\bar\lambda} e^{\frac{\bar\lambda}{2}z^2}$$

$$\therefore \quad \widehat{Yh}(\lambda) = \partial_\lambda \hat{h}(\lambda)$$

$$\widehat{Hh}(\lambda) = \langle \underbrace{\textstyle\sum (a_i^* a_i + \tfrac{1}{2})\, \varphi_{\bar\lambda}}_{} | h \rangle$$

$$\left(\frac{r}{2} + \sum_i z_i \partial_{z_i}\right) e^{\frac{\bar\lambda}{2}z^2} = \left(\frac{r}{2} + \sum_i z_i \bar\lambda z_i\right) e^{\frac{\bar\lambda}{2}z^2}$$

$$= \left(\frac{r}{2} + 2\bar\lambda\partial_{\bar\lambda}\right) \varphi_{\bar\lambda}$$

$$\widehat{Hh}(\lambda) = \left(\frac{r}{2} + 2\lambda\partial_\lambda\right) \hat{h}(\lambda)$$

$$\boxed{X = -\lambda^2\partial_\lambda - \frac{r}{2}\lambda, \quad Y = \partial_\lambda, \quad H = 2\lambda\partial_\lambda + \frac{r}{2}}$$

One can check these define a repn. of $\mathfrak{sl}_2$ on the polynomials in λ.

$$H\lambda^n = \left(2n + \frac{r}{2}\right)\lambda^n$$

so the lowest weight is $\frac{r}{2}$.

February 8, 1986:

Here is a direct construction of unitary representations of $\widetilde{SU(1,1)}$ having lowest weight s for each $s > 0$.

$$SU(1,1) = \left\{ \begin{pmatrix} a & b \\ \bar{b} & \bar{a} \end{pmatrix} \middle| \ |a|^2 - |b|^2 = 1 \right\}$$

$$su(1,1) = \text{Lie } SU(1,1) = \left\{ \begin{pmatrix} ia & b \\ \bar{b} & -ia \end{pmatrix} \middle| \ a \text{ real} \right\}$$

has the basis

$$\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} = i \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = iH$$

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = X + Y$$

$$\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} = -iX + iY$$

In order to have a unitary representation these operators must be skew adjoint which means that we have

$$H = H^* \qquad\qquad X = -Y^*$$

We start with the r-fold tensor power of the metaplectic representation, really the cyclic sub representation of this generated by $|0\rangle$. This gives the desired unitary representation of weight $s = \frac{r}{2}$ for $r = 1, 2, \cdots$. Then we will see the formulas make sense for all $s > 0$.

$\mathcal{H}_{\frac{r}{2}}$ is spanned by the vectors

$$X^n \cdot 1 = \left(-\frac{z^2}{2}\right)^n$$

where $z^2 = \sum_{i=1}^r z_i^2$. Recall $X = -\frac{1}{2}\sum_1^r a_i^{*2}$, $Y = \frac{1}{2}\sum_1^r a_i^2$

$H = \left(\sum_i a_i^* a_i\right) + \frac{r}{2}$. We will use the basis

$$v_{s+2n} = \frac{X^n}{n!} \cdot 1$$

for $\mathcal{H}_s$, where $s = \frac{r}{2}$. These vectors are orthogonal and we now compute their norm-squared:

$$e^{-t\frac{z^2}{2}} = e^{tX}1 = \sum_{n\geq 0} t^n v_{s+2n}$$

$$\sum_n |t|^{2n} \| v_{s+2n} \|^2 = \left\| e^{-t\frac{z^2}{2}} \right\|^2 = (1-|t|^2)^{-s}$$

$$= \sum \frac{s(s+1)\cdots(s+n-1)}{n!} (|t|^2)^n$$

$$\therefore \quad \boxed{\| v_{s+2n} \|^2 = \frac{s(s+1)\cdots(s+n-1)}{n!}}$$

interpreted as 1 if $n=0$.

We need the effect of X, Y, H on this basis.

Clearly

$$\boxed{\begin{aligned} H v_{s+2n} &= (s+2n) v_{s+2n} \\ X v_{s+2n} &= (n+1) v_{s+2n+2}. \end{aligned}}$$

Note

$$Y e^{tX} 1 = \tfrac{1}{2} \sum \partial_i^2 e^{-t\frac{z^2}{2}}$$

$$= \tfrac{1}{2} \sum \partial_i \left(-t z_i e^{-t\frac{z^2}{2}} \right)$$

$$= \tfrac{1}{2} \sum (t^2 z_i^2 - t) e^{-t\frac{z^2}{2}} = (t^2 X - ts) e^{tX} 1$$

$$\therefore \quad Y \frac{X^n}{n!} 1 = -X \frac{X^{n-2}}{(n-2)!} 1 - s \frac{X^{n-1}}{(n-1)!} 1$$

$$\boxed{Y v_{s+2n} = -(s+n-1) v_{s+2n-2}}$$

Check sl_2 relations

$$(XY - YX) v_{s+2n} = X\left[-(s+n-1) v_{s+2n-2} \right] - Y\left[(n+1) v_{s+2n+2} \right]$$

$$= -(s+n-1)(n) v_{s+2n} + (n+1)(s+n) v_{s+2n}$$

$$= (s+2n) v_{s,2n}$$

In the preceding calculation you made an error with the notation v_{s+2n} which is imprecise. Better notation:

$$\boxed{\begin{aligned}
v_{s,n} &= \frac{X^n}{n!} v_s \\
\|v_{s,n}\|^2 &= \frac{s(s+1)\cdots(s+n-1)}{n!} \\
H v_{s,n} &= (s+2n) v_{s,n} \\
X v_{s,n} &= (n+1) v_{s,n+1} \\
Y v_{s,n} &= -(s+n-1) v_{s,n-1}
\end{aligned}}$$

Next check $Y = -X^*$ ($H = H^*$ obvious for s real since the eigenvalues are real + eigenvectors are orthonormal.)

$$\langle v_{s,n-1} | Y v_{s,n} \rangle = -(s+n-1)\|v_{s,n-1}\|^2$$
$$= -(s+n-1)\frac{s(s+1)\cdots(s+n-2)}{(n-1)!}$$

$$\langle X v_{s,n-1} | v_{s,n} \rangle = n\|v_{s,n}\|^2$$
$$= n\,\frac{s(s+1)\cdots(s+n-1)}{n!}$$

clear.

Conclusion: The above formulas define an irreducible unitary representation of $SU(1,1)$ for all $s > 0$.

New viewpoint: Let $G = \widetilde{SU(1,1)}$, K the inverse image of the diagonal matrices in $SU(1,1)$. Given a unitary character χ of K one obtains an equivariant line bundle L_χ over G/K = disk equipped with an inner product. It seems that L_χ also has an invariant connection and hence a holomorphic structure. Then one can form the Hilbert space of holom. sections whose length^2 is integrable wrt the invariant volume on G/K. This would give a unitary representation. The only catch is whether there are square integrable holomorphic sections.

Let's try carrying this out for the metaplectic representation, more generally for the representations $\mathcal{H}_s$ which I can handle maybe by the same method. For each t in the disk I consider the line in $\mathcal{H}_s$ spanned by $\varphi_t = e^{tX} v_s$. This gives a holomorphic subbundle L of the trivial bundle over the disk with fibre $\mathcal{H}_s$. At least for $s = \frac{k}{2}$ it is a G equivariant subbundle, probably also in general. Let $\check{L}$ be the dual line bundle. Any element of $\mathcal{H}_s$ gives rise to a holomorphic section of $\check{L}$. The question is when are such sections square integrable?

Look at the element v_s. To find the value of the associated section at the point t, we project v_s into L_t; the length squared function is then

$$t \longmapsto \left| \frac{\langle v_s | \varphi_t \rangle}{\|\varphi_t\|} \right|^2 = \frac{1}{\|\varphi_t\|^2} = (1-|t|^2)^s$$

On the other hand the volume on the disk is $\dfrac{d^2 z}{(1-|z|^2)^2}$

$$\left(\quad w = \frac{az+b}{\bar{b}z+\bar{a}} \qquad dw = \frac{(\bar{b}z+\bar{a})a\,dz - (az+b)\bar{b}\,dz}{(\bar{b}z+\bar{a})^2} = \frac{dz}{(\bar{b}z+\bar{a})^2} \right.$$

$$1-|w|^2 = \frac{|\bar{b}z+\bar{a}|^2 - |az+b|^2}{|\bar{b}z+\bar{a}|^2} = \frac{|b|^2|z|^2 + \cancel{\bar{b}z a} + \cancel{b\bar{z}\bar{a}} + |a|^2 - |a|^2|z|^2 - \cancel{az\bar{b}} - \cancel{\bar{a}\bar{z}b} - |b|^2}{|\bar{b}z+\bar{a}|^2}$$

$$1 - |w|^2 = \frac{1 - |z|^2}{|b\bar{z} + \bar{a}|^2}$$

$$\therefore \quad dw\, d\bar{w} = \frac{dz\, d\bar{z}}{(\bar{b}z + \bar{a})^2 (\overline{\bar{b}z + \bar{a}})^2} = \frac{dz\, d\bar{z}}{|\bar{b}z + a|^4}$$

$$\frac{dw\, d\bar{w}}{(1 - |w|^2)^2} = \frac{dz\, d\bar{z}}{(1 - |z|^2)^2} \qquad)$$

Thus the section associated to v_s is square integrable provided

$$\infty > \int (1 - |t|^2)^s \frac{d^2 t}{(1 - |t|^2)^2} = \pi \int_0^1 (1 - r^2)^{s-2}\, d(r^2)$$

and this happens only for $s > 1$.

What this probably means is that only for $s > 1$ do we have a square integrable representation. Maybe this is what one means by $s = 1$ being a "mock" discrete series representation.

February 10, 1986

General discussion of Gaussian integrals with complex exponent. Given a real vector space X and a suitable function $f(x)$ on it, we would like to make sense out of the family of integrals

$$\int f(x)\, e^{-\frac{1}{2}\tau(x)}\, dx$$

where τ ranges over the space of complex-valued quadratic forms on X having positive real part.

The first point is that the set of τ is a Siegel upper half plane; it is the symmetric space associated to a symplectic group, namely the autos. of $X \oplus X^*$. I propose to adopt the symplectic viewpoint as much as possible, in particular, to try to avoid constructions depending on the Lagrangian subspaces X, X^* of $X \oplus X^*$.

We can interpret τ as an irreducible state of the Weyl algebra $\mathrm{Weyl}(X \oplus X^*)$; it gives rise to an irreducible representation of the Weyl algebra on a Hilbert space $\mathcal{F}_\tau$, and there is a distinguished unit vector in $\mathcal{F}_\tau$. Thus over the UHP we have a Hilbert space bundle $\mathcal{F} = \{\mathcal{F}_\tau\}$, which is a Weyl algebra module bundle and which contains a distinguished section of norm 1 everywhere. This bundle and section are equivariant for the action of the symplectic group.

In finite dimensions the Weyl algebra has a unique irreducible module S up to isom. Hence we have a canonical isomorphism

$$\mathcal{F} = \mathcal{L} \otimes S$$

where $\mathcal{L}$ is a line bundle over the UHP. Since S is not a repn. of the symplectic grp but rather its double cover the metaplectic group, $\mathcal{L}$ is equivariant for the metaplectic group. Let's try to determine $\mathcal{L}$ more precisely.

Fix a basepoint 0 in the UHP and identify $\mathcal{F}_0$ with S at that point. As we move over the UHP we can compare $\mathcal{F}_\tau$ with S. In the former we have the unit vector $|0\rangle_\tau$ and in the latter we have $|0\rangle$. The projective isom. of $\mathcal{F}_\tau$ with S gives a line $L_\tau \subset S$ and fixing an isom. of $\mathcal{F}_\tau$ with S is the same as choosing a generator for L_τ. choosing a symplectic group elt sending 0 to τ, it can be lifted unique up to sign in the metaplectic group and then one gets two generators of opposite sign in L_τ. Unfortunately the stabilizer of 0 acts non-trivially on L_0, something like the square root of the determinant of a unitary matrix. So $\mathcal{L}$ should be the equivariant line bundle on the UHP for the symplectic group associated to $\sqrt{\det}$ on the unitary group. Weight $\frac{1}{2}$. So $\mathcal{L}$ is not equivariantly trivial, however, using an orthogonal complement of "k in g" one should get a connection in $\mathcal{L}$, and hence one can construct a trivialization of $\mathcal{L}$.

Better, $\mathcal{L}$ should be a holomorphic line bundle with a metric. It should be possible to cancel the curvature starting from a basepoint, and thus construct a holomorphic trivialization of $\mathcal{L}$. Once this is done any element of S determines a holom. section of $\mathcal{F}$, whence a function on the UHP by taking inner product with $|0\rangle_\tau$ (const. in $\mathcal{F}$).

February 11, 1986

Yesterday I described a good framework for Gaussian integrals with complex exponent:

X real v.s., $L^2(X)$ repn of $W^{eyl}(V)$, $V = X \oplus X^*$

$$\int f(x)\, e^{-\frac{1}{2}\tau(x)}\, dx/\text{norm} = \langle f \,|\, \text{irred. Gauss. state} \rangle$$

Siegel UHP $=$ irred Gauss. states of $\mathrm{Weyl}(V)$

$$\tau \longmapsto \mathcal{F}_\tau$$

$$\mathcal{F}_\tau \xleftarrow{\sim} \mathcal{L}_\tau \otimes S \quad , \quad S \text{ fixed}$$

$\mathcal{L} = \mathcal{L}_\tau = (\text{determinant}^{-1/2})$ line bundle over UHP

Trivializing $\mathcal{L}$ gives family of Gaussian integrals linear fnls on $f \in S$.

Question: If X has a complex structure, what can you say about Gaussian integrals with S^1-symmetry such as

$$\int e^{-\tau|z|^2} f(z)\, d^2z/\text{norm} \quad , \quad \mathrm{Re}\,\tau > 0 \text{ [redacted] } ?$$

First we want to calculate carefully in the case $X = \mathbb{C}$. Start with the operators: Consider $L^2(\mathbb{C})$ with the operators (momentum + position)

$$\partial_z, \partial_{\bar z} \quad , \quad z, \quad \bar z$$

I want a holomorphic representation, so I fix the Gaussian state $e^{-|z|^2} = e^{-z\bar z}$.

$$a = \frac{1}{\sqrt{2}}(\partial_{\bar z} + z) \qquad a^* = \frac{1}{\sqrt{2}}(-\partial_z + \bar z)$$

$$b = \frac{1}{\sqrt{2}}(\partial_z + \bar z) \qquad b^* = \frac{1}{\sqrt{2}}(-\partial_{\bar z} + z)$$

The only Gaussian states in $L^2(\mathbb{C})$ which are S^1 invariant are of the form $e^{-\tau|z|^2}$ with $\mathrm{Re}(\tau) > 0$. Let's find what operators kill this. Need

$$z = \frac{1}{\sqrt{2}}(a + b^*) \qquad \bar{z} = \frac{b+a^*}{\sqrt{2}}$$

$$\partial_{\bar{z}} = \frac{1}{\sqrt{2}}(a - b^*) \qquad \partial_z = \frac{b-a^*}{\sqrt{2}}$$

$e^{-\tau|z|^2}$ is killed by

$$\partial_z + \tau\bar{z} = \frac{1}{\sqrt{2}}\Big((b-a^*) + \tau(b+a^*)\Big) = \frac{1}{\sqrt{2}}\Big[(1+\tau)b - (1-\tau)a^*\Big]$$

$$\partial_{\bar{z}} + \tau z = \frac{1}{\sqrt{2}}\Big((a-b^*) + \tau(a+b^*)\Big) = \frac{1}{\sqrt{2}}\Big[(1+\tau)a - (1-\tau)b^*\Big]$$

Now if we use the holom. repn. with $a^* =$ mult by z, $b^* =$ mult by w, we see $e^{-\tau|z|^2}$ corresponds to

$$e^{\lambda z w} \qquad \lambda = \frac{1-\tau}{1+\tau}.$$

Note $\mathrm{Re}(\tau) > 0 \iff |\lambda| < 1$.

We consider the circle action on $L^2(\mathbb{C})$ given by $T_\zeta(f(z)) = f(\zeta z)$ for $\zeta \in \mathbb{T}$. Clearly

$$T\hat{z}T^{-1} = \zeta\hat{z}$$

$$T\hat{\bar{z}}T^{-1} = \zeta^{-1}\hat{\bar{z}}$$

$$T\partial_z T^{-1} = \zeta^{-1}\partial_z$$

$$T\partial_{\bar{z}} T^{-1} = \zeta\partial_{\bar{z}}$$

$\hat{\ }$ denotes the op. of multiplying

hence
$$T\begin{pmatrix} a \\ b^* \end{pmatrix}T^{-1} = \zeta\begin{pmatrix} a \\ b^* \end{pmatrix}, \quad T\begin{pmatrix} a^* \\ b \end{pmatrix}T^{-1} = \zeta^{-1}\begin{pmatrix} a^* \\ b \end{pmatrix}$$

This implies that on the holom. repn. one has

$$T_\zeta f(z, w) = f(\zeta^{-1} z, \zeta w)$$

Now we can determine the infinitesimal symplectic transformations which commute with the S^1 action. (Note: If X is a real vector space with a complex structure, then $\mathrm{Hom}_{\mathbb{C}}(X, \mathbb{C}) \xrightarrow{\sim} \mathrm{Hom}_{\mathbb{R}}(X, \mathbb{R}) = X^*$; [redacted] this isomorphism is given by $\lambda \longmapsto \mathrm{Re}(\lambda)$, and the inverse sends $\mu : X \to \mathbb{R}$ into $\lambda(x) = \mu(x) - i\lambda(Jx)$. Thus the real symplectic space $X \oplus X^*$ is in fact a complex symplectic space. One would therefore expect those real symplectic transformations which are compatible with the circle action to form a complex symplectic group. [redacted] Thus we would expect the dimension (if $\dim_{\mathbb{C}} X = n$)

$$\dim sp(2n) = \frac{2n(2n+1)}{2} = 2n^2 + n.$$ [redacted] (think of quadratic Hamiltonians)

However, we will see otherwise.)

So we look at the generators of the complexified Lie algebra of the symplectic group. [redacted] These are $(n=1)$.

$$a^{*2},\ a^*b^*,\ b^{*2},\ a^2,\ ab,\ b^2,\ a^*a + \tfrac{1}{2},\ a^*b,\ b^*a,\ b^*b + \tfrac{1}{2}$$

The ones commuting with the S^1 action are

$$a^*b^*,\quad ab,\quad a^*a + \tfrac{1}{2},\quad b^*b + \tfrac{1}{2}$$

which gives a four dimensional Lie algebra. In general we have

$$a_i^* b_j^*,\quad a_i b_j,\quad a_i^* a_j + \tfrac{1}{2}\delta_{ij},\quad b_i^* b_j + \tfrac{1}{2}\delta_{ij}$$

which gives a $4n^2$-dim'l Lie alg.

We continue with the task of finding the symplectic automorphisms which are compatible with the circle action on $V = X \oplus X^*$ associated to a complex structure on X. Notice that this circle action is

$$(*) \qquad \begin{pmatrix} e^{it} & 0 \\ 0 & e^{-it} \end{pmatrix}$$

so that the symplectic pairing (which is complex bilinear on V) is preserved. We are interested in real symplectic autos. of V commuting with this circle action. The complex symplectic group of V consists of real symplectic autos commuting with the circle action

$$\begin{pmatrix} e^{it} & 0 \\ 0 & e^{it} \end{pmatrix}$$

The goal will be to show the group of real symplectic autos commuting with $(*)$ is isomorphic to $U(n,n)$, where $n = \dim_{\mathbb{C}} X$. Let complexify V:

$$V_c = X_c \oplus X_c^*$$

This is a complex symplectic space with circle action having weights $e^{\pm it}$, hence the eigenspaces

$$V_c = V_c^+ \oplus V_c^-$$

are complementary maximal isotropic subspaces. In our former calculations

V_c^+ spanned by a, b^* ; V_c^- spanned by a^*, b.

The symplectic autos. of V_c over $\mathbb{C}$ commuting with the circle action are of the form

$$\begin{pmatrix} g & 0 \\ 0 & (g^t)^{-1} \end{pmatrix} \quad \text{rel to} \quad V_c = V_c^+ \oplus V_c^-$$

where g is any auto of V_c^+. Now $V_c^\pm$ are transformed into each other under the cx. conjugation on V_c fixing V, and we want real autos. This means that $(g^t)^{-1}$ is same kind of cx. conjugate of g, or that $g^{-1} = g^*$, whence we have a kind of unitary group. Let's now work out the details using our operators.

V_c^+ spanned by a_i, b_i^*

V_c^- ———— a_i^*, b_i

Symplectic form

$$\Omega(x^+ + x^-, y^+ + y^-) = (x^+)^t \varepsilon y^- - (y^+)^t \varepsilon x^-$$

where ε changes the signs of the coefficients of the b_i. $\therefore$ $\varepsilon = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. Then a symplectic transf. preserving the $\pm$ decomposition will be of the form $\begin{pmatrix} g & 0 \\ 0 & h \end{pmatrix}$ where

$$\Omega(gx^+ + hx^-, gy^+ + hy^-) = \Omega(x^+ + x^-, y^+ + y^-)$$

$$\begin{array}{ccc} \| & & \| \\ & & (x^+)^t \varepsilon y^- - (y^+)^t \varepsilon x^- \end{array}$$

$$(gx^+)^t \varepsilon hy^- - (gy^+)^t \varepsilon hx^- = (x^+)^t g^t \varepsilon hy^- - (y^+)^t g^t \varepsilon h x^-$$

$$\iff \quad g^t \varepsilon h = \varepsilon \quad \text{or} \quad h = \varepsilon (g^t)^{-1} \varepsilon$$

For $\begin{pmatrix} g & 0 \\ 0 & h \end{pmatrix}$ to be real means $h = \bar{g}$, so we get the condition $g = \varepsilon (g^*)^{-1} \varepsilon$ which defines the unitary group $U(n,n)$.

Alternatively

$$\mathrm{ad}(a^* b^*) \begin{pmatrix} a & b^* \end{pmatrix} = \begin{pmatrix} -b^* & 0 \end{pmatrix} = \begin{pmatrix} a & b^* \end{pmatrix} \begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix}$$

$$\mathrm{ad}(a b) \begin{pmatrix} a & b^* \end{pmatrix} = \begin{pmatrix} a & b^* \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

$$\mathrm{ad}(a^* a) \begin{pmatrix} - & - \end{pmatrix} = \begin{pmatrix} a & b^* \end{pmatrix} \begin{pmatrix} -1 & 0 \\ 0 & 0 \end{pmatrix}$$

$$\mathrm{ad}(b^* b) \begin{pmatrix} \cdot & \cdot & \cdot \end{pmatrix} = \begin{pmatrix} a & b^* \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

It's clear from this that the skew adjoint operators in this Lie algebra correspond to matrices

$$\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \ni \quad \alpha^* = -\alpha, \quad \beta^* = \gamma, \quad \delta^* = -\delta.$$

February 13, 1986

Still I would like to properly understand Gaussian integrals such as

$$\int e^{-\int \bar{\psi}(\partial_t+\omega)\psi \, dt (?)} \mathcal{D}\bar{\psi}\mathcal{D}\psi$$

on the space of complex valued loops $\psi: S^1 \longrightarrow \mathbb{C}$.

I have attempted to understand Gaussian integrals using Weyl algebra methods. How can I apply these to the above integral? First we should write it as a product of integrals of the form

$$\int e^{-\tau|z|^2} d^2z \left(\frac{\tau}{\pi}\right)$$

Here we use the complex coordinates $\psi \longmapsto \psi_k$ (kth Fourier coefficient) with respect to which the operator $\partial_t + \omega$ becomes diagonal with eigenvalues $\tau = ik + \omega$.

So next we want to determine a completion of the space of polynomial functions on which the integral is a linear functional. What comes to mind is the following. The integral on polynomials is easily worked out. The only non-zero moments are

$$\int e^{-\tau|z|^2} z^k \bar{z}^k \, \tau \frac{d^2z}{\pi} = \frac{k!}{\tau^k}$$

In other words, if we use the standard Hilbert space completion of the polynomials in which an orthonormal basis is $\frac{z^k}{\sqrt{k!}} \frac{\bar{z}^l}{\sqrt{l!}}$, then the integral on this orthonormal basis is

$$\int e^{-\tau|z|^2} \frac{z^k}{\sqrt{k!}} \frac{\bar{z}^l}{\sqrt{l!}} \tau \frac{d^2 z}{\pi} = \delta_{kl} \frac{1}{\tau^k} .$$

This linear functional is representable as an inner product, provided

$$\sum_{k,l} \left| \delta_{kl} \frac{1}{\tau^k} \right|^2 = \sum_k \frac{1}{|\tau|^{2k}} < \infty$$

i.e. for $|\tau| > 1$. Notice this condition is quite different from the familiar one that $\mathrm{Re}(\tau) > 0$.

Remark: If V is a symplectic vector space, then any Lagrangian subspace X determines an irreducible cyclic module over the Weyl alg $W(V)$, namely

$$W(V) \otimes_{W(X)} \mathbb{C} \simeq W(V)/W(V)X \underset{\text{additive}}{\simeq} S(V/X).$$

Given complementary Lagrangian subspaces X, Y one has a pairing

$$\begin{array}{c} W(V)/_{Y}W(V) \otimes W(V)/W(V)X \\ \downarrow \\ W(V)/YW(V) \otimes_{W(V)} W(V)/W(V)X \\ \| \\ \mathbb{C} \xrightarrow{\sim} W(V)/YW(V) + W(V)X \end{array}$$

The reason this is $\mathbb{C}$ is that one has an isom $W(V) = W(Y) \otimes W(X) = S(Y) \otimes S(X)$ compatible with left $S(Y)$-multiplication and right $S(X)$-mult.; so $YW(V)_{\wedge}^{+W(V)X} = YS(Y) \otimes S(X) + S(Y) \otimes S(X)X$ gives all polys without constant term. The above pairing is equivalent to the natural pairing of

$$S(X) \simeq W(V)/YW(V) \quad \text{and} \quad S(Y) \simeq W(V)/W(V)X$$

derived from the duality between X and Y. For example when $V = \underbrace{\mathbb{C}a}_{X} + \underbrace{\mathbb{C}a^*}_{Y}$

$W(V)/YW(X)$ has basis $\langle 0 | a^k$

$W(V)/W(V)X$ —————— $a^{*k}|0\rangle$

and the pairing evaluates $\langle 0 | a^k a^{*k} | 0 \rangle$ by normal ordering $a^k a^{*k}$ and taking the constant terms which gives $k!$

Further remarks: ① When V has a real structure one can take ~~[illegible]~~ $Y = \bar{X}$ if this is complementary to X (i.e. totally non-real), then $W(V)/YW(V)$ is the conjugate vector space to $W(V)/W(V)X$, hence one has a hermitian form on the latter. It's positive when X satisfies the well-known positive condition.

② If $V = Y \oplus X$ as above, and X' is another Lagrangian subspace complementary to Y, then we have non-degenerate pairings of $W(V)/W(V)X$ and $W(V)/W(V)X'$ with $W(V)/YW(V)$. Thus modulo convergence we have an isomorphism of the irred. modules corresponding to X, X'. This is exactly what happens when we have $Y = \mathbb{C}a^*$, $X = \mathbb{C}a$ and $X' = \mathbb{C}(a - \lambda a^*)$, and identify formally $W(V)/W(V)X'$ with $W(V)/W(V)X = \mathbb{C}[z]$ and the cyclic vector $e^{\lambda z^2/2}$.

③ Idea: I eventually want to think of the infinite dimensional Weyl (C^*?) algebra with its inequivalent representations as analogous to the C^*-algebra of a foliation. This raises the question: What is the cyclic cohomology of the Weyl algebra?

Suppose $A = \mathbb{C}[a^*, a]$ with $[a, a^*] = 1$ and set $B = A \otimes A^{op} = \mathbb{C}[\underbrace{a^* \otimes 1}_{\lambda^*}, \underbrace{a \otimes 1}_{\lambda}, \underbrace{1 \otimes a^*}_{\mu^*}, \underbrace{1 \otimes a}_{\mu}]$

Then $\left[\begin{smallmatrix}\lambda^* \\ \lambda\end{smallmatrix}, \begin{smallmatrix}\mu^* \\ \mu\end{smallmatrix}\right] = 0$, $[\lambda, \lambda^*] = 1$, but $[\mu^*, \mu] = 1$ as the order of mult. in A^{op} is reversed. [redacted]

A is both a left and right B-module:

A left B-module :
$$\lambda^* = a^*\circ \qquad \lambda = a\circ \qquad \mu^* = \circ a^* \qquad \mu = \circ a$$

A right B-module
$$\lambda^* = \circ a^* \qquad \lambda = \circ a \qquad \mu^* = a^*\circ \qquad \mu = a\circ$$

Note that as left A-module, the element $1 \in A$ is killed by $\lambda^* - \mu^*$ and $\lambda - \mu$. As [redacted]

$$[\lambda-\mu, \lambda^*-\mu^*] = [\lambda,\lambda^*] + [\mu,\mu^*] = 1-1=0$$

$\lambda-\mu$, $\lambda^*-\mu^*$ span a Lagrangian subspace X of V where $V =$ [redacted] span of $\lambda^*, \lambda, \mu^*, \mu$ in B, so that $\mathrm{Weyl}(V)=B$. Also we have

$$A = B/BX \quad \text{as left } B\text{-module}$$

$$A = B/XB \quad \text{as right } B\text{-module.}$$

February 15, 1986

Compute cyclic cohomology of a Weyl algebra $A = \mathbb{C}[a_i^*, a_i]$. Set $B = A \otimes A^{op} = \mathbb{C}[\underbrace{a_i^*\otimes 1}_{\lambda_i^*}, \underbrace{a_i \otimes 1}_{\lambda_i}, \underbrace{1\otimes a_i^*}_{\mu_i^*}, \underbrace{1\otimes a_i}_{\mu_i}]$ where the λ's and μ's commute and

$$[\lambda_i, \lambda_j^*] = \delta_{ij} \qquad [\mu_i, \mu_j^*] = -\delta_{ij}$$

B is the Weyl algebra gen. by V = space with basis $\lambda_i^*, \lambda_i, \mu_i^*, \mu_i$. ~~[illegible]~~ A has both a left and right B-module structure (these don't commute). If $X \subset V$ is the subspace spanned by elts $\lambda_i^* - \mu_i^*$, $\lambda_i - \mu_i$ then X is Lagrangian since

$$[\lambda_i - \mu_i, \lambda_j^* - \mu_j^*] = \delta_{ij} - \delta_{ij} = 0 .$$

One has that X kills $1 \in A$ for both left + right structure, also

$$A \simeq B/BX \qquad \text{as left } B\text{-module}$$

$$A \simeq B/XB \qquad \text{— right —}$$

We now compute the Hochschild homology $\mathrm{Tor}_*^B(A,A)$ using a Koszul resolution. Start with the Koszul cx:

$$\longrightarrow \Lambda^2 X \otimes S(X) \xrightarrow{\partial} X \otimes S(X) \xrightarrow{\partial} S(X) \longrightarrow k \longrightarrow 0$$

where $$\partial = \sum_j i_j \otimes (\text{left mult by } x_j)$$
where x_j is a basis for X, and the i_j are the interior products on ΛX belonging to the dual basis elts. This Koszul complex is acyclic, and tensoring with B which is free over $S(X)$ gives a resolution of right B-modules

$$\longrightarrow \Lambda^2 X \otimes B \xrightarrow{\partial} \Lambda^1 X \otimes B \xrightarrow{\partial} B \longrightarrow \underset{\overset{\|}{A}}{B/XB} \longrightarrow 0$$

Then the Hochschild homology is that of the complex obtained by tensoring $\otimes_B A$:

$$\longrightarrow \Lambda^2 X \otimes A \longrightarrow X \otimes A \longrightarrow A$$

It is therefore the Koszul homology of A with respect to the family of commuting endos:

$$\lambda_i^* - \mu_i^* \ : f \longmapsto a_i^* f - f a_i^* = [a_i^*, f]$$
$$\lambda_i - \mu_i \ : f \longmapsto a_i f - f a_i = [a_i, f]$$

But this is easily determined as follows; suppose to simplify the is one pair a^*, a. Usual normal ordering gives A the basis $(a^*)^k a^l$ which identifies A with polynomials in two variables additively. Relative to this identification $[a_i, ?]$ is differentiation with respect to the first variable, and $[a_i^*, ?]$ is minus diffn. with respect to the second variable. But the Koszul complex of $k[X_1, .., X_n]$ wrt the endos. ∂_{X_i} is isomorphic to the DR cx of $k[X_1, .., X_n]$, and so the homology is zero except for a k in degree n.

<u>Conclude:</u> The Hochschild homology of $A = \mathbb{C}[a_i^*, a_i]$ with $2n$ operators is

$$H_i(A) = \begin{cases} 0 & i \neq 2n \\ k & i = 2n. \end{cases}$$

The cyclic homology is

$$HC_i(A) = \begin{cases} k & i = 2n, 2n+2, 2n+4, \dots \\ 0 & \text{otherwise} \end{cases}$$

Now arises the <u>Question</u>: Can one find a simple description of the $2n$ cyclic cocycle on A detecting the generator of $HC_{2n}(A)$?

Let's adopt Connes' idea of looking for a differential algebra containing A and a suitable linear functional. The first thing that comes to mind is Irving Segal's quantized differential forms. When A is the Weyl algebra on L, these are elements of $A \otimes \Lambda(L^*)$ i.e. multilinear alternating maps from L to A. However, Segal shows Poincaré's lemma holds so these aren't what I want.

When $A = \mathbb{C}[a^*, a]$, the quantized forms give the complex

$$\begin{tikzcd} & A \arrow[dr, "{[a,\ ]}"] & \\ A \arrow[ur, "{[a^*,\ ]}"] \arrow[dr, "{[a,\ ]}"'] & \oplus & A \\ & A \arrow[ur, "{-[a^*,\ ]}"'] & \end{tikzcd}$$

Note that $[\alpha, ?]$ is a derivation of A with values in itself. I want to see a product on the above complex making it a diff alg. We have to have

$$df = [a,f]\,da^* - [a^*,f]\,da$$

if we want d to be made of the maps $[a,\ ]$, $[a^*,\ ]$ and do the correct thing on $f = a, a^*$. Then if we want d to be a derivation it seems we want to have da, da^* commuting with A.

Also

$$\begin{aligned} d^2(f) &= d\big([a,f]\,da^* - [a^*,f]\,da\big) \\ &= \big([a[a,f]]\,da^* - [a^*,[a,f]\,da\big)\,da^* \\ &\quad - \big([a,[a^*,f]]\,da^* - [a^*[a^*,f]]\,da\big)\,da \end{aligned}$$

and the only way this can be zero is for

$$(da^*)^2 = (da)^2 = da\,da^* + da^*\,da = 0$$

This produces Segal's quantized differential forms.

February 22, 1986

Recall an old idea for producing cyclic cocycles. Consider a vector bundle E with connection ∇ over M, and form the graded algebra $R^\circ = \Omega^\circ(M, \mathrm{End}\, E)$ with the derivation of degree 1, $D\alpha = [\nabla, \alpha]$ and the element $K = \nabla^2$ of degree 2. These satisfy

$$DK = 0 \qquad D^2\alpha = [K, \alpha]$$

which are formulas I have seen analogues of for "asymptotic" Dirac operators. (In this analogous setup R° is the ~~[illegible]~~ graded algebra (over $\mathbb{C}[h]$) constructed from Getzler's filtration on the algebra of diff'l operators on spinors, D is $[h\not{D}, \ ]$ where $\not{D} = \gamma^\mu \nabla_\mu$, and $K = h^2 \not{D}^2$.)

Returning to $R^\circ = \Omega^\circ(M, \mathrm{End}\, E)$ I propose to ~~construct~~ construct cyclic cocycles on $\Omega^\circ(M, \mathrm{End}\, E)$, by producing them as left-invariant differential forms on the gauge group $\mathcal{G}$ of autos. of E^N. This is done by working over $\mathcal{G} \times M$ with the family of connections

$$\nabla_t = \delta + t\theta + \nabla$$

$$\nabla_t^2 = K + tD\theta + (t^2 - t)\theta^2.$$

Recall the "transgression" formula

$$\mathrm{tr}\,(K + D\theta)^n - \mathrm{tr}\, K^n = (\delta + d)\int_0^1 dt\; n\, \mathrm{tr}\left(\theta\left(K + tD\theta + (t^2-t)\theta^2\right)^{n-1}\right)$$

which holds in $C^\circ(\tilde{g}, R^\circ)/[\,,\,] \longrightarrow C^\circ(\tilde{g}, \Omega^\circ(M))$.

What I hoped to do was to use this formula to produce cocycles in the Dirac case.

This means that I want to forget about the $\mathbb{Z}$ grading on $R^\cdot$ and I would like to suppose only that R° is $\mathbb{Z}_2$-graded, and that it is equipped with a trace τ such that $\tau \circ D = 0$. Then the above formula becomes

$$\tau((K+D\theta)^n) - \tau(K^n) = \delta \int_0^1 dt\ n\,\tau\left(\theta(K+tD\theta+(t^2-t)\theta^2)^{n-1}\right)$$

(Recall proof. One has

$$\partial_t\ \tau(\nabla_t^2)^n = n\,\tau\big(\underbrace{\partial_t(\nabla_t^2)}_{[\nabla_t,\theta]}\ \nabla_t^{2(n-1)}\big)$$

$$= n\,\tau\left([\delta + t\theta + \nabla, \theta(\nabla_t^2)^{n-1}]\right)$$

$$= \delta\ n\,\tau\left(\theta(\nabla_t^2)^{n-1}\right) \qquad \text{and then do } \int_0^1 dt.)$$

However the formula is not useful unless

$$\tau((K+D\theta)^n) \overset{?}{=} \tau(K^n).$$

This holds for $n=1$, but fails for $n=2$.

Let's verify this carefully: $\quad 2\int_0^1 (t^2-t)dt = 2\left[\frac{1}{3}-\frac{1}{2}\right] = -\frac{1}{3}$

$$\tau(K+D\theta)^2 - \tau(K^2) = \delta\ \tau\left(2\theta K + \theta D\theta - \tfrac{1}{3}\theta^3\right)$$

We know already that $\delta\,\tau(\theta^3) = 0$. Also

$$\delta\,\tau(\theta\cdot D\theta) = \tau\left(-\theta^2\cdot D\theta + \theta\cdot D(-\theta^2)\right)$$

$$= \tau\left(-\theta^2\cdot D\theta - \theta\cdot D\theta\cdot\theta + \theta^2\cdot D\theta\right) = +\tau(\theta^2\cdot D\theta)$$

$$0 = \tau(D(\theta^3)) = \tau\left(D\theta\cdot\theta^2 - \theta\cdot D\theta\cdot\theta + \theta^2\cdot D\theta\right) = 3\tau(\theta^2\cdot D\theta)$$

so $\quad \delta\tau(\theta\cdot D\theta) = 0.\quad$ On the other hand

$$\delta\tau(2\theta K) = \tau(-2\theta^2 K) \quad \text{and}$$

$$\tau(K+D\theta)^2 - \tau(K^2) = \tau(\underbrace{K\cdot D\theta}_{D(K\theta)} + \underbrace{D\theta\, K}_{D(\theta K)} + (D\theta)^2)$$

$$[D,\theta[D,\theta]] = [D,\theta]^2 - \theta[K,\theta]$$

$$\Rightarrow \quad \tau[D,\theta]^2 = \tau(\theta[K,\theta]) = \tau(\theta K\theta - \theta^2 K) = \tau(-2\theta^2 K)$$

Apparently the term $\tau[D,\theta]^2$ is non-zero in general.

The above calculation shows clearly that the cochain

$$\int_0^1 dt\; 2\tau\{\theta(K + t D\theta + (t^2-t)\theta^2)\}$$

is not a good thing to look at. This might explain why ~~~~ I had such difficulty in the past finding the appropriate cocycles to attach to Dirac operators.

March 1, 1986

I want to consider the problem of explaining Connes construction of cyclic cocycles attached to Dirac operators in terms of ideas such as transgression, and possibly a proof of the periodicity theorem.

Let's start with the odd diml case. Here the index of the family of Dirac operators is an odd K-class on $B\mathcal{G}$, which provides odd cohomology classes, which in turn transgress to even cohomology classes on $\mathcal{G}$. (If one can do the transgression in K-theory, then one has an even K-class on $\mathcal{G}$.) The 0-diml cohomology class is an index-type map $\pi_0 \mathcal{G} \longrightarrow \mathbb{Z}$, so it is not obtainable from the Lie algebra cohomology. Nevertheless I would like to carefully understand this map, so as to study the transgression process.

Consider the simplest case: $\mathcal{H} = L^2(S^1)$ with the family of Dirac ops

$$A = \frac{1}{i}(\partial_x + \alpha) \qquad \alpha \in \underbrace{C^\infty(S^1, i\mathbb{R})}_{\mathcal{A}}$$

and $\mathcal{G} = C^\infty(S^1, U(1))/U(1)$. Here $\mathcal{G}$ acts freely on $\mathcal{A}$, ~~and~~ we have

$$\mathcal{A}/\mathcal{G} \xrightarrow{\sim} U(1) = S^1$$

given by the monodromy.

In this example the family of operators over $B\mathcal{G}$

is easy to see concretely: One has the Hilbert bundle over $B\mathcal{G} = \mathcal{A}/\mathcal{G} = S^1$ obtained ~~by~~ from the natural repn. of $\mathcal{G}$ on $\mathcal{H}$. It is the bundle with fibre $\mathcal{H}$ over S^1 obtained by glueing the ends of $I \times \mathcal{H}$ with a gauge transf. g of degree 1. The family of Dirac operators is then obtained by taking a path going from a fixed on A_0 to its transform under g.

What might be important here is that the family of operators is so canonical that there aren't many ways to analytically associate differential forms to this family.

There is the example of the η-invariant which up to sign is given by

$$e^{i\pi\eta_A} = \text{monodromy}.$$

Let's discuss the K-theory aspects. We have this family of Diracs over $\mathcal{A}/\mathcal{G} = S^1$ and the index is an odd K-class on $\mathcal{A}/\mathcal{G}$. If we were to succeed in representing this odd K-class as a map $\mathcal{A}/\mathcal{G} \longrightarrow$ unitaries $\equiv 1$ mod trace class operators, then by taking the determinant we get a map $S^1 \longrightarrow S^1$ whose degree determines the K-class.

In general a family of odd Dirac operators determines an odd K-class, hence ~~odd~~ cohomology classes on the parameter space. I want to construct forms representing these classes, in particular I want I want a closed 1-form.

Notice that because $H^1(X, \mathbb{Z}) = [X, S^1]$, the one dimensional integral class we seek to represent is represented by a map of the parameter space to S^1. One representative is undoubtedly

$$A \longmapsto e^{i\pi\eta_A}$$

so the 1-form we want is essentially $d\eta$.

Review the formulas about η:

$$\eta(A,s) = \sum \operatorname{sgn}(\lambda)\,|\lambda|^{-s}$$

$$= \operatorname{tr}\left(\frac{A}{|A|}|A|^{-s}\right) = \operatorname{tr}\left(A\,(A^2)^{-\left(\frac{s+1}{2}\right)}\right)$$

$$= \frac{1}{\Gamma\left(\frac{s+1}{2}\right)} \int_0^\infty \underbrace{\operatorname{tr}\left(Ae^{-tA^2}\right)}_{\sim\, c_{-1/2}\,t^{-1/2} \text{ in its asymp. exp.}} t^{+\left(\frac{s+1}{2}\right)} \frac{dt}{t}$$

as $s \to 0$

$$\sim \frac{1}{\Gamma\left(\frac{1}{2}\right)}\, c_{1/2} \underbrace{\int_0^1 t^{+\frac{s}{2}} \frac{dt}{t}}_{+\frac{2}{s}} = \frac{1}{\Gamma\left(\frac{1}{2}\right)}\, c_{1/2}\left(+\frac{2}{s}\right)$$

This last formula shows why the fact that $\eta(A,s)$ is regular at $s=0$ is deep. On the other hand

$$\delta\eta(A,s) = -s\,\operatorname{tr}\left(\delta A\,(A^2)^{-\left(\frac{s+1}{2}\right)}\right)$$

$$= \frac{-s}{\Gamma\left(\frac{s+1}{2}\right)} \int_0^\infty \underbrace{\operatorname{tr}\left(\delta A\, e^{-tA^2}\right)}_{\sim\, c_{-1/2}\,t^{-1/2}} t^{+\left(\frac{s+1}{2}\right)} \frac{dt}{t}$$

as $s \to 0$

$$\sim \frac{-s}{\Gamma\left(\frac{1}{2}\right)}\, c_{-1/2} \cdot \left(+\frac{2}{s}\right) = -\frac{2}{\sqrt{\pi}}\, c_{-1/2}$$

is regular at $s=0$, which is one of the steps in the APS proof (residue of the apparent pole is constant under deformation)

The formula

$$\delta\eta_A = -\frac{2}{\sqrt{\pi}} \text{"}\lim_{t\to 0}\text{"} \operatorname{tr}(\delta A . e^{-tA^2})/t^{1/2}$$

suggests that $\operatorname{tr}(\delta A\, e^{-tA^2})$ might be a closed 1-form. This one can verify on the resolvent side as follows. I want to shift notation to $L = iA$. Then

$$d \operatorname{tr}\left(\frac{1}{1-L^2} dL\right) = +\operatorname{tr}\left(\frac{-1}{1-L^2} d(L^2) \frac{1}{1-L^2} dL\right)$$

$$= -\operatorname{tr}\left(\frac{1}{1-L^2}(dL.L + L.dL)\frac{1}{1-L^2} dL\right)$$

$$= -\operatorname{tr}\left(\frac{1}{1-L^2} dL . L . \frac{1}{1-L^2} dL\right) - \operatorname{tr}\left(L\left(\frac{1}{1-L^2} dL\right)^2\right)$$

These two terms cancel as $\operatorname{tr}(XY) = -\operatorname{tr}(YX)$ when X, Y are odd degree matrix forms.

A more convincing demonstration is to find a function whose d gives the 1-form:

$$d \log \det\left(\frac{1+L}{1-L}\right) = \operatorname{tr}\left\{\frac{dL}{1+L} - \frac{1}{1-L}(-dL)\right\}$$

$$= \operatorname{tr}\left(\frac{1}{1+L} + \frac{1}{1-L}\right) dL = \operatorname{tr}\left(\frac{2}{1-L^2} dL\right)$$

A better way to put this is as follows. We are considering the Cayley transform

$$L \longmapsto \frac{1+L}{1-L}$$

from skew adjoint matrices to unitaries and taking the trace of the MC form

$$g^{-1}dg = \left((1-L)^{-1}(1+L)\right)^{-1} d\left((1-L)^{-1}(1+L)\right)$$

$$= (1+L)^{-1}\cancel{(1-L)}\left\{-\cancel{(1-L)^{-1}}(-dL)(1-L)^{-1}(1+L) + \cancel{(1-L)^{-1}}dL\right\}$$

$$= (1+L)^{-1}dL\left\{1 + (1-L)^{-1}(1+L)\right\}$$

$$= \frac{1}{1+L}\,dL\,(1-L+1+L)\,\frac{1}{1-L}$$

$$= \frac{1}{1+L}\,2dL\,\frac{1}{1-L}$$

Therefore

<u>Prop</u>: The forms $\operatorname{tr}\left(\frac{2}{1-L^2}dL\right)^{2k+1}$ on skew-adjoint operators are closed.

March 2, 1986

The problem is to study the (anti-) transgression of the closed 1-form

$$\mathrm{tr}\left(\frac{1}{1-L^2}\,dL\right)$$

at least in the case of the family of operators $L = \partial_x + \alpha$ over S^1. Choose a gauge transf. g so that

$$g^{-1}(\partial_x + ia).g \;=\; \overbrace{\partial_x + \alpha}^{L}$$

i.e. $$g^{-1}(\partial_x g) + ia \;=\; \alpha$$

where ia is the constant term of α:

$$2\pi i\, a = \int_{S^1} \alpha\, dx$$

Then $$\langle x|\frac{1}{1-L^2}|y\rangle = \langle x|\, g^{-1}\frac{1}{1-(\partial_x+ia)^2}\, g\,|y\rangle$$

$$= g(x)^{-1}\langle x|\frac{1}{1-(\partial_x+ia)^2}|y\rangle\, g(y)$$

and

$$\mathrm{tr}\left(\frac{1}{1-L^2}\,\delta L\right) = \int \langle x|\frac{1}{1-L^2}|x\rangle\,\delta\alpha(x)\,dx$$

$$= \int \langle x|\frac{1}{1-(\partial_x+ia)^2}|x\rangle\;\delta\alpha(x)\,dx$$

$$= \frac{1}{2\pi}\sum_n \frac{1}{1+(n+a)^2}\cdot \delta\overbrace{\int \alpha(x)\,dx}^{2\pi i a}$$

$$= i\left(\sum_n \frac{1}{1+(n+a)^2}\right)\delta a$$

Thus the 1-form, or more generally the family of 1-forms

$$\mathrm{tr}\left(\frac{1}{\lambda - L^2}\,\delta L\right), \qquad \mathrm{tr}\left(e^{+tL^2}\,\delta L\right)$$

come from 1-forms on the circle [illegible] via the monodromy map

$$\mathcal{A}/\mathcal{G} \longrightarrow S^1 \qquad \alpha \longmapsto e^{\int \alpha\, dx} = e^{2\pi i a}$$

$$\mathrm{tr}\left(\frac{2t}{1-t^2L^2}\,\delta L\right) = i\underbrace{\left(\sum_n \frac{2t}{1+t^2(n+a)^2}\right)}\delta a$$

$$\text{as } t \to 0 \quad \sim \int_{-\infty}^{\infty} \frac{2\,dx}{1+x^2} = 2\arctan x\Big]_{-\infty}^{+\infty} = 2\pi$$

March 3, 1986

Here are some important points:

$$\begin{array}{ccc} H^1(a/\mathcal{G},\mathbb{Z}) & = & [a/\mathcal{G}, S^1] \\ \wr\| & & \\ \mathrm{Hom}(\pi_0\mathcal{G},\mathbb{Z}) & & \end{array}$$

The 1-forms $\mathrm{Tr}\left(\frac{2t}{1-t^2L^2}\,dL\right)$ are all cohomologous and represent the class in $H^1(a/\mathcal{G},\mathbb{Z})$ we are interested in. The limit as $t\to 0$ is the "classical" limit and it is presumably given by differential geometry in general. Over S^1 we found

$$\lim_{t\to 0}\mathrm{Tr}\left(\frac{2t}{1-t^2L^2}\,dL\right) = 2\pi i\, da$$

$$L = \partial_x + \alpha \qquad ia = \text{const term of } \alpha$$

The problem is to pick a function on a whose d is this 1-form. Here's a way to proceed: Identity

$$\partial_t\, \underbrace{\mathrm{tr}\left(\frac{2t}{1-t^2L^2}\,dL\right)}_{\omega_t} = d\, \underbrace{\mathrm{tr}\left(\frac{2}{1-t^2L^2}\,L\right)}_{f_t}$$

$$[\omega_t]_{t_0}^{t_1} = d\int_{t_0}^{t_1} f_s\, ds$$

Formally it's clear that $\omega_t \to 0$ as $t \to +\infty$ and that f_t does also. Hence proceeding carelessly we have

$$-\omega_t = d\left(\int_t^\infty f_s \, ds\right)$$

This can't be correct because $\int_t^\infty f_s \, ds$ if defined is a function on $\mathcal{A}/\mathcal{G}$.

I think what's happening can be seen already with the η invariant:

$$\eta(A_s) = \frac{1}{\Gamma(\frac{s+1}{2})} \int_0^\infty \underbrace{tr(A e^{-tA^2})}_{= O(t^{1/2})} t^{\frac{s+1}{2}} \frac{dt}{t}$$

? according to Bismut, Freed

The η-invariant is defined by doing an integral from 0 to ∞. We know it gives a discontinuous function.

So what I think happens is that the integral $\int^\infty f_s \, ds$ is defined where L is non singular, otherwise it jumps. What I would like to do is to maybe smooth this discontinuity by a periodicity trick of the sort used by Connes. Somehow have to kill the topological obstruction.

March 4, 1986

I consider the family of Diracs on S^1

$$L = \partial_x + ia \qquad a \in \mathbb{R} = \mathcal{A}$$

with $\mathbb{Z} = \mathcal{G}$ acting as gauge transformations. Analytically we have defined closed 1-forms on $\mathbb{R}/\mathbb{Z}$ using the formula

$$\omega_t = \mathrm{tr}\left(\frac{2t}{1-t^2L^2}\, dL \right)$$

These forms are cohomologous, the simplest one being the limiting one as $t \downarrow 0$ which gives

$$\omega_0 = 2\pi i\, da.$$

I want to transgress the class of ω_t:

$$\begin{array}{ccc} H^1(\mathcal{A}/\mathcal{G}, \mathbb{Z}) & = & [\mathcal{A}/\mathcal{G}, S^1] \\ \downarrow & & \\ \mathrm{Hom}(\pi_0\mathcal{G}, \mathbb{Z}) & & \end{array}$$

This means lifting ω_t to $\mathcal{A}$ writing it as d of something f and then restricting f to a $\mathcal{G}$ orbit. On the set where L is invertible one has a natural candidate for f defined analytically but this f is discontinuous for $a \in \mathbb{Z}$.

The hope is to somehow "double" the operator à la Connes and reduce to the invertible case modulo using periodicity.

It seems that I am forced to handle the following problem. Given $g \in G$ and $L \in \mathcal{A}$, we then get a loop of operators by choosing a path from L to $g^{-1}.L.g$. To this loop of operators we must associate an integer. This is the APS spectral flow.

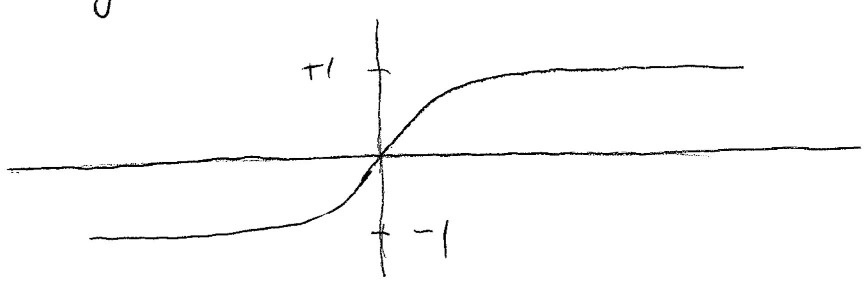
So now I understand the transgression analytically. The nice idea here is Atiyah's one of using the division of singular Fredholm operators in $\mathcal{F}_1$.

Alternative: Replace L by $\frac{1}{i}L(1-L^2)^{-1/2}$ which will be an involution in the Calkin algebra. Now to an involution F and invertible g one has an index given by

$$\frac{1}{2}\,tr\left(g^{-1}[F,g]\right)$$

Maybe this formula makes sense in our context.

In general let $f(x)$, $-\infty \leq x \leq +\infty$ look like



e.g. $\frac{x}{\sqrt{1+x^2}}$ or $\tanh x$. Then consider

$$\frac{1}{2}\,tr\ \ g^{-1}[f(A),g] = \frac{1}{2}tr\left(g^{-1}f(A)g - f(A)\right)$$

where $A = \frac{1}{i}L = (-i)(\partial_x + ia)$. We see this is independent of the choice of f, because the difference $f_1(A) - f_2(A) = (f_1 - f_2)(A)$ will be of trace class.

We can thus choose $f(A)$ to be ~~the projector~~ involution which is -1 on strictly negative eigenspaces and $+1$ on positive or zero eigenspaces. If $g \cdot e^{inx} = e^{i(n+r)x}$, $g = e^{irx}$, then $g^{-1}f(A)g$ is the same involution except r eigenvalues are shifted. Thus

$$\frac{1}{2}\operatorname{tr}\left[g^{-1}f(A)g - f(A)\right] = r$$

Better: $\quad Ae_n = (n+a)e_n \qquad\qquad e_n = e^{inx}$

$$(g^{-1}Ag)e_n = g^{-1}A\, e_{n+r} = (n+r+a)e_n$$

$$\sum_n f(n+r+a) - f(n+a) = \lim_{N\to+\infty} \sum_{-N}^{N} f(n+r+a) - f(n+a)$$

$$= \lim_{N\to\infty} \sum_{N-r+1}^{N} f(n+r+a) - \sum_{-N}^{-N+r-1} f(n+a)$$

$$= 2r.$$

At this point what I have apparently done is to completely remove paths from the problem. So before I needed a path joining L to $g^{-1}Lg$, and then looked at the spectral flow along the path (or loop) now one just computes a commutator and a trace.

Question: Have we actually defined a cyclic 1-cocycle attached to L?

Let $\tau(a_0, a_1) = \mathrm{tr}(a_0[F, a_1])$. [redacted]

Assume $[F, a]$ is trace class for all a so this is well-defined. Then

$$\mathrm{tr}(a_0 a_1 [F, a_2]) - \mathrm{tr}(a_0[F, a_1 a_2]) + \mathrm{tr}(a_2 a_0 [F, a_1])$$

$$= -\mathrm{tr}(a_0[F, a_1]a_2) + \mathrm{tr}(a_2 a_0 [F, a_1]) = 0$$

in general. But

$\mathrm{tr}(a_0[F, a_1]) + \mathrm{tr}(a_1[F, a_0]) =$ [redacted] $\mathrm{tr}([F, a_0 a_1])$

so it seems one must [redacted] assume $\mathrm{tr}\,[F, a] = 0$ to get cyclic symmetry. Or else one can suppose $F^2 = 1$ for

$$\underbrace{\mathrm{tr}(F[F, a_0][F, a_1])}_{\text{skew in } a_0, a_1} = \mathrm{tr}(a_0[F, a_1] - F a_0 F[F, a_1])$$

$$= 2\,\mathrm{tr}(a_0 [F, a_1])$$

as $F[F, a_0] + [F, a_0]F = 0$

March 10, 1986

Suppose $T: \mathcal{H}^+ \longrightarrow \mathcal{H}^-$ invertible, and let the algebra $\mathcal{A}$ act on $\mathcal{H} = \mathcal{H}^+ \oplus \mathcal{H}^-$ preserving the grading. Suppose $[a,T]$ of trace class. I claim that

$$\varphi(a) = tr(T^{-1}[a,T])$$

is a trace on $\mathcal{A}$.

Proof 1:

$$0 = tr([a, T^{-1}[b,T]])$$

$$= tr(-T^{-1}[a,T]\,T^{-1}[b,T]) + tr(T^{-1}[a,[b,T]])$$

shows that the 2nd term is symmetric in a,b. But this term is also

$$tr(T^{-1}[a,[b,T]]) = tr(T^{-1}[[a,b],T]) + tr(T^{-1}[b,[a,T]])$$

~~Since the second term is symmetric~~, so it follows that $0 = tr(T^{-1}[[a,b],T]) = \varphi([a,b])$

showing that φ is a trace.

Proof 2: Set $F = \begin{pmatrix} 0 & T^{-1} \\ T & 0 \end{pmatrix}$. ~~$\varepsilon = \begin{pmatrix} 1 & \\ & -1 \end{pmatrix}$~~

Then

$$tr_s(F[F,a]) = tr_s \begin{pmatrix} 0 & T^{-1} \\ T & 0 \end{pmatrix} \begin{pmatrix} 0 & -T^{-1}[T,a]T \\ [T,a] & 0 \end{pmatrix}$$

$$= tr_s \begin{pmatrix} T^{-1}[T,a] & 0 \\ 0 & -[T,a]T^{-1} \end{pmatrix} = 2\, tr(T^{-1}[T,a]).$$

Now

$$tr_s(F[F,ab]) = tr_s(F[F,a]b) + tr_s(Fa[F,b])$$

$$tr_s(F[F,ba]) = tr_s(Fb[F,a]) + tr_s(F[F,b]a)$$

$$tr_s(F[F,[a,b]]) = tr_s(-[F,b][F,a]) + tr_s([F,a][F,b])$$

$$= 2\,\mathrm{tr}_s\left([F,a][F,b]\right)$$

$$= 2\,\mathrm{tr}_s\left([F, a[F,b]]\right) = 0.$$

because $[F,[F,b]] = F(Fb-\cancel{bF}) + (\cancel{Fb}-bF)F = [F^2, b] = 0$

Claim also that this trace is stable under perturbations in F.

$$\delta\varphi(a) = \delta\,\mathrm{tr}\left(T^{-1}[a,T]\right)$$

$$= \mathrm{tr}\left(-T^{-1}\delta T\, T^{-1}[a,T]\right) + \mathrm{tr}\left(T^{-1}[a,\delta T]\right)$$

$$= \mathrm{tr}\left(-T^{-1}[a,T]T^{-1}\cdot\delta T\right) + \text{———}$$

$$= \mathrm{tr}\left([a,T^{-1}]\,\delta T\right) + \mathrm{tr}\left(T^{-1}[a,\delta T]\right)$$

$$= \mathrm{tr}\left([a,T^{-1}\delta T]\right) = 0$$

$$\cancel{\delta\,\mathrm{tr}_s(F[F,a]) = \overset{①}{\mathrm{tr}_s(\delta F[F,a])} + \overset{②}{\mathrm{tr}_s(F[\delta F,a])}}$$

$$\cancel{0 = \overset{③}{\mathrm{tr}_s([F,a]\,\delta F)} + \overset{④}{\mathrm{tr}_s(F[\delta F,a])}}$$

~~We need to use~~ $\cancel{\delta F\cdot F + F\cdot\delta F = 0}$.

~~whence~~ $\cancel{\mathrm{tr}_s\left(\overset{④}{[\delta F,a]F} + \overset{①}{\delta F[F,a]} + \overset{③}{[F,a]\delta F} + \overset{②}{F[\delta F,a]}\right) = 0}$

$\cancel{① + ④ = 0, \quad ② + ③ = 0 \qquad ③ = ④}$

$$\delta\,\mathrm{tr}_s\left(F[F,a]\right) = \mathrm{tr}_s\left(\delta F\,[F,a]\right) + \mathrm{tr}_s\left(F[\delta F,a]\right)$$

$$0 = \mathrm{tr}_s\left([F,a]\,\delta F\right) + \mathrm{tr}_s\left(F[\delta F,a]\right)$$

$$\delta\,\mathrm{tr}_s\left(F[F,a]\right) = -2\,\mathrm{tr}_s\left([F,a]\,\delta F\right) = -2\,\mathrm{tr}_s\left([F, a\,\delta F]\right) = 0$$

since $[F,\delta F] = F\delta F + \delta F.F = \delta(F^2) = \delta 1 = 0.$

March 8, 1986

Discussion of the transgression problem.

On the space of operators one has even diml cohomology classes; these are the character classes of the index virtual bundle. I can construct representatives of these classes as differential forms by means of a graph construction.

In the odd case we have ~~skew~~ adjoint operators L. To such an L assign its Cayley transform

$$g = \frac{1+L}{1-L} = (1-L)^{-1}(1+L)$$

which is unitary. Then

$$g^{-1}dg = (1+L)^{-1}(1-L)\left[(1-L)^{-1}dL(1-L)^{-1}(1+L)+(1-L)^{-1}dL\right]$$

$$= (1+L)^{-1}\,dL\left\{(1-L)^{-1}(1+L)+1\right\}$$

$$= (1+L)^{-1}\,2\,dL\,(1-L)^{-1}$$

and so we get closed forms

$$\operatorname{tr}(g^{-1}dg)^{2k+1} = \operatorname{tr}\left(\frac{2dL}{1-L^2}\right)^{2k+1}$$

When you want to transgress these forms to the group of gauge transformations, you write the form ω as $d\eta$, which is possible as the space of operators is contractible. Then you restrict η to a $\mathcal{G}$ orbit, actually pull-back by an orbit map: $\mathcal{G} \longrightarrow \mathcal{a}$. In order to get a left-invariant form on $\mathcal{G}$, it is necessary to choose an invariant η. To do this we want to contract over G-orbit equivariantly. There are two obvious directions $L \to tL$ with $t \downarrow 0$ and $t \uparrow \infty$. The former gives the classical limit in the case of Dirac operators and the latter is defined only for invertible L.

Actually it is probably worthwhile carrying out this process in the even case for the 2 form.

March 11, 1986:

Construction of cyclic cocycles.

Given an involution F one maps invertibles g to involutions by

$$g \longrightarrow gFg^{-1}.$$

On the space of involutions one has the 2-form

$$\operatorname{tr} F(dF)^2$$

which is closed and invariant. (closed because $(dF)^3$ anti-commutes with F and so

$$[F, F(dF)^3] = 2(dF)^3.)$$

as

$$\delta(gFg^{-1}) = \delta g\, F g^{-1} - gFg^{-1}\delta g\, g^{-1}$$

$$= g[\underbrace{g^{-1}\delta g}_{\theta}, F]g^{-1}$$

The pull-back of the two form to $\mathcal{G}$ is

$$\operatorname{tr} F[\theta, F][\theta, F] = \operatorname{tr} F[F,\theta]^2.$$

The corresponding cyclic cocycle is

$$\varphi(a,b) = \operatorname{tr}\big(F[F,a][F,b]\big) = 2\operatorname{tr}\big(a[F,b]\big)$$

provided the traces make sense.

Now we have seen that

$$\varphi(a,b) = 2\operatorname{tr}\big(a[F,b]\big)$$

is a cyclic 1-cocycle provided that $[F,b] \in \mathcal{L}^1$ always and that $\operatorname{tr}\big([F,a]\big) = 0$ always, but without requiring $F^2 = 1$. ($F^2 = 1 \implies [F, F[F,a]] = 2[F,a]$ and so $\operatorname{tr}[F,a] = 0$).

Let's compute an example. $\mathcal{H} = L^2(S^1)$ with orthonormal basis $\frac{e^{inx}}{\sqrt{2\pi}}$, F diagonal operator in this basis with

$$F\, e^{inx} = f(n) e^{inx}$$

Set $\qquad \varphi(a,b) = tr\left(a[F,b]\right), \qquad a, b \in C^\infty(S^1)$.

We evaluate for $\quad a = e^{imx}, \; b = e^{inx}$

$$[F, e^{inx}]\, e^{ipx} = [f(n+p) - f(p)]\, e^{i(n+p)x}$$

We only get a non-zero trace when $m+n=0$.

$$\varphi(e^{-imx}, e^{inx}) = \sum_p \left(f(n+p) - f(p)\right)$$

$$= \sum_p \left(f(n+p) - f(n-1+p)\right) + \dots + \left(f(1+p) - f(p)\right)$$

$$= n\left(f(+\infty) - f(-\infty)\right)$$

Note that $[F, e^{ix}] \in \mathcal{L}^1 \implies \sum_p' |f(1+p) - f(p)| < \infty$

$\implies f(+\infty) - f(-\infty)$ defined.

It's clear that

$$\varphi(a,b) = \text{const} \int_{S^1} a\, db$$

We might ask how $\varphi(a,b) = tr\left(a[F,b]\right)$ depends on F. If δF is of trace class, then

$$\delta\varphi(a,b) = tr\left(a.\delta F.b - ab.\delta F\right)$$
$$= -tr\left(\delta F.[a,b]\right)$$

which is a cyclic 1-coboundary.

Now suppose we are given ~~the Dirac operator on an odd dim'l manifold~~ an unbounded self adjoint operator D and we want to associate cyclic cocycles to it.