April 7 – 22, 1984

698 – 749

April 7, 1984

Review the calculations: Let ω^μ be a frame in T^*, X_μ the dual frame in T. In practice one uses either ω^μ = an orthonormal frame or $\omega^\mu = dx^\mu$. Put

$$d\omega^\alpha = \frac{1}{2}\hat{\Gamma}^\alpha_{\mu\nu}\,\omega^\mu\omega^\nu$$

Then
$$i(X_\nu)\,i(X_\mu)\,d\omega^\alpha = \hat{\Gamma}^\alpha_{\mu\nu}$$
$$\|$$
$$X_\mu\, i(X_\nu)\omega^\alpha - X_\nu\, i(X_\mu)\omega^\alpha - i\,[X_\mu, X_\nu]\,\omega^\alpha$$

so
$$[X_\mu, X_\nu] = -\hat{\Gamma}^\alpha_{\mu\nu}\,X_\alpha$$

Suppose given a metric:
$$(\omega^\mu, \omega^\nu) = g^{\mu\nu} \quad \left(\text{or} \quad |\xi_\mu\omega^\mu|^2 = g^{\mu\nu}\xi_\mu\xi_\nu\right)$$

Then we can form the Clifford algebra $C(T^*)$ with respect to this metric. Let γ^μ denote the element of $C(T^*)$ corresponding to ω^μ, so that
$$[\gamma^\mu, \gamma^\nu]_+ = 2g^{\mu\nu}$$

~~The Clifford algebra acts in $\Lambda(T^*)$ by assigning~~

Better: The Clifford algebra is equipped with a canonical map $c: T^* \longrightarrow C(T^*)$ satisfying $c(\xi)^2 = |\xi|^2$. If $\gamma^\mu = c(\omega^\mu)$, then
$$c(\xi) = c(\xi_\mu\omega^\mu) = \gamma^\mu\xi_\mu$$
and
$$c(\xi) = |\xi|^2 \iff [\gamma^\mu, \gamma^\mu]_+ = 2g^{\mu\nu}.$$

Let E be a $C(T^*)$-module, i.e. a vector bundle on which the algebra bundle acts. For example we can take $E = \Lambda T^*$ with $c(\xi)$ the operator

$$c(\xi) = i(\xi) + e(\xi)$$

where $e(\xi)$ is [redacted] exterior multiplication by ξ and $i(\xi)$ is interior multiplication by the linear functional $(\xi, ?)$ on T^*. As

$$(\xi, \eta) = g^{\mu\nu} \xi_\mu \eta_\nu$$

we see $(\xi, ?)$ is pairing with the vector field $g^{\mu\nu}\xi_\mu X_\nu$. [redacted]

Thus we have

$$c(\xi) = \xi_\mu \left(i(g^{\mu\nu} X_\nu) + e(\omega^\mu) \right)$$

or

$$\gamma^\mu = [redacted]\ g^{\mu\nu} i(X_\nu) + e(\omega^\mu)$$

as an operator on ΛT^*.

By letting $C(T^*)$ act on $1 \in \Lambda^0 T^*$ we get an isomorphism:

$$C(T^*) \xrightarrow{\sim} \Lambda T^*$$

$$\alpha \longmapsto \alpha \cdot 1$$

~~Next let D be a connection on T^*: D is an operator~~ $\cancel{\Gamma(T^*) \xrightarrow{D} \Gamma(T^* \otimes T^*)}$

~~satisfying~~ $\cancel{D(fs) = df\, s + f Ds}$. ~~We can write~~

$\cancel{D = \omega^\mu D_\mu}$, ~~where~~ $\cancel{D_\mu = i(X_\mu) D}$. ~~We put~~

$$\cancel{D_\mu \omega^\nu = \Gamma^\nu_{\mu\lambda} \omega^\lambda}$$

Better: Let's consider a <u>connection</u>

$$D : \Gamma(E) \longrightarrow \Gamma(T^* \otimes E)$$

$$D = \omega^\mu D_\mu \qquad D_\mu = i(X_\mu) D$$

D satisfies $D(fs) = df \cdot s + f\,Ds$ and can then be extended to $\Gamma(\Lambda T^* \otimes E)$. The curvature is $F = D^2$ which can be viewed as a 2-form with $\mathrm{End}\,E$-values.

$$
\begin{aligned}
D^2 s &= D(\omega^\mu D_\mu s) = d\omega^\mu D_\mu s - \omega^\mu D D_\mu s \\
&= (d\omega^\alpha D_\alpha - \omega^\mu \omega^\nu D_\nu D_\mu) s \\
&= \tfrac{1}{2}\omega^\mu \omega^\nu \left([D_\mu, D_\nu] + \hat{\Gamma}^\alpha_{\mu\nu} D_\alpha \right)
\end{aligned}
$$

If we write

$$F = \tfrac{1}{2}\omega^\mu\omega^\nu F_{\mu\nu} \qquad F_{\mu\nu} \in \Gamma(\mathrm{End}\,E)$$

then we have

$$F_{\mu\nu} = [D_\mu, D_\nu] + \hat{\Gamma}^\alpha_{\mu\nu} D_\alpha \qquad \text{or}$$

$$i(X_\nu) i(X_\mu) F = [D_{X_\mu}, D_{X_\nu}] - D_{[X_\mu, X_\nu]}$$

Next consider the case of a connection D on T^*. In terms of the frame ω^μ we can write

$$D_\mu \omega^\nu = \Gamma^\nu_{\mu\lambda} \omega^\lambda$$

The corresponding connection on T can be found as follows

$$0 = D_\mu (X_\nu, \omega^\alpha) = (D_\mu X_\nu, \omega^\alpha) + \underbrace{(X_\nu, D_\mu \omega^\alpha)}_{\Gamma^\alpha_{\mu\nu} = (X_\nu,\, \Gamma^\alpha_{\mu\lambda}\omega^\lambda)}$$

$$\therefore \quad D_\mu X_\nu = -\Gamma^\alpha_{\mu\nu} X_\alpha$$

The torsion of the connection is

$$T(X,Y) = D_X(Y) - D_Y(X) - [X,Y]$$

$$-T(X_\mu, X_\nu) = +\left(\Gamma^\alpha_{\mu\nu} - \Gamma^\alpha_{\nu\mu} - \hat{\Gamma}^\alpha_{\mu\nu}\right) X_\alpha$$

~~Now suppose we have a $C(T^*)$-mod~~

(Two things one might add: What it means for D to preserve the metric and what the curvature is in this index notation.)

Now suppose D is a connection on T^* preserving the metric, whence it extends to $C(T^*)$. Let E be a $C(T^*)$-module, and suppose E is equipped with a connection D compatible with Clifford multiplication and the connection on T^*, i.e.

$$[D_\mu, c(\xi)] = c(D_\mu \xi)$$

or

$$[D_\mu, \gamma^\alpha] = \Gamma^\alpha_{\mu\nu}\gamma^\nu$$

Consider the Dirac operator $\not{D}$ which is the composition

$$\Gamma(E) \xrightarrow{D} \Gamma(T^* \otimes E) \xrightarrow{c} \Gamma(E)$$

i.e.

$$\not{D} = \gamma^\mu D_\mu .$$

Covariant Laplacean is the composition

$$\Gamma(E) \xrightarrow{D_E} \Gamma(T^* \otimes E) \xrightarrow{D_{T^*\otimes E}} \Gamma(T^* \otimes T^* \otimes E) \rightarrow \Gamma(E)$$

where the last arrow is contraction with the metric.

$$\cancel{D_{T^*\otimes E} D_E = D_{T^*\otimes E}\,\omega^\nu D^E_\nu = D^{T^*}\omega^\nu D^E_\nu + \omega^\nu D^E D^E_\nu}$$

$$\cancel{= \omega^\mu \Gamma^\nu_{\mu\alpha}\,\omega^\lambda D_\nu + \omega^\nu \omega^\mu}$$

$$\omega^\mu\otimes\left[D_\mu\left(\omega^\nu\otimes D_\nu e\right)\right] = \omega^\mu\otimes\{D_\mu\omega^\alpha\otimes D_\alpha e + \omega^\nu\otimes D_\mu D_\nu e\}$$
$$= \omega^\mu\otimes\{\Gamma^\alpha_{\mu\nu}\omega^\nu\otimes D_\alpha e + \omega^\nu\otimes D_\mu D_\nu e\}$$
$$= \omega^\mu\otimes\omega^\nu\otimes\left(D_\mu D_\nu + \Gamma^\alpha_{\mu\nu}D_\alpha\right)e$$

So the <u>covariant Laplacean</u> $D\cdot D$ is
$$D\cdot D = g^{\mu\nu}\left(D_\mu D_\nu + \Gamma^\alpha_{\mu\nu}D_\alpha\right)$$

Now we compute $\not{D}^2$.

$$\not{D}^2 = \gamma^\mu D_\mu \gamma^\nu D_\nu = \gamma^\mu\gamma^\nu D_\mu D_\nu + \gamma^\mu[D_\mu,\gamma^\alpha]D_\alpha$$
$$= \gamma^\mu\gamma^\nu\left(D_\mu D_\nu + \Gamma^\alpha_{\mu\nu}D_\alpha\right)$$
$$= \tfrac{1}{2}\left([\gamma^\mu,\gamma^\nu]_+ + [\gamma^\mu,\gamma^\nu]_-\right)\left(D_\mu D_\nu + \Gamma^\alpha_{\mu\nu}D_\alpha\right)$$
$$= g^{\mu\nu}\left(D_\mu D_\nu + \Gamma^\alpha_{\mu\nu}D_\alpha\right) + \tfrac{1}{2}\left(\tfrac{1}{2}[\gamma^\mu,\gamma^\nu]_-\right)\left([D_\mu,D_\nu] + (\Gamma^\alpha_{\mu\nu}-\Gamma^\alpha_{\nu\mu})D_\alpha\right)$$
$$= \underbrace{g^{\mu\nu}\left(D_\mu D_\nu + \Gamma^\alpha_{\mu\nu}D_\alpha\right)}_{D\cdot D} + \tfrac{1}{2}\left(\tfrac{1}{2}[\gamma^\mu,\gamma^\nu]_-\right)\Big\{F_{\mu\nu} + \underbrace{(\Gamma^\alpha_{\mu\nu}-\Gamma^\alpha_{\nu\mu}-\hat{\Gamma}^\alpha_{\mu\nu})}_{\text{torsion}}D_\alpha\Big\}$$

$\therefore$ <u>Prop</u>: $\not{D}^2 = D\cdot D$ + 0th order $\iff$ torsion $= 0$.

In ~~[illegible]~~ this case
$$\not{D}^2 = D\cdot D + \tfrac{1}{2}\gamma^\mu\gamma^\nu F_{\mu\nu}$$
where F is the curvature.

Notice that wrt Getzler's filtration on the algebra of differential operators on the spinors, where γ^μ, D_μ have order 1, the torsion term $\frac{1}{2}\gamma^\mu\gamma^\nu T^\alpha_{\mu\nu}D_\alpha$ for $T\neq 0$ would have order 3. Thus for his analysis it appears essential that we use the Levi-Civita connection.

Let's go over Getzler's filtration. The bundle E on which $C(T^*)$ acts can be written as a tensor product

$$E = S \otimes V \qquad V = \mathrm{Hom}_{C(T^*)}(S, E)$$

where $C(T^*)$ acts on S, so that

$$c(\xi)(u \otimes v) = c(\xi)u \otimes v$$

The connection D on E is obtained from a connection on V and the induced connection on S from the connection on T^*. The curvature of D on E is then the sum of the two curvatures for S and V.

Let's describe the curvature for S. Let's work with an orthonormal frame ω^μ to simplify. The curvature of T^* will be denoted

$$R = \tfrac{1}{2}\omega^\mu\omega^\nu R_{\mu\nu} \qquad R_{\mu\nu} \in \Gamma\,\mathrm{End}(T^*)$$

Then $R_{\mu\nu}$ is a skew-symmetric endomorphism of T^*, and can be written

$$R_{\mu\nu,kl}\ \omega^k i(X_l)$$

Now given a skew symmetric matrix A_{kl} we want the corresponding quadratic element of $C(T^*)$ such that bracketing $c(T^*) \subset C(T^*)$ by this quadratic element gives the skew-symm. transf. A. This quadratic element is

$$\frac{1}{4} A_{kl}\,\gamma^k\gamma^l$$

In symbols, we have the embedding

$$\mathrm{Lie}\, SO(n) \hookrightarrow C_n$$

$$A_{kl} \longmapsto \tfrac{1}{4} A_{kl}\, \gamma^k \gamma^l$$

It follows [redacted] (pretty much by definition) that the action of Lie $SO(n)$ on S is defined [redacted] via this map. Thus the curvature of the connection on S induced from the given connection on D is

$$\frac{1}{8}\, \omega^\mu \omega^\nu R_{\mu\nu, kl}\, \gamma^k \gamma^l .$$

The Dirac operator squared on $E = S \otimes V$ is then

$$\not{D}^2 = \underbrace{\left(D_i^2 + \Gamma^\alpha_{ii} D_\alpha\right)}_{D\cdot D} + \frac{1}{8} R_{\mu\nu,kl}\, \gamma^\mu \gamma^\nu \gamma^k \gamma^l + \frac{1}{2} \gamma^\mu \gamma^\nu F^V_{\mu\nu}$$

~~[illegible]~~ Supposedly the R-term reduces to the scalar curvature.

Let's now consider just the case of spinors so that the last term is not there. I look at diffl operators on S. This algebra is generated by $\Gamma(C(T^*))$, which is generated by the γ^μ's over the functions $C^\infty(M)$, and by the operators D_μ. In fact any such diffl operator can be written uniquely

$$\sum_{I,\alpha} f_{I,\alpha}\, \gamma^I D^\alpha$$

Now introduce, following Getzler, the filtration which defines the order of $\gamma^I D^\alpha$ to be $|I| + |\alpha|$. What are the commutation relations among the generators?

$$[\gamma^\mu, f] = 0$$
$$[D_\mu, f] = X_\mu f$$
$$[\gamma^\mu, \gamma^\nu]_+ = \delta_{\mu\nu}$$
$$[D_\mu, \gamma^\nu] = \Gamma^\nu_{\mu\lambda}\gamma^\lambda$$
$$[D_\mu, D_\nu] = \underbrace{F_{\mu\nu}}_{} - \hat{\Gamma}^\alpha_{\mu\nu} D_\alpha$$

$$\tfrac{1}{4} R_{\mu\nu k l}\gamma^k\gamma^l$$

(Actually the last relation shows it might be more convenient to work with $\omega^\mu = dx^\mu$ for then $\hat{\Gamma}^\alpha_{\mu\nu} = 0$. But as with a universal enveloping algebra it is not necessary that the commutation relations allow one to move things to one side.)

If we take the associated graded algebra, then we get an algebra over the functions with the generators $\tilde{\gamma}^\mu$, $\tilde{D}_\mu$ satisfying the commutation relations

$$[\tilde{\gamma}^\mu, \tilde{\gamma}^\nu]_+ = 0 \quad , \quad [\tilde{D}_\mu, \tilde{\gamma}^\nu] = 0$$
$$[\tilde{D}_\mu, \tilde{D}_\nu] = \tfrac{1}{4} R_{\mu\nu k l}\tilde{\gamma}^k\tilde{\gamma}^l$$

And in this algebra we wish to compute the heat operator for

$$\tilde{\not{D}}^2 = \tilde{D}_\mu^2$$

But notice that this "algebra" comes from something simpler. Throw away the $\tilde{\gamma}^\mu$ and replace the elements

$$\tilde{F}_{\mu\nu} = \tfrac{1}{4} R_{\mu\nu k l}\tilde{\gamma}^k\tilde{\gamma}^l$$

by arbitrary nilpotent elements in a commutative ring. These do commute because the $\tilde{\gamma}^k$ generate an exterior algebra, hence degree 2 elements in this algebra commute.

So I learn that my idea that the limiting situation is indeed the covariant Laplacean for a particle in a constant magnetic field is essentially correct.

Let's go back to the Dirac operator for a constant magnetic field and try to completely work out the limiting form of the super heat kernel. We work in $\mathbb{R}^2 = \mathbb{C}$ with the line bundle having the curvature $\omega \, dz d\bar{z} = -2i\omega \, dx dy$. Then

$$D = dx \, D_x + dy \, D_y \qquad \text{with} \qquad [D_x, D_y] = -2i\omega$$

and the Laplacean is $D_x^2 + D_y^2$. So its essentially a harmonic oscillator $q^2 + p^2$, $[p,q] = [\frac{1}{i}D_y, \frac{1}{i}D_x] = [D_x, D_y] = \frac{2\omega}{i}$ so the spectrum is $(2n+1)h = (2n+1)2\omega$, $n \in \mathbb{Z}$. We have

$$D_x^2 + D_y^2 = (D_x - iD_y)(D_x + iD_y) - \underbrace{i[D_x, D_y]}_{2\omega}$$

$$[D_x + iD_y, -D_x + iD_y] = 2i \, 2\omega$$

so put $a = \frac{1}{2\sqrt{\omega}}(D_x + iD_y) = \frac{1}{\sqrt{\omega}} D_{\bar{z}}$ and then

$$-(D_x^2 + D_y^2) = 4\omega\left(a^* a + \tfrac{1}{2}\right)$$

We computed before that for $\omega = 1$

$$\langle z | e^{-t a^* a} | w \rangle = \frac{1}{\pi} \frac{1}{1 - e^{-t}} \, e^{-\frac{1}{2} \frac{1+e^{-t}}{1-e^{-t}} |z-w|^2} \, \underbrace{P(z,w)}_{\text{parallel transport from } w \text{ to } z}$$

$$\sim \frac{1}{\pi} \frac{1}{t} \, e^{-\frac{1}{t}|z-w|^2} \, P(z,w)$$

April 8, 1984

Notes from conversations in Witten's office with Atiyah. Witten claims there is a physical way to see the (or a) connection on the determinant line bundle over a surface.

One can start from the Lagrangian

$$(1) \qquad \mathcal{L} = \frac{1}{g^2}\int d^3x\, F_{\mu\nu}^2 + \int d^3x\, \bar{\psi}(i\not{D} + im)\psi .$$

When one does the fermion integration one gets

$$\det(i\not{D} + im)$$

which becomes e^{inW} in the large m limit where W is the Chern-Simons term.

Canonical quantization of the Lagrangian

$$(2) \qquad \mathcal{L} = \frac{1}{g^2}\int d^3x\, F_{\mu\nu}^2 + (?)n\, W$$

using the gauge $A_0 = 0$ gives

$$F_{0i}^a = \frac{\partial A_i^a}{\partial t} = -i\frac{\delta}{\delta A_i^a} + n\,\varepsilon_{ij}A_j^a .$$

which says that the ~~[illegible]~~ operator belonging to $\frac{\partial A_i^a}{\partial t}$ is a covariant derivative. (The above formula comes from the canonical quantization formula

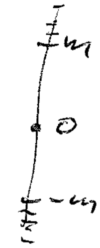
$$-i\frac{\delta}{\delta A_i^a} = \frac{\delta \mathcal{L}}{\delta\left(\frac{\partial A_i^a}{\partial t}\right)} = \frac{\partial A_i^a}{\partial t} + n\,\varepsilon_{ij}A_j^a .)$$

The conclusion is that the presence of the fermion term in (1) for large m, or the Chern-Simons term in (2) is to change the Hamiltonian from $-\Delta + \|F\|^2$ to $-\Delta_C + \|F\|^2$ where C is a line bundle with connection.

From the Hamiltonian viewpoint one has

$$(1)' \qquad H = \frac{1}{g^2}\int d^2x \left(-i\frac{\delta}{\delta A^a_{(x)}}\right)^2 + \|F\|^2 + \int_{d^2x} \bar{\psi}(i\not{D}+\beta m)\psi$$

Here $i\not{D} + \beta m = \begin{pmatrix} m & D \\ \bar{D} & -m \end{pmatrix}$ has spectrum



The above is a second quantized Hamiltonian; the $\bar{\psi}, \psi$ are operators on a Fock space, but for large m only the ground line remains. Thus in the large m limit we get

$$H = -\Delta_C + \|F\|^2$$

where Δ_C is the Laplacian on this line bundle.

Witten hadn't yet made the connection on the line bundle, or the line bundle itself explicit. His idea was to study the spectrum of

$$H = -\Delta + \lambda M$$

where M is a spatially varying mass matrix with a unique ground ~~state~~. The idea is that as $\lambda \to \infty$ the spectrum of H would converge to that of $-\Delta_C$, where C is the line bundle of ground states.

Consider a Dirac operator, say the Dirac operator on a Riemannian $Spin^c$ manifold with coeffs. in a bundle E with connection. According to Connes there are explicit cyclic cocycles on the ring $\Gamma(\mathrm{End}\,E)$ we can attach to this Dirac operator. Let's consider the simpler cocycles over $\Gamma(\mathcal{O}_M)$, which should be easier to understand. There is a whole sequence of these cocycles, but they should all come from an even differential forms. Thus I feel that the cyclic cocycles are derived gadgets - there should be something more basic which explains them. There are two possibilities, I feel, for the more basic gadget I seek - the limiting heat kernel, or the physicists' Lagrangian.

~~[illegible]~~

There is a problem with trying to take the classical limit of the super heat kernel $e^{-tH-\theta Q}$ where $Q^2=-H$, $Q=\emptyset$. One idea I had was that the answer should be something like

$$e^{-tp^2}\,e^{+tF-\theta D} \qquad\qquad F = D^2$$

Over $\mathbb{R}^n$ we have an algebra consisting of elements x^μ, γ^μ, D_μ satisfying $[x^\mu,x^\nu]=[x^\mu,\gamma^\nu]=0$, $[D_\mu,x^\nu]=\delta_\mu^\nu$, $[\gamma^\mu,\gamma^\nu]_+ = 2\delta^{\mu\nu}$, $[D_\mu,\gamma^\nu]=0$, $[D_\mu,D_\nu]=F_{\mu\nu}(x)$. One has

$$Q = h\gamma^\mu D_\mu$$

$$Q^2 = h^2 D_\mu^2 + \frac{h^2}{2}\gamma^\mu\gamma^\nu F_{\mu\nu}$$

In the limit I want

$$Q = \omega^\mu D_\mu = D$$

$$Q^2 = -p_\mu^2 + \frac{1}{2}\omega^\mu\omega^\nu F_{\mu\nu}$$

which is algebraically impossible.

April 9, 1984

Yesterday I became discouraged with the idea of a classical limit for the super heat kernel. The problem is

$$Q = h\gamma^\mu D_\mu \longrightarrow \omega^\mu D_\mu$$

$$Q^2 = h^2 D_\mu^2 + \frac{h^2}{2}\gamma\gamma F \longrightarrow -p_\mu^2 + \frac{1}{2}\omega^\mu\omega^\nu F_{\mu\nu}$$ see 756

is algebraically impossible. The problem is that Q is of order 2 in Getzler's sense, and Q^2 is of order 2. So there is no contradiction, only that you can't hope to build Q into an algebra over $\mathbb{C}[h]$ with a specialization at $h=0$ such that the relation $\overline{Q^2} = \bar{Q}^2$ holds.

Therefore I want to go back over the Lagrangian and path integral business of the physicists. After all the Lagrangian is a classical gadget supposedly.

Let's start with the path integral for the kernel of e^{-tH} where $H = h^2 D_\mu^2$, $D_\mu = \partial_\mu + A_\mu$, where we are over $\mathbb{R}^n$ and the A_μ are matrices. Then I believe that one has

$$\langle x | e^{-\beta H} | y \rangle = \int_{\substack{x(0)=y \\ x(\beta)=x}} \mathcal{D}x(t)\, e^{-\int_0^\beta \frac{\dot{x}^2}{4h^2}dt}\ \underbrace{T\{e^{-\int_0^\beta A_\mu \dot{x}^\mu dt}\}}_{\substack{\text{parallel transport} \\ \text{along } x(t)}}$$

The physicists teach us that we can write parallel transport using a fermion path integral. This goes as follows. Suppose we have a connection over $0 \le t \le \beta$, say $\partial_t + A$. Then one writes the

action
$$S = \int_0^\beta \bar{\eta}(\partial_t + A)\eta \, dt$$

which is to be interpreted as a skew-symmetric form on the space of vector functions $(\eta(t), \bar{\eta}(t))$. Then the path integral of e^{-S}, as a fermion path integral, suitably interpreted, is a Gaussian element of the exterior algebra of the vector space $V(\beta)^* \oplus V(0)$, which represents the parallel translation map

$$\Lambda V(0) \xrightarrow{\sim} \Lambda V(\beta)$$

So we have a path integral expression then

$$\langle x | e^{-tH} | y \rangle = \int \mathcal{D}x(t)\, \mathcal{D}\bar{\eta}(t)\, \mathcal{D}\eta(t)\; e^{-\left(\int \left[\frac{\dot{x}^2}{4h} + \bar{\eta}(\partial_t + A)\eta\right] dt\right)}$$

Next we wish to consider the Dirac operator. The Lagrangian which gives the heat kernel for the Dirac operator is (over $\mathbb{R}^n$) supposed to be

$$\frac{1}{h^2}\left(\frac{\dot{x}^2}{4} + \frac{1}{4}\psi\dot{\psi}\right) + \bar{\eta}\left(\partial_t + A_\mu \dot{x}^\mu - \frac{1}{2}\psi^\mu\psi^\nu F_{\mu\nu}\right)\eta$$

A curious point is that this appears quartic in the fermion variables $\psi, \eta, \bar{\eta}$. Let's go over where this is supposed to come from.

The Dirac operator acts on $S \otimes E$, and as pointed out by the FW paper one can extend it naturally to $S \otimes \Lambda E$, which is the natural fermion Fock space associated to the fermion variables $\psi, \eta, \bar{\eta}$. Thus we really do get a quartic interaction

At this point I want to know how to do the path integral over $\psi(t)$. For fixed $x(t), \eta(t), \bar{\eta}(t)$

this path integral will be Gaussian:

$$\int \mathcal{D}\psi \; e^{-\int_0^\beta \left(\frac{1}{4h^2}\psi\dot\psi - \frac{1}{2}\psi^\mu\psi^\nu(\bar\eta F_{\mu\nu}\eta)\right)dt}$$

Notice that $\bar\eta F_{\mu\nu}\eta$ is a skew-symmetric matrix with values in a commutative ring, namely, the even part of the exterior algebra of the space of $\eta(t), \bar\eta(t)$. (?)

In any case we show first understand how to do a path integral

$$\int \mathcal{D}\psi(t)\; e^{-\int_0^\beta \left(\frac{1}{2}\psi\dot\psi - \frac{1}{2}\psi^\mu\psi^\nu F_{\mu\nu}(t)\right)dt}$$

where $F_{\mu\nu}(t)$ is a skew-symmetric matrix of functions of t with values in a commutative algebra.

Basically, this path integral is equivalent to the effect on the spinors of the element

$$T\left\{e^{\int_0^\beta F_{\mu\nu}(t)\,dt}\right\}$$

of the orthogonal group. Why? A Gaussian integral can be evaluated at the critical point

$$0 = \delta\int_0^\beta \left(\frac{1}{2}\psi\dot\psi - \frac{1}{2}F_{\mu\nu}\psi^\mu\psi^\nu\right)dt$$

$$= \int_0^\beta \left[\frac{1}{2}(\delta\psi\,\dot\psi + \psi\,\delta\dot\psi) - F_{\mu\nu}(\delta\psi^\mu\psi^\nu + \psi^\mu\delta\psi^\nu)\right]dt$$

$$= \frac{1}{2}\left[\psi\,\delta\psi\right]_0^\beta + \int_0^\beta \delta\psi^\mu\left\{\dot\psi^\mu - F_{\mu\nu}\psi^\nu\right\}dt$$

So the variational equations are

$$\dot\psi^\mu = F_{\mu\nu}\psi^\nu$$

and the solution is

$$\psi(t) = T\left\{ e^{\int_0^t F(t)\,dt} \right\} \psi(0)$$

A perplexing point: Evaluate action at the critical point

$$S = \frac{1}{2}\int_0^\beta (\psi\dot{\psi} - F\psi\psi)\,dt$$

$$= \frac{1}{2}\int_0^\beta (\psi^\mu F_{\mu\nu}\psi^\nu - F_{\mu\nu}\psi^\mu\psi^\nu)\,dt = 0$$

I don't think it helps to do the ψ integral first, because the result will be a time-ordered exponential

$$T\left\{ e^{\int_0^\beta (\bar{\eta} F_{\mu\nu} \eta)} \right\}$$

where the non-commutativity of the orthogonal group will be involved. This won't be a Gaussian expression in $\bar{\eta}, \eta$, just like if I do the $\eta, \bar{\eta}$ integral, I get something which isn't Gaussian in ψ.

It seems to be a good idea to correlate my two approaches to fermion path integrals.

The old approach is time independent and based on the formula

$$\frac{\int \mathcal{D}\bar{\psi}\,\mathcal{D}\psi\; e^{-\bar{\psi}A\psi + \bar{J}\psi + \bar{\psi}J}}{\int \mathcal{D}\bar{\psi}\,\mathcal{D}\psi\; e^{-\bar{\psi}A\psi}} = e^{\bar{J}A^{-1}J}$$

The new approach somehow interprets the path integral

$$\int \mathcal{D}\bar{\psi}(t)\,\mathcal{D}\psi(t)\; e^{-\int_0^\beta \bar{\psi}(\partial_t + B)\psi\, dt}$$

as the parallel transport $T\left\{e^{-\int_0^\beta B\,dt}\right\}$ with respect to the connection $\partial_t + B$.

One might call the first Hamiltonian and the second Lagrangian.

Let's try relating the two using the fundamental solution for $\partial_t + B$ which is forward, e.g.

$$\theta(t)\, e^{-tB}$$

is B is constant.

Next let's look at the orthogonal variants of the above formulas. Suppose we consider a Gaussian integral with exponent

$$\frac{1}{2}\psi^\mu \omega_{\mu\nu}\psi^\nu + J_\mu \psi^\mu$$

$$= \frac{1}{2}\psi\omega\psi + \frac{1}{2}J\psi - \frac{1}{2}\psi J$$

$$= \frac{1}{2}\left[\psi\omega\psi + J\omega^{-1}\omega\psi - \psi\omega\,\omega^{-1}J - J\omega^{-1}J\right] + \frac{1}{2}J\omega^{-1}J$$

$$= \frac{1}{2}\left(\psi + J\omega^{-1}\right)\omega\left(\psi - \omega^{-1}J\right) + \frac{1}{2}J\omega^{-1}J. \quad \therefore$$

$$\boxed{\frac{\int \mathcal{D}\psi\; e^{\frac{1}{2}\psi\omega\psi + J\psi}}{\int \mathcal{D}\psi\; e^{\frac{1}{2}\psi\omega\psi}} = e^{\frac{1}{2}J\omega^{-1}J}}$$

Another idea is that the Fourier transform of a degenerate Gaussian should involve some kind of δ function. For example the F.T. of 1 is $\delta(0)$. So what might this mean in the context of the formula

$$\frac{\int \mathcal{D}\psi \, e^{\frac{1}{2}\psi\omega\psi + J\psi}}{\int \mathcal{D}\psi \, e^{\frac{1}{2}\psi\omega\psi}} = e^{\frac{1}{2}J\omega^{-1}J}$$

Think of ω as being diagonal with diagonal entries $\omega_1, \dots, \omega_n$. Initially we suppose the ω_i are invertible, but then we let $\omega_1, \dots, \omega_k$ go to zero. Let's rewrite the formula as

$$\int \mathcal{D}\psi \, e^{\frac{1}{2}\psi\omega\psi + J\psi} = e^{\frac{1}{2}J\omega^{-1}J} \underbrace{\int \mathcal{D}\psi \, e^{\frac{1}{2}\psi\omega\psi}}_{\omega_1 \cdots \omega_n}$$

Slight difficulty: Diagonal means 2×2 blocks $\begin{pmatrix} 0 & \omega_i \\ -\omega_i & 0 \end{pmatrix}$ and then $\frac{1}{2}\psi\omega\psi = \frac{1}{2}(\psi_1, \psi_2)\begin{pmatrix} 0 & \omega \\ -\omega & 0 \end{pmatrix}\begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} = \omega\psi_1\psi_2$. Now in 2 dims.

$$\int \mathcal{D}\psi_1 \mathcal{D}\psi_2 \, e^{\omega\psi_1\psi_2 + J_1\psi_1 + J_2\psi_2} = \omega - J_1 J_2 = \omega(1 - \omega^{-1} J_1 J_2)$$

So it is clear that what one is getting from the F.T. is the δ-function $(-J_1 J_2)$ for each $\omega_i = 0$, and the inverse Gaussian $e^{\frac{1}{2}J\omega^{-1}J}$ in the directions where ω is invertible.

The lesson seems to be this: We start with the [redacted] skew-symmetric forms on [redacted] V which we write $\frac{1}{2}\psi^\mu \omega_{\mu\nu}\psi^\nu$, where ψ^μ is a basis for the dual space V^*. The kernel K of this form is [redacted] a [redacted] subspace of V. The F.T.

$$\int \mathcal{D}\psi \; e^{\frac{1}{2}\psi\omega\psi + J\psi}$$

is an element of ΛV which will be of the form

$$\left(\text{gen of } \Lambda^{max}(K)\right) \cdot \text{ inverse Gaussian in } \Lambda(V/K)$$

Thus one gets a sort of interpretation of

$$\int \mathcal{D}\psi \; e^{\frac{1}{2}\psi\omega\psi}$$

as the line in the exterior algebra belonging to the kernel of [redacted] ω.

We have seen that the parallel translation term belonging to a path is complicated for the square of the Dirac operator. The action is

$$S = \int \left[\frac{\dot{x}^2}{4h^2} + \frac{\psi\dot{\psi}}{4h^2} + \bar{\eta}\left(\partial_t + \dot{x}^\mu A_\mu - \tfrac{1}{2}\psi^\mu\psi^\nu F_{\mu\nu}\right)\eta \right] dt$$

and the square of $\not{D}$ is

$$\not{D}^2 = h^2 D_\mu^2 + \frac{h^2}{2}\gamma^\mu\gamma^\nu F_{\mu\nu}$$

Notice that $h^2 D_\mu^2$ results from the part

$$\frac{\dot{x}^2}{4h^2} + \bar{\eta}\left(\partial_t + \dot{x}^\mu A_\mu\right)\eta$$

of the Lagrangian. I think that the term $\frac{\psi\dot{\psi}}{4h^2}$ is needed to quantize ψ^μ into $h\gamma^\mu$.

To check this, suppose $\dim E = 1$ whence the η integral will give an ordinary exponential

$$e^{-\int(\dot{x}^\mu A_\mu - \frac{1}{2}\psi^\mu\psi^\nu F_{\mu\nu})dt}$$

(we've ~~cut~~ to the $\bar{\eta}\eta = 1$ sector). This means for the ψ integration we have a quadratic Lagrangian:

$$\frac{\psi\dot{\psi}}{4h^2} - \frac{1}{2}\psi^\mu\psi^\nu F_{\mu\nu}$$

Equation of the critical point is

$$\frac{\dot{\psi}^\mu}{2h^2} - F_{\mu\nu}\psi^\nu = 0 \qquad\qquad \dot{\psi}^\mu = 2h^2 F_{\mu\nu}\psi^\nu$$

and so the parallel translation operator on spinors obtained by doing the ψ integration is going to be obtained by lifting the path $t \mapsto 2h^2 F_{\mu\nu}(x(t))$ in the orthogonal Lie algebra to a path in the spinor group. But now recall the Lie algebra embedding

$$\begin{aligned} \text{Lie } SO(n) &\hookrightarrow C_n \\ A_{\mu\nu} &\longmapsto \tfrac{1}{4} A_{\mu\nu}\gamma^\mu\gamma^\nu \end{aligned}$$

Therefore we see that the ψ integral gives the parallel transport operator

$$T\left\{ e^{\int_0^\beta \frac{1}{2} h^2 \gamma^\mu\gamma^\nu F_{\mu\nu}} \right\}$$

which checks nicely.

April 10, 1984

Let A be an algebra over k, and let $\tilde{A}$ be a graded subalgebra of $A \otimes k[h]$ which is an algebra over $k[h]$. Then

$$\tilde{A} = \bigoplus_{n \geq 0} A_n h^n$$

where A_n is an increasing algebra filtration of A:

$$A_p \cdot A_q \subset A_{p+q} \quad , \quad A_{n-1} \subset A_n .$$

We then can identify the $h \neq 0$ and $h = 0$ "specializations" as follows:

$$\tilde{A}/h\tilde{A} = gr(A) = \bigoplus_{n \geq 0} A_n/A_{n-1}$$

$$\tilde{A} [redacted] [h^{-1}] \simeq A \otimes k[h, h^{-1}]$$

The first example will be to take $A = k[x, \partial_x]$ of differential operators and let $\tilde{A}$ be the subalgebra of $A \otimes k[h]$ generated by the functions of x, [redacted] the element h, and the elements $h\partial_\mu$. Since the generators of $\tilde{A}$ are homogeneous, it follows $\tilde{A}$ is a graded subalg. So we have an increasing alg. filtration $\{A_n\}$ of A. It's clear that A_n contains all elements of the form $f(x)\partial^\alpha$ with $|\alpha| \leq n$. Let $A_n' =$ span of $f(x)\partial^\alpha$ with $|\alpha| \leq n$. Then to see $A_n' = A_n$ by induction I use

$$\begin{tikzcd}
0 \arrow[r] & A_{n-1}' \arrow[r] \arrow[d, "\wr", "\text{ind.}"'] & A_n' \arrow[r] \arrow[d] & A_n'/A_{n-1}' \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & A_{n-1} \arrow[r] & A_n \arrow[r] & A_n/A_{n-1} \arrow[r] & 0
\end{tikzcd}$$

so all one needs is $gr' \twoheadrightarrow gr$. But $gr(A) = \tilde{A}/h\tilde{A}$

is generated by the images of the given generators for $\tilde{A}$. So we know that $gr(A)$ is generated by $k[x]$, $\overline{h\partial_\mu}$. But $[h\partial_\mu, f(x)] = h(\partial_\mu f)(x)$ so $[\overline{h\partial_\mu}, f(x)] = \overline{h\,\partial_\mu f} = 0$, so we see that A_n/A_{n-1} is a quotient of A_n'. This shows that $A_n' = A_n$ by induction which means that the A_n' is an increasing algebra filtration.

I went thru the above just to see that I could get the basic algebra filtration formula

$$A_p \cdot A_q \subset A_{p+q}$$

easily.

Now I want to apply this to Getzler's case, namely where A will be the algebra of differential operators on a Clifford module $E = S \otimes V$. This time we let $\tilde{A}$ be the subalgebra of $A \otimes k[h]$ generated by h, $\Gamma(\mathrm{End}\,V)$, the elements $h\gamma^\mu$, hD_μ. I could say more intrinsically $hc(\omega)$, $\omega \in \Gamma(T^*)$, and hD_X where $X \in \Gamma(T)$. These are homogeneous generators, and hence $\tilde{A}$ corresponds to an algebra filtration A_n. The associated graded algebra

$$\tilde{A}/h\tilde{A} = gr(A)$$

is generated by $\Gamma(\mathrm{End}\,V)$, $\overline{h\gamma^\mu}$, $\overline{hD_\mu}$. We have the following relations.

$$[h\gamma^\mu, h\gamma^\nu]_+ = h^2\, 2g^{\mu\nu} \Longrightarrow [\overline{h\gamma^\mu}, \overline{h\gamma^\nu}] = 0$$

$$[hD_\mu, h\gamma^\alpha] = h\Gamma^\alpha_{\mu\nu} h\gamma^\nu \Longrightarrow [\overline{hD_\mu}, \overline{h\gamma^\alpha}] = 0$$

$$\left.\begin{aligned} [hD_\mu, hD_\nu] &= h^2\{F^E_{\mu\nu} - \Gamma^\alpha_{\mu\nu} D_\alpha\} \\ F^E_{\mu\nu} &= R_{\mu\nu k l}\tfrac{1}{4}\gamma^k\gamma^l + F^V_{\mu\nu} \end{aligned}\right\} \Longrightarrow \begin{aligned}[\overline{hD_\mu}, \overline{hD_\nu}] \\ = \tfrac{1}{4} R_{\mu\nu k l}\, \overline{h\gamma^k}\,\overline{h\gamma^l}\end{aligned}$$

$\alpha \in \Gamma(\mathrm{End}\,V)$ $\quad [hD_\mu, \alpha] = h[D_\mu, \alpha] \Longrightarrow [\overline{hD_\mu}, \bar{\alpha}] = 0.$

Question: Is there any relation between this kind of $h \to 0$ asymptotic analysis, and the poles of Γ and ζ functions?

Notice that in the flat case with the operator
$$H = -h^2 \Delta + V(x)$$
that this is not homogeneous in $\tilde{A}$. Here $A = k[x, \partial_x]$ and $\tilde{A}$ is generated by $k[x]$ and $p = \frac{h}{i}\partial_x$ inside $A \otimes k[h]$. Thus
$$H = p^2 + V(x)$$
is not homogeneous yet it has a nice specialization at $h=0$ (in fact any element of $\tilde{A}$ does). In this specialization x, p commute so
$$e^{-tH} = e^{-tp^2} e^{-tV} \qquad \text{at } h = 0.$$

Similarly when we consider $\not{D} = h\gamma^\mu D_\mu + L$ and
$$-H = \not{D}^2 = h^2 D_\mu^2 + \frac{h^2}{2}\gamma\gamma F + h\,\gamma[D, L] + L^2$$
and we work with $A =$ **diffl.** ops. on $S \otimes E$, $\tilde{A} \subset A \otimes k[h]$ spanned by $k[x]$, $\operatorname{End} E$, $h\gamma^\mu$, hD_μ, then H has terms of degree $0, 1, 2$ and specializes nicely at $h=0$ to
$$-p_\mu^2 + (dx^\mu D_\mu + L)^2$$

April 11, 1984

I want to look again at the local index thm. for families of Dirac operator on a fixed Riem. manifold M. Let Y be a parameter manifold, let E be a vector bundle over $Y\times M$, and suppose given a connection on E. Let S be the spinors on M. Using the vertical part of the connection, i.e. the partial connection in the M-direction, we get a differential operator on $S\otimes E = pr_2^*(S)\otimes E$, which is a family of Dirac ops. on M parametrized by Y.

Let's give some formulas. The interest being local on Y we fix a point $y_0 \in Y$, and trivialize E horizontally: $E = pr_2^*(E_0)$, $E_0 = E|\{y_0\}\times M$. Then the family becomes a family of Dirac operators on E_0 over M depending on $y\in Y$. We can write it

$$\not{D}^y = \gamma^\mu D_\mu^y = \gamma^\mu D^0_\mu + \gamma^\mu A^y_\mu$$

where $dx^\mu A^y_\mu$ is a family of elements of $\Omega^1(M, \operatorname{End} E_0)$.

What is the index of this family? It is given by the Hilbert bundle $\mathcal{H}$ over Y with $\mathcal{H}_y = L^2(M, S\otimes E_y)$ with its natural $\mathbb{Z}_2$-grading, together with the Dirac operator family $\not{D}_y$ on $\mathcal{H}_y$. Let's denote this operator by $L: \mathcal{H} \circlearrowleft$; really Hilbert bundle endomorphism instead of operator.

To define the character of the index we need a connection in $\mathcal{H}$. This is furnished by the Y-direction part of the given connection on E. Again using the trivialization

$E = pr_2^* E_0$, the connection in the Y-direction can be described by

$$D' = dy^j(\partial_{y_j} + B_j)$$

where B_j^y is an Endomorphism of E_0 depending on y. The total connection on $E = pr_2^*(E_0)$ over $Y \times M$ is

$$D' + D'' = dy^j(\partial_{y_j} + B_j) + dx^\mu A_\mu + dx^\mu D^0_\mu$$

So D' can be interpreted as a connection on the Hilbert bundle $\mathcal{H}$. Together with L it gives a superconnection, and the character of the index is

$$tr_s\left(e^F\right) \qquad\qquad F = (D' + L)^2$$

Here F is viewed as a form on Y with values in $\operatorname{End}\mathcal{H}$ and the supertrace is taken over $\mathcal{H}$ to give a differential form on Y.

From the trivialization $E = pr_2^* E_0$, we get that $\mathcal{H}$ is trivial with fibre $\mathcal{H}_0 = L^2(M, S \otimes E_0)$. So our superconnection is an [redacted] operator on [redacted] $\Omega(Y) \otimes \mathcal{H}_0$, namely

$$d_y + dy^j B_j + \underbrace{\gamma^\mu(D^0_\mu + A_\mu)}_{\{\not{D}\, y\}}$$

[redacted]

The curvature of this superconnection is

$$F = (D' + L)^2 = (D')^2 + [D', L] + L^2$$

$$= \tfrac{1}{2} dy^j dy^k\left(\partial_j B_k - \partial_k B_j - [B_j, B_k]\right) + dy^j \gamma^\mu\left[\partial_j + B_j, D_\mu\right]$$

$$+ \; D_\mu^2 + \tfrac{1}{2}\gamma^\mu\gamma^\nu[D_\mu, D_\nu]$$

I want to evaluate $tr_s(e^F)$ at y_0. I can assume the trivialization $E = pr_2^* E_0$ chosen so that $B_j = 0$ at y_0. Then F becomes

$$F = D_\mu^2 + \frac{1}{2}\gamma^\mu\gamma^\nu[D_\mu, D_\nu] + \gamma^\mu(-dy^j)\partial_j A_\mu + \frac{1}{2}dy^j dy^k F'_{jk}$$

which has exactly the same shape as the square of a Dirac operator with potential:

$$(\gamma^\mu D_\mu + \tilde{L})^2 = D_\mu^2 + \frac{1}{2}\gamma^\mu\gamma^\nu[D_\mu, D_\nu] + \gamma^\mu[D_\mu, \tilde{L}] + \tilde{L}^2$$

The latter would be an operator on $S \otimes E$; the former is an operator on $S \otimes E \otimes \Lambda T^*_{y_0}$ commuting with right multiplication by ΛT^*_y.

Let's consider again calculating $tr_s(e^F)$ at y_0. $F = (D' + L)^2$ is the sum of three terms: L^2 = the square of the Dirac operator $(\gamma^\mu D_\mu)^2$, $[D', L]$ = the derivative of the Dirac operator $dy^j \gamma^\mu \partial_j A_\mu$, and finally the curvature $(D')^2$. Now we replace L by hL and let $h \downarrow 0$. We know that the DR class of $tr_s(e^F)$ doesn't change, so this limiting form is what we use as defn. of the character of the index bundle.

The analysis is therefore exactly the same as will enter into the local index thm. for $\gamma^\mu D_\mu + \tilde{L}$, and so gives

$$tr_s\left(e^{(D' + hL)^2}\right) \longrightarrow \left(\int_M tr(e^{D^2})\hat{A}(M)\right)_{y_0}$$

where D^2 is the curvature of the total connection on E.

Summary: We fix a Dirac operator $\not{D}^0$ and want to explain the character for the index of any family of Dirac operators containing $\not{D}^0$. What does the family $\{\not{D}^y\}$ give us infinitesimally? We need a connection in the Y-direction in order to define the character in the first place. This connection has at $y=0$ a curvature $\frac{1}{2} dy^j dy^k \Omega_{jk}$, where $\Omega_{jk} \in \Gamma(\operatorname{End} E^0)$. Relative to this connection we can consider the variation in the ~~Dirac operator~~ connection D^y as y varies near $y=0$. This gives at $y=0$ a 1-form on Y with values in $\Omega^1(M, \operatorname{End} E^0)$:

$$dy^j \partial_{y_j} D^y = dy^j dx^\mu \partial_{y_j} A_\mu$$

Finally we have the curvature $F^0 = (D^0)^2 \in \Omega^2(M, \operatorname{End} E^0)$. Thus we get the curvature of the total conn. over $Y \times M$

$$\Omega + dA + F^0 \in \Lambda^2 T_0^* \otimes \Omega^0 \oplus \Lambda^1 T_0^* \otimes \Omega^1 \oplus \Lambda^0 T_0^* \otimes \Omega^0$$

where Ω^i stands for $\Omega^i(M, \operatorname{End} E^0)$. The character of the index is the element of ΛT_0^* given by

$$\int_M \operatorname{tr}_E\left(e^{\Omega + dA + F^0}\right).$$

It seems that this might give the complete index information about $\not{D}^0$ that I have been looking for. The first point is that it contains the form on M

$$\operatorname{tr}_E\left(e^{F^0}\right)$$

which produces the index.

Notice that if we know the matrix forms e^{tF^0} then we can derive $\operatorname{tr}\left(e^{F^0 + dA + \Omega}\right)$ by the perturbation expansion.

April 12, 1984

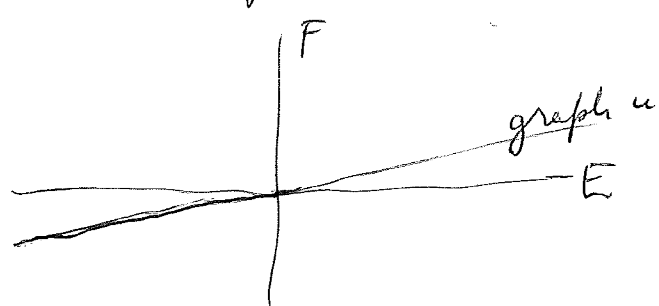
Grassmannian graph construction. Let $u: E \to F$ be a vector bundle homomorphism over M, let $Y \subset M$ be the closed set where u is not invertible. Consider over M the Grassmannian bundle $G_d(E\oplus F)$ of d-planes in $E \oplus F$, where $d = \dim E$. Then we have

$$\begin{tikzcd} G_d(E\oplus F) \arrow[d, "\pi"] \\ M \arrow[u, bend left, "s_u"] \end{tikzcd}$$

where s_u is the section which assigns to $m \in M$ the subspace $\text{graph}(u_m) \subset E_m \oplus F_m$. Over the Grass. bdle we have a canonical exact sequence

$$0 \longrightarrow S \longrightarrow \pi^*(E\oplus F) \longrightarrow Q \longrightarrow 0.$$

From the picture



one sees that the graph projects isomorphically onto E, and that F projects isomorphically onto $E\oplus F/\text{graph } u$. Thus we have canon. isoms.

$$s_u^*(S) \simeq E \qquad s_u^*(Q) \simeq F.$$

Next let φ be a characteristic class for vector bundles. Then

$$\varphi(E) = s_u^*(\varphi(S)) = \pi_* s_{u*} s_u^* \varphi(S)$$
$$= \pi_*\left((s_u)_* 1 \cdot \varphi(S)\right)$$

Now I replace u by tu where t is a parameter and let $t \to \infty$. I suppose that Y is a submanifold, so that $\dim E = \dim F$, and generically u is an isomorphism. Then off Y the graph of tu approaches the subspace F. Let's put

$$Z_\infty = \lim_{t \to \infty} \text{ of the cycle } s_{tu}(M) \text{ in } G_d(E \oplus F).$$

We see that Z_∞ contains the image of the section s_F of $G_d(E\oplus F)/M$ corresponding to the subbundle $F \subset E \oplus F$. We can write as cycles

$$Z_\infty = \tilde{Y} \cup s_F(M)$$

where $\tilde{Y}$ sits over Y. Here the idea is that as we approach Y, the map u acquires a kernel, so that upon considering $t\,u(e(m))$ with t, m moving appropriately ($t \to \infty$, $m \to y \in Y$) we get all kinds of interesting limits.

By homotopy invariance for cycles:

$$\varphi(E) = \pi_*\left(Z_\infty \cdot \varphi(S)\right)$$
$$= \pi_*\left((\tilde{Y} + (s_F)_* 1)\, \varphi(S)\right)$$
$$= \pi_*(\tilde{Y} \cdot \varphi(S)) + \pi_* s_{F*} s_F^* \varphi(S)$$

So we get

$$\boxed{\varphi(E)-\varphi(F) = \pi_*(\tilde{\gamma}\cdot\varphi(S))}$$

Next let us realize $\varphi(S)$ by an explicit differential form on $G_d(E\oplus F)$. Then we can pull it back via s_{tu} and watch what happens as $t \to \infty$. To construct a representative for $\varphi(S)$, we need a connection on S. This would come from connections on E, F and a metric, which would allow one to project back onto the graph$(u) \subset E \oplus F$. So we need the same data as for the superconnection approach.

Let's consider the case where E and F are trivial bundles of the same rank d, equipped with standard connection. Then we have a map

$$M \xrightarrow{s_{tu}} G_d(\mathbb{C}^{2d}) \qquad m \longmapsto \text{graph } u(m).$$

and I want to pull back the Chern character form for the subbundle. Recall this is $tr(e^F)$ where $F = e\,de\,de$ is the curvature of the Grassmannian connection.

Take $d=1$, whence we have $\mathbb{P}_1 = \mathbb{C}\cup\{\infty\}$ with $z\in\mathbb{C}$ corresponding to the line spanned by $(1,z)$. Then

$$e = \begin{pmatrix}1\\ z\end{pmatrix}\frac{1}{1+|z|^2}\begin{pmatrix}1 & \bar{z}\end{pmatrix}$$

More generally, let e be the projection onto the graph of $T: V \longrightarrow W$ inside of $V \oplus W$. Then clearly we have

$$e = \begin{pmatrix} 1 \\ T \end{pmatrix} (1+T^*T)^{-1} \begin{pmatrix} 1 & T^* \end{pmatrix}$$

$$(1-e) = \begin{pmatrix} -T^* \\ 1 \end{pmatrix} (1+TT^*)^{-1} \begin{pmatrix} -T & 1 \end{pmatrix}$$

Both ~~are idempotents~~, e projects on graph $T = \begin{pmatrix} 1 \\ T \end{pmatrix} V$ whereas the other expression projects on $\begin{pmatrix} -T^* \\ 1 \end{pmatrix} W$, which one knows is the $\perp$ space to graph T.

The curvature form is

$$e\,de\,de = de\,(1-e)\,de$$

Now

$$de = \begin{pmatrix} 0 \\ dT \end{pmatrix}(1+T^*T)^{-1}\begin{pmatrix} 1 & T^* \end{pmatrix} + \begin{pmatrix} 1 \\ T \end{pmatrix}(1+T^*T)^{-1}\begin{pmatrix} 0 & dT^* \end{pmatrix}$$
$$+ \begin{pmatrix} 1 \\ T \end{pmatrix} d(1+T^*T)^{-1}\begin{pmatrix} 1 & T^* \end{pmatrix}$$

so

$$(1-e)\,de = \begin{pmatrix} -T^* \\ 1 \end{pmatrix}(1+TT^*)^{-1}\, dT\,(1+T^*T)^{-1}\begin{pmatrix} 1 & T^* \end{pmatrix}$$

$$F = de(1-e)de = \begin{pmatrix} 1 \\ T \end{pmatrix}(1+T^*T)^{-1} dT^* (1+TT^*)^{-1}\, dT\,(1+T^*T)^{-1}\begin{pmatrix} 1 & T^* \end{pmatrix}$$

$$\boxed{e\,de\,de = \begin{pmatrix} 1 \\ T \end{pmatrix}(1+T^*T)^{-1} dT^* (1+TT^*)^{-1}\, dT\,(1+T^*T)^{-1}\begin{pmatrix} 1 & T^* \end{pmatrix}}$$

So in particular we see that the curvature of $\mathcal{O}(-1)$ on $\mathbb{P}^1$ is $\dfrac{d\bar{z}\,dz}{(1+|z|^2)^2}$

Finally let us consider the bundle map $h = z : \mathbb{1} \longrightarrow \mathbb{1}$ over $\mathbb{C}$. Then when we pull back the first Chern form for $\mathcal{O}(-1)$ on $\mathbb{P}^1$ by the map $z \longmapsto tu(z) = tz$, we get the form

$$\frac{t^2\, d\bar{z}\, dz}{(1+t^2|z|^2)^2}$$

So we get something quite different from the Gaussian form obtained via the superconnection theory.

On the other hand things shouldn't really be that different, for if I were to take a map T between trivial bundles of the same rank, and form the superconnection

$$D + L = d + i\begin{pmatrix} 0 & T^* \\ T & 0 \end{pmatrix}$$

then the curvature is

$$(D+L)^2 = i\begin{pmatrix} 0 & dT^* \\ dT & 0 \end{pmatrix} - \begin{pmatrix} T^*T & 0 \\ 0 & TT^* \end{pmatrix}$$

so that we have the same ingredients.

Computation for your class. The connection on T^* is relative to the frame ω^μ given by

$$D = d + \Gamma = \omega^\mu(\underbrace{d_\mu + \Gamma_\mu}_{D_\mu}) \qquad d_\mu = i(X_\mu)d$$

$$D_\mu(\omega^a \xi_a) = \Gamma^{a}_{\mu\nu}\,\omega^\nu \xi_a + \omega^a X_\mu \xi_a$$

Thus

$$\Gamma_\mu = \Gamma_{\mu a}^{\ \ \ b}\, e(\omega^a)\, i(X_b) \qquad \text{on } T^*$$

Thus the connection on ΛT^* can be written

$$D = d + \Gamma \qquad d = \text{connection flat relative to the frame } \omega^\mu$$

$$\Gamma = \omega^\mu \Gamma_{\mu a}{}^{b}\, e(\omega^a) . i(X_b). \qquad \text{1-form values in } \mathrm{End}\,\Lambda T^*.$$

The curvature can be written

$$D^2 = d\Gamma + \Gamma^2 = \tfrac{1}{2}\omega^\mu \omega^\nu R_{\mu\nu a}{}^{b}\, e(\omega^a)\, i(X_b)$$

Now assume the frame is orthonormal, whence $\Gamma_{\mu a}{}^{b}$, $R_{\mu\nu a}{}^{b}$ are skew-symmetric in a, b. The connection on the spinors is obtained by using

$$\mathrm{Lie}\, SO(n) \hookrightarrow C_n = \mathrm{End}(S_n)$$

$$A_{ab} \longmapsto \tfrac{1}{4} A_{ab} \gamma^a \gamma^b .$$

Thus the connection on S = trivial bundle with fibre S_n is $D = d + \tilde{\Gamma}$ where

$$\tilde{\Gamma} = \frac{1}{4}\omega^\mu \Gamma_{\mu a}{}^{b} \gamma^a \gamma^b$$

and the curvature is

$$\tilde{R} = \frac{1}{8}\omega^\mu \omega^\nu R_{\mu\nu a}{}^{b} \gamma^a \gamma^b .$$

It follows that the Dirac operator on the spinors is

$$\not{D}^2 = D_\mu^2 + \frac{1}{8} R_{\mu\nu ab}\, \gamma^\mu \gamma^\nu \gamma^a \gamma^b$$

where since the frame is orthonormal, there is no need to distinguish upper & lower indices.

Recall that R_{ijkl} is skew under $i \leftrightarrow j$ or $k \leftrightarrow l$ is symmetric under $(ij) \leftrightarrow (kl)$ and also that if skew symmetrized on ijk it gives zero. Now

$$R_{ijkl}\, \gamma^i \gamma^j \gamma^k \gamma^l \in \Lambda^0 + \Lambda^2 + \Lambda^4$$

where $\Lambda^p T^*$ is identified with the space of monomials in the γ's of length p.

The component in Λ^4 is the sum over $i\,j\,k\,l$ distinct, so is zero as R_{ijkl} skew symmetrized on i, j, k is zero.

The component in Λ^2 is the sum over

$$i, j, i, k \qquad \text{distinct}$$

+ three similar possibilities.

$$R_{ijil}\,\gamma^i\gamma^j\gamma^i\gamma^l = -\underbrace{R_{ijil}}_{\text{symm}}\,\underbrace{\gamma^j\gamma^l}_{\text{skew}} = 0$$

The component in Λ^0 is

$$\frac{1}{8}\left(-R_{ijij} + R_{ijji}\right) = -\frac{1}{4}R_{ijij} = -\frac{1}{4}(\text{scalar curv.})$$

(Check sign:

$$-\not{D}^2 = -D_\mu^2 + \frac{s}{4} \geq \frac{s}{4}$$

hence $s > 0 \implies$ NO harmonic spinors.)

April 13, 1984: (Becky comes today)

April 15, 1984:

Last night it occurred to me that although $Q = h\gamma^\mu D_\mu$ is not in the algebra of asymptotic diffl. operators, perhaps it could be adjoined since its square $Q^2 = -H$ does belong to this algebra.

I think what I want to have is for bracketing with Q to preserve the algebra A of asymp. diffl ops. This seems reasonable from the supersymmetry viewpoint. The algebra A is generated by $f(x)$, $h\gamma^\mu$, hD_μ.

$$[Q, h\gamma^\alpha]_+ = [h\gamma^\mu D_\mu, h\gamma^\alpha]$$
$$= h^2 2g^{\mu\alpha} D_\mu + h^2\gamma^\mu \Gamma^\alpha_{\mu\nu}\gamma^\nu \longrightarrow h\gamma^\mu h\gamma^\nu \Gamma^\alpha_{\mu\nu}$$

$$[Q, hD_\nu] = h^2[\gamma^\mu D_\mu, D_\nu]$$
$$= -h^2\Gamma^\mu_{\nu\alpha}\gamma^\alpha D_\mu + h^2\gamma^\mu\left\{\tfrac14 R_{\mu\nu a}{}^b\gamma^a\gamma^b + F_{\mu\nu} - \hat\Gamma^\alpha_{\mu\nu}D_\alpha\right\}$$

Now $\gamma^\mu R_{\mu\nu a}{}^b\gamma^a\gamma^b = -R_{\nu\mu a}{}^b\gamma^\mu\gamma^a\gamma^b$ and we know that R antisymmetrized on 3 indices is zero, so

$$-R_{\nu\mu a}{}^b\gamma^\mu\gamma^a\gamma^b = -R_{\nu\mu\mu}{}^b\gamma^b + R_{\nu\mu a}{}^\mu\gamma^a = 2R_{\mu\nu\mu}{}^b\gamma^b$$

involves only one γ. Thus

$$[Q, hD_\nu] = -h^2\Gamma^\mu_{\nu\alpha}\gamma^\alpha D_\mu +$$
$$- h^2\hat\Gamma^\alpha_{\mu\nu}\gamma^\mu D_\alpha + h\left\{\tfrac12 R_{\mu\nu\mu}{}^b h\gamma^b + h\gamma^\mu F_{\mu\nu}\right\}$$

$$= -h^2(\underbrace{\Gamma^\mu_{\nu\alpha} - \hat\Gamma^\mu_{\nu\alpha}}_{\Gamma^\mu_{\alpha\nu}\text{ because torsion is zero}})\gamma^\alpha D_\mu + \underline{\qquad\qquad}$$

$$[Q, f] = h[\gamma^\mu D_\mu, f] = h\gamma^\mu[D_\mu, f] = h\gamma^\mu X_\mu f$$

So we see that bracketing with Q preserves the algebra of asymptotic diff'l operators. The limiting commutation relations are

$$[Q,f] = \overline{h\gamma^\mu}\, X_\mu f \quad (= df)$$

$$[Q, \overline{h\gamma^\alpha}] = \overline{h\gamma^\mu}\,\overline{h\gamma^\nu}\,\Gamma^\alpha_{\mu\nu} = \tfrac{1}{2}\Gamma^\alpha_{\mu\nu}\,\overline{h\gamma^\mu}\,\overline{h\gamma^\nu}$$

$$[Q, \overline{hD_\nu}] = -\Gamma^\mu_{\alpha\nu}\,\overline{h\gamma^\alpha}\,\overline{hD_\mu}$$

The first two relations say that, on the algebra generated by the f and $\overline{h\gamma^\mu}$ which we can identify with the differential forms under $\overline{h\gamma^\mu} = \omega^\mu$, that $[Q,\ ]$ is the same as d. This suggests that $[Q,\ ]$ is associated to some sort of connection. The third relation says that if we identify $\overline{hD_X}$ with the vector field X, then $[Q,\ ]$ is the actual covariant differentiation on sections of T:

$$D_\mu X_\nu = -\Gamma^\alpha_{\mu\nu} X_\alpha$$

Here's the picture then of the $h=0$ limit, or associated graded algebra, of Getzler's filtered algebra. First of all it is an algebra over $\Omega(M)$, the algebra of diff forms on M. $\Omega(M)$ is actually in the center of $gr(A)$. Moreover with respect to $[Q,\ ]$ on $gr(A)$, this central subalgebra is closed. Better, the odd degree derivation $[Q,\ ]$ of $gr(A)$ induces d on $\Omega(M)$.

Next $gr(A)$ is a twisted version of $\Omega(M)\otimes_{\Omega^0(M)}\Gamma(S(T))$, where the twisting comes from the skew-form on $\Gamma(T)$ with values in $\Omega^2(M)$ given by the curvature. So we get an increasing filtration on $gr(A)$ corresponding to the actual degree as a differential operator. On $\Omega(M) \otimes \Gamma(S_k T)$ one should see $[Q, \ ]$ as the connection operator $D = \omega^\mu D_\mu$ extended from T to $S_k T$.

April 18, 1984

Let's consider the process of going from the cyclic cohomology of $\Omega^0(M)$ to that of $\Omega^0(M, \mathrm{End}\,E)$ and then restricting to $\Omega^0(M) \xhookrightarrow{\ \cdot\, id_E\ } \Omega^0(M, \mathrm{End}\,E)$. This should be analogous to going from a Dirac op. to [redacted] the one obtained by tensoring it with E. Of course we need to use a connection D on E.

There are two possible versions of this process that I know (see p. 633 + for earlier work.) I start with the one I found using transgression ideas.

Recall that the really nice way to think of the cyclic cohomology of $\Omega^0(M)$ is via the form

$$(*) \qquad \int_0^1 dt\ \mathrm{tr}\left(e^{t d\theta + (t^2-t)\theta^2}\theta\right) \ \in C(\mathfrak{g}, \Omega_M)$$

where $\mathfrak{g} \to \mathfrak{gl}_n\, \Omega^0(M)$ and $\theta \in C^1(\mathfrak{g}, \Omega^0_M) \otimes M_n$ is the Maurer-Cartan form of this ~~embedding~~ representation. The reason I say this is a really nice way to do things, is that the above form sets up the basic isomorphism

$$(**) \qquad HC_p(\Omega^0_M) \simeq \Omega^p/d\Omega^{p-1} \oplus H^{p-2}_{DR} \oplus \cdots\cdots$$

Now $\Omega^0(M, \mathrm{End}\,E)$ is Morita equivalent to $\Omega^0(M)$, so it has the same cyclic cohomology. There exists therefore a form generalizing $(*)$ which [redacted] sets up the isomorphism of $HC(\Omega^0(M, \mathrm{End}\,E))$ with $(**)$. This form is

$$(+) \qquad \int_0^1 dt\ \mathrm{tr}\left(e^{D^2 + t[D,\theta] + (t^2-t)\theta^2}\theta\right) \quad \in C(\mathfrak{g}, \Omega_M)$$

where here $\theta \in C^1(\mathfrak{g}, \Omega^0(M, \mathrm{End}\,E)) \otimes M_n$ is the MC form belonging to ~~an embedding~~ a representation $\mathfrak{g} \to \mathfrak{gl}_n(\Omega^0(M, \mathrm{End}\,E))$.

Next take the representation $\mathfrak{gl}_n\, \Omega^0(M) \longrightarrow \mathfrak{gl}_n\, \Omega^0(M, \mathrm{End}\,E)$.

belonging to $f \mapsto f\cdot id_E$. Then $[D,\theta] = d\theta$ and $\theta, d\theta$ commute with D^2. Thus restricting the above form$^{(+)}$ to $gl(\Omega^0(M))$ yields

$$(1) \qquad tr(e^{D^2}) \int_0^1 tr\left(e^{t\,d\theta + (t^2-t)\theta^2}\theta\right).$$

Here I have used that the trace in (+) is as forms with values in $\operatorname{End} E \otimes M_r$, and that D^2 has values in $\operatorname{End} E$, whereas $\theta, d\theta$ have values in M_r. Thus the traces multiply.

The formula (1) shows the virtues of my formalism.

The second version is that of Connes which I will carry thru for a bundle E which is represented by an idempotent matrix e, and which is equipped with the Grassmannian connection.

Connes idea is that the way cyclic cocycles for $\Omega^0(M)$ extend to $\Omega^0(M, \operatorname{End} E)$ is via the "cycle"

$$\begin{tikzcd}
\Omega^0(M,\operatorname{End}E) \arrow[d, hook, "\rho"] & & & \\
\Omega^0(M)\otimes M_q \arrow[r, "d"] & \Omega^1(M)\otimes M_q \arrow[r, "d"] & \cdots \arrow[r] & \Omega^p(M)\otimes M_q \arrow[r] \arrow[d, "\int_\gamma"] & {} \\
 & & & \mathbb{C} &
\end{tikzcd}$$

Thus the cyclic cocycle $\int_\gamma tr\, \theta(d\theta)^p$ on $gl_r(\Omega^0(M))$ will become

$$\int_\gamma tr\, \rho(\theta)(d\rho(\theta))^p \qquad \text{on} \qquad gl_r(\Omega^0(M, \operatorname{End} E))$$

Suppose then we restrict to $\Omega^0(M) \subset \Omega^0(M, \mathrm{End}\,E)$.
We get

$$\int_\gamma tr\left[\Theta e\,(d(\Theta e))^p\right]$$

which we can calculate using his δ homomorphism as follows. Recall that the latter is defined using the free diff'l algebra generated by an idempotent e:

$$A \overset{\rho}{\subset} \hat{\Omega}(A) \otimes \hat{\Omega}(ke^{\bullet}) \qquad\qquad a \longmapsto a \otimes e = ae$$

In this case we have

$$\begin{array}{ccc} \Omega^0(M) & \subset & \Omega^\cdot(M) \otimes \hat{\Omega}(ke^{\bullet}) \\ & & \downarrow \\ & & \Omega^\cdot(M) \otimes \left(\hat{\Omega}(ke^{\bullet})/[\ ,\ ]\right) \quad \text{basis} \quad e(de)^{2j} \\ & & \downarrow \quad \text{universal } e(de)^{2j} \text{ goes to } tr\left(e(de)^{2j}\right) \\ & & \Omega^\cdot(M) \end{array}$$

So what we seem to get is the formula that from $\int_\gamma tr\,\Theta\,d\Theta^p$ we get the cyclic cocycle

$$S^{j}\left\{\int_\gamma \frac{(e\,de^2)^j}{j!}\right\} \qquad \text{where } \int_\gamma \frac{(e\,de^2)^j}{j!} \text{ stands for } \int \frac{(e\,de^2)^j}{j!}\,\cancel{tr}\,\Theta(d\Theta)^{p-2j}$$

April 20, 1984

harmonic oscillator heat kernel

$$H = \frac{1}{2}(-\partial_x^2 + x^2) = a^*a + \frac{1}{2} \qquad a = \frac{1}{\sqrt{2}}(\partial_x + x)$$
$$a^* = \frac{1}{\sqrt{2}}(-\partial_x + x)$$

$$\Phi_0 = e^{-\frac{x^2}{2}}\pi^{-1/4}$$

$$\Phi_n = \frac{1}{\sqrt{n!}}(a^*)^n\Phi_0 = \frac{1}{\sqrt{n!}}\left(\frac{-1}{\sqrt{2}}\right)^n e^{\frac{x^2}{2}}\partial_x^n e^{-x^2}\pi^{-1/4}$$

is an orthonormal sequence in $L^2(\mathbb{R})$; $a^*a\,\Phi_n = n\Phi_n$.

Put
$$K_t(x,y) = \sum_n e^{-nt}\,\Phi_n(x)\,\Phi_n(y)$$

This is the kernel of e^{-ta^*a} if $\{\Phi_n\}$ is complete; by evaluation it will follow that $K_t(x,y) \longrightarrow \delta(x,y)$ as $t \to 0$, whence follows completeness.

$$K_t(x,y) = \pi^{-1/2} e^{\frac{x^2}{2}+\frac{y^2}{2}} \sum_n \frac{1}{n!} e^{-nt}\left(\frac{1}{2}\right)^n \partial_x^n\partial_y^n \; e^{-(x^2+y^2)}$$

$$= \pi^{-1/2} e^{\frac{1}{2}(x^2+y^2)} \; e^{\frac{1}{2}e^{-t}\partial_x\partial_y} \; e^{-(x^2+y^2)}$$

But we have the following from F.T.

$$e^{\frac{1}{2}\partial^t P \partial} \; e^{-\frac{1}{2}x^t Q x} = \det(1+PQ)^{-1/2} \, e^{-\frac{1}{2}x^t Q(1+PQ)^{-1}x}$$

In the above example

$$P = \frac{1}{2}\begin{pmatrix} 0 & e^{-t} \\ e^{-t} & 0 \end{pmatrix} \qquad Q = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} \qquad PQ = \begin{pmatrix} 0 & e^{-t} \\ e^{-t} & 0 \end{pmatrix}$$

$$\det(1+PQ) = \det\begin{pmatrix} 1 & e^{-t} \\ e^{-t} & 1 \end{pmatrix} = 1 - e^{-2t}$$

$$(1+PQ)^{-1} = \frac{1}{1-e^{-2t}}\begin{pmatrix} 1 & -e^{-t} \\ -e^{-t} & 1 \end{pmatrix}$$

so

$$K_t(x,y) = \frac{\pi^{-1/2}\, e^{\frac{1}{2}(x^2+y^2) - \frac{1}{1-e^{-2t}}(x^2+y^2-2e^{-t}xy)}}{\sqrt{1-e^{-2t}}}$$

This is the kernel of e^{-ta^*a}; to get e^{-tH} multiply by $e^{-t/2}$. Then $e^{-t/2}/\sqrt{1-e^{-2t}} = (2\sinh t)^{-1/2}$ and so

$$\boxed{\langle x|e^{-tH}|y\rangle = \frac{1}{(2\pi t)^{1/2}}\left(\frac{t}{\sinh t}\right)^{1/2} e^{-\frac{1}{2\sinh t}\left((\cosh t)(x^2+y^2)-2xy\right)}}$$

Holomorphic function representation. Define

$$T : L^2(\mathbb{R}) \xrightarrow{\sim} \text{holom. fns on } \mathbb{C} \text{ with } \|f\|^2 = \int \frac{d^2z}{\pi} e^{-|z|^2}|f(z)|^2$$

by

$$(Tf)(z) = \sum \frac{z^n}{\sqrt{n!}}(\Phi_n|f)$$

$$= \int dx \underbrace{\sum \frac{z^n}{\sqrt{n!}}\Phi_n(x)}_{T(z,x)} f(x)$$

Then

$$T(z,x) = \sum_n \frac{z^n}{n!}\left(\frac{-1}{\sqrt{2}}\right)^n e^{\frac{x^2}{2}}(\partial_x)^n e^{-x^2}\,\pi^{-1/4}$$

$$= \pi^{-1/4} e^{\frac{x^2}{2}}\, e^{-\frac{1}{\sqrt{2}}z\partial_x}\left(e^{-x^2}\right)$$

$$= \pi^{-1/4} e^{\frac{x^2}{2}}\, e^{-(x-\frac{1}{\sqrt{2}}z)^2}$$

$$\boxed{T(z,x) = \pi^{-1/4}\cdot e^{-\frac{x^2}{2}+\sqrt{2}zx-\frac{z^2}{2}}}$$

In the holomorphic function representation

$$a^*a\ z^n = z\frac{d}{dz}z^n = nz^n$$

hence $\quad e^{-ta^*a} z^n = e^{-tn} z^n = (e^{-t}z)^n$

and so $\quad \boxed{(e^{-ta^*a} f)(z) = f(e^{-t}z)}$

April 21, 1964

Let E be a v.b. over M, and $B = \Omega^\circ(M, \mathrm{End} E)$. Then the algebra B is Morita equivalent to $A = \Omega^\circ(M)$, and so has the same cyclic cohomology. If D is a connection on E, then I believe that the isom. on cyclic cohomology is implemented by the cochain

$$(1) \qquad \int_0^1 dt \; tr\left(e^{D^2 + t[D,\theta] + (t^2-t)\theta^2}\theta\right) \quad \in \quad C^\cdot(\mathfrak{g}, \Omega^\cdot(M))$$

where $\mathfrak{g} = \mathfrak{gl}(B)$. If we restrict this cochain to $\mathfrak{gl}(A) \subset \mathfrak{gl}(B)$, where $A \subset B$ is $f \mapsto f\, id_E$, then we get the cochain

$$tr_E(e^{D^2}) \cdot \int_0^1 dt \; tr\left(e^{t d\theta + (t^2-t)\theta^2}\theta\right)$$

This fact supports my belief that (1) is a good formula.

But now suppose E occurs as a direct summand of a trivial vector bundle: $E = \mathrm{Im}(e)$, $e \in M_r(A)$, $e^2 = e$. Then we have an embedding

$$B = e M_r(A) e \quad \subset \quad M_r(A)$$

of algebra which is non-unital. It induces a Lie algebra embedding

$$\mathfrak{gl}(B) \overset{j}{\subset} \mathfrak{gl}(M_r(A)) = \mathfrak{gl}(A)$$

whence the Chern-Simons cochain

$$\int_0^1 dt \; tr\left(e^{t d\theta + (t^2-t)\theta^2}\theta\right)$$

, I use to describe the cyclic cohomology of A, can be restricted to a [redacted] cochain on $gl(B)$:

$$(2) \qquad j^* \int_0^1 dt \; tr\left(e^{t d\theta + (t^2-t)\theta^2}\theta\right) \quad \in C^\bullet(gl(B), \Omega^\bullet(M))$$

The problem now is to compare the two cochains (1) and (2), where in the former we take $D = e.d.e$ to be the Grassmannian connection. Hopefully the two forms will coincide; this is what we should try to prove.

An important ingredient of the proof will be, I think, that on the Grassmannian the invariant forms realize the cohomology exactly, so that formulas like $ch(S) + ch(Q) =$ [redacted] r [redacted] hold on the form level. Let's go over this, but using the idempotent matrix for e over M which corresponds to a map from M into the Grassmannian. Write $E \oplus E' = 1^{\oplus r}$, where $E = \mathrm{Im}\, e$, $E' = \mathrm{Im}(1-e)$.

The [redacted] induced connection on E (resp. E') by the taut. connection d on the trivial bundle is $e \cdot d \cdot e$ (resp $(1-e)\cdot d \cdot (1-e)$), and its curvature is $e(de)^2$ (resp $(1-e)(d(1-e))^2 = (1-e)(de)^2$.). Thus

$$ch(E, e.d.e) + ch(E', (1-e).d.(1-e)) = tr\, e^{e(de)^2} + tr\, e^{(1-e)(de)^2}$$

$$= tr\, e . e^{(de)^2} + tr\,(1-e).e^{(de)^2} = tr\, e^{(de)^2} = r$$

and the last equality comes from the fact that $tr\,(de)^{2j} = 0$, $j > 0$, using cyclic symmetry of the trace.

Another way to describe this is to use the direct sum of the induced connections

$$D = e.d.e + (1-e).d.(1-e)$$
$$= d + e.de - de.e$$

with the curvature

$$D^2 = -e\,de^2 - de.e.de = -e\,de^2 - (1-e)de^2 = -de^2.$$

Then $tr(e^{D^2}) = r$ as above.

So we try to prove that (1), (2) are the same. It seems a good idea to consider an arbitrary connection D on the trivial bundle $\underline{1}^{\oplus r}$ and to consider the form

$$\int_0^1 dt \ tr\left(e^{D^2 + t[D,\theta] + (t^2-t)\theta^2}\theta\right)$$

where $\theta \in C^1(gl_k(B), M_k(B))$ has values which are endos of E. In better words, θ is a matrix of endomorphisms of $\underline{1}^{\oplus r} = E \oplus E'$ which carry E into itself and are 0 on E'. Thus $e\theta = \theta e = \theta$.

When $D = d$ we get the form (2). What do I need in order to get the form (1) which I now write

$$\int_0^1 dt \ tr\left(e^{e(de)^2 + t\, e(d\theta)e + (t^2-t)\theta^2}\theta\right) \quad ?$$

Think in terms of the splitting $E \oplus E'$, and try to get only "triangular" endos., i.e. we want D^2, $[D,\theta]$ to carry E into <u>itself</u>.

Earlier work p. 481-482

In order to produce Lie algebra cohomology for the group $\mathcal{G}$ of gauge transformations, we have used the process of transgression in the fibre bundle $\mathcal{A} \to \mathcal{A}/\mathcal{G}$. A form on $\mathcal{A}/\mathcal{G}$ is pulled up to $\mathcal{A}$ where it becomes cohomologous to zero within the $\mathcal{G}$-invariant forms, then the cobounding cochain is restricted to a $\mathcal{G}$-orbit. It is natural to ask whether this means something analytically.

We start with the class $ch_k(\text{index})$ over $\mathcal{A}/\mathcal{G}$ which is realized as the integral over M of a form $(\hat{A}(M) \cdot ch(\tilde{E}))_{n+2k}$, where $ch(\tilde{E})$ is computed using the connection on $pr_2^*(E)$ over $\mathcal{A} \times M$ which descends. Upon lifting to $\mathcal{A}$ we can move linearly from this connection to to tautological connection which has curvature of types $(1,1)$ and $(0,2)$ over $\mathcal{A} \times M$. This means that if $k > n/2$ then the new form $\left(\hat{A}(M) \cdot ch(pr_2 E^*, \text{taut. conn})\right)_{n+2k}$ has filtration $\geq \frac{1}{2}(n+2k) > n$ and so is zero, providing us with the required transgression form.

So we ask what this vanishing might mean analytically. In this case we consider the tautological family of Dirac operators indexed by $\mathcal{A}$, and the Chern character form belonging to this family is

$$\mathrm{Tr}\left(e^{h^2 \not{D}^2 + d(h\not{D})}\right) = \mathrm{Tr}\left(e^{L^2 + [D,L] + D^2}\right)$$

The local index thm. for families says as $h \to 0$ the form on the left should have zero component of degrees $> n$. This seems clear as there would be two many h factors.

April 22, 1984

Let's review what we learned in ~~December~~ November + December about the transgression.

We consider on the bundle $pr_2^*(E)$ over $\mathcal{a}\times M$ the tautological connection. This is invariant under the $\mathcal{G}$-action but doesn't descend to the orbit space. Using a connection in the principal bundle $\mathcal{a}\to\mathcal{a}/\mathcal{G}$ we can produce a modified connection which descends.

Now restrict to a $\mathcal{G}$-orbit, whence we have the bundle $pr_2^*(E)$ over $\mathcal{G}\times M$ with two $\mathcal{G}$-invariant connections. To fix the notation we assume that $\mathcal{G}$ acts to the right on $\mathcal{a}$ (recall that if $g\in\mathcal{G}$, then $A\mapsto g^*A$ is a natural right action on the connection forms on the principal bundle P of E.) Using a local trivialization of E we can write connections on E in the form $D=d+A$, $d=d_M$ whence the action is

$$D\longrightarrow g^{-1}Dg \qquad \text{or} \qquad A\longrightarrow g^{-1}dg+g^{-1}Ag = g^*A$$

Let $\delta=d_{\mathcal{G}}$, and let $\hat{g}$ denote the tautological automorphism of $pr_2^*(E)$ over $\mathcal{G}\times M$, i.e. $\hat{g}$ at (g,m) is $g(m)$ viewed as an autom. of E_m, or as a matrix in the local trivialization. The tautological connection is

$$\delta+d+\hat{g}^*A_0 = \delta+\hat{g}^{-1}(d+A_0)\hat{g}$$

where A_0 is the point such that $g\longrightarrow g^*A_0$ gives the $\mathcal{G}$-orbit in question. The modified connection is obtained by adding the MC form $\hat{g}^{-1}\delta\hat{g}$ to this:

$$\delta+\hat{g}^{-1}\delta\hat{g}+\hat{g}^{-1}(d+A_0)\hat{g} = \hat{g}^{-1}(\delta+d+A_0)\hat{g}$$

In other words the modified connection is the transform by $\hat{g}$ of the pull-back of $d+A_0$ on E over M, hence it descends to $d+A_0$.

If we transform the tautological and modified connections via $\hat{g}$ we get the two connections

$$\delta + d + A_0 \quad = \quad [redacted]\; D$$

$$\delta + d + A_0 + \hat{g}\, d(\hat{g}^{-1}) \quad = \quad D + \theta$$

where D is the pull-back via pr_2 of D_0 on E and θ is the right-invariant MC form. Let's now perform the autom. $g \longrightarrow g^{-1}$ of $\mathcal{G}$, so that θ becomes the left-invariant MC form.

Now we [redacted] construct the transgression forms using the path of connections $D + t\theta$ on $pr_2^*(E)$ over $\mathcal{G} \times M$, and get the form

$$(*) \qquad \int_0^1 dt \; tr\left(e^{D^2 + t[D,\theta] + (t^2 - t)\theta^2} \theta \right)$$

as usual. [redacted]

Conclusion: The transgression process [redacted] mentioned in my letter to Singer leads to the same forms I have been considering for the cyclic theory. So the only point worth worrying about is the nature of the Hilbert space operators underlying these forms.

So let us now begin the analysis. We have a family of Dirac operators parametrized by $\mathbb{R} \times \mathcal{G}$, namely the family associated to the connection $D + t\theta$ where $D = d_t + \delta + d + A_0$ is the pull back of D_0 on E

under the projection $\mathbb{R}\times\mathcal{Y}\times M \longrightarrow M$.

We use the vertical part of this connection to construct the family of operators. This gives $L = \not{D}_0$ independent of the parameter point in $\mathbb{R}\times\mathcal{Y}$. We thus get the superconnection

$$d_t + \delta + t\theta + L$$

with curvature

$$L^2 + t[L,\theta] + (t^2 - t)\theta^2 + dt\,\theta .$$

The character form is

$$\mathrm{Tr}_s\left(e^{\beta(L^2 + t[L,\theta] + (t^2-t)\theta^2 + dt\,\theta)}\right)$$

where β is put in to keep track of the different Chern character components. ? WAIT.

Let's go back to the local index thm. for families and put β in. Consider a family of Dirac operators $L: y \longmapsto \not{D}_y$ on M and a connection D on the Hilbert bundles. Better we have $D^{tot} = D + D^{v}$ and consider the superconnection $D + L$ acting on the trivial bundle over Y with fibre the spinors over M with coeffs. in E.

Over Y we have the Chern character form

$$\mathrm{Tr}_s\left(e^{\beta(h^2L^2 + [D, hL] + D^2)}\right)$$

which is a closed even form for each β, h. The coh. class of this form on Y doesn't change as we vary h. If we let $h \longrightarrow 0$, we can evaluate in the classical limit and get

$$\int_M \left(\frac{i}{2\pi\beta}\right)^{n/2} \hat{A}(M,\beta)\,\mathrm{tr}\, e^{\beta(D^{tot})^2}$$

where the form $\hat{A}(M,\beta)$ is $\hat{A}$ applied to β· curvature. From this sort of formula one sees that the component of degree $2k$ is multiplied by β^k. Thus it seems that cohomologically the β-degree gives the grading of forms by degree.

Let's rewrite the character form as

$$(1) \qquad Tr_s\left(e^{\beta(L^2 + dy^\mu[D_\mu, L] + \frac{1}{2}dy^\mu dy^\nu F_{\mu\nu})}\right)$$

and assume that it has an asymptotic expansion as $\beta \downarrow 0$ in powers of β. Then suppose we work at a point of Y and rescale the variables $dy^\mu = \frac{\psi^\mu}{\sqrt{\beta}}$ whence we have

$$(2) \qquad Tr_s\left(e^{\beta L^2 + \sqrt{\beta}\psi^\mu[D_\mu, L] + \frac{1}{2}\psi^\mu\psi^\nu F_{\mu\nu}}\right).$$

As $\beta \longrightarrow 0$, $\sqrt{\beta}$ is like Planck's constant before, so we obtain the limit

$$(3) \qquad \int_M \left(\frac{i}{2\pi}\right)^{n/2} \hat{A}(M)\, tr\left(e^{(D^{tot})^2}\right)$$

with the appropriate interpretation of $(D^{tot})^2$; (it involves ψ^μ's now). Actually I should call $\psi^\mu = dx^\mu$, so that I am just rescaling on the tangent space at the point of interest in Y. Then $D^{tot} = dx^\mu D_\mu + D^\vee$.

What all this means is that if (2) is expanded in terms of the $\psi^\mu = dx^\mu$'s, we get

$$(1) = (2) = \sum \alpha_I\, dx^I = \sum \alpha_I\, \beta^{\frac{1}{2}|I|}\, dy^I$$

where $\lim_{\beta\to 0} \alpha_I(\beta)$ exists and is given essentially by (3). Thus the terms of the asymptotic expansion of the Chern character form (1) look like:

power of β / degree

	0	2	4	6
0	*	0	0	0
1	*	*	0	0
2	*	*	*	0
3	*	*	*	*

and the local index formula (3) gives the leading edge. Notice that this is completely consistent with the idea that, [redacted] like the finite dimensional case [redacted]

$$tr_s\left(e^{\beta(D+L)^2}\right)$$

the coefficient of β^k is a form with components of degrees $0, 2, \ldots, 2k$, but only the highest degree is important cohomologically.

The next step seems to be to do some integration with respect to β, either Laplace or Mellin transform.

$$\int_0^\infty d\beta \;\; e^{\beta(D+L)^2}\, e^{-\beta\lambda} = \frac{1}{\lambda-(L+D)^2}$$

$$\int_0^\infty d\beta \;\; e^{\beta(L+D)^2}\, \beta^{s-1} = \frac{\Gamma(s)}{[-(L+D)^2]^s}$$

$$\boxed{\int_0^\infty d\beta\, \beta^k e^{-\beta\lambda} = \frac{\Gamma(k+1)}{\lambda^{k+1}}}$$

The former seems more interesting. What does it give in the classical limit $L \mapsto hL$, $h \to 0$? This requires the Laplace transform of a power of β: