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March 28, 1984

Yesterday I concluded that I should find the appropriate path integral for the heat operator: $e^{-\beta H}$. Suppose $H = \frac{p^2}{2} + V(q)$. Then

$$\langle q | e^{-\varepsilon H} | q' \rangle = \int \frac{dp}{2\pi h} \langle q | p \rangle e^{-\varepsilon \frac{p^2}{2}} e^{-\varepsilon V(q')} \langle p | q' \rangle$$

$$= \int \frac{dp}{2\pi h} e^{\frac{i}{h} p(q-q') - \varepsilon\left(\frac{p^2}{2} + V(q')\right)}$$

$$= \int \frac{dp}{2\pi h} e^{\left[\frac{i}{h} p\left(\frac{q-q'}{\varepsilon}\right) - \frac{p^2}{2} - V(q')\right]\varepsilon}$$

which leads to the path integral

$$\langle q | e^{-\beta H} | q' \rangle = \int \mathcal{D}p(t)\mathcal{D}q(t)\, e^{\int_0^\beta \left(\frac{i}{h} p\dot{q} - H\right) dt}$$

where $q(0) = q'$, $q(\beta) = q$.

As we let $h \to 0$ the important part is $\int p\dot{q}\,dt$ which is stationary when

$$\delta \int p\dot{q}\,dt = \int (\delta p\, \dot{q} + p\, \delta\dot{q})\,dt = 0 \implies \dot{q} = \dot{p} = 0.$$

This means we get zero if $q \neq q'$. To get a more interesting limit put

$$q(t) = q' + h\,x(t) \qquad \Rightarrow \qquad q = q' + h\,x(\beta)$$

$$\dot{q} = h\dot{x}$$

so

$$\int_0^\beta \left[\frac{i}{h} p\dot{q} - H(q,p)\right] dt = \int_0^\beta \left[i p\dot{x} - H(q',p)\right] dt + O(h)$$

$$\cancel{= i p\, x(\beta) - \beta\left(\frac{p^2}{2} + V(q)\right) + O(h)}$$

Let's do the $\mathcal{D}p(t)$ integration which is Gaussian to get

$$\int \mathcal{D}x(t)\, e^{\int_0^\beta [-\frac{1}{2}\dot{x}^2 - V(q')]dt \,+ O(h)}$$

$$= e^{-\beta V(q')}\, e^{-\frac{x(\beta)^2}{2\beta}} \cdot \text{norm const.} \times (1+O(h))$$

$$= e^{-\frac{1}{2\beta}\left(\frac{q-q'}{h}\right)^2 - \beta V(q')}$$

I want to do something similar for fermions and need something like the Fourier transform analysis. In the fermion setup the Hilbert space is $\Lambda[\psi^\mu]$ the functions of the ψ's. The operators are $\hat{\psi}^\mu$ = exterior mult. by ψ^μ and $\hat{\bar{\psi}}^\mu$ = interior multiplication by the linear functional $h\langle\psi^\mu|$. I need a F.T.

$$\Lambda[\psi^\mu] \xrightarrow{\sim} \Lambda[\bar{\psi}^\mu] \qquad \text{reverse } e_\mu, \iota_\mu$$

$$\Lambda^p W^* \simeq \Lambda^{n-p} W$$

so we need an elt of $\Lambda^n W$, i.e. $\int \mathcal{D}\psi$. Let's try the transform

$$f(\psi) \longmapsto \tilde{f}(\bar{\psi}) = \int \mathcal{D}\psi\; e^{\frac{1}{h}\bar{\psi}\psi} f(\psi)$$

Then
$$\int \mathcal{D}\psi\; e^{\frac{1}{h}\bar{\psi}\psi} \frac{\partial}{\partial\psi^\mu} f(\psi)$$

$$= \int \mathcal{D}\psi \left\{ \frac{\partial}{\partial\psi^\mu}\left[e^{\frac{1}{h}\bar{\psi}\psi} f(\psi)\right] + \frac{1}{h}\bar{\psi}^\mu e^{\frac{1}{h}\bar{\psi}\psi} f(\psi) \right\}$$

$$= (-1)^n \frac{1}{h} \bar{\psi}^\mu \int \mathcal{D}\psi\; e^{\frac{1}{h}\bar{\psi}\psi} f(\psi)$$

Also we have

$$\int \mathcal{D}\bar\psi \; e^{\frac{1}{h}\psi\bar\psi} \; e^{\frac{1}{h}\bar\psi\psi'} = \int \mathcal{D}\bar\psi \; e^{\frac{1}{h}(\psi-\psi')\bar\psi}$$

= [redacted] $\pm \frac{1}{h^n} \prod_\mu (\psi^\mu - \psi'^\mu) = \frac{1}{h^n}\,\delta(\psi-\psi')$.

Next let's use this transform to analyze the operator $H = \omega \cdot h\frac{\partial}{\partial\psi}\cdot\psi\cdot$ [redacted] .

$$\omega h \frac{\partial}{\partial\psi}\psi f = \omega h \frac{\partial}{\partial\psi}\int \mathcal{D}\bar\psi \; e^{\frac{1}{h}\psi\bar\psi}\int \mathcal{D}\psi' \; e^{\frac{1}{h}\bar\psi\psi'}\,\psi' f(\psi')$$

= [redacted] $\int \mathcal{D}\bar\psi\,\mathcal{D}\psi' \; e^{-\frac{1}{h}\bar\psi(\psi-\psi')}\,\omega\bar\psi\psi' f(\psi')$

Then use $e^{-\omega h\frac{\partial}{\partial\psi}\cdot\psi} = 1$ [redacted] $\omega\cdot h\frac{\partial}{\partial\psi}\cdot\psi + O(\omega^2)$ and so I think it's clear that we get a kind of path integral representation for $e^{-\beta H}$ as

$$\int \mathcal{D}\bar\psi(t)\,\mathcal{D}\psi(t) \; e^{\int_0^\beta\left(-\frac{1}{h}\bar\psi\dot\psi - \omega\bar\psi\psi\right)dt}$$

Note that $-\frac{1}{h}\bar\psi\dot\psi = \frac{i}{h}\left(i\bar\psi\dot\psi\right)$ which is consistent with our former identification of $i\bar\psi\dot\psi$ with the fermion analogue of $p\dot q$.

Now I want to use this path integral repn. to compute the "limiting heat operator". The idea somehow was to put $\psi(t) = \psi(0) + h\,\chi(t)$, whence the exponent becomes

$$-\int\left(\bar{\psi}\dot{\chi} + \omega\bar{\psi}\psi_0\right)dt \quad + O(h)$$

Then one has a Gaussian to do wrt $\bar{\psi}, \chi$, so one looks at the critical points:

$$\dot{\chi} + \omega\psi_0 = 0$$

$$\dot{\bar{\psi}} = 0$$

Unfortunately the first condition makes the exponent zero, so something is missing.

I am searching for the limiting heat operator in the fermion situation. Let's first look at the boson case: $H = \frac{p^2}{2} + V(q)$. Algebraically we work in a Weyl algebra $W(q,p,h)$ where $[p,q] = \frac{h}{i}$. The heat operator in this algebra is $e^{-\beta H}$, and its classical limit is the function $e^{-\beta H(q,p)}$ of q,p.

So the limiting heat operator is $e^{-\beta H}$ calculated in the algebra of functions on T^*. So far q,p have been treated symmetrically. But if we want a kernel on the tangent bundle, then we have to treat $S(q)$, the functions of the q's as being given and staying unchanged as $h \longrightarrow 0$. The actual kernel $\langle q | e^{-\beta H} | q' \rangle$ is obtained by using the eigenfns. of this algebra of functions on M.

If I want the analogue for fermions, then the Weyl-Heisenberg alg is now $C(\psi, \bar{\psi}, h)$ where $[\psi, \bar{\psi}] = h$. It specializes at $h = 0$ to $\Lambda(\psi, \bar{\psi})$. In order to identify this with a convolution algebra

One has to single out the comm. subalgebra $\Lambda(\psi)$ and use its eigenfunctions $|J\rangle = \delta(\psi - J)$ for variables J. Note in 1-dim

$$\psi \cdot \delta(\psi - J) = \psi(\psi - J)$$
$$= -\psi J = J\psi = J(\psi - J)$$

and in several dimensions

$$\delta(\psi - J) = \prod_\mu (\psi^\mu - J^\mu)$$

We have

$$\int \mathcal{D}\psi \; \delta(\psi - J)\, f(\psi) = \int \mathcal{D}\psi\, (\psi - J)(a + b\psi)$$
$$= \int \mathcal{D}\psi\, (\psi a - Jb\psi) = a + bJ$$

hence $\langle J| \cdots = \int \mathcal{D}\psi\; \delta(\psi - J) \cdots$. One has

$$\langle J | J' \rangle = \int \mathcal{D}\psi\, (\psi - J)(\psi - J') = J - J' = \delta(J - J').$$

Now it should be possible to calculate the kernel $\langle J | e^{-\beta H} | J' \rangle$ where $H = \hbar\omega\, a^* a = \hbar\omega \psi \frac{\partial}{\partial\psi}$.

(Interesting Point: This kernel is like a kernel as a distribution, undefined for values of J, J'; but makes sense operational when smeared with "functions" of J, J'.)

$$\langle J | e^{-\beta\hbar\omega \psi \frac{\partial}{\partial\psi}} | J' \rangle = \int \mathcal{D}\psi\, (\psi - J)\, e^{-\beta\hbar\omega \psi \frac{\partial}{\partial\psi}} (\psi - J')$$
$$= \int \mathcal{D}\psi\, (\psi - J)\left(e^{-\beta\hbar\omega}\psi - J'\right)$$
$$= e^{-\beta\hbar\omega} J - J'$$

We want to appreciate the limit of this as $h \to 0$, so we rewrite it

$$\langle J \,|\, e^{-\beta h\omega \psi \frac{\partial}{\partial\psi}} \,|\, J' \rangle = e^{-\beta h \omega} J - J'$$

$$= \left(e^{-\beta h\omega} - 1\right) J + J - J'$$

$$= h \left\{ \frac{e^{-\beta h\omega} - 1}{h} J + \frac{J-J'}{h} \right\}$$

Now let's compare this with the Fourier transf. wrt $\bar{\psi}$ dual to $\frac{J-J'}{h}$ of $e^{-\beta H}$, where $H = \omega\bar{\psi}\psi \in \Lambda(\psi, \bar{\psi})$. Here ψ is to be identified with J.

$$\int \mathcal{D}\bar{\psi} \; e^{\frac{J-J'}{h}\bar{\psi}} \; e^{-\beta\omega\bar{\psi} J} = -\frac{J-J'}{h} - \beta\omega J$$

which agrees up to sign. (H should be $\omega\psi\bar{\psi}$ to correspond to $h\omega\psi\frac{\partial}{\partial\psi}$, and there are other ambiguities to be worked out eventually.)

Anyway what is clear in this whole game is that we want to work with the functions on T^*, not with the <u>convolution algebra</u> of T.

Summary of the preceding week: I started with the idea ~~the~~ the limiting heat kernel should contain <u>all</u> the index information associated to a Dirac operator. In particular I should get from the limiting kernel all the cyclic cocycles on $\Gamma(\mathrm{End}\,E)$ belonging to the connection on E. I should also be able to start with the limiting kernel for $\gamma^\mu \partial_\mu$ and reduce it by an idempotent matrix e so as to obtain the limiting heat kernel for $e \cdot \gamma^\mu \partial_\mu \cdot e$.

This seemed to be too naive. For example from e^{D^2} I couldn't see how to get the cyclic cocycles on $\Gamma(\mathrm{End}\, E)$. Also I seem to want to have the connection as part of the data.

Thus I was led to the Friedan–Windey idea of enriching the heat kernel to $e^{-tH-\theta Q}$. Then in calculating with superfields I found the Lagrangian has all the index information. So now the project will be to derive the limiting form of the FW heat operator and connect things with the Lagrangian.

March 29, 1984

After Bott + Graeme: Let $\mathbb{R}^{1,1}$ be the super line with the coordinates t, θ. One knows

$$\mathrm{Map}(\mathbb{R}^{0,1}, M) = \text{the supermanifold given by } M \text{ with its DR algebra}$$

This has coordinates x^μ, ψ^μ so that "fns." on this supermanifold are $f(x^\mu, \psi^\mu) \in \Omega^*(M)$. Somehow one writes

$$X^\mu = x^\mu + \theta\psi^\mu$$

to denote an variable "element" of this supermanifold.

Next

$$\mathrm{Map}(\mathbb{R}^{1,1}, M) = \mathrm{Map}(\mathbb{R}^{1,0} \times \mathbb{R}^{0,1}, M)$$

can be interpreted either as paths in the supermanifold $\mathrm{Map}(\mathbb{R}^{0,1}, M)$, or as the supermanifold given the path space $\mathrm{Map}(\mathbb{R}^{1,0}, M)$ equipped with its DR complex. A typical "element" of $\mathrm{Map}(\mathbb{R}^{1,1}, M)$ is denoted

$$X^\mu = x^\mu(t) + \theta\psi^\mu(t)$$

Now we know that $\mathbb{R}^{1,1}$ is a supergroup, or rather we think this is true. Let's make this precise.

Suppose we start with the Lie superalgebra $\mathfrak{g}$ generated by a single odd degree element Q. The Lie superalg. $\mathfrak{g}$ has the basis Q, Q^2. Given a commutative superalg R a degree zero element of $R \otimes \mathfrak{g}$ is of the form

$$t Q^2 + \theta Q \qquad t \in R^0, \ \theta \in R^1$$

and the bracket of two of these elements is

$$[t_1 Q^2 + \theta_1 Q, t_2 Q^2 + \theta_2 Q] = [\theta_1 Q, \theta_2 Q] =$$

$$\theta_1 Q \theta_2 Q - \theta_2 Q \theta_1 Q = -2\theta_1\theta_2 Q^2$$

This means that the Lie algebra $(R \otimes \mathfrak{g})^0$ is a central extension

$$0 \longrightarrow R^0 \cdot Q^2 \longrightarrow (R \otimes \mathfrak{g})^0 \longrightarrow R^1 Q \longrightarrow 0$$

and so is nilpotent. Hence there is an obvious group we [redacted] can attach to it, namely, the group generated by (formal) exponentials

$$e^{tQ^2 + \theta Q} \qquad t \in R^0, \text{ [redacted] } \theta \in R^1$$

with the group law

$$\boxed{e^{t_1 Q^2 + \theta_1 Q} \; e^{t_2 Q^2 + \theta_2 Q} = e^{(t_1 + t_2 - \theta_1 \theta_2) Q^2 + (\theta_1 + \theta_2) Q}}$$

which comes from $e^A e^B = e^{A + B + \frac{1}{2}[A,B] + \cdots}$ or more directly

$$e^{t_1 Q^2 + \theta_1 Q} \; e^{t_2 Q^2 + \theta_2 Q} = e^{(t_1 + t_2) Q^2} \underbrace{(1 + \theta_1 Q)(1 + \theta_2 Q)}_{1 + (\theta_1 + \theta_2) Q + \underbrace{\theta_1 Q \theta_2 Q}_{-\theta_1 \theta_2 Q^2}}$$

$$= e^{(t_1 + t_2) Q^2 + (\theta_1 + \theta_2) Q - \theta_1 \theta_2 Q^2}$$

[redacted]

So now we have a functor from commutative superalgebras to groups. One can think as follows: a map from a supermanifold $(Z, \mathcal{O}_Z)$ into $\mathbb{R}^{1,1}$ can be identified with an even "function" t on Z together with an odd "function" θ and we know how to make these into a non-commutative group as above. Notice that $t \in \mathcal{O}_Z^{ev}$ determines an actual function $Z \to \mathbb{R}^{1,0}$ which is the reduced map associated to $(t, \theta) : (Z, \mathcal{O}_Z) \to \mathbb{R}^{1,1}$.

Next consider the convolution superalgebra of $\mathbb{R}^{1,1}$. This can be defined in terms of representations of this supergp:

Every representation is a supermodule over the convolution superalgebra. The representation is clearly determined by Q, so that the convolution superalgebra should contain functions of Q. Think in terms of Q^2 being self adjoint. Then the convolution algebra contains $A(Q^2) + B(Q^2)Q$. Note that it commutative as an algebra, but non-commutative as a superalgebra. just like the universal enveloping superalgebra of $\mathfrak{g}$.

Also if we write a "function" $f(t,\theta)$ on $\mathbb{R}^{1,1}$ in the form $f(t,\theta) = a(t) + \theta b(t)$, then

$$\int dt \int d\theta \;\; f(t,\theta)\, e^{tQ^2 + \theta Q}$$

$$= \int dt \int d\theta \;\; (a + \theta b)(1 + \theta Q) e^{tQ^2}$$

$$= \int dt\, (b + aQ) e^{tQ^2} = \int dt\, b(t) e^{tQ^2} + Q \int dt\, a(t)\, e^{tQ^2}$$

Now that we have identified the convolution algebra we can identify left and right infinitesimal translation corresponding to the element Q in the Lie algebra. Let's consider left multiplication by εQ.

$$\varepsilon Q \int dt\, (b + aQ) e^{tQ^2} = \varepsilon \int dt\, (bQ e^{tQ^2} + a \underbrace{Q^2 e^{tQ^2}}_{\frac{d}{dt} e^{tQ^2}})$$

$$= \varepsilon \int dt\, (-a' + bQ) e^{tQ^2}$$

This corresponds to $f = a + \theta b \longrightarrow b - \theta a' = (\partial_\theta - \theta \partial_t) f$

so

$$\boxed{\varepsilon Q \cdot f = \varepsilon(\partial_\theta - \theta \partial_t) f}$$

Next consider right mult.

$$\int dt\, (b + aQ) e^{tQ^2} \varepsilon Q = \varepsilon \int dt\, (bQ - aQ^2) e^{tQ^2}$$

$$= \varepsilon \int dt \, (a' + bQ) e^{tQ^2}$$

This corresponds to $\quad f = a + \theta b \longrightarrow b + \theta a' = (\partial_\theta + \theta \partial_t) f$
so we have

$$\boxed{f \cdot \varepsilon Q = \varepsilon (\partial_\theta + \theta \partial_t) f}$$

At this point [redacted] we understand something about the "path integral with fermions" which is supposed to yield the super heat operator. The integral is some kind of integral of a "function" on the supermanifold $\mathrm{Map}(\mathbb{R}^{1,1}, M)$. Now as Bott points out this is

$$\mathrm{Map}(\mathbb{R}^{0,1}, \mathrm{Map}(\mathbb{R}^{1,0}, M))$$

so the "functions" on this supermanifold are differential forms on the path space. In particular the action should be a differential form on the path space. I guess it is clear that

$$S = \int (\dot{x}^2 + \psi \dot{\psi}) \, dt$$

can be interpreted as the sum of a zero form and a 2-form on the path space.

Problem: What does one mean by a critical point for a differential form?

The critical points of the action are supposed to be the classical paths, hence the above question.

Actually it [redacted] seems to be more illuminating to take the case of a pure fermion theory, say a single fermion to keep things simple. Then what this is analogous to is a system with a single degree q of freedom, and the action in this case is a function

$$S = \int L(q, \dot{q})\, dt$$

on the space of paths $q(t)$ in $\mathbb{R}$. So if V is the vector space of such paths, an action such as

$$\int \left(\tfrac{1}{2}\dot{q}^2 - \tfrac{1}{2}\omega^2 q^2\right) dt$$

is simply a quadratic form on V. The typical fermion action

$$\int \psi \dot{\psi}\, dt$$

is a skew-symmetric form on the space of paths $\psi(t)$ with values in $\mathbb{R}$. So a more general question would be: Given an element in $\Lambda(W^*)$ what does one mean by its critical points?

Now given $f \in S(W^*)$ its critical points are those points ξ of W such that $df \in W^* \otimes S(W^*)$ vanishes when we use the evaluation at ξ map

$$\left|\begin{array}{lcl} W^* \otimes S(W^*) & \longrightarrow & W^* \\ dx^\mu \otimes g_\mu & \longrightarrow & dx^\mu g_\mu(\xi). \end{array}\right.$$

However we've already decided that it is impossible to evaluate an element of $\Lambda(W^*)$. Nevertheless we can divide out by the ideal in $\Lambda(W^*)$ generated by the derivatives.

March 30, 1984

Critical points for a fermion [redacted] action. The action is an element S of $\Lambda(W^*)$, [redacted] and one is given a subspace W' of W [redacted] consisting of the variations the action is subject to. À la Grothendieck one considers a point of $\Lambda(W^*)$ to be a homomorphism of ΛW^* to a commutative superalgebra. So a point will be critical when it kills the derivatives $\partial_{w'} S$ of S with respect to the elements in W'. The critical points are the points of the quotient algebra of $\Lambda(W^*)$ by the ideal generated by the $\partial_{w'} S$, $w' \in W'$. (Here $\partial_w = i_w$)

For example, let's consider a quadratic action $S \in \Lambda^2(W^*)$. Then $\{\partial_{w'} S\}$ is a subspace of W^* which is the annihilator of the subspace

$$Z = \{ w \in W \mid \partial_w \partial_{w'} S = 0 \ \text{[redacted]} \ \forall w' \in W' \}.$$

[redacted] Actually, to be precise put $i_{W'} S = \{\partial_{w'} S\}$, then clearly $Z = (i_{W'} S)^\perp$, so that morally $Z^\perp = i_{W'} S$, hence

$$W^* / i_{W'} S = W^*/Z^\perp \simeq Z^* .$$

Hence the critical points are the 'points' of

$$\Lambda(W^*) \Big/ \begin{matrix}\text{ideal gen} \\ \text{by } i_{W'} S\end{matrix} = \Lambda(Z^*).$$

So one can think of the critical points as being the [redacted] subspace Z of W consisting of those elements orthogonal to the space of variations W' wrt the skew form S.

Let's be specific and consider an action

$$\int_a^b \left(\psi^\mu \mathring{\psi}^\mu + \tfrac{1}{2}\omega_{\mu\nu}\psi^\mu\psi^\nu\right) dt . \qquad \omega_{\mu\nu} = -\omega_{\nu\mu}$$

Here W will be the space vector functions $f(t) = (f^\mu(t))$ on the interval $a \le t \le b$. On W one has the linear functionals $\psi^\mu(t) : f \longrightarrow f^\mu(t)$, $\mathring{\psi}^\mu(t) : f \longmapsto \mathring{f}^\mu(t)$. One takes their wedge product to get the skew form

$$\psi^\mu(t)\mathring{\psi}^\mu(t) : \quad f_1, f_2 \longmapsto f_1^\mu(t)\mathring{f}_2^\mu(t) - f_2^\mu(t)\mathring{f}_1^\mu(t)$$

for each t, μ, then one sums over μ and integrates over t. The possible variations form the subspace W' of W consisting of those f vanishing at a and b.

Take the skew form on a pair f, f_1, where $f_1 \in W'$.

$$f, f_1 \longmapsto \int_a^b \left(f^\mu \mathring{f}_1^\mu - f_1^\mu \mathring{f}^\mu + \omega_{\mu\nu} f^\mu f_1^\nu\right) dt$$

$$= \underbrace{\left[f^\mu f_1^\mu\right]_a^b}_{\substack{\shortparallel \\ 0 \text{ as } f_1 \in W}} + \int_a^b \left(-2\mathring{f}^\mu f_1^\mu + \omega_{\mu\nu} f^\mu f_1^\nu\right) dt$$

If we want this to vanish for all $f_1 \in W$, then we get

$$-2\mathring{f}^\mu + \omega_{\nu\mu} f^\nu = 0$$

Let's check this against the physicists mode of calculation

$$\delta \int \left(\psi^\mu \mathring{\psi}^\mu + \tfrac{1}{2}\omega_{\mu\nu}\psi^\mu\psi^\nu\right) dt = \int \Big[\delta\psi^\mu \mathring{\psi}^\mu + \psi^\mu \delta\mathring{\psi}^\mu + \tfrac{1}{2}\omega_{\mu\nu}\,\delta\psi^\mu\psi^\nu + \tfrac{1}{2}\omega_{\mu\nu}\psi^\mu\delta\psi^\nu\Big] dt$$

$$= \left[\psi^\mu \delta\psi^\mu\right]_a^b + \int \left[\delta\psi^\mu \mathring{\psi}^\mu - \mathring{\psi}^\mu\delta\psi^\mu + \omega_{\mu\nu}\,\delta\psi^\mu\psi^\nu\right] dt$$

$$= \cancel{\left[\psi^\mu \delta\psi^\mu\right]_a^b} + \int \delta\psi^\mu\left(2\mathring{\psi}^\mu + \omega_{\mu\nu}\psi^\nu\right) dt$$

which gives the equations

$$2\dot{\psi}^\mu + \omega_{\mu\nu}\psi^\nu = 0$$

which are clearly the same.

Let's consider a scalar particle described by the Laplacean $\frac{1}{2}D_\mu^2 = \frac{1}{2}(\partial_\mu + iA_\mu)^2$, where A_μ is real. The Hamiltonian in this case is $\frac{1}{2}(p-A)^2$, whence

$$\dot{x} = \frac{\partial H}{\partial p} = p - A$$

so

$$L = p\dot{x} - H = (\dot{x}+A)\dot{x} - \frac{\dot{x}^2}{2} = \frac{\dot{x}^2}{2} + A\dot{x}$$

or more precisely $L = \frac{1}{2}(\dot{x}^\mu)^2 + A_\mu \dot{x}^\mu$. Now the action $\frac{i}{h}S = \frac{i}{h}\int_0^t \left(\frac{1}{2}\dot{x}^2 + A\dot{x}\right)dt$ should lead to $e^{+i\frac{h}{2}D_\mu^2 t}$

and the action

$$-S = \int_0^\beta \left(\frac{i}{h}p\dot{x} - \frac{(p-A)^2}{2}\right)dt \quad \text{should lead to} \quad e^{\beta\frac{h^2}{2}D_\mu^2}.$$

We should have

$$\langle q|e^{\beta h^2 \frac{1}{2}D_\mu^2}|q'\rangle = \int\limits_{\substack{q(0)=q' \\ q(\beta)=q}} \mathcal{D}q\mathcal{D}p \; e^{\int_0^\beta \left(\frac{i}{h}p\dot{q} - \frac{(p-A)^2}{2}\right)dt}.$$

Now the stationary phase method applied to the above as $h\to 0$, says we must make $\int p\dot{q}\,dt$ stationary, which forces $\dot{q}=0$, and hence the limit is zero for $q\neq q'$. To get a sensible limit we put $q' + hx(t) = q(t) \implies q = q(\beta) = q' + hx(\beta)$

whence

$$S = \int \left[\frac{i}{h} p h \dot{x} - \frac{(p - A(q' + hx))^2}{2}\right] dt$$

$$= \int \left[ip\dot{x} - \frac{(p-A)^2}{2}\right] dt \quad + O(h) \qquad A = A(q')$$

Now do the p ~~integration~~: first change $p \to p + A$

$$S = \int \left[ip\dot{x} + iA\dot{x} - \frac{p^2}{2}\right] dt$$

and so we get

$$\langle q' + hv | e^{\beta h^2 \frac{1}{2} D_\mu^2} | q' \rangle = \int_{\substack{x(0)=0 \\ x(\beta)=v}} \mathcal{D}x(t)\, e^{\int_0^\beta \left[-\frac{1}{2}\dot{x}^2 + iA\dot{x}\right] dt} (1 + O(h))$$

This last integral is Gaussian, so it can be done by finding the critical point (actually one would have done this before p integration)

$$\frac{d}{dt} \frac{\partial}{\partial \dot{x}} \left(-\frac{1}{2}\dot{x}^2 + iA\dot{x}\right) = \frac{d}{dt}(-\dot{x} + iA) = -\ddot{x} = 0$$

$\Rightarrow \quad \dot{x}$ constant $\quad \Rightarrow \quad x(t) = \frac{v}{\beta} t$, so the above integral is

$$e^{-\frac{1}{2}\frac{v^2}{\beta} + iAv} \times \text{norm. constant} \times (1 + O(h))$$

$$v = \frac{q - q'}{h} \qquad A = A(q')$$

Presumably this result would come out of gauge transformations.

At the moment I feel that it might be possible to find the limiting heat kernel results from the path integral formalism. A basic problem is how Planck's constant should appear in the action. For example we know in the boson setup that

$$S = \int_0^\beta \left(\frac{i}{\hbar} p\dot{q} - H\right) dt$$

is such that $\int \mathcal{D}p\mathcal{D}q \; e^S$ gives the heat operator $e^{-\beta H}$. Then if $H = \frac{p^2}{2} + V(q)$ and I do the p-integration I get the action

$$S = \int_0^\beta \left(-\frac{1}{2}\left(\frac{\dot{q}}{\hbar}\right)^2 - V(q)\right) dt .$$

But now what do I do for the fermions? Now the Lagrangian is $\int \frac{1}{2}\psi\dot{\psi}\, dt$ or more generally $\int \frac{1}{2}(\psi^\mu \dot{\psi}^\mu + \omega_{\mu\nu}\psi^\mu\psi^\nu)dt$, before $\hbar$ is put in. In this case the heat operator $e^{-\beta H}$ is what?

March 31, 1984

Friedan + Windey represent $e^{-\tau H - \hat{\tau} Q}$ by

$\int \mathcal{D}X \, \mathcal{D}\bar{N}\mathcal{D}N \; e^{-S}$ where $S = -\int_0^1 dt \int d\theta \left[\frac{1}{4g^2}(DX)D(DX) + \bar{N}DN \right]$

where $g = \tau^{1/2} - \theta\hat{\tau}$. Now I am interested in $e^{-h^2 H}$ so we see their $\tau^{1/2}$ = my h. Thus their action resembles the action

$$S = -\int_0^\beta \left[\frac{1}{2}\left(\frac{\dot{q}}{h}\right)^2 + V(q) \right] dt$$

which is such that $\int \mathcal{D}q \, e^S$ represents $e^{-\beta H}$ $H = \frac{p^2}{2} + V(q)$.

The conclusion I draw from this is that the $\bar{N}DN$ part of the Lagrangian doesn't involve h. In other words I can do the $N, \bar{N}$ integration and I obtain a "function" of X with matrix values. (Strictly: The paths go between given endpoints x', x and the $N, \bar{N}$ integral will [redacted] have its values in $\mathrm{Hom}(E_{x'}, E_x)$.)

[redacted] Thus we get a "function" of X with matrix values [redacted] i.e. a differential form on the space of paths $x(t)$.

Let's calculate the kinetic part of the FW action, so as to see what the critical points are.

$$DX = (\theta\partial_t - \partial_\theta)(x + \theta\psi) = \theta\dot{x} - \psi$$

$$DX\,\partial_t X = (\theta\dot{x} - \psi)(\dot{x} + \theta\dot{\psi}) = -\psi\dot{x} + \theta(\dot{x}^2 + \psi\dot{\psi})$$

$$g^{-2} = (h - \theta\hat{\tau})^{-2} = \left[h\left(1 - \theta\frac{\hat{\tau}}{h}\right)\right]^{-2} = h^{-2}\left(1 + 2\theta\frac{\hat{\tau}}{h}\right)$$

$$\int_0^1 dt \int d\theta \, \frac{1}{g^2} DX \, \partial_t X = \int dt \int d\theta \, \frac{1}{h^2}\left(1+2\theta\frac{\hat\tau}{h}\right)\left[-\psi\dot x + \theta(\dot x^2+\psi\dot\psi)\right]$$

$$= \frac{1}{h^2}\int dt \int d\theta \left[-\psi\dot x + \theta(\dot x^2+\psi\dot\psi) - 2\theta\frac{\hat\tau}{h}\psi\dot x\right]$$

$$= \frac{1}{h^2}\int dt \left[\dot x^2 + \psi\dot\psi - 2\frac{\hat\tau}{h}\psi\dot x\right]$$

Recall this action is to yield $e^{-h^2H-\hat\tau Q}$ and we are trying to get something reasonable as $h \to 0$. It seems therefore that we want to put $\boxed{\hat\tau = h\varepsilon}$ to get something nice. (If we leave the h in the denominator then as $h\to 0$ we get $e^{-\hat\tau Q} = (1-\hat\tau\gamma^\mu p_\mu)$ which is pretty singular; the critical points of $\int(\psi\dot x)\,dt$ are $\dot\psi = 0$, $\dot x = 0$. On the other hand if we let $\hat\tau = o(h)$, e.g. put $\hat\tau = 0$ as in FW, then we get just the heat operator $e^{-h^2H} = e^{-P^2}$; the critical points of $\int(\dot x^2+\psi\dot\psi)\,dt$ are $\ddot x = 0$, $\dot\psi = 0$.)

What are the critical points of

$$S = \frac{1}{2}\int dt \left(\dot x^2 + \psi\dot\psi - 2\varepsilon\psi\dot x\right) \ ?$$

$$\delta S = \int dt \, \frac{1}{2}\left(2\dot x\,\delta\dot x + \delta\psi\,\dot\psi + \psi\,\delta\dot\psi - 2\varepsilon\,\delta\psi\,\dot x - 2\varepsilon\psi\,\delta\dot x\right).$$

$$= \int dt \left(-\ddot x\,\delta x + \delta\psi\,\dot\psi - \varepsilon\,\delta\psi\,\dot x + \varepsilon\dot\psi\,\delta x\right)$$

$$= \int dt \left[(-\ddot x + \varepsilon\dot\psi)\,\delta x + (-\varepsilon\dot x - \dot\psi)\,\delta\psi\right] = 0$$

$$\Longrightarrow \quad \ddot x = \varepsilon\dot\psi, \quad \dot\psi + \varepsilon\dot x = 0 \quad \Longrightarrow \quad \ddot x = 0$$

So $\dot x$ is constant and $\dot\psi = -\varepsilon\dot x$

Suppose we start with the path integral

$$\int \mathcal{D}x\,\mathcal{D}\psi\; e^{-S}$$

where $S = \frac{1}{\hbar^2}\int_0^\beta \frac{1}{4}\left(\dot{x}^2 + \psi\dot{\psi} - 2\varepsilon\psi\dot{x}\right)dt$ (the $\frac{1}{4}$ is so we obtain $e^{\beta\partial^2}$). The action is quadratic in x and ψ, and hence the path integral can be evaluated by finding the critical value of S. The critical points can be found in the standard way, but to be different write $L = \frac{1}{2}(\dot{x}^2 + \psi\dot{\psi}) - \varepsilon\psi\dot{x}$; then

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) - \frac{\partial L}{\partial x} = \ddot{x} - \varepsilon\dot{\psi} = 0$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\psi}}\right) - \frac{\partial L}{\partial \psi} = \frac{d}{dt}\left(-\frac{1}{2}\psi\right) - \left(\frac{1}{2}\dot{\psi} + \varepsilon\dot{x}\right) = -\dot{\psi} - \varepsilon\dot{x} = 0$$

(where $\frac{\partial}{\partial\psi}(-\varepsilon\psi\dot{x}) = +\varepsilon\frac{\partial\psi}{\partial\psi}\dot{x} = \varepsilon\dot{x}$). Thus the critical points are

$$\dot{\psi} = -\varepsilon\dot{x} \qquad \ddot{x} = \varepsilon\dot{\psi} = 0$$

so $\dot{x}$ is constant, and so is $\dot{\psi}$ whence

$$\psi(t) = \psi_0 - \varepsilon\dot{x}t$$

$$\psi\dot{\psi} = (\psi_0 - \varepsilon\dot{x}t)(-\varepsilon\dot{x}) = -\psi_0\varepsilon\dot{x}$$

$$\varepsilon\psi = \varepsilon\psi_0$$

Then the restriction of S to these critical points is

$$S = \frac{1}{\hbar^2}\int_0^\beta \frac{1}{4}\left(\dot{x}^2 - \psi_0\varepsilon\dot{x} - 2\varepsilon\psi_0\dot{x}\right)dt$$

$$= \frac{1}{\hbar^2}\beta\,\frac{1}{4}\left(\dot{x}^2 - \varepsilon\psi_0\dot{x}\right)$$

Now the path integral $\int \mathcal{D}x\, \mathcal{D}\psi\, e^{-S}$ is supposed to represent a superheat operator in some way, rather the kernel of this operator. We've seen this path integral reduces to one over a much simpler supermanifold, namely the set of paths $x(t)$, $0 \le t \le \beta$ with $\ddot{x} = 0$ together with the space of ψ_0. I don't yet understand how the fermion sides of this integral is to be interpreted as an operator. However we can replace $\{x(t)\}$ by the possible endpoints:

$$x(t) = q' + \dot{x} t \qquad \dot{x} = \frac{q-q'}{\beta}$$

whence the path integral gives

$$\text{norm. const} \times e^{-\frac{1}{4\beta}\left(\frac{q-q'}{h}\right)^2 + \frac{\varepsilon}{4h}\left(\frac{q-q'}{h}\right)\psi_0}$$

Let's now compute the actual super heat operator $e^{-\beta H - \eta Q}$, where $Q = i\gamma^\mu p_\mu$, $H = -Q^2 = p^2$

$$\langle q | e^{-\beta H - \eta Q} | q' \rangle = \int \frac{d^n p}{(2\pi h)^n}\, e^{ip\left(\frac{q-q'}{h}\right) - \beta p^2 - \eta \cdot i\gamma^\mu p_\mu}$$

Put $v = \frac{q-q'}{h}$ $\qquad t = \eta\gamma$

$$\int \frac{d^n p}{(2\pi h)^n} e^{ipv - \beta p^2 - itp} = \int \frac{d^n p}{(2\pi h)^n}\, e^{ip(v-t) - 2\beta \frac{p^2}{2}}$$

$$= \frac{(\sqrt{2\pi})^n}{(2\pi h)^n} \frac{1}{(\sqrt{2\beta})^n}\, e^{-\left(\frac{1}{2}\right)\frac{1}{2\beta}(v-t)^2} = \frac{1}{(4\pi\beta h^2)^{n/2}}\, e^{-\frac{1}{4\beta}\left[v^2 - 2v\eta\gamma\right]}$$

$$= \frac{1}{(4\pi\beta h^2)^{n/2}} e^{-\frac{1}{4\beta}\left(\frac{g-g'}{h}\right)^2 + \frac{1}{2\beta}\eta\left(\frac{g-g'}{h}\right)\gamma}$$

Hence comparing this with $*$ we see that

$$\text{norm const} = \frac{1}{(4\pi\beta h^2)^{n/2}}$$

$$\frac{\varepsilon}{4h}\psi_0 = \frac{1}{2\beta}\eta\gamma$$

[redacted] In order to satisfy the latter it seems we ought to take

$$\psi_0 = h\gamma \quad , \quad \eta = \frac{\beta}{2}\varepsilon$$

(At this point I seem to find that FW (4.6)

$$S = \int_0^\tau dt\left(\frac{1}{4}\dot{x}^\mu\dot{x}^\mu + \frac{1}{4}\psi^\mu\dot{\psi}^\mu - \frac{1}{2}\frac{\hat{\tau}}{\tau}\psi^\mu\dot{x}^\mu\right)$$

does not give the operator $e^{-\tau H - \hat{\tau} Q}$ but is wrong. Namely $\int \mathcal{D}[X]\, e^{-S}$ gives $e^{-\tau H - \eta Q}$ where $\eta = \frac{\tau}{2}\frac{\hat{\tau}}{\tau} = \frac{\hat{\tau}}{2}$)

Summarize: My goal is to obtain the kernel of the super heat operator

$$e^{-\beta H - \varepsilon Q} \qquad Q = i\gamma^\mu p_\mu \;, \quad H = -Q^2 = p^2$$

One has

$$\langle g | e^{-\beta H - \varepsilon Q} | g' \rangle = \frac{1}{(4\pi\beta h^2)^{n/2}} e^{-\frac{1}{4\beta}v^2 + \frac{1}{2\beta}\varepsilon\gamma^\mu v_\mu}$$

where $v = \frac{g-g'}{h}$. On the other hand if we form a path integral $\int \mathcal{D}x\mathcal{D}\psi\, e^{-S}$, where

~~[illegible]~~

$$S = \frac{1}{h^2}\int_0^\beta \left(\tfrac{1}{4}\dot{x}^2 + \tfrac{1}{4}\psi\dot{\psi} - \tfrac{\varepsilon}{\beta}\psi\dot{x}\right)dt$$

and $x(t)$ runs over paths going from q' to q, then evaluating the path integral by evaluating the action at the stationary point yields

$$\begin{pmatrix}\text{norm}\\ \text{const}\end{pmatrix} e^{-\frac{1}{4\beta}\left(\frac{q-q'}{h}\right)^2 + \frac{1}{2\beta}\varepsilon\frac{\psi_0}{h}\left(\frac{q-q'}{h}\right)} \qquad \psi_0 = \psi(q')$$

Therefore if I identify $\frac{\psi_0}{h}$ with γ and the normalization constant with $\frac{1}{(4\pi\beta h^2)^{n/2}}$, then the above action gives the superheat kernel.

Now the program will be to bring in the gauge fields and the "internal degrees" of freedom" $\bar{\eta}, \eta$. The action should be

$$\boxed{S = \frac{1}{h^2}\int_0^\beta \left(\tfrac{1}{4}\dot{x}^2 + \tfrac{1}{4}\psi\dot{\psi} - \tfrac{\varepsilon}{\beta}\psi\dot{x}\right)dt + \int_0^\beta dt\ \bar{\eta}\left(\partial_t + \dot{x}A - \tfrac{1}{2}\psi\psi F\right)\eta}$$

I want to handle the way I did the scalar operator D_μ^2 (665-6). Here

$$-S = \int_0^\beta \left(\frac{i}{h}p\dot{x} - \frac{(p-A)^2}{2}\right)dt$$

To do the p integration use critical point

$$i\frac{\dot{x}}{h} - (p-A) = 0 \qquad \text{or} \qquad p = i\frac{\dot{x}}{h} + A$$

$$\frac{i}{h}\dot{x}\left(i\frac{\dot{x}}{h} + A\right) - \frac{1}{2}\left(i\frac{\dot{x}}{h}\right)^2 = \frac{1}{2}\left(i\frac{\dot{x}}{h}\right)^2 + i\frac{\dot{x}}{h}A$$

New action is $\displaystyle\int_0^\beta \left(-\frac{1}{2}\left(\frac{\dot{x}}{h}\right)^2 + iA\frac{\dot{x}}{h}\right)dt$

The idea is now to write $x(t) = x(0) + hv(t)$

and then to let $h \to 0$. This gives

$$\int_0^\beta \left(-\tfrac{1}{2}\dot{v}^2 + iA(x(0)+hv(t))\,\dot{v}\right)dt$$

$$= \int_0^\beta \left(-\tfrac{1}{2}\dot{v}^2 + i A_0 \dot{v}\right)dt \;+ O(h)$$

and now do the integral via critical points.

Thus to treat the action on the preceding page I want to rescale x, ψ in such a way that we can see what the asymptotic expansion in h is. (I think we are going to have to replace: $\dot{x} = h\dot{v}$, $\psi = h\gamma$ whence the second term in S will have h's we won't want to set $= 0$.

April 1, 1984

I would like to attempt again to pin down what should be meant by the limiting heat kernel, especially in a pure fermion situation, e.g. $L = \psi\dot{\psi}$, where there's no obvious kernel one can attach to an operator.

Let's begin with the idea that for physical reasons (e.g. because physicists are very smart and have great insight) a system in quantum mechanics is described by an algebra of operators, called observables, that there is an operator H in this algebra such that the average value of an [redacted] **observable A at inverse** temperature β is $\mathrm{tr}(e^{-\beta H}A)/\mathrm{tr}(e^{-\beta H})$, and finally that as Planck's constant goes to zero the whole setup goes over into the classical mechanics description.

My idea here is the following. There is a path integral describing the heat operator $e^{-\beta H}$. More precisely the Schwartz kernel $\langle q|e^{-\beta H}|q'\rangle$ is represented by a path integral over all paths going from q' to q. [redacted] When I calculated with the fermion $L = \bar{\psi}\dot{\psi} - \omega\bar{\psi}\psi$ I found a kernel $\langle J|e^{-\beta H}|J'\rangle$ in some sense. But it has to be interpreted as a distribution. If one adopts the analogous viewpoint toward $\langle q|e^{-\beta H}|q'\rangle$ one sees that one is interested in $\mathrm{tr}(e^{-\beta H}A)$ for all operators A. Then the $h \to 0$ limit, which I handled before by putting $q = q' + hv$, now might be treated by restricting A to being an observable.

In any case we can try to test this out in [redacted] a fermion theory with real fermions.

Let's get a typical real fermion Lagrangian. The following seems to be a simple example. Let's [redacted] take the action in p.673 which is supposed to yield $e^{-\beta H - \varepsilon Q}$. We have to integrate over $x(t)$, $\psi(t)$, $\eta(t)$, $\bar{\eta}(t)$. First ask what

$$\int \mathcal{D}\bar\eta \mathcal{D}\eta \, e^{-\int dt \; \bar\eta (\partial_t \; \alpha)\eta}$$

looks like. The action being quadratic one looks at the critical points

$$(\partial_t \; \alpha)\eta = 0 \qquad +\partial_t \bar\eta + \alpha \bar\eta = 0$$

and replaces the integral by the integral over the critical points. It's more or less clear that by the time the endpoint conventions are worked out one should just be getting the parallel transport operator

$$T\left\{ e^{\int_0^\beta \alpha \, dt} \right\}$$

extended to the exterior algebra.

A simple case is where η has one component in which case A, F are 1×1 matrices and so

$$T\left\{ e^{\int (-\dot x A + \frac{1}{2}\psi\psi F) dt} \right\} = e^{\int_0^\beta (-\dot x A + \frac{1}{2}\psi\psi F) dt}$$

Thus in the case of an abelian gauge field the $\bar\eta, \eta$ integration can be done and yields a representation for $e^{-\beta H - \varepsilon Q}$ using the action

$$S = \frac{1}{h^2}\int_0^\beta \left(\tfrac{1}{4}\dot x^2 + \tfrac{1}{4}\psi\dot\psi - \tfrac{\varepsilon}{\beta}\psi \dot x\right) dt + \int_0^\beta \left(+\dot x A \; \tfrac{1}{2}\psi\psi F\right) dt$$

Recall that the $\frac{1}{h^2}$ is justified by several reasons: It gives the right answer for $i\gamma^\mu \partial_\mu = Q$, and also comes out of the Friedan-Windey formulas. Now we want to integrate this over ψ, and the Lagrangian is quadratic. Hence the integral over ψ ought to be easy, modulo an interpretation of what it means.

Let's therefore consider a quadratic real fermion Lagrangian

$$L = \tfrac{1}{2}\psi\dot{\psi} - \tfrac{1}{2}\omega_{\mu\nu}\psi^\mu\psi^\nu + J_\mu\psi^\mu$$

The equations of motion are

$$\frac{d}{dt}\left(\frac{\partial L}{\partial\dot{\psi}^\mu}\right) = -\tfrac{1}{2}\dot{\psi}^\mu = \frac{\partial L}{\partial\psi^\mu} = \tfrac{1}{2}\dot{\psi}^\mu - \tfrac{1}{2}\omega_{\mu\nu}\psi^\nu + \tfrac{1}{2}\omega_{\nu\mu}\psi^\nu - J_\mu$$

or $\boxed{\dot{\psi}^\mu = \omega_{\mu\nu}\psi^\nu + J_\mu}$. If we substitute the equations of motion into L we get

$$\tfrac{1}{2}\psi(\omega\psi + J) - \tfrac{1}{2}\psi\omega\psi + J\psi = \tfrac{1}{2}J\psi$$

Let's take $J = 0$. Then the equations of motion tell us that the field ψ develops according to the infinitesimal orth. transformation ω. But I don't understand what this means, because the action $\int L\,dt$ is zero.

April 2, 1984

Motion in a uniform magnetic field. This is an abelian gauge field with constant curvature.

Quite generally given a connection, let's fix the gauge by using radial parallel transport from the origin. Then we have $D = dx^\mu(\partial_\mu + A_\mu)$, where $A_\mu = 0$ at $x=0$ and $x^\mu D_\mu = x^\mu \partial_\mu \iff x^\mu A_\mu = 0$.

So if we write

$$A_\mu = x^\nu a_{\nu\mu} + O(x^2)$$

we have $A = dx^\mu A_\mu = a_{\nu\mu} x^\nu dx^\mu + O(x^2)$

so $F = dA + A^2 = a_{\nu\mu} dx^\nu dx^\mu + O(x)$ and so we conclude that $a_{\nu\mu} = \frac{1}{2}F_{\mu\nu}$ at $x=0$. The conclusion is that using radial parallel transport to trivialize the bundle yields

$$\boxed{A_\mu = \frac{1}{2}x^\nu F_{\nu\mu}^{(0)} + O(x^2)}$$

In the abelian case A is a one-form, so $A^2 = 0$. If $F = dA$ is constant, i.e. $F_{\mu\nu}$ is constant, then one can choose a gauge transformation to make $A = \frac{1}{2}x^\nu F_{\nu\mu}dx^\mu$.

We want to work out the super heat kernel for $\not{D}_A$. $F_{\nu\mu}$ is a skew-symmetric matrix and our metric gives us an inner product, so by an orthog. transformation we can reduce to a direct sum of 2-plane situations.

Let's go over the classical mechanics of motion in a magnetic field. The Hamiltonian is

$$H = \tfrac{1}{2}(p-A)^2$$

(This A is real, so that I have dropped an i which has to be reintroduced later). Then

$$\dot{x} = \frac{\partial H}{\partial p} = p - A$$

so

$$L = p\dot{x} - H = (\dot{x}+A)\dot{x} - \tfrac{1}{2}\dot{x}^2 = \tfrac{1}{2}\dot{x}^2 + A\dot{x}$$

Lets consider $A = -\omega y\,dx$ on $\mathbb{R}^2$. Then

$$L = \tfrac{1}{2}(\dot{x}^2+\dot{y}^2) - \omega y \dot{x}$$

$$p_x = \frac{\partial L}{\partial \dot{x}} = \dot{x} - \omega y \qquad \dot{p}_x = \ddot{x} - \omega \dot{y} = \frac{\partial L}{\partial x} = 0$$

$$p_y = \frac{\partial L}{\partial \dot{y}} = \dot{y} \qquad \dot{p}_y = \ddot{y} = \frac{\partial L}{\partial y} = -\omega \dot{x}$$

$$\ddot{x} = \omega \dot{y}\,, \quad \ddot{y} = -\omega \dot{x}$$

$$\therefore \quad x = x_0 + \mathrm{Re}(\alpha e^{-i\omega t}) = x_0 + |\alpha|\cos(\omega t + \phi)$$

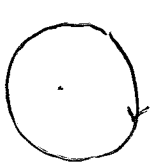
$$y = y_0 + \mathrm{Im}(\alpha e^{-i\omega t}) = y_0 - |\alpha| \sin(\omega t + \phi)$$

Thus we see that for a field with

$$F = d(-\omega y\,dx) = \omega\,dx\,dy$$

we get rotational motion of frequency ω around arbitrary centers.

Same equations starting from



$$H = \tfrac{1}{2}\left[(p_x + \omega y)^2 + p_y^2\right]$$

$$\dot{x} = \frac{\partial H}{\partial p_x} = p_x + \omega y \qquad \dot{p}_x = -\frac{\partial H}{\partial x} = 0$$

$$\dot{y} = \frac{\partial H}{\partial p_y} = p_y \qquad \dot{p}_y = -\frac{\partial H}{\partial y} = -(p_x + \omega y)\omega$$

so $\qquad \ddot{y} = \dot{p}_y = -\dot{x}\omega \qquad$ and $\qquad \ddot{x} = (\dot{p}_x + \omega\dot{y}) = \omega\dot{y}$

as above.

Next we want to do the quantum mechanics. Then $p-A$ becomes $\frac{1}{i}\partial_\mu - A_\mu$: $\frac{1}{i}\partial_x + \omega y$, $\frac{1}{i}\partial_y$ and the Hamiltonian for a scalar particle is

$$H = \frac{1}{2}\left[\left(\frac{1}{i}\partial_x + \omega y\right)^2 + \left(\frac{1}{i}\partial_y\right)^2\right] = -\frac{1}{2}\left[(\partial_x + i\omega y)^2 + (\partial_y)^2\right]$$

The spectrum of this operator on $L^2(\mathbb{R}^2)$ can be found as follows. Observe that H commutes with ∂_x (and with $\partial_y + i\omega x$:

$$[\partial_y + i\omega x, \partial_x + i\omega y] = e^{-i\omega xy}[\partial_x, \partial_y]e^{i\omega xy} = 0)$$

Let's use the $[\partial_x, H] = 0$ fact, to decompose an eigenspace for H according to eigenfunctions of ∂_x; i.e. look for joint eigenfunctions. A joint eigenfn. has the form $e^{ikx}u(y)$ where u is an eigenfns for the operator

$$-\frac{1}{2}\left[(ik + i\omega y)^2 + \partial_y^2\right] = -\frac{1}{2}\partial_y^2 + \frac{1}{2}(\omega y + k)^2$$

This is essentially the harmonic oscillator Hamiltonian. So we conclude that the spectrum of H is $\{(n+\frac{1}{2})\omega; n \geq 0\}$.

If $u_n(y)$ is an n-th eigenfunction for $-\frac{1}{2}\partial_y^2 + \frac{1}{2}\omega^2 y^2$, then a complete family of eigenfunctions for H is

$$e^{ikx}u_n\left(y + \frac{k}{\omega}\right) \qquad k \in \mathbb{R},\ n \in \mathbb{N}$$

For example if $n=0$, then $u_0(y) = \text{const}\ e^{-\frac{\omega}{2}y^2}$ so we obtain the following basis for the $\frac{1}{2}\omega$ eigenspace

$$e^{ikx}e^{-\frac{\omega}{2}(y+\frac{k}{\omega})^2} = e^{-\frac{\omega}{2}y^2 + ikx - \cancel{2}\frac{\omega}{\cancel{2}}y\frac{k}{\omega} - \frac{\omega}{2}\frac{k^2}{\omega^2}}$$

or $\qquad e^{-\frac{\omega}{2}y^2 + ikz} \qquad\qquad z = x+iy$

We can check the space spanned by these functions is stable under ∂_x, $\partial_y + i\omega x$. One has

$$e^{\frac{\omega}{2}y^2}\begin{Bmatrix}\partial_x \\ \partial_y + i\omega x\end{Bmatrix} e^{-\frac{\omega}{2}y^2 + ikz} = \begin{Bmatrix}\partial_x \\ \partial_y \underbrace{- \omega y + i\omega x}_{i\omega z}\end{Bmatrix} e^{ikz}$$

so it's clear.

Let's take a more enlightened viewpoint. We have a line bundle L over $\mathbb{R}^2 = \mathbb{C}$ equipped with unitary connection, hence it has a holomorphic structure and the Dirac operator can be identified with $\bar\partial - \bar\partial^*$. I should really adopt a viewpoint as consistent with translation invariance of the curvature as possible. Also one has rotation invariance. Actually one has the Heisenberg group acting on the line bundle L.

Anyway, suppose we have a line bundle L over $\mathbb{R}^2 = \mathbb{C}$ equipped with a connection $D = (D_x, D_y)$, having constant curvature $[D_x, D_y] = -i\omega$. Then on $S \otimes L = L \oplus L$ we have the Dirac op

$$\not{D} = \gamma_x D_x + \gamma_y D_y = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} D_x + \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} D_y$$

$$= \begin{pmatrix} 0 & D_x - iD_y \\ D_x + iD_y & 0 \end{pmatrix} \qquad\qquad -[D_x, iD_y] = -\omega$$

Now

$$-\not{D}^2 = \begin{pmatrix} -(D_x - iD_y)(D_x + iD_y) & 0 \\ 0 & -(D_x + iD_y)(D_x - iD_y) \end{pmatrix} = -\Delta + \begin{pmatrix} -\omega & 0 \\ 0 & \omega \end{pmatrix}$$

where $-\Delta = -(D_x^2 + D_y^2) \geq 0$. In fact we can see that $-\Delta - \omega \geq 0 \Rightarrow -\Delta \geq \omega$, and this minimum is achieved on the L^2-holomorphic sections.

(Now this is an interesting development. I started out with idea that $D_x^2 + D_y^2$ is related to the harmonic oscillator, because of a specific choice of gauge: $D_x = \partial_x + i\omega y$, $D_y = \partial_y$. But then I reach the ~~situation~~ where the actual holom. representation for the harmonic oscillator occurs. Maybe I can use this to represent things nicely.)

The goal is to represent the super heat kernel belonging to $\not{D}$ in as convenient a form as possible. The super heat operator is $e^{+t\not{D}^2 + \theta\not{D}}$ and its kernel $\langle z | e^{t\not{D}^2 + \theta\not{D}} | z' \rangle \in \mathrm{Hom}(S \otimes L_{z'}, S \otimes L_z)$.

We will want to compare the value of this kernel with the radial parallel transpoint from z' to z. By translational invariance it is enough to take $z' = 0$ and then the result should be rotationally invariant (maybe covariant).

Let consider first $e^{t\not{D}^2}$ and use that this is easily related (bottom preceding page) to $e^{t\Delta}$, $\Delta = D_x^2 + D_y^2$. I want to use the radial $\parallel$ transport gauge:

$$D_x = \partial_x + \tfrac{1}{2} i\omega y, \quad D_y = \partial_y - \tfrac{1}{2} i\omega x$$

$$D_x + i D_y = 2\partial_{\bar{z}} + \tfrac{1}{2} i\omega y + \tfrac{1}{2}\omega x$$

$$= 2\partial_{\bar{z}} + \tfrac{1}{2}\omega z \qquad \text{kills } e^{-\frac{\omega}{4}|z|^2}$$

and $f(z) e^{-\frac{\omega}{4}|z|^2}$ where $f(z)$ is holomorphic.

$$D_x - i D_y = 2\partial_z + \tfrac{1}{2} i\omega y - \tfrac{1}{2}\omega x = 2\partial_z - \tfrac{1}{2}\omega\bar{z}$$

Then

$$(D_x - iD_y)(D_x + iD_y) = \left(2\partial_z - \tfrac{1}{2}\omega\bar{z}\right)\left(2\partial_{\bar{z}} + \tfrac{1}{2}\omega z\right)$$

$$\begin{array}{c} \parallel \\ \Delta + \omega \end{array} \qquad = 4\partial^2_{z\bar{z}} + \omega\left(z\partial_z - \bar{z}\partial_{\bar{z}}\right) - \tfrac{1}{4}\omega^2|z|^2 + \omega$$

Better approach to the spectrum of $D_x^2 + D_y^2$ is to use creation and annihilation operators.

$$\left[D_x + iD_y, -D_x + iD_y\right] = 2i[D_x, D_y] = 2i(-i\omega) = 2\omega$$

so we can put $\quad a = \frac{1}{\sqrt{2\omega}}(D_x + iD_y), \quad a^* = \frac{1}{\sqrt{2\omega}}(-D_x + iD_y)$

and then $\quad \omega a^* a = \frac{1}{2}(-D_x + iD_y)(D_x + iD_y)$

$$= -\tfrac{1}{2}(D_x^2 + D_y^2) - \tfrac{\omega}{2} .$$

Discussion (as opposed to calculation). We start with the line bundle L over $\mathbb{R}^2$ with connection given by an invariant non-degenerate 2 form, e.g.

$$D_x = \partial_x + \tfrac{1}{2}i\omega y, \quad D_y = \partial_y - \tfrac{1}{2}i\omega x, \quad [D_x, D_y] = -i\omega$$

The operators D_x, D_y generate a Heisenberg algebra which extends to an action of the Heisenberg group on L covering the translation group on $\mathbb{R}^2$. On the other hand we have also the operators on the sections of L given by a $\tilde{D}_x, \tilde{D}_y$ which commute with D_x, D_y.

Formulas: Write things in creation and annihilation form.

Start again: Let L be the line bundle over $\mathbb{R}^2$ with connection corresponding to an invariant non-degenerate form (say a Kähler form relative to $\mathbb{R}^2 = \mathbb{C}$.) Then there are two Heisenberg groups acting on sections of L and these actions commute. One comes from lifting invariant vector fields on $\mathbb{R}^2$ to operators on sections via the connection. The other comes from lifting invariant vector fields to autos. of the line bundle with connection.

The point is that this means that the sections of L are naturally going to appear as the tensor product of the irreducible repns. of the two Heisenberg groups. So it would seem that we have a way to assign to sections of L operators on the irreducible representation.

Let's work out ~~the~~ formulas. Start with the connection

$$D_x = \partial_x + i\omega y \quad , \quad D_y = \partial_y$$

Then the commuting pair is

$$\partial_x \quad , \quad \partial_y + i\omega x \quad .$$

April 3, 1984

Let's consider the trivial line bundle L over $\mathbb{C}$ with the connection given by the operators

$$\partial_{\bar z} + \tfrac{1}{2} z \quad , \quad \partial_z - \tfrac{1}{2}\bar z$$

This line bundle has the holomorphic section $e^{-\frac{1}{2}|z|^2}$. So I can use $f(z) \longrightarrow f(z)e^{-\frac{1}{2}|z|^2}$ to identify L as a holomorphic line bundle with the trivial bundle equipped with the Gaussian metric. The space of sections of L is thus the functions on $\mathbb{C}$ with connection given by the operators

$$\partial_{\bar z} \quad , \quad \partial_z - \bar z$$

and with L^2 norm

$$\|f\|^2 = \int \frac{d^2 z}{\pi} e^{-|z|^2} |f(z)|^2$$

Then we have the following [redacted] commuting sets of operators

$$\begin{cases} a = \partial_{\bar z} \\ a^* = \bar z - \partial_z \end{cases} \qquad\qquad \begin{cases} b = \partial_z \\ b^* = z - \partial_{\bar z} \end{cases}$$

I want to calculate the kernel of the operator e^{-ta^*a}, where by kernel I mean

$$K_t(z,w) = \sum \psi_n(z)\, e^{-t\lambda_n}\, \overline{\psi_n(w)}$$

where ψ_n is an orthonormal basis of eigenfunctions. For example as $t \longrightarrow \infty$, we know that e^{-ta^*a} approaches the projection onto the holomorphic functions, for which we have the orth. basis $\frac{z^n}{\sqrt{n!}}$. Thus this projection has the kernel

$$\sum_n \frac{z^n \bar w^n}{n!} = e^{z\bar w}$$

The natural way to describe a harmonic oscillator is to use coherent states, namely

$$e^{ka^*+lb^*}1$$

We have

$$e^{lb^*}1 = e^{lz}e^{-l\partial_z}1 = e^{lz}$$

$$e^{ka^*}e^{lz} = e^{k\bar z}e^{-k\partial_z}e^{lz} = e^{k\bar z}e^{l(z-k)}$$

$$\Rightarrow \langle z|e^{ka^*+lb^*}1\rangle = e^{k\bar z+lz-lk}$$

Then

$$e^{-ta^*a}e^{ka^*+lb^*}1 = e^{ke^{-t}a^*+lb^*}e^{-ta^*a}1$$

$$\langle z| \quad \rangle = e^{ke^{-t}\bar z+lz-lke^{-t}}$$

Now the reproducing formula using coherent states is

$$f(z) = \int\frac{d^2k}{\pi}\frac{d^2l}{\pi}e^{-|k|^2-|l|^2}\langle z|e^{ka^*+lb^*}1\rangle\langle e^{ka^*+lb^*}1|f\rangle$$

So

$$(e^{-ta^*a}f)(z) = \int\frac{d^2k\,d^2l}{\pi^2}e^{-|k|^2-|l|^2}e^{ke^{-t}\bar z+lz-lke^{-t}}\int\frac{d^2w}{\pi}e^{-|w|^2}e^{\overline{k\bar w+lw-lk}}f(w)$$

and hence the heat kernel is

$$\int\frac{d^2k\,d^2l}{\pi^2}e^{-|k|^2-|l|^2+ke^{-t}\bar z+lz-lke^{-t}+\bar kw+\bar l\bar w-\bar l\bar k}$$

We do the integral over l first.

$$\left[-|l|^2+l(z-ke^{-t})+\bar l(\bar w-\bar k)-\text{prod}\right]+\text{prod}$$

This gives

$$\int\frac{d^2k}{\pi}e^{-|k|^2+ke^{-t}\bar z+\bar kw+(z-ke^{-t})(\bar w-\bar k)}$$

$$-(1-e^{-t})|k|^2+ke^{-t}(\bar z-\bar w)+\bar k(w-z)+z\bar w$$

$$z\bar{w} + (1-e^{-t})\left[-|k|^2 + k\frac{e^{-t}}{1-e^{-t}}(\bar{z}-\bar{w}) + \bar{k}\frac{1}{1-e^{-t}}(w-z) - \text{prod} + \text{prod}\right]$$

This gives for the kernel of e^{-ta^*a}

$$\frac{1}{1-e^{-t}} \cdot e^{z\bar{w} + \frac{e^{-t}}{1-e^{-t}}(\bar{z}-\bar{w})(w-z)}$$

Now what I have to do is rewrite this in an invariant way using the connection, really the parallel transport along the line from w to z.

April 4, 1984

I am considering over $\mathbb{C}$ the trivial holomorphic line bundle with the metric

$$|1|^2 = e^{-|z|^2}$$

The connection form is $d'(-|z|^2) = -\bar{z}dz$, so the connection is

$$D = d - \bar{z}dz = dz(\partial_z - \bar{z}) + d\bar{z}\,\partial_{\bar{z}}$$

and the curvature is

$$D^2 = d(-\bar{z}dz) = dz\,d\bar{z} = \frac{2}{i}dx\,dy$$

I want parallel translation in this line bundle. This means that given a path $z(t)$ we want a lifting $u(t)$ into the line bundle which is horizontal:

$$\frac{D}{dt}u(t) = \left(\frac{d}{dt} - \bar{z}\frac{dz}{dt}\right)u = 0 \quad \Rightarrow \quad \dot{u} - \bar{z}\dot{z}u = 0$$

$$\Rightarrow \quad u(t) = e^{\int_0^t \bar{z}\dot{z}\,dt}\,u(0).$$

Thus the parallel translation along a curve γ is

$$e^{\int_\gamma \bar{z}dz}.$$

Take the straight line from w to z:

$$z(t) = w + t(z-w) \qquad \dot{z} = z-w$$

$$\int_0^1 \bar{z}\,dz = \int_0^1 (\bar{w} + t(\overline{z-w}))(z-w)\,dt = \bar{w}(z-w) + \tfrac{1}{2}|z-w|^2$$

Recall that we computed the operator e^{-ta^*a} and found

$$\left(e^{-ta^*a}f\right)(z) = \int \frac{d^2w}{\pi} \frac{e^{-\frac{e^{-t}}{1-e^{-t}}|z-w|^2 + z\bar{w} - |w|^2}}{1-e^{-t}} f(w).$$

Write this using $\parallel$ transport

$$\left(e^{-ta^*a}f\right)(z) = \int \frac{d^2w}{\pi} \frac{1}{1-e^{-t}} e^{-\left(\frac{e^{-t}}{1-e^{-t}}+\frac{1}{2}\right)|z-w|^2 + \bar{w}(z-w) + \frac{1}{2}|z-w|^2} f(w)$$

$$\boxed{\left(e^{-ta^*a}f\right)(z) = \int \frac{d^2w}{\pi} \frac{1}{1-e^{-t}} e^{-\frac{1+e^{-t}}{2(1-e^{-t})}|z-w|^2} e^{\int_w^z \bar{z}\,dz} f(w)}$$

Next I would like to relate this formula with the path integral. Here the connection is

$$D = dz\, D_z + d\bar{z}\, D_{\bar{z}} = dz(-a^*) + d\bar{z}\, a$$

and we want the operator

$$-a^*a = D_z D_{\bar{z}} = \tfrac{1}{4}(D_x - iD_y)(D_x + iD_y)$$

$$= \tfrac{1}{4}(D_x^2 + D_y^2) + \tfrac{1}{4} i \underbrace{[D_x, D_y]}_{-2i} = \tfrac{1}{4}(D_x^2+D_y^2) - \tfrac{1}{2}$$

The operator $\frac{1}{4}D_\mu^2$ should be associated to the Hamiltonian

$$H = \tfrac{1}{4}(p - iA)^2 \qquad D_\mu = \partial_\mu + A_\mu = i(p_\mu - iA_\mu)$$

so the action for e^{-tH} is

$$\int \left[ip\dot{x} - \tfrac{1}{4}(p-iA)^2\right] dt$$

Find stationary pt w.r.t. p:

$$i\dot{x} - \tfrac{1}{2}(p-iA) = 0 \qquad p = i(2\dot{x}+A)$$

$$i\,i(2\dot{x}+A)\dot{x} - \tfrac{1}{4}(2i\dot{x})^2 = -2\dot{x}^2 - A\dot{x} + \dot{x}^2$$

so the action is

$$S = \int_0^t (\dot{x}^2 + A\dot{x})\, dt$$

and supposedly the kernel for $e^{+\frac{t}{4}D_\mu^2}$ is obtained by using the path integral of e^{-S}.

So now we want to use a connection with curvature $-2i\,dx\,dy$, and which is nice at 0, so that radial parallel transport is trivial. The connection is:

$$\partial_x + iy \quad , \quad \partial_y - ix$$

so we get the action

$$S = \int_0^t \left[(\dot{x}^2 + \dot{y}^2) + i(y\dot{x} - x\dot{y})\right] dt$$

and the equations for the critical point

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{x}} = 2\ddot{x} + i\dot{y} = \frac{\partial L}{\partial x} = -i\dot{y} \quad \Rightarrow \quad \ddot{x} = -i\dot{y}$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{y}} = 2\ddot{y} - i\dot{x} = \frac{\partial L}{\partial y} = i\dot{x} \quad \Longrightarrow \quad \ddot{y} = i\dot{x}$$

Thus $\ddot{x} = (-iy)^{\cdot} \qquad (-iy)^{\cdot\cdot} = \dot{x}$ so the general solution starting at $x = y = 0$ is

$$x = A(e^t - 1) + B(e^{-t} - 1)$$

$$-iy = A(e^t - 1) - B(e^{-t} - 1)$$

So
$$x = \mathring{x} + (-A-B)$$
$$y = \mathring{y} + (-A+B)i$$

and
$$S = \int_0^t dt \left(\cancel{\mathring{x}^2} + \mathring{y}^2 + i(\underbrace{\mathring{y} + i(-A+B)}_{\cancel{i\mathring{x}}})\mathring{x} - i(\underbrace{\mathring{x} + (-A-B)}_{-i\mathring{y}})\mathring{y} \right]$$

$$= \int_0^t dt \left[i^2(-A+B)(Ae^t - Be^{-t}) + i^2(A+B)(Ae^t + Be^{-t}) \right]$$

$$= -\int_0^t dt \; 2AB\,(e^t + e^{-t}) = -2AB\,(e^t - e^{-t})$$

$$x + iy = 2B(e^{-t} - 1)$$
$$x - iy = 2A(e^t - 1)$$
$$|z|^2 = 4AB\,(e^t - 1)(e^{-t} - 1)$$

Thus
$$S = -\frac{|z|^2}{2} \frac{e^t - e^{-t}}{(e^t - 1)(e^{-t} - 1)} = -\frac{|z|^2}{2} \frac{(1 - e^{-t})(1 + e^{-t})}{(1 - e^{-t})(e^{-t} - 1)}$$
$$= +\frac{|z|^2}{2}\left(\frac{1 + e^{-t}}{1 - e^{-t}}\right)$$

which then leads to the exponential factor in the kernel
$$\langle z | e^{-ta^*a} | 0 \rangle = \cdots\cdots \; e^{-\frac{|z|^2}{2}\frac{1+e^{-t}}{1-e^{-t}}}$$

April 6, 1984

Let ω^μ be a local moving frame in T^*, and suppose given a metric

$$|\xi_\mu \omega^\mu|^2 = g^{\mu\nu}\xi_\mu \xi_\nu$$

Let E be a vector bundle and let $\mathcal{D}$ be a first order operator on E such that the (leading) symbol of $\mathcal{D}^2$ is multiplication by the metric. Then the symbol of $\mathcal{D}$:

$$T^* \otimes E \longrightarrow E \qquad\qquad \xi \otimes s \longmapsto \sigma(\mathcal{D}, \xi)s$$

extends to an action of the Clifford algebra $C(T^*)$ on E. Let γ^μ denote the endomorphism of E corresponding to the section ω^μ of T^*: $\gamma^\mu = \sigma(\mathcal{D}, \omega^\mu)$. $\boxed{[\gamma^\mu, \gamma^\nu]_+ = 2g^{\mu\nu}}$

Let $D : \Gamma(E) \longrightarrow \Gamma(T^* \otimes E)$ be a connection on E, and write $D = \omega^\mu D_\mu$ where $D_\mu = i(X_\mu)D$ and X_μ is the dual frame in T. Then the operator

$$\not{D} = \gamma^\mu D_\mu$$

on E has the same symbol as $\mathcal{D}$.

A natural condition to impose on the connection D is that it be compatible with a connection on T^* preserving the metric, which we shall also denote by D. For the moment we do not assume this connection is the Levi-Civita connection belonging to the metric. In terms of the basis ω^μ we can describe the connection

$$D_\mu \omega^\nu = \Gamma^\nu_{\mu\lambda}\omega^\lambda.$$

If ω^μ is an orthonormal frame, then for D to preserve the metric connection means $\Gamma^\nu_{\mu\lambda} = -\Gamma^\lambda_{\mu\nu}$.

For the connections on T^*, E to be compatible with Clifford multiplication means that

$$D_\mu(\gamma^\nu s) = \gamma^\nu(D_\mu s) + \underbrace{\mathrm{Cliff}(D_\mu \omega^\nu)\cdot s}_{\Gamma^{\nu}_{\mu\lambda}\,\gamma^\lambda s}$$

or

$$\boxed{[D_\mu, \gamma^\nu] = \Gamma^{\nu}_{\mu\lambda}\gamma^\lambda}$$

Finally we want $[D_\mu, D_\nu]$ and this is essentially the curvature of the connection. Recall the curvature is the operator $F = D^2$:

$$\Gamma(E) \xrightarrow{D} \Gamma(T^* \otimes E) \xrightarrow{D} \Gamma(\Lambda^2 T^* \otimes E).$$

Thus $F = D^2 = D(\omega^\mu D_\mu) = d\omega^\mu D_\mu - \omega^\mu D D_\mu$. Let's put

$$d\omega^\alpha = \frac{1}{2}\hat{\Gamma}^{\alpha}_{\mu\nu}\,\omega^\mu\omega^\nu \quad , \quad \text{Then} \quad \omega^\mu\omega^\nu D_\nu D_\mu$$

$$F = \frac{1}{2}\omega^\mu\omega^\nu F_{\mu\nu} = \frac{1}{2}\hat{\Gamma}^{\alpha}_{\mu\nu}\,\omega^\mu\omega^\nu D_\alpha + \frac{1}{2}\omega^\mu\omega^\nu[D_\mu, D_\nu]$$

and so we have the commutation relation

$$\boxed{\begin{array}{l} [D_\mu, D_\nu] = F_{\mu\nu} - \hat{\Gamma}^{\alpha}_{\mu\nu} D_\alpha \\ \text{where} \quad d\omega^\alpha = \frac{1}{2}\hat{\Gamma}^{\alpha}_{\mu\nu}\,\omega^\mu\omega^\nu \end{array}}$$

Next we compute the square of the Dirac operator:

$$\not{D}^2 = \gamma^\mu D_\mu \gamma^\nu D_\nu = \gamma^\mu\gamma^\nu D_\mu D_\nu + \gamma^\mu \Gamma^{\alpha}_{\mu\lambda}\gamma^\lambda D_\alpha$$

$$= \gamma^\mu\gamma^\nu\left\{D_\mu D_\nu + \Gamma^{\alpha}_{\mu\nu} D_\alpha\right\}$$

$$= \left\{\frac{1}{2}[\gamma^\mu, \gamma^\nu]_+ + \frac{1}{2}[\gamma^\mu, \gamma^\nu]_-\right\}\left\{D_\mu D_\nu + \Gamma^{\alpha}_{\mu\nu} D_\alpha\right\}$$

$$\boxed{\not{D}^2 = g^{\mu\nu}\left(D_\mu D_\nu + \Gamma^\alpha_{\mu\nu} D_\alpha\right) + \tfrac{1}{2}\gamma^\mu\gamma^\nu\left\{[D_\mu, D_\nu] + \left(\Gamma^\alpha_{\mu\nu} - \Gamma^\alpha_{\nu\mu}\right) D_\alpha\right\}}$$

Next compute the trace Laplacean

$$\Gamma(E) \xrightarrow{D} \Gamma(T^*\otimes E) \xrightarrow{D_{T^*\otimes E}} \Gamma(T^*\otimes T^*\otimes E)$$

$$\begin{aligned} D_{T^*\otimes E} \cdot D_E &= \omega^\mu D_\mu (\omega^\nu D_\nu) \\ &= \omega^\mu \omega^\nu D_\mu D_\nu + \omega^\mu \Gamma^\alpha_{\mu\lambda} \omega^\lambda D_\alpha \\ &= \omega^\mu \omega^\nu \left\{ D_\mu D_\nu + \Gamma^\alpha_{\mu\nu} D_\alpha \right\} \end{aligned}$$

The trace Laplacean is obtained by using $(\omega^\mu, \omega^\nu) = g^{\mu\nu}$ whence

$$\boxed{\text{trace Laplacean}_{\wedge}^{D\cdot D} = g^{\mu\nu}\left(D_\mu D_\nu + \Gamma^\alpha_{\mu\nu} D_\alpha\right)}$$

Thus we conclude that $\not{D}^2 = D\cdot D$ + 0th order **provided that**

$$\hat{\Gamma}^\alpha_{\mu\nu} = \Gamma^\alpha_{\mu\nu} - \Gamma^\alpha_{\nu\mu}.$$

This is probably equivalent to torsion $= 0$. We can check this brutally as follows:

The torsion is $T(X,Y) = \nabla_X Y - \nabla_Y X - [X,Y]$ where $\nabla_X = i_X D$. Now we compute the torsion for $X = X_\mu$, $Y = X_\nu$.

$$i_{X_\nu}\omega^\alpha = \delta^\alpha_\nu \implies i_{D_\mu X_\nu}\omega^\alpha + \underbrace{i_{X_\nu} D_\mu \omega^\alpha}_{i_{X_\nu}\Gamma^\alpha_{\mu\lambda}\omega^\lambda = \Gamma^\alpha_{\mu\nu}} = 0$$

$$\implies \boxed{D_\mu X_\nu = -\Gamma^\alpha_{\mu\nu} X_\alpha}$$

$$d\omega^\alpha = \frac{1}{2}\hat{\Gamma}^\alpha_{\rho\sigma}\,\omega^\rho\omega^\sigma$$

$$\iota_{X_\nu}\iota_{X_\mu} d\omega^\alpha = \frac{1}{2}\hat{\Gamma}^\alpha_{\rho\sigma}\left[\delta^\rho_\mu\delta^\sigma_\nu - \delta^\rho_\nu\delta^\sigma_\mu\right]$$

$$= \frac{1}{2}\left(\hat{\Gamma}^\alpha_{\mu\nu} - \hat{\Gamma}^\alpha_{\nu\mu}\right) = \hat{\Gamma}^\alpha_{\mu\nu}$$

$$\iota_{X_\nu}\iota_{X_\mu} d\omega^\alpha = \cancel{X_\mu\,\iota_{X_\nu}\omega^\alpha} - \cancel{X_\nu\,\iota_{X_\mu}\omega^\alpha} - \iota_{[X_\mu,X_\nu]}\,\omega^\alpha$$

Thus

$$\boxed{d\omega^\alpha = \frac{1}{2}\hat{\Gamma}^\alpha_{\mu\nu}\,\omega^\mu\omega^\nu \iff [X_\mu, X_\nu] = -\hat{\Gamma}^\alpha_{\mu\nu}X_\alpha}$$

$$\boxed{D_\mu\omega^\alpha = \Gamma^\alpha_{\mu\lambda}\,\omega^\lambda \iff D_\mu X_\nu = -\Gamma^\alpha_{\mu\nu}X_\alpha}$$

Hence $T(X_\mu,X_\nu) = \left(-\Gamma^\alpha_{\mu\nu} + \Gamma^\alpha_{\nu\mu} + \hat{\Gamma}^\alpha_{\mu\nu}\right)X_\alpha$

which completes the check. $\therefore$ We have proved

<u>Prop</u>. Given a $C(T^*)$-bundle E, ~~[illegible]~~ a connection D on E compatible with a connection D on T^* preserving the metrics, form $\not{D} = \gamma^\mu D_\mu$. Then $\not{D}^2$ differs from the **covariant** Laplacean $D\cdot D$ on E by a zeroth order operator $\iff$ The connection on T^* is the Levi-Civita connection. In this case we have the Weitzenbock formula

$$\not{D}^2 = D\cdot D + \frac{1}{2}\gamma^\mu\gamma^\nu F_{\mu\nu}$$

Let's further check that Torsion $= 0$ is equivalent to the connection inducing d on forms: The composition

$$\Gamma(T^*) \xrightarrow{D} \Gamma(T^*\otimes T^*) \xrightarrow{\wedge} \Gamma(\Lambda^2 T^*)$$

should be d. But

$$D\omega^\alpha = \omega^\mu \otimes D_\mu \omega^\alpha = \omega^\mu \otimes \Gamma^\alpha_{\mu\nu}\,\omega^\nu \xrightarrow{\ \wedge\ } \Gamma^\alpha_{\mu\nu}\,\omega^\mu \wedge \omega^\nu = \tfrac{1}{2}\left(\Gamma^\alpha_{\mu\nu} - \Gamma^\alpha_{\nu\mu}\right)\omega^\mu\omega^\nu$$

so this is clear.

Our next project will be to rescale the metric $g^{\mu\nu} \longrightarrow h^2 g^{\mu\nu}$. Actually this isn't too meaningful because we must also deform the Clifford multiplication.

To fix the ideas let's take $E = \Lambda T^*$ with connection obtained from a connection D on T^* preserving the metric $g^{\mu\nu}$. As we scale the metric we keep D fixed.

We consider the standard left Clifford multiplication on ΛT^*, namely $\gamma^\mu = i_{\omega^\mu} + e_{\omega^\mu}$, where i_{ω^μ} is interior multiplication by the linear functional on T^* given by $(\omega^\mu, ?)$. As $(\omega^\mu, \omega^\nu) = g^{\mu\nu}$, this linear functional is $g^{\mu\nu} X_\nu$, i.e. contracting with this vector. So

$$\gamma^\mu = g^{\mu\nu}\, i_{X_\nu} + e_{\omega^\mu}$$

and so when we rescale we get

$$\gamma^\mu = h^2 g^{\mu\nu}\, i_{X_\nu} + e_{\omega^\mu}$$

The Dirac operator $\gamma^\mu D_\mu = h^2 g^{\mu\nu} i_{X_\nu} D_\mu + e_{\omega^\mu} D_\mu$ should be the operator $d^* + d$ when the torsion is zero. As $h \to 0$, it approaches the DR operator d.

For the next step it is useful to choose ω^μ to be an orthonormal frame: $g^{\mu\nu} = \delta^{\mu\nu}$, and then D_μ is the derivative in the direction of the unit vector X_μ. After rescaling we have the ortho normal frame $\omega^{\mu,h} = \frac{1}{h}\omega^\mu$ since

$$\left(\tfrac{1}{h}\omega^\mu, \tfrac{1}{h}\omega^\nu\right)_{(h)} = h^2\left(\tfrac{1}{h}\omega^\mu, \tfrac{1}{h}\omega^\nu\right) = (\omega^\mu, \omega^\nu) = \delta^{\mu\nu}.$$

Then hX_μ is an orth. frame in T and $D^h_\mu = hD_\mu$.
The corresponding anti-commuting involutions to the frame $\omega^{\mu,h}$ is

$$\gamma^{\mu,h} = \frac{1}{h}\left(h^2\iota_{X_\mu} + e_{\omega^\mu}\right) = \iota_{hX_\mu} + e_{\frac{1}{h}\omega^\mu}.$$

We will be interested in the algebra generated by the operators x^i, $D^h_\mu = hD_\mu$, $h\gamma^{\mu,h} = h^2\iota_{X_\mu} + e_{\omega^\mu} = \gamma^\mu$.

Commutation relations

$$[hD_\mu, x^i] = h\underbrace{[D_\mu, x^i]}_{X_\mu x^i} \longrightarrow 0$$

$$[\gamma^\mu, \gamma^\nu] = 2h^2\delta^{\mu,\nu} \longrightarrow 0$$

$$[hD_\mu, hD_\nu] = h^2[D_\mu, D_\nu]$$
$$= h^2\left\{F_{\mu\nu} - \hat{\Gamma}^\alpha_{\mu\nu} D_\alpha\right\}$$

Now $F_{\mu\nu}$ is the derivation of ΛT^* extending the curvature endomorphism $R_{\mu\nu}$ of T^*. So we have

$$\cancel{F_{\mu\nu} = R_{\mu\nu\alpha\beta}\, e_{\omega^\alpha}\, \iota_{X_\beta}}$$
$$\cancel{= R_{\mu\nu\alpha\beta}\, e_{\omega^\alpha}\, \iota_{X_\beta} + \iota}$$

$$F_{\mu\nu} = R_{\mu\nu k\ell}\, e_{\omega^k}\, \iota_{X_\ell}$$
$$= \frac{1}{2} R_{\mu\nu k\ell}\left(\iota_{X_k} + e_{\omega^k}\right)\left(\iota_{X_\ell} + e_{\omega^\ell}\right)$$

where we use that

$$(\iota_{X_k} + e_{\omega k})(\iota_{X_\ell} + e_{\omega \ell}) = \iota_{X_k} e_{\omega \ell} + e_{\omega k}\, \iota_{X_\ell}$$

$$= - e_{\omega \ell}\, \iota_{X_k} + e_{\omega k}\, \iota_{X_\ell}$$

and the fact that $R_{\mu\nu}$ is skew-symmetric. This can also be written

$$h^2 F_{\mu\nu} = \frac{1}{2} R_{\mu\nu k \ell} \left(h^2 \iota_{X_k} + e_{\omega k}\right)\left(h^2 \iota_{X_\ell} + e_{\omega \ell}\right)$$

$$= \frac{1}{2} R_{\mu\nu k \ell}\, \gamma^k \gamma^\ell$$

I am considering then the algebra generated by h, x^i, hD_μ, γ^μ with the relations that h is central, the x^i commute, and

$$[hD_\mu, x^i] = h X_\mu x^i$$

$$[\gamma^\mu, \gamma^\nu]_+ = 2h^2 g^{\mu\nu} \qquad [x^i, \gamma^\mu] = 0$$

$$[hD_\mu, \gamma^\nu] = h\, \Gamma^\nu_{\mu\lambda} \gamma^\lambda$$

$$[hD_\mu, hD_\nu] = \frac{1}{2} R_{\mu\nu k \ell}\, \gamma^k \gamma^\ell - h\, \hat{\Gamma}^\alpha_{\mu\nu} (hD_\alpha)$$