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March 13, 1984

Yesterday I seemed to find that for a Kahler manifold, the forms with operator $d+d^*$ splits as a sum of the bundles $T^{0,*}$ equipped with the operators $\bar\partial + \bar\partial^*$. I want to check this now in the flat cases.

Take complex dimension 1. We are identifying the Clifford algebra C_2 with generators γ^1, γ^2 with the exterior algebra with generators $dx\, dy$ so that

$$\begin{matrix} 1 & & 1 \\ \gamma^1 & & dx \\ \gamma^2 & \longleftrightarrow & dy \\ \gamma^1\gamma^2 & & dx\,dy \end{matrix}$$

Now the Clifford algebra C_2 viewed as a left module over itself splits into the eigenspaces for a^*a. These are

eigenvalue 0 : a^*, aa^*

eigenvalue 1 : a, a^*a

Let's find what these are in the exterior algebra. Recall

$$\gamma^1 = a^* + a \qquad \gamma^1 + i\gamma^2 = 2a \implies a \leftrightarrow \tfrac{1}{2}(dx + i\,dy) = \tfrac{dz}{2}$$
$$\gamma^2 = ia^* - ia \qquad \gamma^1 - i\gamma^2 = 2a^* \implies a^* \longleftrightarrow \tfrac{1}{2}(dx - i\,dy) = \tfrac{1}{2}d\bar z$$

$$\gamma^1\gamma^2 = -ia^*a + iaa^* \implies i\gamma^1\gamma^2 = a^*a - aa^* = 2a^*a - 1 = 1 - 2aa^*$$

$$a^*a = \tfrac{1}{2}(1 + i\gamma^1\gamma^2) \implies a^*a \longleftrightarrow \tfrac{1}{2} + \tfrac{i}{2}dx\,dy = \tfrac{1}{2} + \frac{d\bar z\, dz}{4}$$
$$aa^* = \tfrac{1}{2}(1 - i\gamma^1\gamma^2) \implies aa^* \longleftrightarrow \tfrac{1}{2} - \tfrac{i}{2}dx\,dy = \tfrac{1}{2} + \frac{dz\, d\bar z}{4}$$

$$\not{D} = \gamma^1\partial_x + \gamma^2\partial_y = (e_1 + \iota_1)\partial_x + (e_2 + \iota_2)\partial_y$$

e_j = ext. mult by dx^j
ι_j = int mult.

Now I want to check that the forms $fa + ga^*a$ are stable under $\not{D}$. This is clear, but it would be nice to see what it looks like on the level of forms. Thus

I want to look at $f\,\frac{1}{2}(dx+i\,dy) + g\,\frac{1}{2}(1+i\,dx\,dy)$ and see what $\not{D}$ does to it.

$$\not{D}\, f(dx+i\,dy) = (e_1\partial_x + e_2\partial_y + \iota_1\partial_x + \iota_2\partial_y)(f(dx+i\,dy))$$

$$= i\,\partial_x f\,dx\,dy \text{ [illegible] } i\,\partial_y f\,dx\,dy + \partial_x f + i\partial_y f$$

$$= (\partial_x f + i\partial_y f)(1 + i\,dx\,dy)$$

$$\not{D}\, g(1+i\,dx\,dy) = (e_1\partial_x + e_2\partial_y + \iota_1\partial_x + \iota_2\partial_y)\,g(1+i\,dx\,dy)$$

$$= dx\,\partial_x g + dy\,\partial_y g + i\partial_x g\,dy - i\partial_y g\,dx$$

$$= (\partial_x g - i\partial_y g)(dx + i\,dy)$$

So I am now at the point where I see that the grading in the almost complex case of $C(T^*)$ is stable under $\not{D}$, and gives something like the $\bar\partial + \bar\partial^*$ operators for the different Dolbeault complexes.

The next point will be to find out what the circle action looks like on the level of kernels. The point is that we have an actual circle action on forms, hence it is represented by a δ-function kernel. The operator $\Delta = (d+d^*)^2$, hence $e^{-t\Delta}$, commutes with this action, so this says something about the forms on $M\times M$ and T_M which are the Schwartz kernels.

We have now decided to think of our basic operator as the Dirac operator $\not{D} = \gamma^\mu D_\mu$ on the Clifford algebra bundle $C(T^*)$, rather than as the Hodge-de Rham operator $d+d^*$ on the exterior algebra bundle $\Lambda(T^*)$. Consequently we should think through the Schwartz kernel theorem, whereby operators on $\Gamma(\Lambda(T^*))$ are realized by forms on $M \times M$. This is done using the following ideas

$$\mathrm{End}(\Gamma(\Lambda(T^*))) = \Gamma(\Lambda T^*) \otimes \Gamma(\Lambda T^*)^*$$
$$\cup \qquad\qquad\qquad \cup$$
$$\mathrm{End}_{\substack{\text{smooth}\\ \text{kernel}}}(\Gamma(\Lambda T^*)) = \Gamma(\Lambda T^*) \otimes \Gamma((\Lambda T^*))^{\text{smooth dual}}$$

$$= \Gamma(M, \Lambda T^*) \otimes \Gamma(M, (\Lambda T^*)^* \otimes \omega)$$

$$= \Gamma(M \times M, \mathrm{Hom}(pr_2^* \Lambda T^*, pr_1^* \Lambda T^*) \otimes pr_2^* \omega)$$

Now to simplify I suppose that M is oriented + Riemannian so that ω is a trivial line bundle; recall that ω = is the line bundle of odd forms of highest degree. The point now becomes why is

$$\mathrm{Hom}(pr_2^* \Lambda T^*, pr_1^* \Lambda T^*) \cong pr_1^* \Lambda T^* \otimes pr_2^* \Lambda T^*.$$

This isomorphism is the way we see forms on the product interpreted as operators on forms. What is involved is that one has a way to identify the algebra ΛT^* with its dual.

In abstract terms given A, a finite dimensional algebra, suppose there is an element $\lambda \in A^*$ such that the map $A \longrightarrow A^*$ $\quad a \longmapsto \lambda \circ a$ is an isomorphism. In other words every linear functional is of the form $x \longmapsto \lambda(ax)$ for some! $a \in A$, or that [redacted] right A-module A^* is isomorphic to the right A-module A. Then we get an isomorphism

$$\begin{array}{ccc} \mathrm{End}(A) = A \otimes A^* & \xleftarrow{\sim} & A \otimes A \\ (x \mapsto f\lambda(gx)) & \longleftarrow\!\shortmid & f \otimes g \end{array}$$

and so every endo. of A is represented by a kernel in $A \otimes A$. What is the trace?

$$\begin{array}{ccccc} A \otimes A & \longrightarrow & A \otimes A^* & \longrightarrow & \text{[redacted]}\ k \\ f \otimes g & \longmapsto & f \otimes \lambda\cdot g & \longmapsto & \lambda(gf) \end{array}$$

So now the idea is to do this business with the Clifford algebra.

Idea: Suppose we start with any connection in T^* preserving the metric. Then we can form the operator $\not{D}$ on $C(T^*) = \Lambda(T^*)$ and ask how it compares with $d + d^*$, which we believe to be the Dirac operator associated to the [redacted] Levi-Civita connection. The difference of two connections is a one-form with values in skew-symmetric endos. of T^*, and this must be the torsion.

<u>Question</u>: To what extent can we regard $d + d^*$ as a [redacted] Dirac operator with coefficients? Is $d + d^*$ obtained by twisting $\not{D}$? If so I could use a flat connection on T^* to do the local computations.

Let's think about my class today. Let's review Thursday's lecture in the case of a complex line bundle E over M. One supposes given a connection in E which preserves a hermitian inner product. Using this hermitian product we define the Clifford algebra bundle $C(E)$ and the module bundle over it $\Lambda(E)$. If $\pi: E \longrightarrow M$ is the projection then on $\pi^*(\Lambda E)$ we have an odd degree endomorphism L whose effect at $\xi \in E$ is $i(e_\xi + \iota_\xi)$ on $\pi^*(\Lambda E)_\xi = \Lambda E_{\pi(\xi)}$. The connection on E induces one on ΛE, and we pull this back to a connection D on $\pi^*(\Lambda E)$. Then the goal was to compute the Chern character form

$$tr_s\left(e^{(D+L)^2}\right).$$

We do this locally using a trivialization $E = M \times \mathbb{C}$ compatible with the hermitian inner product on E. Let $z = pr_2 : E \longrightarrow \mathbb{C}$. Then we have

$$L = i(z a^* + \bar{z} a)$$

$$D = d + \theta\, a^* a \qquad \text{on } M \times \Lambda\mathbb{C}$$

where the connection on E is $d + \theta$ relative to the trivialization. Then

$$(D+L)^2 = \overbrace{\omega}^{d\theta} a^* a + \overbrace{i\,(dz + \theta z)}^{-J} a^* + \overbrace{i(d\bar{z} - \theta \bar{z})}^{\bar{J}} a - |z|^2$$

$$= \omega a^* a + a^* J + \bar{J} a$$

and we found

$$tr_s\, e^{(D+L)^2} = \left(1 - e^{\omega}\right) e^{-\bar{J}\omega^{-1} J}\, e^{-|z|^2}$$

$$= \frac{1-e^{\omega}}{\omega} \underbrace{\left(\omega + J\bar{J}\right) e^{-|z|^2}}_{\text{Thom form}}$$

From now on we will be concerned with the Thom form:

$$e^{-|z|^2}\left(\omega + (dz+\theta z)(d\bar z - \theta \bar z)\right)$$

This lives on $M \times \mathbb{C}$ so that θ is really $\pi^*\theta$ and $\omega = d\theta = \overset{\text{really}}{\pi^*}(d\theta)$

$$= e^{-|z|^2}\left(d\theta + \theta(z\,d\bar z + \bar z\,dz) + dz\,d\bar z\right)$$

$$= d\left(e^{-|z|^2}\theta\right) + e^{-|z|^2}\frac{dz}{z}\,z\,d\bar z$$

$$= d\left\{e^{-|z|^2}\left(\theta + \frac{dz}{z}\right)\right\} = d\left\{e^{-|z|^2}\left(\frac{dz}{z} + \pi^*\theta\right)\right\}$$

This has the following interpretation. A complex line bundle E has an associated principal $\mathbb{C}^\times$-bundle which is $E - 0$ section $= E - M$. The connection on E is equivalent to a connection form $\tilde\theta$ on $E-M$ which has the form in a trivialization $E - M \simeq M \times \mathbb{C}^\times$

$$\tilde\theta = \underset{\substack{\uparrow \\ \text{MC form on } \mathbb{C}^\times}}{\frac{dz}{z}} + \pi^*\theta \qquad \longleftrightarrow \qquad \frac{dz}{z} = \frac{d(re^{i\theta})}{re^{i\theta}} = \frac{dr}{r} + i\,d\theta$$

Thus for a complex line bundle E with connection form $\tilde\theta$ on $E-M$, we have

$$\boxed{\text{Thom form} = d\left\{e^{-\|\ \|^2}\tilde\theta\right\}}$$

where $\|\ \|^2$ is any hermitian metric. (It is necessary that $\|\ \|^2$ be divisible by z, so that any $\|z\|^2 = g|z|^2$, with g any function on M, works.

Notice that from the viewpoint of algebraic geometry one should define $\iota, 1$ using ΛE^*. In effect π^*E has a canonical section, so one gets the Koszul complex

$$\cdots \longrightarrow \mathcal{O}_E \underset{\mathcal{O}_M}{\otimes} E^* \longrightarrow \mathcal{O}_E \longrightarrow i_* \mathcal{O}_M \longrightarrow 0$$

which shows that

$$Td^{-1}(E) = \frac{ch\ i^! i_! 1}{i^* i_* 1} = \frac{ch(\lambda_{-1} E^*)}{c_n E} = \frac{1 - e^{-c_1 L}}{c_1 L}$$

for a complex line bundle. In [redacted] the above one should change

$$L = i(za + \bar{z} a^*)$$

$$D = d + \theta^* a^* a \qquad\qquad \theta^* = -\bar\theta$$

because D should be the connection on ΛE^* and $d + \theta$ is the connection on E. This means that we would then get

$$\text{form}\left(ch(i_! 1)\right) = \frac{1 - e^{-\omega}}{\omega}\,(\omega + J\bar{J})\, e^{-|z|^2}$$

which looks a little better.

Who

New idea: Write the Thom form U on the tangent bundle as the Fourier transform of a form on the cotangent bundle. Because we should think of forms on the tangent bundle as a convolution algebra, and we are exponentiating in this convolution algebra. Thus when I take the classical limit of $e^{-4h^2\Delta}$ I should obtain a 1-parameter semi-group of Thom forms [redacted] $U_t = e^{-t\Phi}$, where Φ is a (≤ 2) form on the tangent bundle but this exponential is calculated in the convolution algebra.

[redacted] Convolution takes place in the tangent spaces, so the convolution algebra of forms on the tangent bundle [redacted] contains the algebra of forms on the manifold, and is locally the tensor product of the convolution algebra of forms on the [redacted] fibre with the forms on the base. (So we can [redacted] discuss it for any vector bundle. Also we can describe it using the principal bundle. If we pull up to the principal bundle, it becomes a tensor product of $\Omega(P)$ with the convolution algebra of the vector space.)

Go back to the formula for the Thom form in the [redacted] case of a complex vector bundle E.

$$[\text{redacted}] \int \mathcal{D}\psi^*\mathcal{D}\psi \; e^{\omega\psi^*\psi + \psi^*\underbrace{(-i)(dz+\theta z)}_{J} + \underbrace{i(d\bar z-\theta\bar z)}_{\bar J}\psi - |z|^2}$$

Suppose I rescale $z \to t^{-1/2}z$ so as to get the forms to concentrate around the zero section.

$$e^{-\frac{|z|^2}{t}} \int \mathcal{D}\psi^*\mathcal{D}\psi \; e^{\omega\psi^*\psi + t^{-1/2}\psi^* J + t^{-1/2}\bar J\psi}$$

Now rescale in the integral $\psi \to t^{1/2}\psi$, but beware of the way $\mathcal{D}\psi^*\mathcal{D}\psi$ behaves, and we get

$$e^{-\frac{|z|^2}{t}} \frac{1}{t^n} \int \mathcal{D}\psi^*\mathcal{D}\psi \; e^{t\omega\psi^*\psi + \psi^* J + \bar{J}\psi}$$

In order to get this to appear as a full Fourier transform I have to change the z to a dual variable ω.

$$e^{-\frac{|z|^2}{t}} \frac{1}{t^n} = \int e^{-t|\omega|^2 + \omega\bar{z} - \bar{\omega}z} \frac{d^{2n}\omega}{\pi^n}$$

The exponent becomes

$$t\omega\psi^*\psi + \psi^*(-i)(dz + \theta z) + i(d\bar{z} - \theta\bar{z})\psi - t|\omega|^2 + \omega\bar{z} - \bar{\omega}z$$

We are integrating over ω, ψ which are dual to z and dz. Perhaps this is true at a point where $\theta = 0$; and in general ψ is dual to $dz + \theta z$. So therefore we see that we are taking the Fourier transform of the Gaussian

$$e^{t\left(-|\omega|^2 + \omega\psi^*\psi\right)}$$

We should next work out what this becomes in the case of the tangent bundle. I want real variables as opposed to complex ones.

Perhaps it's more important to see whether we can get the other index thms.

March 15, 1984

Conversation with Atiyah.

There might be a problem with proving a local index theorem for connections in a Riemannian manifold with non-zero torsion. Atiyah Singer + Donnelly run into a ~~situation~~ of this sort with torus bundles; this work deals with η-invariants and cusps - a higher dim'l version of phenomena encountered by Hirzebruch on Hilbert modular surfaces. In this case the non-zero torsion didn't matter.

Characterization of torsion-zero. A connection on T^* has torsion zero $\Leftrightarrow$ each horizontal tangent space for the connection in T^* is Lagrangian. Witten's 2 form on ΩM is closed $\Leftrightarrow$ some symmetrization of the torsion is zero.

I asked if there is a generalization of Kähler manifolds where the flat $S^1 \subset C(T^*)$ becomes a flat $S^3 \subset C(T^*)$. Michael says there is a theory of almost-quaternionic manifolds - they are not usually complex manifolds, for example, quaternion projective space. The S^3 tends to be twisted. These manifold tend to be interesting in twistor theory.

Bott's description of Todd: Take the left-invariant forms on a Lie group, pull them back by the exponential map, and write them in terms of the constant vector fields on the Lie algebra. The operator relating these two is

$$\frac{e^{ad(X)}-1}{ad(X)} \qquad \text{at} \quad X \in \mathfrak{g}.$$

In this connection Michael mentioned Harishchandra's thm. describing biinvariant differential operators on G; if central functions on G are identified with W-invariant functions on T, then a biinvariant operator corresponds to a constant coefficient W-invariant operator on T conjugated by the Weyl denominator:

$$U(\mathfrak{g})^{\mathfrak{g}} \simeq S(\mathfrak{h})^{W}$$

Michael points out that one of these Gaussian forms on a vector bundle extends smoothly to the sphere bundle obtained by 1-point compactifying each fibre.

Michael also mentioned in connection with the forced harmonic oscillator that there is a ~~situation~~, described by the words "constant magnetic field", which one handle by changing the symplectic structure.

March 15, 1984 (cont)

Let V be a real vector space (f.d.). Then if we choose a Haar measure on V one can form the convolution algebra of Schwartz functions $\mathcal{S}(V)$. The measures themselves form an algebra under convolution, and the Haar measure allows one to assign measures to functions. The measures can be convolved, because given $d\mu_1$ and $d\mu_2$, then one gets $d\mu_1 \otimes d\mu_2$ on $V \times V$, which can then be pushed forward under the addition map $V \times V \longrightarrow V$. Seems to work for distributions which are tempered.

Next lets generalize this to the case of currents which are tempered. Given two tempered currents we can form their external product which is a tempered distribution on $V \times V$ and then [redacted] push forward. One has to be careful because [redacted] a Schwartz function on V doesn't pull back to one on $V \times V$.

In any case things should work with currents with compact support. Also one sees that [redacted] when one works with currents, then it is clear that we are using odd forms in the sense of de Rham.

So my problem is now to describe the convolution algebra of forms and to produce a Fourier transform for it. The convolution algebra of forms should be isomorphic under Fourier transform to the algebra of forms.

I want to produce a Fourier transform between rapidly decreasing forms on V and V^*. Actually what we should get is an isomorphism between currents on V and forms on V^*. For example distributions on V when applied to $e^{-i(x,\xi)}$ give functions on V^*, so this has to be jazzed up to go from currents on V to forms on V^*. Look at the currents which are supported at the origin and are at most δ-function currents. These should form an exterior algebra canonically isomorphic to ΛV, which in turn I can think of as the constant 1-forms on V^*.

Next lets try to identify δ-function p-currents, which should be $\Lambda^p V$, with singular forms of the type $\delta(x)\omega$ with $\omega \in \Lambda^{n-p} V^*$. Implicit in the notation $\delta(x)$ is $\delta(x) d^n x$, so that one has trivialized $\Lambda^n V^*$. Thus the Fourier transform now looks as follows on $\{\delta(x)\omega\}$. It assigns to $\delta(x)\,\omega$ with $\omega \in \Lambda^{n-p} V^*$, the element $\tilde{\omega}$ of $\Lambda^p V$ corresponding under cap product

$$\Lambda^p V \otimes \Lambda^n V^* \xrightarrow{\sim} \Lambda^{n-p} V^*$$

and then interprets $\tilde{\omega} \in \Lambda^p V$ as a constant p form on V^*. Let then x^μ be coords on V, ξ_μ the dual coords on V^*. Then $\delta(x)\, dx^{\mu_1} \dots dx^{\mu_{n-p}}$ should go into $\pm d\xi_{\nu_1} \dots d\xi_{\nu_p}$ where the ν's are complementary to the μ's. The best way to do this it seems is to use integration

$$\int d^n x \; e^{dx^\mu d\xi_\mu + i x^\mu \xi_\mu} \, \delta(x)\, dx^{\mu_1} \dots dx^{\mu_{n-p}} .$$

March 16, 1984

One of the problems I have is as follows. I have been looking at the Hodge DR operator $d+d^*$ and trying to see the limiting heat kernel as a n-form on the tangent bundle. Now in the general situation we want to consider a Dirac operator with coefficients in a bundle E, and then the limiting kernel will be some kind of form on the tangent bundle with coefficients in $\mathrm{End}(E)$. Since the operator $d+d^*$ is the Dirac operator with coefficients in the spinors, we must therefore see $\mathrm{End}(S) = C(T^*)$ as part of the forms on T^*.

What I seem to be aiming at is a way to decompose forms on the tangent bundle. Operators on $C(T^*)$ are the basic object, and the $d+d^*$ operator uses the left Clifford multiplication, whereas one also has the right Clifford multiplication. It is the left Clifford multiplications that are being rescaled and deformed to the exterior algebra. Let's think of operators on $C(T^*)$ in terms of Schwartz kernels which are sections over $M\times M$ of $pr_1^* C(T^*) \otimes pr_2^*(C(T^*)^* \otimes \lambda(T^*))$. ~~[illegible]~~ I can't see the limiting process this way. However if I use the self-duality of the Clifford algebra I can think of my kernels as sections over $M\times M$ with values in

$$C(T^*_{M\times M}) \otimes pr_2^*\lambda(T^*) = pr_1^* C(T^*) \otimes pr_2^*(C(T^*)) \otimes pr_2^*\lambda(T^*)$$

Now the rescaling process should be clearer, as it only takes place in the normal direction.

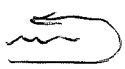
[redacted] Let's try to understand the limiting kernel and Getzler's filtration in the case of a Dirac operator with arbitrary coefficients on a flat Riemannian manifold. So take $M = \mathbb{R}^n$, n even, $D_\mu = \partial_\mu + A_\mu$ where A_μ are endos. of a trivial bundle E, and then the Dirac operator is $\not{D} = h\gamma^\mu D_\mu$ operating on $S \otimes E$. The Laplacean is

$$\not{D}^2 = h^2 D_\mu^2 + \frac{h^2}{2}\gamma^\mu\gamma^\nu F_{\mu\nu}.$$

So what I want to look at now is the heat operator $e^{t\not{D}^2}$ which can be expanded using the perturbation series. As usual it is easier to write the resolvent geom. series

$$\frac{1}{\lambda - \not{D}^2} = \frac{1}{\lambda - h^2 D_\mu^2} + \frac{1}{\lambda - h^2 D_\mu^2}\left(+\frac{h^2}{2}\gamma^\mu\gamma^\nu F_{\mu\nu}\right)\frac{1}{\lambda - h^2 D_\mu^2} + \cdots$$

$$e^{t\not{D}^2} = \frac{1}{2\pi i}\oint e^{\lambda t}\frac{1}{\lambda - \not{D}^2}\,d\lambda$$

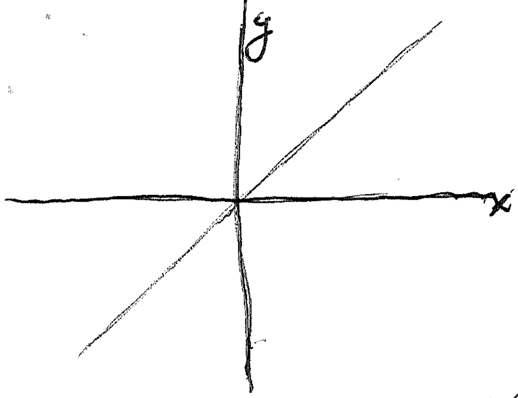


What I want to do is to take the Schwartz kernel $\langle x | e^{t\not{D}^2} | y \rangle$ put $x = y + hv$, and then let $h \to 0$. The first case to understand is where $A_\mu = 0$ so that $\not{D}^2 = h^2\partial_\mu^2$ is the usual Laplacean, whence

$$\langle x | e^{th^2\partial_\mu^2} | y \rangle = \frac{e^{-\frac{(x-y)^2}{4th^2}}}{\left(\sqrt{4th^2\pi}\right)^n} \longmapsto \frac{1}{h^n}\frac{e^{-\frac{|v|^2}{4t}}}{(4\pi t)^{n/2}}$$

Thus we are working on $\mathbb{R}^n$ and we want $e^{th^2\partial_\mu^2}$, as an operator on forms, expressed as an integral operator with kernel an n-form on $\mathbb{R}^n \times \mathbb{R}^n$. Let's first take $n = 1$. We expect the kernel to be a 1-form on $\mathbb{R} \times \mathbb{R}$ of the form

$$\frac{e^{-\frac{(x-y)^2}{4th^2}}}{\sqrt{4\pi t h^2}}(-dx + dy)$$



In fact this form is clearly closed as it is the pull back of a 1-form on the line under the map $x-y$. It also integrates to 1 in the vertical direction.

Let's now check that it gives the right operator on forms, i.e. it should reproduce constant forms.

$$\int_y \frac{e^{-\frac{(x-y)^2}{4t\hbar^2}}}{\sqrt{4\pi t\hbar^2}}(-dx+dy) \text{ [redacted] } (a+b\,dy)$$

$$= \int_y \text{———} \left(a\,dy + b\,dy \text{ [redacted] } dx \right) = a + b\,dx$$

so it does work.

At this point I have to understand how to pull this form back to the tangent bundle. What's bothering me is that I'm not sure about heading perpendicular to the diagonal [redacted] or not. It seems that we want to have $y = x + h\sigma$, whence the form is

$$\frac{e^{-\frac{\sigma^2}{4t}}}{\sqrt{4\pi t}}\,d\sigma$$

as it should be.

Clearly we take the product to handle $\mathbb{R}^n$.

Let's now try to handle the case of coefficients. Here the basic idea will be that in the $h \to 0$ limit I can assume F constant and $A=0$. Then $D_\mu^2 = \partial_\mu^2$, so perturbation series becomes

$$e^{th^2(\partial^2+\sigma F)} = e^{th^2\partial^2} + \int_0^t dt_1\, e^{(t-t_1)h^2\partial^2} h^2\sigma F\, e^{t_1 h^2\partial^2}$$

$$+ \int_0^t dt_1 \int_0^{t_1} dt_2\, e^{(t-t_1)h^2\partial^2} h^2\sigma F\, e^{(t_1-t_2)h^2\partial^2} h^2\sigma F\, e^{t_2 h^2\partial^2} + \cdots$$

$$= e^{th^2\partial^2}\left(1 + t\, h^2\sigma F + \frac{t^2}{2}\left(h^2\sigma F\right)^2 + \cdots\right) = e^{t(h^2\partial^2 + h^2\sigma F)}$$

where σF stands for $\frac{1}{2}\gamma^\mu\gamma^\nu F_{\mu\nu}$.

Now I have to somehow bring in the idea that $h\gamma^\mu \longrightarrow dx^\mu$ as $h \longrightarrow 0$.

March 17, 1984

Consider the heat operator $e^{-t\Delta}$ on forms $\Gamma(\Lambda T^*)$. A section $\alpha \in \Gamma C(T^*)$ of the Clifford bundle determines left and right multiplication operators L_α, R_α on $\Gamma C(T^*) = \Gamma(\Lambda T^*)$. Then we get 4 operators by composing with $e^{-t\Delta}$ on the left or right. The question will be what happens to these operators as we take the classical limit.

Let's look at this in Euclidean spaces. There is a question as to whether I should represent operators by Schwartz kernels with values in the Clifford algebra over the product.

Basic Principle: We are deforming an algebra. Think of the deformation as a bundle of algebras over the h-line. For example, take the universal enveloping algebra W of the Heisenberg algebra. Then we get an algebra with generators p, q, h and relations $[p,q] = h/i$, $[p,h] = [q,h] = 0$. This is naturally an algebra over $k[h]$. Moreover

$$W \otimes_{k[h]} k[h,h^{-1}] \simeq k[x,\partial_x] \otimes k[h,h^{-1}] \qquad k = \mathbb{C}$$

so that all the specializations of W for values $h \in \mathbb{C}$, $h \neq 0$ are isomorphic, and representable as operators on the polys. in x

Now let's discuss the corresponding situation where polynomials in x are replaced by $\mathbb{C}[x] \otimes C_n \simeq \mathbb{C}[x] \otimes \Lambda \mathbb{R}^n$. I have to describe the generators of my algebra. These generators can be specialized for any value of h. So at the moment I think I want $q, p, \psi^\mu, \bar\psi^\mu$, and h; the basic relations are the above ones for q, p and

$$[\psi^\mu, \bar\psi^\nu]_+ = 0 \quad , \quad [\psi^\mu, \psi^\nu]_+ = 2h^2 \delta_{\mu\nu} \quad , \quad [\bar\psi^\mu, \bar\psi^\nu]_+ = 2\delta_{\mu\nu}.$$

and the fact that the q,p algebra commutes with the $\psi, \bar\psi$ alg.

Let us calculate in the case of Euclidean space the [redacted] kernels representing $e^{-t\Delta}R_\alpha$, $e^{-t\Delta}L_\alpha$ and see if they have nice limits as $t \to 0$. Here $\alpha \in C_n$ is a constant coefficient [redacted] element on $\Gamma(C(T^*))$ and so [redacted] L_α, R_α commute with Δ.

Now I am interested in operators on $\Gamma(C(T^*)) = \Omega^0(M)\otimes C_n$ and the algebra of operators is clearly $\Omega^0(M\times M)\otimes C_n \otimes C_n$, where the **two factors of** C_n are respectively the left and right multiplications. Here $\Omega^0(M\times M)$ is thought of as the integral operators on functions.

Question: I know that if I express $e^{-t\Delta}$ as a form on the product, then it is given by the differential form

$$\frac{e^{-\frac{(x-y)^2}{4t}}}{(4\pi t)^{n/2}} \prod_1^n (dy_i - dx_i)$$

which makes sense as $t\to 0$ on the tangent bundle if I put $y_i = x_i + \sqrt{t}\, v_i$. Suppose I consider the operators such as $e^{-t\Delta}L_\alpha R_\beta$. When do their associated forms converge as $t \to 0$?

I need a convenient representation for [redacted] End$(\Lambda \mathbb{C}^n)$ in order to answer this question. Clearly we get something like the constant forms on the product. Take $n=1$. Then we have four constant forms on the product $1, dx, dy, dx\,dy$. [redacted] We combine with $\dfrac{e^{-\frac{(x-y)^2}{4t}}}{\sqrt{4\pi t}}$ and ask up putting $y = x + \sqrt{t}\, v$ that we get a finite limit as $t \to 0$. This.

$$\begin{aligned} 1 &\longmapsto 1 \\ dx &\ dx \\ dy &\ dx + \sqrt{t}\, dv \\ dx\,dy &\ \sqrt{t}\, dx\, dv \end{aligned}$$

so the only forms that remain finite are $dx - dy$, $dx\,dy$

Actually we see a natural filtration which describes the limiting behavior

$$0 \longrightarrow \langle dx - dy,\ dxdy \rangle \longrightarrow \langle 1, dx, dy, dxdy \rangle \longrightarrow \langle 1, dx \rangle \longrightarrow 0.$$

Now I ought to be able to describe this in general. Suppose we have n variables $x^1, \ldots, x^n$. Then we write

$$\Lambda[dx^\mu, dy^\mu] \simeq \Lambda[dx^\mu, \sqrt{t}\, d\sigma^\mu] = \Lambda[dx^\mu] \otimes \Lambda[\sqrt{t}\, d\sigma^\mu]$$

and if we divide by $(\sqrt{t})^n$, then the only part that remains finite as $t \to 0$ is $\Lambda[dx^\mu] \cdot d\sigma^1 \ldots d\sigma^n$.

Let's go onto the case of a surface with coordinates x^1, x^2. The problem will be whether the kernel for the operator $e^{-t\Delta}\pi$ when rescaled remains finite, where π is the projection on the 0 eigenspace for the circle action. Now π is a projector on $\Lambda[dx^1, dx^2]$ which the γ^1, γ^2 left multiplication. I have to find what π is and express it as a form on the product, and then see that it all works.

We know that π has for its image the forms $1 + i\,dx^1 dx^2$, $dx^1 + i\,dx^2$ and has for kernel the forms $1 - i\,dx^1 dx^2$, $dx^1 - i\,dx^2$. The kernel on the product is

$$\frac{1}{2i}(1 + i\,dx^1 dx^2)(1 + i\,dy^1 dy^2) - \frac{1}{2i}(dx^1 + i\,dx^2)(dy^1 - i\,dy^2)$$

Anyway the $\underline{1}$ here is disastrous which indicates that your ideas about how to get the $\bar\partial$-Laplacean kernel from the d-Laplacean are wrong.

March 20, 1984

Today I want prove the index theorem on a torus, in fact a 2-torus, for the Dirac operator with coefficients in vector bundle with connection, and eventually super-connection. I would like to start with the heat operator for the usual Laplacean $\Delta = \partial_\mu^2$ and represent the vector bundle by an idempotent matrix. The curvature contribution then comes from the Grassmannian connection.

Let $\gamma^\mu \partial_\mu$ be the Dirac operator on the torus M, and let e be an idempotent matrix function on M. Then we have the Dirac operator on the v. bundle $E = \operatorname{Im} e$

$$e \cdot \gamma^\mu \partial_\mu \cdot e = \gamma^\mu \, e \cdot \partial_\mu \cdot e$$

and we want to compute its index [redacted]. One has

$$(\gamma^\mu D_\mu)^2 = D_\mu^2 + \tfrac{1}{2}\gamma^\mu \gamma^\nu F_{\mu\nu} \qquad F = e\,de\,de$$

At this point one has to find some analytical way of dealing with the index, and the only possibility is via a trace. The problem for me is to work within the trivial bundle and avoid the standard process of working locally on M with a trivialization of E.

Possibility: The limiting heat kernel is supposed to contain all the index information and is therefore maybe equivalent to the cyclic cocycles belonging to the operator.

Let's look at the problem of finding the limiting heat kernel. We have the standard perturbation expansion for $e^{h\not D^2}$ or $(\lambda + h\not D^2)^{-1}$.

$$\not{D} = \frac{h}{i}(\gamma^\mu D_\mu)$$

$$e^{-t\not{D}^2} = e^{th^2 D_\mu^2} + \int_0^t dt_1\, e^{(t-t_1)h^2 D_\mu^2} \frac{h^2}{2}\gamma^\mu\gamma^\nu F_{\mu\nu}\, e^{t_1 h^2 D_\mu^2} + \cdots$$

$$\frac{1}{\lambda + \not{D}^2} = \frac{1}{\lambda - h^2 D_\mu^2} + \frac{1}{\lambda - h^2 D_\mu^2}\left(+\frac{h^2}{2}\gamma^\mu\gamma^\nu F_{\mu\nu}\right)\frac{1}{\lambda - h^2 D_\mu^2} + \cdots$$

Now $\displaystyle \langle x | \frac{1}{(\lambda - h^2\partial_\mu^2)^k} | x + hv\rangle = \int \frac{d^n\xi}{(2\pi)^n}\, e^{-i\xi x} \frac{1}{(\lambda + h^2\xi^2)^k}\, e^{+i\xi(x+hv)}$

$$= \frac{1}{h^n}\int \frac{d^n p}{(2\pi)^n}\, \frac{e^{ipv}}{(\lambda + p^2)^k}$$

When I take the limiting heat kernel, I want to be able to replace D_μ^2 by its leading term ∂_μ^2 and fix $F_{\mu\nu}$ at the point x. I want to work in the algebra generated by the operators x^μ, $p_\mu = \frac{h}{i}\partial_\mu$, $h\gamma^\mu$ and then specialize at $h = 0$. Maybe we want to reduce this algebra, or actually the matrix algebra over it, by the idempotent matrix e which is a function of x.

Another idea we learn from Friedan + Windey is to work with the heat operator

$$e^{-tH - \tau\not{D}} = e^{-tH}(1 - \tau\not{D})$$

where τ is an odd quantity. This must correspond to Feynman's propagator

$$\frac{1}{\gamma\cdot p + \varepsilon m} = \frac{\gamma\cdot p + \varepsilon m}{p^2 + m^2}$$

and must relate nicely to your $\gamma^\mu D_\mu + \varepsilon L$. What this means is that I also want to look at both

$$e^{-t\not{D}^2} \text{ and } e^{-t\not{D}^2}\not{D} \qquad \text{or} \qquad \frac{1}{\lambda + \not{D}^2} \text{ and } \frac{1}{\lambda + \not{D}^2}\not{D}$$

So let's consider $i\not{D} = +h\gamma^\mu D_\mu + \varepsilon L$ and see if we can get the filtration straight. It is not in the algebra generated by the $x^\mu, p_\mu, h\gamma^\mu$. However its square

$$-\not{D}^2 = h^2 D_\mu^2 + \frac{h^2}{2}\gamma\gamma F + h\varepsilon\gamma^\mu [D_\mu, L] + L^2$$

is in the algebra.

Let's ignore this and go back to the problem of constructing the limiting heat kernel. Start with

$$e^{-t\not{D}^2} = e^{th^2D_\mu^2} + \int_0^t dt_1\, e^{(t-t_1)h^2D_\mu^2}\, \frac{h^2}{2}\gamma^\mu\gamma^\nu F_{\mu\nu}\, e^{t_1 h^2 D_\mu^2} + \cdots$$

and

$$\langle x | e^{th^2\partial_\mu^2} | x + hv\rangle = \frac{1}{h^n}\int \frac{d^np}{(2\pi)^n} e^{ipv - t|p|^2}$$

$$= \frac{1}{(4\pi t h^2)^{n/2}}\, e^{-\frac{v^2}{4t}}$$

In the limiting case the terms in the above expansion should commute to lower order, so that we get something like

$$\langle x | e^{-t\not{D}^2} | x + hv\rangle \sim \frac{e^{-\frac{v^2}{4t}}}{(4\pi t h^2)^{n/2}}\, e^{th^2\frac{1}{2}\gamma^\mu\gamma^\nu F_{\mu\nu}(x)}$$

$$\sim \frac{1}{h^n}\int \frac{d^np}{(2\pi)^n}\, e^{ipv + t\left(-p^2 + \frac{h^2}{2}\gamma^\mu\gamma^\nu F_{\mu\nu}(x)\right)}$$

There is a problem as to what this means since it is not the full asymptotic expansion. But the idea is that it lives in the algebra gen. by $x, p, h\gamma$ and is the leading term as $h \to 0$.

The problem is really whether & how one can define a limiting heat kernel for

$$-i\not{D} = h\gamma^\mu D_\mu .$$

Whether we use the heat operator $e^{-t\not{D}^2}$ or the resolvent $\frac{1}{\lambda+\not{D}^2}$ is irrelevant, or should be, except that the resolvent we expect to be rather singular.

The heat operator $e^{-t\not{D}^2}$ and [redacted] the resolvent are operators on $L^2(M, S\otimes E)$; I'm thinking of M as a torus, hence S is the vector space of n-diml. spinors. What should the Schwartz kernels of these operators look like? In general one can represent[redacted] operators on functions [redacted] by functions on the product using the given volume on M. Let's therefore consider the Schwartz kernel of $e^{-t\not{D}^2}$ as a function on the product $M\times M$ with coefficients in the bundle $\mathrm{End}(S\otimes E)$. Better, think of S and E as trivial bundles then we actually have a function with values in the algebra

$$\mathrm{End}(S\otimes E) = \mathrm{Cliff}\otimes \mathrm{End}\,E$$

Now what sort of limit does one take?

Start again: We consider a Dirac operator $Q = -i(h\gamma^\mu D_\mu)$ over a torus M operating on functions with values in $S\otimes E$. We write the Schwartz kernel for e^{-tQ^2} as a function on M with values in

$$\mathrm{End}(S\otimes E) = \mathrm{Cliff}\otimes\mathrm{End}\,E$$

To keep things simple I can suppose $E = \mathbb{1}$ so that

I am dealing with the Dirac operator associated to a $U(1)$ gauge field. Now

$$-Q^2 = h^2 D_\mu^2 + \frac{h^2}{2}\gamma^\mu\gamma^\nu F_{\mu\nu}$$

lives in the algebra of operators generated by the elts h, x^μ, $p_\mu = \frac{h}{i}\partial_\mu$, $h\gamma^\mu$. It seems that this algebra can be specialized at $h=0$ to the functions on the cotangent bundle tensored with an exterior alg with generators $h\gamma^\mu$. Call these dx^μ, whence we seem to find that the limiting algebra is the algebra of horizontal forms on the cotangent bundle (with values in End E.)

Now this line of argument suggests that the limiting kernel should be the form

$$e^{-t|p|^2 + t\frac{1}{2}dx^\mu dx^\nu F_{\mu\nu}}.$$

But we can also proceed directly using perturbation series

$$\frac{1}{\lambda+Q^2} = \frac{1}{\lambda - h^2D_\mu^2} + \frac{1}{\lambda-h^2D_\mu^2}\left(\frac{h^2}{2}\gamma^\mu\gamma^\nu F_{\mu\nu}\right)\frac{1}{\lambda-h^2D_\mu^2} + \cdots$$

All these have Schwartz kernels which are operators on $M\times M$ with values in $\mathrm{End}(S\otimes E)$. Because

$$h^n \langle x|\frac{1}{\lambda - h^2\partial_\mu^2}|x+hv\rangle = \int \frac{d^np}{(2\pi)^n} e^{ipv}\frac{1}{\lambda-p^2}$$

we expect also that

$$h^n\langle x|\frac{1}{\lambda+Q^2}|x+hv\rangle$$

has a nice limit as $h\to 0$. Except at this point we also want to rescale in the Clifford algebra. It is not clear what this means.

But the problem is the following. We know more or less that the leading term of

$$\langle x | \frac{1}{\lambda + Q^2} | x + hv \rangle \quad \text{is} \quad \langle x | \frac{1}{\lambda - h^2 D_\mu^2} | x + hv \rangle = \frac{1}{h^n} \frac{e^{-\frac{|v|^2}{4t}}}{(4\pi t)^{n/2}} + \cdots$$

Consequently the first two terms in the perturbation series are of size $\frac{1}{h^n}$ and $\frac{h^2}{h^n}$; so unless one works with $h\gamma^\mu$ there is no hope for a limiting kernel.

March 21, 1984

The problem for me is to see if it is at all sensible to think in terms of a limiting heat kernel which lies in the convolution algebra of the tangent bundle. If so, then this limiting heat kernel should be equivalent to all the cyclic cocycles of the operator.

What form do the cyclic cocycles take? The Dirac operator with coefficients in E represents a K-homology class on $\Gamma \operatorname{End} E$, more precisely, an element of $KK(\Gamma \operatorname{End} E, \mathbb{C})$. The Chern character of this K-homology class is some sort of cyclic cocycle class. It really ought to be represented by the matrix form $\hat{A}(M) \cdot (e^{D^2})$, and then when we cup with the canon. elt of $\underline{KK(C(M), \Gamma \operatorname{End} E)}$, we get $\hat{A}(M)\operatorname{tr}(e^{D^2})$.

Let's consider on flat space a Dirac op

$$Q = -ih\gamma^\mu D_\mu$$

$$-Q^2 = h^2 D_\mu^2 + \frac{h^2}{2}\gamma^\mu\gamma^\nu F_{\mu\nu}$$

$$e^{-tQ^2} = e^{th^2D_\mu^2} + \int_0^t dt_1\, e^{t_1 h^2 D_\mu^2}\left(\frac{h^2}{2}\gamma\gamma F\right) e^{(t-t_1)h^2 D_\mu^2} + \cdots$$

$$\frac{1}{\lambda+Q^2} = \frac{1}{\lambda - h^2 D_\mu^2} + \frac{1}{\lambda - h^2 D_\mu^2}\left(\frac{h^2}{2}\gamma\gamma F\right)\frac{1}{\lambda - h^2 D_\mu^2} + \cdots\cdots$$

$$\langle x | \frac{1}{(\lambda - h^2\partial_\mu^2)^k} | y\rangle = \int \frac{d^n\xi}{(2\pi)^n} e^{+i\xi(x-y)} \frac{1}{(\lambda + h^2\xi^2)^k}$$

$$\cancel{\langle x | \frac{1}{(\lambda - h^2\partial_\mu^2)^k} | x+hv\rangle = \int \frac{d^n p}{(2\pi h)^n} e^{-ipv} \frac{1}{(\lambda+p^2)^k}}$$

$$= \frac{1}{h^n}\int \frac{d^n p}{(2\pi)^n} e^{ip\left(\frac{x-y}{h}\right)} \frac{1}{(\lambda+p^2)^k}$$

$$\langle x | e^{th^2 \partial_\mu^2} | y \rangle = \frac{1}{h^n} \int \frac{d^n p}{(2\pi)^n} \, e^{ip\frac{x-y}{h}} \, e^{-tp^2}$$

Here is the point: I want a limiting kernel as $h \to 0$ for e^{-tQ^2} or $\frac{1}{\lambda + Q^2}$. Now the only way I can get this is by making $h\gamma^\mu$ have a finite nonzero limit as $h \to 0$. Otherwise the higher terms in the perturbation expansion die.

So what do I have to do to construct the limiting kernel? Take

$$\langle x | e^{-tQ^2} | y \rangle$$

which has values in $C_n \otimes \text{End}\, V$, then use $\Lambda\, dx^\mu \xrightarrow{\sim} C_n$ where $dx^\mu \longleftrightarrow h\gamma^\mu$ to write this as a differential form, and also put $y = x + hv$. At this point we get on the tangent bundle a horizontal form with matrix coefficients, we can then multiply by h^n and let $h \to 0$ and we get

$$\int \frac{d^n p}{(2\pi)^n} \, e^{-ipv - tp^2 + \frac{t}{2} dx^\mu dx^\nu F_{\mu\nu}} .$$

This is seen more easily with the resolvent

$$\frac{1}{\lambda + p^2} + \frac{1}{\lambda + p^2} \left(\tfrac{1}{2} dx\, dx\, F \right) \frac{1}{\lambda + p^2} + \cdots$$

$$= \frac{1}{\lambda + p^2 - \frac{1}{2} dx\, dx\, F}$$

Now the curious thing is the t in front of the curvature which I didn't expect. If I restrict the kernel to the zero section and do the p integration I get

$$\frac{1}{(4\pi t)^{n/2}}\, e^{tF} \qquad\qquad F = \tfrac{1}{2} dx^\mu dx^\nu F_{\mu\nu}$$

Actually I am missing the normalization $(2i)^{n/2}$ that comes from the

$$tr_s(\gamma^1 \cdots\cdots \gamma^n) = (2i)^{n/2}$$

Thus we get $\dfrac{(2i)^{n/2}}{(4\pi t)^{n/2}} = \left(\dfrac{i}{2\pi}\right)^{n/2} \dfrac{1}{t^{n/2}}$ which when we integrate over the manifold M will give the index. Notice the t drops out.

One of the things I want to do with the limiting heat kernel is to reconstruct the heat kernel from it. I would like to be able to write down the limiting kernel by using differential forms, and then show the actual kernel is obtained by the iteration process. This should give the existence of the actual heat operator as well as the index results.

But now suppose we have a line bundle for coefficients, in fact a flat line bundle. The limiting kernel lies in forms with values in End E which is canonically trivial for a line bundle. If there is no curvature, then the limiting kernel does see anything of the given connection on E. So the reconstruction process must use the connection.

A possibility is that we also get a limiting kernel for the operator

$$e^{-tQ^2 - \varepsilon Q} = e^{-tQ^2}(1 - \varepsilon Q) \qquad \Big|\ \begin{matrix}\varepsilon^2 = 0 \\ \varepsilon \text{ odd}\end{matrix}$$

March 23, 1984 (Yesterday Carl was 19)

Let D be a connection on E and consider the Dirac operator $\gamma^\mu D_\mu$ on $S \otimes E$. I seem to believe that the form $\hat{A}(M)\, e^{D^2} \in \Omega^{ev}(M, \operatorname{End} E)$ gives all the index information belonging to (E, D). In particular it should somehow give rise to all the cyclic cocycles on $\Omega^0(M, \operatorname{End} E)$ belonging to $\gamma^\mu D_\mu$ viewed as an operator relative to this ring of endomorphisms.

Now it occurred to me that as there were problems connected with constructing cyclic cocycles on $\Omega^0(M, \operatorname{End} E)$ from the connection D, this approach might be too naive. So it would be a good idea to review what was learned from our previous work.

There were ~~too~~ approaches: One is based on the construction of secondary classes using Bott's thm.. The other is based on Connes work, realizing D as a Grassmannian connection: $E = \operatorname{Im} e$, $D = e \cdot d \cdot e$.

The first goes as follows. Cyclic cocycles on $\Gamma(\operatorname{End} E)$ are invariant closed differential forms on U_N of this ring. So we look at the MC form

$$\theta \in C^1(\tilde{\mathfrak{g}}, \Omega^0(M, \operatorname{End} E)) \subset \Omega(G \times M, pr_2^* \operatorname{End} E)$$

as a flat G-connection on $pr_2^* E$ over $G \times M$, whence

$$d_{\mathfrak{g}} \theta + \theta^2 = 0.$$

Then extend this G-connection to a full connection using D on E. D on E lifts to $\tilde{D} = d + D$ on $pr_2^* E$ and then the extension is $\tilde{D} + \theta$. We use the family $\tilde{D} + t\theta$ of connections on $pr_2^* E$, which has curvature

$$(\tilde{D} + t\theta)^2 = \tilde{D}^2 + t[\tilde{D}, \theta] + t^2 \theta^2 = D^2 + t[D, \theta] + (t^2 - t)\theta^2$$

$$\operatorname{tr} e^{(\tilde{D}+\theta)^2} - \operatorname{tr} e^{\tilde{D}^2} = d\left\{\int_0^1 dt \operatorname{tr}\left(e^{D^2+t[D,\theta]+(t^2-t)\theta^2}\theta\right)\right\}$$

The form in braces on the right is a form on $G\times M$, and when integrated over cycles in M leads to differential forms on G which are left-invariant. We see that we get a pretty messy mixture of D^2, $[D,\theta]$, θ for our cyclic cocycles. So it doesn't look as simple as knowing e^{D^2}.

Connes approach goes like this. Suppose that $E = \operatorname{Im} e$ where e is a projector on a trivial bundle $\mathbb{N}$, and $D = e.d.e$. Then we obtain a map of $\mathcal{a} = \Omega^0(M, \operatorname{End} E) = e M_N(A) e$, $A = \Omega^0(M)$, to a diff'l graded algebra, namely the matrices [redacted] over $\Omega(M)$:

$$\begin{array}{ccccc}
\Omega^0(M,\operatorname{End} E) & & & & \\
\cap\ \rho & & & & \\
M_N(\Omega^0(M)) & \xrightarrow{d} & M_N(\Omega^1(M)) & \xrightarrow{d} & \cdots\cdots
\end{array}$$

[redacted] Now given a closed trace on $M_N \otimes \Omega(M)$, i.e. a closed p-dim'l current γ on M, then it gives rise to a cyclic cocycle on $\Omega^0(M, \operatorname{End} E)$, namely

$$\underset{\mathcal{a}}{\overset{}{a_0,\ldots,a_p}} \longmapsto \int_\gamma \operatorname{tr} \rho(a_0)\, d\rho(a_1)\cdots d\rho(a_p).$$

I remember vaguely feeling that these two approaches probably lead to the same cocycles, except that the first one produces many S-iterated versions.

It seems already that there is a problem with the cyclic cocycles belonging to $\gamma^\mu\partial_\mu$. What is it that I want to be able to do? I want to consider a given Dirac operator on $S\otimes E$. Then suppose given an $\mathrm{End}(E)$ module V, I want to be able to tensor with V. So $V = E^*\otimes F$ and a connection on V compatible with the connection on E, is equivalent to a connection on F. So the tensoring process gives the Dirac operator on $S\otimes F$ associated to this connection.

So what I need is enough information about the Dirac operator on $S\otimes E$ so as to compute the index density for the operator on $S\otimes F$.

Let's regard as minimal information about $\gamma^\mu\partial_\mu$, that its index information is equivalent to the current on M given by $\int_M$. Then minimal information about $e\cdot\gamma^\mu\partial_\mu\cdot e$ should consist of the cocycle

$$X_0,\dots,X_p \longmapsto \int_M \mathrm{tr}\, X_0\, dX_1 \cdots dX_p \qquad X_i \in \Gamma(\mathrm{End}\,E) = e\,M_N(A)\,e.$$

We should be able to rewrite this entirely in terms of the connection $D = e\cdot d\cdot e$ on $\Gamma(\mathrm{End}\,E)$.

$$dX = d(eX) = de\cdot X + e\,dX$$

$$dX = d(eXe) = (de\,e)X + DX + X(e\,de)$$

Recall what we learned from Connes and is in his Ch. II. Take the cocycle

$$\int \mathrm{tr}\,\rho(a_0)\,d\rho(a_1)\cdots d\rho(a_p) \qquad a_i \in \Gamma\{\mathrm{End}\,E\}$$

and restrict this cocycle to the subalgebra ae, where now $a \in C^\infty(M)$. Then $\rho(a) = ae$ and so we that we are in the same sort of formalism used to define the S-operator. In fact his cocycle is obtained from the currents

$$\int_M (e\,de^2)^i \qquad i = 0, 1, \ldots$$

by the S-operation.

Next do this process in the case of my transgression formulas. θ becomes $e\theta$ where the new θ has values in $M_N(A)$. We get

$$D^2 = (e \cdot d \cdot e)^2 = e(de)^2$$

$$[D, e\theta] = e \cdot d \cdot e\theta + e\theta \cdot d \cdot e$$

$$= +e(d\theta)e - \cancel{e\theta\, d \cdot e} + \cancel{e\theta\, d \cdot e}$$

and so we get

$$\int_0^1 dt \;\; tr\left(e\theta\, e^{e(de)^2 + t\, e(d\theta)e \; + (t^2 - t)\, e\theta^2}\right).$$

But now $e(de)^2$ commutes with $\theta, d\theta$ so this is

$$\int_0^1 dt \;\; tr\left(e^{e(de)^2}\right) tr\left(\theta\, e^{t\, d\theta \, + (t^2-t)\theta^2}\right)$$

where the first trace is taken over E, and the second over $M_N(A)$.

This is evidence for the idea that the Dirac op. on $S \otimes E$ has its index information summarized by $tr(e^{D^2})$, provided we look at it as an operator over $C^\infty(M)$, not $\Gamma(\mathrm{End}\,E)$.

One thing we have learned is that e^{D^2} will probably not be enough to explain the cyclic cocycles over $\Gamma(\mathrm{End}\, E)$. Another thing suggested by the above is that the cyclic cocycle description of things is slightly awkward either from my Lie algebra cohomology viewpoint or Connes DG algebra approach. The difficulty lies already on the level of $\mathbb{R}^n$ and the operator $\gamma^\mu \partial_\mu$.

What I want is a really natural object I can attach to the Dirac operator, which is completely geometric and explains all the index densities we can construct from the Dirac operator.

March 24, 1984

Let's work over $\mathbb{R}^n$ or a torus. Given a vector bundle E with connection D we have determined the classical limit of the heat operator for $\not{D} = \gamma^\mu D_\mu$ on $S \otimes E$ and found that it is essentially the form e^{tF} where $F = \frac{1}{2} dx^\mu dx^\nu F_{\mu\nu}$ is the curvature. This form has values in End E and should tell all about index problems related to $\not{D}$. At least this is what I hope is true.

For example, suppose we have a right End(E)-module bundle V with a connection compatible with the connection on E. Then $V = \mathrm{Hom}(E, F) = E^* \otimes F$ and the connection on V comes from a connection on F. Then tensoring: $(S \otimes E) \otimes_{\mathrm{End}\,E^{op}} (E^* \otimes F) = S \otimes F$, and we get the Dirac operator on $S \otimes F$. I want to obtain the form $e^{tD_F^2}$ from $e^{tD_E^2}$, but we have

$$D^2_{E^* \otimes F} = D^2_{E^*} \otimes 1 + 1 \otimes D_F^2$$

$$e^{tD^2_{E^* \otimes F}} = e^{tD^2_{E^*}} \otimes e^{tD_F^2}$$

so we need somehow to see that

$$e^{tD_E^2} \otimes e^{-tD^2_{E^*}} \longmapsto 1$$

under the tensoring process.

$$K(\tau,\hat\tau) = e^{-\tau H - \hat\tau Q}, \qquad Q^2 = H, \qquad \hat\tau \text{ anti-commuting number}$$

$$= e^{-\tau H}(1-\hat\tau Q)$$

Let $\partial_{\hat\tau}$ denote interior product with $\partial_{\hat\tau}\hat\tau = 1$.

Then $$\partial_{\hat\tau} K(\tau,\hat\tau) = -Q e^{-\tau H} = -Q(1-\hat\tau Q)e^{-\tau H} - Q\hat\tau Q e^{-\tau H}$$

$$= -QK + \hat\tau H e^{-\tau H} = -QK + \hat\tau \partial_\tau K$$

or $$\boxed{(\partial_{\hat\tau} + \hat\tau\partial_\tau)K = -QK}$$

~~Iterating~~ $$(\partial_{\hat\tau} + \hat\tau\partial_\tau)^2 K = Q(\partial_{\hat\tau} + \hat\tau\partial_\tau)K = -HK$$

which is consistent with the formula

$$(\partial_{\hat\tau} + \hat\tau\partial_\tau)^2 = \cancel{\partial_{\hat\tau}^2} + (\partial_{\hat\tau}\hat\tau + \hat\tau\partial_{\hat\tau})\partial_\tau + \cancel{\hat\tau^2\partial_\tau^2}$$

$$= \partial_\tau$$

(In the Friedan-Windey paper $Q = \not{\partial}$ so $Q^2 = -H$ and there are different signs.)

Let change to the F-W signs.

$$K = e^{-\tau H - \hat\tau Q} = (1-\hat\tau Q)e^{-\tau H}$$

$$\partial_{\hat\tau} K = -Q e^{-\tau H} = -Q(1-\hat\tau Q)e^{-\tau H} + \hat\tau Q^2 e^{-\tau H}$$

$$= -QK + \hat\tau\partial_\tau K$$

$$\boxed{(\hat\tau\partial_\tau - \partial_{\hat\tau})K = QK}$$

$$(\hat\tau\partial_\tau - \partial_{\hat\tau})^2 = -\partial_\tau$$

I want next to understand the Lagrangian of Friedan + Windey in the case of Euclidean space but with a gauge field. From their viewpoint they describe the Lagrangian in terms of a superfield

$$X^\mu = x^\mu + \theta\psi^\mu$$

and the super-derivative

$$D = \theta\partial_t - \partial_\theta$$

whose square is $D^2 = -\partial_t$. This theory is supposed to give the Dirac operator by canonical quantization. Then one modifies the flat Lagrangian in order to allow for a background metric and gauge field in a standard way.

In order to allow for the Dirac operator on $S \otimes E$, one introduces fermion quantities $\eta^a, \bar{\eta}_b$, a, b run from 1 to $\dim E$. These are canonically conjugate fermion operators

$$[\bar{\eta}_b, \eta^a]_+ = \delta_b^a$$

To the Lagrangian is added $\bar{\eta}\dot{\eta}$. Since

$$\delta\int(\bar{\eta}\dot{\eta})\,dt = \int(\delta\bar{\eta}\,\dot{\eta} + \bar{\eta}\,\delta\dot{\eta})\,dt$$

$$= \int\left(\delta\bar{\eta}\,\dot{\eta} - \dot{\bar{\eta}}\,\delta\eta\right)dt$$

the equations of motion are $\dot{\eta} = 0$, $\dot{\bar{\eta}} = 0$.

I propose now to go over what FW do, but just for the gauge field, and so I will ignore the part of the Lagrangian dealing with X^μ for the moment.

In order to introduce the background gauge field what we do is to introduce superfields

$$N = \eta + \theta\phi \qquad \bar{N} = \bar{\eta} + \theta\bar{\phi}$$

where $\phi, \bar{\phi}$ are auxiliary fields. The superversion of the kinetic term $\bar{\eta}\dot{\eta}$ is

$$\bar{N}\, D(N) = (\bar{\eta} + \theta\bar{\phi})\underbrace{(\theta\partial_t - \partial_\theta)(\eta + \theta\phi)}_{\theta\dot{\eta} - \phi}$$

$$= -\bar{\eta}\phi - \theta\bar{\phi}\phi + \bar{\eta}\theta\dot{\eta}$$

One integrates $\int dt \int d\theta$, so only the coeff. of θ matters and we get

$$-\int dt\, (+\bar{\phi}\phi + \bar{\eta}\dot{\eta})$$

leading to the equations of motion $\bar{\phi} = \phi = 0$. Eliminating the $\phi, \bar{\phi}$ by these equations gives back

$$-\int dt\, \bar{\eta}\dot{\eta}$$

as desired.

Now when the gauge field is introduced $\bar{N}DN$ is to be replaced by $\bar{N}D_A N$ which is a super version of

$$\bar{\eta}(\dot{\eta} + \dot{x}^\mu A_\mu \eta) = \bar{\eta}\,(\partial_t + \dot{x}^\mu A_\mu)\eta .$$

Here $\partial_t + \dot{x}^\mu A_\mu$ denotes the total change of the η field, assuming a path $x^\mu(t)$ given, and measuring change relative to the given connection A_μ. The appropriate super version seems to be

$$\bar{N}\left(D + (DX^\mu)A_\mu(X)\right)N$$

$$DX^\mu = (\theta\partial_t - \partial_\theta)(x^\mu + \theta\psi^\mu) = \theta\dot{x}^\mu - \psi^\mu$$

$$\therefore \bar{N} D_A N = (\bar{\eta} + \theta\bar{\phi})\left[\theta\partial_t - \partial_\theta + (\theta\dot{x}^\mu - \psi^\mu)\left(A_\mu(x) + \theta\psi^\nu\partial_\nu A_\mu(x)\right)\right](\eta + \theta\phi)$$

$$= \bar{\eta}\left\{\theta\dot{\eta} + \theta\dot{x}^\mu A_\mu \eta - \psi^\mu A_\mu \theta\phi - \psi^\mu\theta\psi^\nu\partial_\nu A_\mu \eta\right\}$$

$$+ \theta\bar{\phi}\left\{-\phi - \psi^\mu A_\mu \eta\right\} \quad + \text{ terms not inv. } \theta$$

$$\int d\theta\, \bar{N} D_A N = -\bar{\eta}\dot{\eta} - \bar{\eta}\,\dot{x}^\mu A_\mu \eta - \bar{\eta}\,\psi^\mu A_\mu \phi - \bar{\eta}\psi^\mu\psi^\nu\partial_\nu A_\mu \eta$$
$$- \bar{\phi}\left\{\phi + \psi^\mu A_\mu \eta\right\}$$

Taking the variation wrt ϕ and $\bar{\phi}$ yields the equations

$$\phi + \psi^\mu A_\mu \eta = 0$$
$$-\bar{\eta}\psi^\mu A_\mu - \bar{\phi} = 0 .$$

Using these to eliminate $\phi, \bar{\phi}$ we get

$$-\bar{\eta}\left\{\dot{\eta} + \dot{x}^\mu A_\mu \eta - \psi^\mu A_\mu \psi^\nu A_\nu \eta + \psi^\mu\psi^\nu\partial_\nu A_\mu \eta\right\}.$$

But recall that in the Lagrangian ψ^μ, ψ^ν anti-commute and

$$F_{\mu\nu} = [D_\mu, D_\nu] = \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu],$$

hence our effective Lagrangian is

$$\boxed{-\bar{\eta}\left\{\dot{\eta} + \dot{x}^\mu A_\mu \eta - \tfrac{1}{2}F_{\mu\nu}\psi^\mu\psi^\nu \eta\right\}}$$

Next I want to generalize this calculation to the case where I have a superconnection $\partial_\mu + A_\mu + L$. Recall that $\bar{N} D_A N$ was obtained from the simpler

$$\int dt\; \bar{\eta}\left(\partial_t + \dot{x}^\mu A_\mu\right)\eta$$

which suggests adding $\int dt\, \bar\eta L \eta$. Thus to $\int dt \int d\theta\, \bar N D_A N$ we want to add the superversion

$$\int dt \int d\theta\, \bar N L(X) N.$$

Now $\bar N L(X) N = (\bar\eta + \theta\bar\phi)(L + \theta\psi^\mu \partial_\mu L)(\eta + \theta\phi)$

$$= \bar\eta(L\theta\phi + \theta\psi^\mu\partial_\mu L\eta) + \theta\bar\phi L\eta$$

so $-\int d\theta\, \bar N L(X) N = -\bar\eta L\phi + \bar\eta\psi^\mu\partial_\mu L\eta - \bar\phi L\eta$

and this is to be added to

$$\int d\theta\, \bar N D_A N = -\bar\eta\dot\eta - \bar\eta\dot x^\mu A_\mu\eta - \bar\eta\psi^\mu A_\mu\phi - \bar\eta\psi^\mu\psi^\nu\partial_\nu A_\mu\eta$$
$$-\bar\phi\{\phi + \psi^\mu A_\mu\eta\}.$$

Taking variations wrt $\phi, \bar\phi$ yields

$$\phi + \psi^\mu A_\mu\eta + L\eta = 0$$
$$\bar\phi + \bar\eta\psi^\mu A_\mu + \bar\eta L = 0.$$

Using this to eliminate $\phi, \bar\phi$ yields

$$-\bar\eta\dot\eta - \bar\eta\dot x^\mu A_\mu\eta + \bar\eta\psi^\mu A_\mu(+\psi^\nu A_\nu\eta + L\eta) - \bar\eta\psi^\mu\psi^\nu\partial_\nu A_\mu\eta$$
$$+\bar\eta\psi^\mu\partial_\mu L\eta + \bar\eta L(+\psi^\mu A_\mu\eta + L\eta)$$

$$= -\bar\eta\dot\eta - \bar\eta\dot x^\mu A_\mu\eta + \tfrac{1}{2}\bar\eta F_{\mu\nu}\psi^\mu\psi^\nu\eta + \bar\eta L^2\eta$$
$$+\bar\eta\psi^\mu(\partial_\mu L + A_\mu L - LA_\mu)\eta$$

$$= -\bar\eta\left\{\partial_t + \dot x^\mu A_\mu - \tfrac{1}{2}F_{\mu\nu}\psi^\mu\psi^\nu - \psi^\mu[D_\mu, L] - L^2\right\}\eta$$

<u>Summary</u>: If I take the superfield Lagrangian

$$\int dt \int d\theta \left(-\bar{N} D_A N + \bar{N} L N \right)$$

and eliminate $\phi, \bar{\phi}$ using the equations of motion:

$$-\phi = (\psi^\mu A_\mu + L)\eta$$
$$-\bar{\phi} = \bar{\eta}(\psi^\mu A_\mu + L)$$

then I obtain the "component" Lagrangian

$$\int dt \; \bar{\eta} \left\{ \partial_t + \dot{x}^\mu A_\mu - \tfrac{1}{2} F_{\mu\nu} \psi^\mu \psi^\nu - \psi^\mu [D_\mu, L] - L^2 \right\} \eta$$

<u>This formula is clearly very good.</u> So I really must try to get to the bottom of this mysterious superformalism. The mystery involves the following ingredients.

1) Introduction and elimination of the auxiliary fields $\phi, \bar{\phi}$.

2) Generalization of the action $S = \int dt \, \mathcal{L}$ to $S = \int dt \int d\theta \; \mathcal{L}$.

Also I have to use Planck's constant to get the result I want.

Fermion Lagrangians + quantization.

Let's develop the theory of a single fermion degree of freedom by analogy with the simple harmonic oscillator.

$$H = \frac{p^2}{2} + \frac{1}{2}\omega^2 q^2 \qquad [p,q] = \frac{h}{i}$$

$$\Big[\underbrace{\frac{ip+\omega q}{\sqrt{2h\omega}}}_{a}, \underbrace{\frac{-ip+\omega q}{\sqrt{2h\omega}}}_{a^*}\Big] = 1$$

$$h\omega\, a^* a = \frac{1}{2}\left(p^2 + \omega^2 q^2 - ip\omega q + \omega q ip\right) = H - \frac{1}{2}h\omega$$

or $$H = h\omega\left(a^* a + \frac{1}{2}\right).$$

Now it seems that the natural way to describe the harmonic oscillator even classically is to use the variables

$$\psi = \frac{1}{\sqrt{2\omega}}(\omega q + ip) \qquad \Longrightarrow \left\{ q = \frac{1}{\sqrt{2\omega}}(\psi + \bar\psi)\right.$$

$$\bar\psi = \frac{1}{\sqrt{2\omega}}(\omega q - ip) \qquad \left(p = \sqrt{\frac{\omega}{2}}\,\frac{1}{i}(\psi - \bar\psi)\right)$$

whence $$\boxed{H = \omega\bar\psi\psi}$$ and

$$\int p\,dq = \frac{1}{2i}\int (\psi - \bar\psi)(d\psi + d\bar\psi)$$

$$= \int \frac{1}{2i}\left(-\bar\psi\, d\psi + \psi\, d\bar\psi\right) = \int i\bar\psi\, d\psi = \int i\bar\psi\dot\psi\, dt$$

Thus the action becomes

$$S = \int p\,dq - H\,dt = \int \left(i\bar\psi\dot\psi - \omega\bar\psi\psi\right) dt \qquad \boxed{L = i\bar\psi\dot\psi - \omega\bar\psi\psi}$$

It's clear that one has upon quantization $\psi = \sqrt{h}\, a$, $\bar\psi = \sqrt{h}\, a^*$ so the canonical commutation relation becomes

$$[\psi, \bar\psi] = h$$

Another way to get this is to compute the variable conjugate to ψ:

$$\frac{\partial L}{\partial \dot\psi} = \frac{\partial}{\partial \dot\psi}(i\bar\psi\dot\psi - \omega\bar\psi\psi) = i\bar\psi$$

and hence we want

$$[i\bar\psi, \psi] = \frac{h}{i} \Rightarrow [\psi, \bar\psi] = h.$$

Next we want to carry this over to fermions. The obvious thing is to put $\psi = \sqrt{h}\, a$, $\bar\psi = \sqrt{h}\, a^*$ where a, a^* are now fermion ann. & creation ops. Then

$$H = \omega\bar\psi\psi \qquad [\psi, \bar\psi]_+ = h$$

(Here, as above, we ignore the $\pm\frac{1}{2}\omega$ which presumably comes from a symmetric or Weyl type quantization.) This gives the correct energy levels.

Now we want a path integral version of this single fermion system. Path integrals arise when one wants to find the time-evolution $e^{-\frac{i}{h}Ht}$ or the heat operator $e^{-\beta H}$ by subdivision of the time or β interval. In the boson setup I represent these operators as kernels relative to the states $|x\rangle$. I want to do something analogous for the fermion system. I also want to see in the case of the quadratic Lagrangian $i\bar{\psi}\dot{\psi} - \omega\bar{\psi}\psi$, the exactness of the classical approximation, i.e. expansion at the critical point.

It would seem a good idea to understand the classical theory with a fermion Lagrangian. Let's recall that symplectic transformations are conveniently represented by quadratic functions, and there should be a parallel theory for orthogonal transformations.

Let's begin with a review of the symplectic transformations which result from a classical Lagrangian via Hamilton's principle. Let t', t be fixed and let

$$S(q,q') = \int_{t'}^{t} L(t, q(t), \dot{q}(t))\, dt$$

where $q(t)$ is the path making the integral stationary, having the endpoints q', q. Then

$$\delta S(q,q') = \int \left(\frac{\partial L}{\partial q}\delta q + \frac{\partial L}{\partial \dot{q}}\delta\dot{q}\right) dt$$

$$= \left[\frac{\partial L}{\partial \dot{q}}\delta q\right]_{t'}^{t} + \int \underbrace{\left(\frac{\partial L}{\partial q} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}}\right)}_{=0} dt$$

$$= p\,\delta q - p'\,\delta q'.$$

Thus the motion of the system gives a transformation $(q',p') \longrightarrow (q,p)$ and one has

$$dS = p\,dq - p'\,dq'$$

which implies that the transformation is symplectic.

Conversely suppose we are given a symplectic transformation $T^*_{M'} \simeq T^*_M$. The graph of this transformation should be a submanifold of $T^*_M \times T^*_{M'} = T^*_{M \times M'}$ which is Lagrangian. Then if it projects non-degenerately on $M \times M'$ one knows it has to be the image of a section of T^* of the form dS where S is a function on $M \times M'$. (Locally.)

Counting: Sp_{2n} acting on $\mathbb{R}^{2n}$ has quadratic forms as Lie algebra. $\therefore$ $\dim Sp_{2n} = \frac{2n(2n+1)}{2} = 2n^2 + n$

Any Lagrangian subspace projecting non-singularly on the q-coords is given by a quadratic form in the q's

$$\therefore \ \dim \text{Lag. subspaces} = \frac{n(n+1)}{2}$$

Stabilizer of the subspace $0 \times \mathbb{R}^n$ is GL_n together with those $\left| \begin{array}{l} q \to q \\ p \to p + tq \end{array} \right.$ preserving $dp\,dq = (dp + t\,dq)\,dq \iff$ t symmetric. Thus

$$\dim \text{stabilizer} = n^2 + \frac{n(n+1)}{2}$$

which checks. Finally we have

$$\dim Sp_{2n} = \dim \text{Lag subspaces in } \mathbb{R}^{2n} \times \mathbb{R}^{2n}.$$

So I see that the Lag. subspaces in $\mathbb{R}^{2n} \times \mathbb{R}^{2n}$ form a natural compactification of Sp_{2n}. And we also have this way of embedding the Lie algebra into the group. Not quite, there's a non-deg. condition.

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Given a quadratic form $S(q, q')$ let's find the condition that it define a symplectic transformation $q', p' \longrightarrow q, p$. One has $p' = -\frac{\partial S}{\partial q'}$ and in this way for q' fixed, p' becomes a function of q. We want this function to be invertible, so that q is a function of p'. The condition is then

$$\det \frac{\partial^2 S}{\partial q \partial q'} \neq 0$$

that S give a symplectic transformation. Similarly given a symplectic transformation, the condition that it comes from an S is that for q' fixed the map $p' \longrightarrow q$ be invertible, so that if

$$\begin{pmatrix} q \\ p \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} q' \\ p' \end{pmatrix}$$ we want b invertible.

Next we want to consider the orthogonal case. In order to proceed by analogy with the symplectic case it seems best to work with hyperbolic quadratic forms. Thus we have a vector space V over a field of the form $W \oplus W^*$ with the quadratic function

$$Q(w + \lambda) = \lambda(w)$$

so that the Clifford algebra $C(V, Q)$ is naturally $\mathrm{End}(\Lambda W)$, or $\mathrm{End}(\Lambda W^*)$.

(In the symplectic case q^μ are fns. on T, and p_μ are functions on T^* or elements of T. The representation is a space of functions on T, or polys. in $S(T^*)$.)

In the orthogonal case we will use 'variables' ψ, $\bar\Psi$ instead of q, p. We want $\bar\Psi\psi$ to corresponds to $p\dot{q}$. Thus

$$V = \underset{\bar\Psi_\mu}{\underset{\cup}{W}} \oplus \underset{\psi^\mu}{\underset{\cup}{W^*}}$$

will be interpreted as operators on $\Lambda W^* = \Lambda[\psi^\mu]$.

Next we want to describe orthogonal transformations $W \oplus W^* = V \simeq V' = W' \oplus W'^*$ by means of a skew form

$$S(\psi, \psi') = \tfrac{1}{2} a_{ij}\psi_i\psi_j + b_{ij}\psi_i\psi_j' + \tfrac{1}{2} c_{ij}\psi_i'\psi_j'$$
$$\in \Lambda^2[W^* \oplus W'^*]$$

Let's start with an orthogonal transformation

$$\begin{pmatrix} \psi \\ \bar\Psi \end{pmatrix} = \begin{pmatrix} A & B \\ C & D \end{pmatrix}\begin{pmatrix} \psi' \\ \bar\Psi' \end{pmatrix}$$

It is useful to think of $\psi^\mu, \bar\Psi_\mu$ as being the coordinate functions on $W \oplus W^* = V$. Thus what I have given is the matrix of the transformation $V' \xrightarrow{\sim} V$ in coordinates. Orthogonality means that

$$\bar\Psi^t\psi = (C\psi' + D\bar\Psi')^t(A\psi' + B\bar\Psi') = \bar\Psi'^t\psi'$$

i.e.

$$\| $$
$$\psi'^t C^t A\psi' + \psi'^t C^t B\bar\Psi' + \bar\Psi'^t D^t A\psi' + \bar\Psi'^t D^t B\bar\Psi'$$

whence we have

$$C^tA + A^tC = 0$$
$$D^tB + B^tD = 0$$
$$C^tB + A^tD = I \quad \text{or} \quad B^tC + D^tA = I$$

i.e.
$$\begin{pmatrix} D^t & B^t \\ C^t & A^t \end{pmatrix}\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix}$$

Now let's solve for Ψ, Ψ' in terms of ψ, ψ'.

$$\psi = A\psi' + B\Psi' \implies \Psi' = B^{-1}\psi - B^{-1}A\psi'$$
$$\Psi = C\psi' + D\Psi' \implies \Psi = C\psi' + D(B^{-1}\psi - B^{-1}A\psi')$$
$$= DB^{-1}\psi + (C - DB^{-1}A)\psi'$$

$$\therefore \begin{pmatrix} \Psi \\ \Psi' \end{pmatrix} = \begin{pmatrix} DB^{-1} & C - DB^{-1}A \\ B^{-1} & -B^{-1}A \end{pmatrix}\begin{pmatrix} \psi \\ \psi' \end{pmatrix}$$

However $\quad D^tB + B^tD = 0 \implies (B^t)^{-1}D^t + DB^{-1} = 0$

$\implies DB^{-1}$ is skew-symm.

Similarly from
$$\begin{pmatrix} A & B \\ C & D \end{pmatrix}\begin{pmatrix} D^t & B^t \\ C^t & A^t \end{pmatrix} = \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix}$$

we get $\quad AB^t + BA^t = 0 \implies B^{-1}A$ is skew-symm.

Finally $\quad B^tC + D^tA = I \implies (B^{-1})^t = C + (B^{-1})^tD^tA$

$= C - DB^{-1}A$

Thus
$$\boxed{\begin{pmatrix} \Psi \\ -\Psi' \end{pmatrix} = \overbrace{\begin{pmatrix} DB^{-1} & C - DB^{-1}A \\ -B^{-1} & B^{-1}A \end{pmatrix}}^{\text{skew symm.}}\begin{pmatrix} \psi \\ \psi' \end{pmatrix}}$$

Now suppose

$$S(\psi, \psi') = \tfrac{1}{2} a_{ij} \psi_i \psi_j + b_{ij} \psi_i \psi'_j + \tfrac{1}{2} c_{ij} \psi'_i \psi'_j$$

with a, c skew symm. Then interpreting $\frac{\partial}{\partial \psi_k}$ as interior product we have

$$\frac{\partial S}{\partial \psi_k} = \tfrac{1}{2} a_{kj} \psi_j - \tfrac{1}{2} a_{ik} \psi_i + b_{kj} \psi'_j$$

$$= a_{kj} \psi_j + b_{kj} \psi'_j$$

$$\frac{\partial S}{\partial \psi'_k} = -b_{ik} \psi_i + \tfrac{1}{2} c_{kj} \psi'_j - \tfrac{1}{2} c_{ik} \psi'_i$$

$$= -b_{jk} \psi_j + c_{kj} \psi'_j$$

so

$$\begin{pmatrix} \frac{\partial S}{\partial \psi_k} \\ \frac{\partial S}{\partial \psi'_k} \end{pmatrix} = \underbrace{\begin{pmatrix} a & b \\ -b^t & c \end{pmatrix}}_{\text{skew}} \begin{pmatrix} \psi \\ \psi' \end{pmatrix}$$

Thus an orthogonal transformation

$$\begin{pmatrix} \psi \\ \bar{\psi} \end{pmatrix} = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} \psi' \\ \bar{\psi}' \end{pmatrix}$$

with B invertible can be realized by a skew form $S(\psi, \psi') = \frac{1}{2} a \psi\psi + b \psi\psi' + \frac{1}{2} c \psi'\psi'$ with b invertible such that one has

$$\bar{\psi} = \frac{\partial S}{\partial \psi} \qquad \bar{\psi}' = -\frac{\partial S}{\partial \psi'}$$

Note: If you are interested in the time-evolution operator $e^{-\frac{i}{\hbar}Ht}$ in the limit as $\hbar \to 0$, then it is given by classical mechanics, i.e. by the symplectic flow on phase space belonging to the Hamiltonian H. But I am interested in the heat operator $e^{-\beta H}$ with β fixed, which is a different limiting process.

Program. Take $e^{-\beta H}$ where $H = \frac{p^2}{2} + \frac{\omega^2}{2} q^2$ is the oscillator Hamiltonian, and describe the $\hbar \to 0$ limit of this operator. Then do the same thing for the fermion case: $H = \omega \bar{\psi} \psi$, where $[\psi, \bar{\psi}]_+ = \hbar$. The real question is what can one say about the limiting behavior of the operator $e^{-\beta H}$ as $\hbar \to 0$?

For each $\hbar$ we have a Clifford algebra, so we have a bundle of algebras over the $\hbar$ line and a section of this bundle. Now the actual algs. are isomorphic and the isomorphism is unique up to inner automorphisms.

The superfield Lagrangian shows that the path integral formalism has to have something valid behind it. However you just spent a lot of time on the classical limit of the operator $e^{-\frac{i}{\hbar}Ht}$ instead of $e^{-\beta H}$.

It seems necessary to have a path integral appropriate for taking the classical limit of $e^{-\beta H}$. This should be quite different from

$$\int e^{\frac{i}{\hbar}S} \qquad\qquad S = \int L\,dt = \int (p\dot{q} - H)\,dt$$