October 15 - November 13, 1984

80-197

Attempt to construct Thom form for a 90-97
complex v.b. E using $PE = SE/S^1$ and the
relation $\xi^n + c_1(E) + \dots + c_n(E) = 0$. Attempt
to prove this relation using equivariant forms 174-180

Review of Atiyah's Lecture on mixed volumes 117-131
and convexity of the moment map. Also
the link with stat. mechanics (Laplace
transform of a measure in R^n)

Kernels (e.g. heat kernels) + blowups. 150's [redacted] October 15, 1984

Construction of the heat operator e^{-tH}. I want to construct the kernel $K(h,t,x,x')$ of this operator invariantly. Goal: Don't make choices dependent upon a coordinate system, rather set things up invariantly and use coordinates to calculate. Method: Follow the filtration idea, maybe introduce h to "analyze" this algebra.

The problem is to construct a forward parametrix for $\partial_t + H$ which means

$$(\partial_t + H)\,K(t,x,x') = \delta(t)\,\delta(x-x') \qquad \text{mod smooth}$$

$$K(t,x,x') = 0 \qquad t<0.$$

I assume H is an elliptic 2nd order operator with positive-definite scalar symbol. I propose to produce a formal fundamental solution of the form

$$K(t,x,x') = \frac{e^{-\frac{1}{t}S(x,x')}}{t^{n/2}} \sum_{k\in\mathbb{Z}} \frac{a_k(x,x')}{t^k}$$

where S, a_k are smooth functions defined in a nbd of $\Delta M \subset M\times M$. Conditions are that $S \geq 0$ with ΔM as a non-degenerate critical submanifold, where $S=0$ and the Hessian of S is the scalar symbol of H, up to a constant. Also a_k is to vanish to order $\geq 2k$ on ΔM.

Now it seems to me that x' is excess baggage since I am not going to compose kernels in this construction. Think in terms of the pole being at $t=0$, $x=x'$; choose coords so that $x'=0$

By the Morse lemma one can suppose that in the coordinate system $S(x) = \frac{1}{4}x^2$. Let's avoid this choice until necessary.

$$-H = a_{ij}\,\partial^2_{ij} + b_j \partial_j + c$$

where a_{ij}, b_j, c are functions of x.

$$\star \qquad \left(\frac{e^{-\frac{S}{t}}}{t^{n/2}}\right)^{-1}(\partial_t + H)\,\frac{e^{-\frac{S}{t}}}{t^{n/2}} = \partial_t + \frac{S}{t^2} - a_{ij}\left(\partial_i - \frac{\partial_i S}{t}\right)\left(\partial_j - \frac{\partial_j S}{t}\right) - \frac{n}{2t} + \cdots\cdots$$

The key idea will now be to consider this operator on terms $\frac{x^\alpha}{t^k}$ which are given the weight $|\alpha| - 2k$. The reason is the basic fact Melrose told me, namely boundedness of

$$\frac{x^\alpha}{t^k}\, e^{-\frac{x^2}{4t}}$$

for $x \in R$, $0 < t \leq 1$, provided $|\alpha| \geq 2k$. In effect one rescales to reduce to the boundedness of

$$\underbrace{\frac{t^{\frac{|\alpha|}{2}}}{t^k}}_{t^{\frac{|\alpha|}{2} - k}}\, y^\alpha\, e^{-\frac{y^2}{4}}$$

So the weight gives the vanishing order ~~as~~ $t \downarrow 0$.

This filtration gives the key to the story. I suppose to see things carefully that $a_{ij}(0) = \delta_{ij}$ and $S(x) = \frac{1}{4}x^2$. Then the conjugated operator $\star$

becomes

$$\partial_t + \frac{x^2}{4t^2} + \frac{-n}{2t} - \partial_i^2 + \frac{1}{t}x_j\partial_j + \frac{n}{2t} - \frac{x^2}{4t^2} - \left(a_{ij}(x) - \delta_{ij}\right)\left(\partial_i - \frac{x^i}{2t}\right)\left(\partial_j - \frac{x^j}{2t}\right) + \cdots$$

The operator

$$\partial_t + \frac{1}{t}x_j\partial_j = \frac{1}{t}\left(t\partial_t + x_j\partial_j\right)$$

acting on $\frac{x^\alpha}{t^k}$ multiplies by $\frac{|\alpha|-k}{t}$. Note that $|\alpha| - k = \frac{|\alpha|}{2} + \left(\frac{|\alpha|}{2} - k\right) \geq \frac{|\alpha|}{2}$ which is >0 unless $|\alpha| = 0$, whence $-k \geq 0$ and so we see that $|\alpha| - k > 0$ unless $\alpha = k = 0$. Thus $t\partial_t + x_j\partial_j$ is invertible on the monomials $\frac{x^\alpha}{t^k}$.

Then look at the other operators

$$t\partial_i^2 \quad , \quad t\, x^\beta\left(\partial_i - \frac{1}{2t}x^i\right)\left(\partial_j - \frac{1}{2t}x^j\right)$$

∂_i^2 lowers $|\alpha|$ by 2 and t lowers k by 1, so that $|\alpha| - 2k$ doesn't change. This operator $t\partial_i^2$ appears to be of the same type as $t\partial_t + x_j\partial_j$

<u>October 16, 1984</u>

I have an operator H, second order differential operator with positive definite scalar symbol and I am trying to construct a solution of

$$(\partial_t + H) K(t,x) = \delta(t)\,\delta(x)$$
$$K(t,x) = 0 \qquad x<0.$$

I look for a formal solution

$$K(t,x) = \frac{e^{-\frac{S(x)}{t}}}{t^{n/2}} \sum_{k\in\mathbb{Z}} a_k(x)\, t^k$$

where S has a non-degenerate minimum at $x=0$ with $S(0)=0$, and its Hessian is determined by the symbol of H at $x=0$.

I suppose S given and then I use the Morse lemma to choose coords. so that $S(x) = \frac{1}{4}x^2$. Then

$$-H = \partial_\mu^2 + a_{\mu\nu}(x)\,\partial_\mu\partial_\nu + b_\mu(x)\,\partial_\mu + c(x)$$

where $a_{\mu\nu}(x) = O(x)$. Set $H_0 = -\partial_\mu^2$. Now conjugate

$$\left(\frac{e^{-x^2/4t}}{(4\pi t)^{n/2}}\right)^{-1}(\partial_t + H)\left(\frac{e^{-x^2/4t}}{(4\pi t)^{n/2}}\right) = \partial_t + \frac{x^2}{4t^2} - \frac{n}{2t} - \left(\partial_\mu - \frac{x^\mu}{2t}\right)^2$$
$$+\, a_{\mu\nu}\left(\partial_\mu - \frac{x^\mu}{2t}\right)\left(\partial_\nu - \frac{x^\nu}{2t}\right) + \cdots$$

☆

$$= \left(\partial_t + \frac{1}{t}x^\mu\partial_\mu - \partial_\mu^2\right) + a_{\mu\nu}(x)\left(\partial_\mu - \frac{x^\mu}{2t}\right)\left(\partial_\nu - \frac{x^\nu}{2t}\right) + \cdots$$

Now we want to look at these operators acting on the space of monomials $x^\alpha t^k$. Ideally we want to construct some formal series

$$\sum c_{\alpha k}\, x^\alpha t^k$$

which is annihilated by the conjugated operator ★

Now yesterday I thought it was enough to keep track of the <u>weight</u> $|\alpha| + 2k$ of the monomial $x^\alpha t^k$. Because

$$t\partial_t + x^\mu \partial_\mu - t\partial_\mu^2$$

preserves this weight, and

$$t\, \overset{x^\beta}{\cancel{g_{\mu\nu}(x)}} \left(\partial_\mu - \frac{x^\mu}{2t}\right)\left(\partial_\nu - \frac{x^\nu}{2t}\right) \qquad \text{degree } |\beta|$$

$$t\, \overset{x^\beta}{\cancel{b_\mu(x)}} \left(\partial_\mu - \frac{x^\mu}{2t}\right) \qquad \text{degree } |\beta| + 1$$

$$t\, \overset{x^\beta}{\cancel{c(x)}} \qquad \text{degree } |\beta| + 2$$

increase the weight.

But then it is necessary to know that $t\partial_t + x^\mu\partial_\mu - t\partial_\mu^2$ acts invertibly on the space of monomials of a given weight. Let us consider this problem carefully:

$$\left(t\partial_t + x^\mu \partial_\mu - t\partial_\mu^2\right) \sum_{|\alpha|+2k=\varepsilon}' c_{\alpha k}\, x^\alpha t^k = 0 \qquad ?$$

Note
$$\sum_{|\alpha|+2k=\varepsilon}' c_{\alpha k}\, x^\alpha t^k = \sum_\alpha c_{\alpha, \frac{\varepsilon-|\alpha|}{2}}\, x^\alpha t^{\frac{\varepsilon-|\alpha|}{2}}$$

$$= t^{\frac{\varepsilon}{2}} \sum_\alpha \underbrace{c_{\alpha, \frac{\varepsilon-|\alpha|}{2}}}_{\tilde{c}_\alpha} \left(\frac{x}{\sqrt{t}}\right)^\alpha$$

$$= t^{\varepsilon/2} f\left(\frac{x}{\sqrt{t}}\right)$$

Set $y = \dfrac{x}{\sqrt{t}}$ and think of $f\left(\frac{x}{\sqrt{t}}\right)$ as $f(y)$. Then

$$\partial_t f\left(\tfrac{x}{\sqrt t}\right) = \frac{\partial f}{\partial y^\mu}(y)\,\frac{\partial y^\mu}{\partial t} = f_\mu(y)\, x^\mu \left(-\tfrac12\right) t^{-3/2}$$

$$t\partial_t f(y) = -\tfrac12\, y^\mu \frac{\partial}{\partial y^\mu} f(y)$$

$$x^\mu \partial_\mu f(y) = x^\mu \frac{\partial}{\partial y^\mu} f(y)\,\frac{1}{\sqrt t} = y^\mu \frac{\partial}{\partial y^\mu} f(y)$$

$$\partial_\mu f(y) = \frac{\partial}{\partial y^\mu} f(y)\,\frac{1}{\sqrt t} \qquad \partial_\mu^2 f(y) = \frac{\partial^2}{\partial y^{\mu\,2}} f(y)\,\frac{1}{t}$$

$$\begin{aligned} 0 &= \left(t\partial_t + \tfrac{\varepsilon}{2} + x^\mu\partial_\mu - t\partial_\mu^2\right) f(y) \\ &= \left(-\tfrac12 y^\mu \frac{\partial}{\partial y^\mu} + \tfrac{\varepsilon}{2} + y^\mu \frac{\partial}{\partial y^\mu} - \frac{\partial^2}{\partial y^\mu}\right) f(y) \\ &= \left(-\partial_{y^\mu}^2 + \tfrac12 y^\mu \partial_{y^\mu} + \tfrac{\varepsilon}{2}\right) f(y) \end{aligned}$$

However it's pretty clear that there should be lots of formal power series solutions.

Hence if I want some sort of uniqueness it seems that I want to restrict

$$* \qquad \sum_{|\alpha|+2k=\varepsilon} c_{\alpha k}\, x^\alpha t^k = t^{\varepsilon/2} f\left(\tfrac{x}{\sqrt t}\right)$$

to be a polynomial in x. On polynomials $f(y)$ the operator $\frac12 y^\mu \partial_{y^\mu} + \frac{\varepsilon}{2}$ is invertible, and $-\partial_{y^\mu}^2$ is zero on the associated graded space for the increasing filtration. So we will be able to invert $t\partial_t + x^\mu\partial_\mu - t\partial_\mu^2$ on the space of $*$ which are polynomials.

It appears that the range $\frac{|\alpha|}{3} \le |\alpha| + 2k$ works, but using such polys. appears dependent on the coordinate system.

October 18, 1984.

Idea last night. Problem is to construct the heat operator e^{-tH} formally at $t=0$. The idea is to start with a "path" $L(t) = 1 - tH + t^2 \cdots$ having the correct first moments. The Feynman-Kac approach is based upon the formula

$$L\left(\frac{t}{N}\right)^N \longrightarrow e^{-tH}$$

which is not really a formal construction in t. But it shows there should be a way ~~to~~ obtain e^{-tH} directly from $L(t)$ without further approximation.

Idea is to use nilpotent elements - t when viewed formally is just a nilpotent element of some algebra. The idea then is that given

$$\gamma(t) = 1 + at + bt^2 + \cdots\cdots$$

one lets $t_1, \dots, t_n$ be elements of square zero and forms

$$\gamma(t_1) \cdots \gamma(t_n) = \prod_{j=1}^n (1 + at_j) = e^{a(t_1 + \cdots + t_n)}.$$

Typical splitting principle idea.

Let E be a complex vector bundle over M. Then we have an obvious circle action on E with fixpt set equal to the zero section. The Berline-Vergne theory tells us how to deform equivariant classes on E to ones supported near the zero section. Is it possible to use the circle action to construct an explicit Thom form?

Note that in the BV theory one chooses an invariant 1-form α on $E - M$ with $\iota_X \alpha = 1$. Such a form is the same thing as a connection form (restrict to $S(E)$) in $S(E)$ relative to $\mathbb{P}E$, and $d\alpha$ represents the class $c_1(\mathcal{O}(-1)) \in H^2(\mathbb{P}E)$. Hence this suggests that a relation exists between the BV forms and the Chern forms of E.

To calculate let's work universally with $U(n)$ acting on $\mathbb{C}^n = V$, and construct equivariant forms on V, SV, $\mathbb{P}V$ for the $G = U(n)$ action. Then

$$V - (0) = \mathcal{O}(-1) - \mathbb{P}E$$

is the principal $\mathbb{C}^\times$-bundle associated to $\mathcal{O}(-1)$. Maybe I should restrict to SV.

The circle action is $e^{i\theta} * v = e^{i\theta} v$ so that

$$Xv = \frac{d}{d\theta}\Big|_{\theta=0} e^{i\theta} v = iv.$$

So in terms of the coordinates z^μ in V we have

$$X z^\mu = i z^\mu, \quad X \bar{z}^\mu = -i \bar{z}^\mu$$

and so

$$X = i z^\mu \frac{\partial}{\partial z^\mu} - i \bar{z}^\mu \frac{\partial}{\partial \bar{z}^\mu}$$

The natural choice for α is

$$\alpha = \frac{1}{2i}\left(\frac{\bar{z}^\mu dz^\mu - z^\mu d\bar{z}^\mu}{|z|^2}\right)$$

If we restrict to SV where $|z| = 1$, and take d we get

$$d\alpha\Big|_{SV} = i\, dz^\mu d\bar{z}^\mu$$

This descends to $\mathbb{P}V$ and represents $c_1\,\mathcal{O}(-1)$ up to normalizations which we now make precise: Here $\mathrm{Lie}\ S^1 = \mathrm{Lie}\ U(1) = i\mathbb{R}$ has the basis $X = i$, so that a connection form θ [redacted] $\in \Omega^1(P)\otimes_{\mathbb{R}}$ [redacted] $i\mathbb{R}$ in a principal S^1-bundle [redacted] is of the form

$$\theta = \alpha i \qquad (\theta = \theta^a X_a)$$

where α is a real 1-form on P. Then the $\boxed{\text{curvature is } d\theta = (d\alpha)i}$ and so the Chern form with the correct integral periods is

$$\frac{i}{2\pi}\, d\theta = -\frac{1}{2\pi}\, d\alpha\,.$$

Now I'd like to check by taking a holom. section s of $\mathcal{O}(-1)$ pulled up to $V - (0)$. Take the tautological section $s(v) = (v, v)$, whence

$$|s|^2 = \bar{z}^\mu z^\mu$$

and the curvature is

$$d'' d' \log |z|^2 = d''\left(\frac{\bar{z}^\nu dz^\nu}{|z|^2}\right)$$

$$= \frac{1}{|z|^4}\left(|z|^2 d\bar{z}^\mu dz^\mu - z^\mu \bar{z}^\nu d\bar{z}^\mu dz^\nu\right)$$

which is pretty messy. However restricting to SV gives

$$d\left(\frac{\bar{z}^\nu dz^\nu}{|z|^2}\right)\Bigg|_{|z|=1} = d\bar{z}^\nu dz^\nu = i\,d\alpha\big|_{SV}$$

for the curvature, which checks.

The good Kahler form on PV is therefore the form obtained by descending

$$\omega = d\alpha\big|_{SV} = i\,dz^\mu d\bar{z}^\mu \quad \text{on } SV.$$

This is the form defining the symplectic structure on PV.

Now the program is as follows. α is an invariant connection for the $U(n)$-action, so we can refine $d\alpha$ to an equivariant form

October 19, 1984

Goal: To understand properly how to construct Thom forms for a complex vector bundle.

I work with equivariant forms for $G = U(n)$ acting on $V = \mathbb{C}^n$. The problem is to do the transgression in the sphere bundle $S(E)$ of the generator of the fibre. This means that I want to take the volume form on $S(V)$ and refine it to an equivariant form. Notation: volume form $\gamma \in \Omega^{2n-1}(SV)^G$, and we have the map

$$\begin{array}{ccc} [S(\mathfrak{g}^*) \otimes \Omega(SV)]^G & \longrightarrow & \Omega(SV)^G \\ \cup\!\!\!\text{\tiny{I}} & & \cup\!\!\!\text{\tiny{I}} \\ \tilde{\gamma} & \longmapsto & \gamma \end{array}$$

and we want to lift γ to $\tilde{\gamma}$. Now $\tilde{\gamma}$ is the transgression cochain, so it isn't closed, but rather $d\tilde{\gamma}$ comes from the Euler form e in $S(\mathfrak{g}^*)^G$.

Yesterday I began to calculate with the natural forms on SV. In fact what are the G-invariant forms? One knows that the invariant forms on $\mathbb{P}V$ are all even-diml: $1, \omega, \dots, \omega^{n-1}$. Why?

$$\mathbb{P}V = G/H \qquad G = U(n), \quad H = U(1) \times U(n-1)$$

$$\Omega(\mathbb{P}V)^G = \left(\Lambda(\mathfrak{g}/\mathfrak{h})^*\right)^H \quad \cancel{}$$

If H = stabilizer of $L \oplus W = V$, then $\mathfrak{g}/\mathfrak{h} \simeq L^* \otimes W$. As $U(n-1)$-module it is just W and

$$(\mathfrak{g}/\mathfrak{h})^* = \mathrm{Hom}_{\mathbb{C}}(W, \mathbb{C}) \simeq W \oplus W^*.$$

Then the theorem of invariants tells us about $(\Lambda W \otimes \Lambda W^*)^{U(n-1)}$ being spanned by the powers of ω.

Now what happens when we take SV? This time let $K = U(n-1)$, so $G/K = SV$ and

$$\Omega(SV)^G = \left(\Lambda (\mathfrak{g}/\mathfrak{k})^*\right)^K$$

Now $\mathfrak{k} \subset \mathfrak{g}$ is

$$\left(\begin{array}{c|c} i\mathbb{R} & -\bar{a}\ \ -\bar{b}\ \ -\bar{c}\ \ \cdots \\ \hline a & \\ b & \mathfrak{k} \\ c & \\ \vdots & \end{array}\right)$$

so as K-module $\mathfrak{g}/\mathfrak{k} = (i\mathbb{R}) \oplus W$ (real v.s.)

so $(\mathfrak{g}/\mathfrak{k})^* = \mathbb{C} \oplus W \oplus W^*$. So we get one extra invariant, so we have

<u>Prop:</u> $\Omega(SV)^G$ has the basis $\alpha, d\alpha, \alpha d\alpha, \ldots, \alpha(d\alpha)^{n-1}$.

At this point one knows that we can do the lifting to equivariant forms, because for one forms invariant and equivariant are equivalent. So $d\alpha$ lifts to

$$\tilde{d}\alpha = (d - \Omega^a \iota_a)\alpha$$

which is an equivariant 2-form coming from PV.

Now I digress to explore the moment map viewpoint. PV is a symplectic manifold and G acts preserving $\omega = d\alpha$. For each $X \in \mathfrak{g}$ one looks for a Hamiltonian H_X, which is a function on PV satisfying

$$dH_X = \iota_X \omega$$

In the present case, where $\omega = d\alpha$ is $(\pm i) \times$ curvature of an equivariant line bundle, H_X is essentially the Higgs field given by

$$\mathcal{L}_X = [\iota_X, D] + \varphi_X$$

operating on sections of the ~~vector~~ line bundle. Actually, I should change notation since X represents the circle generator. Use A for an element in Lie $U(n)$, whence

$$\mathcal{L}_A = [\iota_A, D] + \varphi_A$$

Pulling back to the principal bundle SV, D becomes $d + i\alpha$, so we have

$$i(\iota_A \alpha) + \varphi_A = 0. \qquad \varphi_A = i(-\iota_A \alpha)$$

So

$$dH_A = \iota_A \omega = \iota_A d\alpha = -d(\iota_A \alpha)$$

or $\qquad \varphi_A = i H_A. \qquad \boxed{H_A = -\iota_A \alpha}$

(Clearer: Because $\omega = d\alpha$ we have

$$dH_A = \iota_A \omega = \iota_A d\alpha = +d(-\iota_A \alpha)$$

so that $H_A = -\iota_A \alpha$ is a Hamiltonian. However

$$\mathcal{L}_A - \varphi_A = [\iota_A, D] = [\iota_A, d + i\alpha] = \mathcal{L}_A + i(\iota_A \alpha)$$

so that $\quad \varphi_A = i(-\iota_A \alpha)$. Thus $\varphi_A = i H_A$.)

So now let's find H_A in the case of the line bundle $\mathcal{O}(-1)$ with connection form $i\alpha$, where $\alpha = \frac{1}{i} \bar{z}^\mu dz^\mu$ on SV. We need the vector field X_A on SV. This means we run into the problem of converting the left action of $U(n)$ on SV into a right action, so as to get a left action on functions. So we define

$$(X_A f)(v) = \frac{d}{dt}\Big|_{t=0} f(e^{-tA}v)$$

$$= f(v - tAv)_{[\text{coeff of } t]}$$

$$= \frac{\partial f}{\partial z^\mu}(-Av)_\mu + \frac{\partial f}{\partial \bar z^\mu}\overline{(-Av)_\mu}$$

$$= \left(-a_{\mu\nu} z^\nu \frac{\partial}{\partial z^\mu} - \bar a_{\mu\nu}\bar z^\nu \frac{\partial}{\partial \bar z^\mu}\right) f$$

$$X_A f = \left(\bar a_{\nu\mu} z^\nu \frac{\partial}{\partial z^\mu} + a_{\nu\mu}\bar z^\nu \frac{\partial}{\partial \bar z^\mu}\right) f$$

Thus

$$\boxed{\begin{aligned} X_A &= (-a_{\nu\mu})\, z^\mu \frac{\partial}{\partial z^\nu} + \text{c.c.} && \text{general } A \in gl_n \\ &= \bar a_{\mu\nu}\, z^\mu \frac{\partial}{\partial z^\nu} + \text{c.c.} && A \in \operatorname{Lie} U(n) \end{aligned}}$$

So we are using the contragredient representation on functions.
Then

$$H_A = \iota_{X_A}\left(\frac{1}{i}\bar z^\mu \, dz^\mu\right)$$

$$= \frac{1}{i}\bar a_{\mu\nu} z^\mu \bar z^\nu = \frac{1}{i}\langle Av, v\rangle = i\langle v, Av\rangle$$

Let's return to the main line of investigation. The goal is to find the transgression cochain as a G-equivariant form on SV of degree $2n-1$ restricting to the volume forms, whose differential is the Euler form.

We can consider G-equivariant forms on SV, PV, pt. In particular we have $\tilde d\alpha = (d - \Omega^a \iota_a)\alpha$ representing $c_1(\mathcal{O}(-1)) = -\xi \in H^2_G(PV)$. We know in the equivariant cohomology $H_G^*(PV)$ there is a relation

$$\xi^n + c_1 \xi^{n-1} + \cdots + c_n = 0$$

So it's natural to ask if this holds on the level of equivariant forms. Probably not, instead one should have a formula

$$(\tilde{d}\alpha)^n + c_1(\tilde{d}\alpha)^{n-1} + \cdots + c_n = \tilde{d}\beta$$

where β is an equivariant $(2n-1)$-form on PV. If this is lifted up to SV, one gets

$$\tilde{d}\left[\alpha(\tilde{d}\alpha)^{n-1} + c_1\alpha(\tilde{d}\alpha)^{n-2} + \cdots + c_{n-1}\alpha\right] + c_n = \tilde{d}\beta$$

which gives the transgression form

$$\beta - \alpha\left[(\tilde{d}\alpha)^{n-1} + c_1(\tilde{d}\alpha)^{n-2} + \cdots + c_{n-1}\right].$$

This suggests maybe that $\beta = 0$.

October 20, 1984

$G = U(n)$ acting on $V = \mathbb{C}^n$. Goal: to construct the transgression form in $[S(\mathfrak{g}^*) \otimes \Omega(SV)]^G$ belonging to the Euler class. Basic observation is that an element of $[S(\mathfrak{g}^*) \otimes \Omega(\underline{S}V)]^G$ should be thought of as a form on $\underline{S}V$ depending on an element of $\mathfrak{g}$ in a polynomial way. Let's review yesterday's calculations.

$i\alpha = \bar{z}^\mu dz^\mu = \langle z, dz\rangle$ is the connection form for the S^1-bundle $SV \longrightarrow \mathbb{P}V$. It is G-invariant, hence represents an equivariant form, as the degree is one.

$gl(n)$ acts on V^* via the contragredient representation, which means the vector field on V belonging to $A \in gl(n)$ is

$$X_A = (-a_{\nu\mu}) z^\mu \frac{\partial}{\partial z^\nu} + \text{c.c.}$$

so if A is skew-hermitian

$$X_A = \bar{a}_{\mu\nu} z^\mu \frac{\partial}{\partial z^\nu} + a_{\mu\nu} \bar{z}^\mu \frac{\partial}{\partial \bar{z}^\nu}$$

$\alpha = \frac{1}{i} \bar{z}^\mu dz^\mu$

$\omega = d\alpha = i\, dz^\mu d\bar{z}^\mu$

$i\omega$ = curvature of $O(-1)$ over $\mathbb{P}V$

Thus

$$\iota_{X_A} \alpha = \frac{1}{i} \bar{a}_{\mu\nu} z^\mu \bar{z}^\nu = \frac{1}{i}\langle Az, z\rangle = i\langle z, Az\rangle$$

is the Hamiltonian fn belonging to X_A. Maybe I should call it the momentum J_A belonging to the inf. transf X_A of $\mathbb{P}V$. It has the basic property:

$$\boxed{\begin{aligned} dJ_A &= \iota_{X_A}\omega \qquad\qquad \omega = d\alpha = i\, dz^\mu d\bar{z}^\mu \\ J_A(z) &= i\langle z, Az\rangle \end{aligned}}$$

The differential $\tilde{d}$ in $[S(\mathfrak{g}^*) \otimes \Omega(SV)]^G$ is $d - \Omega^a \iota_a$

where $\Omega^a \iota_a$ means the map

$$\begin{aligned} \Omega(\$V) &\longrightarrow \mathfrak{g}^* \otimes \Omega(\$V) \\ \gamma &\longmapsto (A \mapsto \iota_A \gamma) \end{aligned}$$

Thus
$$\tilde{d}\alpha = \omega - J_A$$
where now I think of A as a variable point of $\mathfrak{g}$. Think of $S(\mathfrak{g}^*) \otimes \Omega(\$V)$ as forms on $\$V$ depending in a polynomial fashion on a variable point A of $\mathfrak{g}$. (forms with values in polynomial functions on $\mathfrak{g}$.)

Next the class $c_1(\mathcal{O}(1)) \in H^2_G(\mathbb{P}V)$ is represented by
$$\xi = -i\tilde{d}\alpha = -i\omega + iJ_A$$

and the classes $c_j(V) \in H^{2j}_G(pt)$ are represented by the polynomial functions
$$c_j(A) = tr(\Lambda^j A).$$
The basic relation in $H^*_G(\mathbb{P}V)$ is
$$\xi^n + c_1 \xi^{n-1} + \cdots + c_n = 0$$
and the LHS is represented by

$$(-i\omega + iJ_A)^n + (trA)(-i\omega + iJ_A)^{n-1} + \cdots + (\det A)$$
$$= \det\left((-i\omega + iJ_A) + A\right)$$

Take $n=1$, whence $\omega = 0$, $\quad iJ_A = -\langle z, Az\rangle$
$$= -A$$

and so one does get zero for $n=1$.

In general what we have is a differential form on PV depending in a polynomial fashion on the skew-symmetric matrix A. Suppose A diagonal with eigenvalues λ_j. Then

$$\iota J_A(z) = -\langle z, Az \rangle = -\lambda_j |z^j|^2$$

so we have

$$[redacted] \det\Big(\overbrace{[redacted]\, dz^\mu d\bar{z}^\mu}^{-i\omega} - \lambda_j |z^j|^2 + A \Big)$$

$$= \prod_{j=1}^{n} \Big(-i\omega - \sum_j \lambda_j |z^j|^2 + \lambda_j \Big).$$

This is obviously not identically zero. The easiest way to see this is to take the case where A is not definite, i.e. some $\frac{1}{i}\lambda_j$ are >0 and some <0. Then there is a z with $\langle z, Az\rangle = 0$, and we have

$$\prod_{j=1}^{n} (-i\omega + \lambda_j)$$

which will be $\neq 0$ as $1, \omega, \omega^2, \dots \omega^{n-1}$ in PV are independent and $\omega^n = 0$.

It is necessary to review the Gaussian [redacted] Thom form which comes out of the superconnection formalism. Now that we [redacted] are thinking in terms of equivariant forms, and these being forms on the manifold depending on an element of $\mathfrak{g}$, it should be easy to make the arguments rigorous, where we assumed formally that ω is invertible. In fact we now have [redacted] that the curvature is really a typical element of $\mathfrak{g}$.

So let's suppose E is a complex vector bundle over M equipped with metric and unitary connection D. The Thom class $\iota_! 1$ in $\tilde{K}(M^E)$ is represented by the super-bundle $\pi^*(\Lambda E^*)$ with odd degree endomorphism

$$L_v : i_v - e_{v^*} \qquad \text{at} \quad v \in E.$$

(This comes from the Koszul complex

$$S(E^*) \otimes \Lambda(E^*)$$

which resolves $\iota_* \mathcal{O}_M$ as an $\mathcal{O}_E$-module :

$$\to \mathcal{O}_E \otimes E^* \to \mathcal{O}_E \to \iota_* \mathcal{O}_M \to 0 .)$$

Now the connection D in E induces a connection in ΛE^* which pulls back to a connection in $\pi^*(\Lambda E^*)$. To compute suppose E trivialized whence

$$D = d + \theta$$

where $\theta_{\mu\nu}$ is a skew-hermitian matrix of 1-forms. The connection on ΛE^* will be $d + \rho(\theta)$ where

$$\rho : \operatorname{End} E \longrightarrow \operatorname{End}(\Lambda E^*)$$

is the exterior algebra of the contragredient representation. If $T : V \to V$, say v_j is a basis, v_j^* the dual basis, then

$$T = v_i (v_i^* T v_j) v_j^*$$

so the contragredient representation sends T to

$$-T^t = (v_j^*)^t (-v_i^* T v_j)(v_i)^t \qquad\qquad F \xrightarrow{v_i} V$$

$$= v_j^* (-v_i^* T v_j) v_i \qquad\qquad F \xleftarrow{v_i^t} V^*$$

and so $\boxed{\rho(T) = a_j^* (-T_{ij}) a_i}$

For T skew-hermitian, then

$$\rho(T) = -T_{ij}\, a_j^* a_i = \bar{T}_{ji}\, a_j^* a_i$$

This gets extended to endomorphism valued forms as

$$\rho(\theta) = \bar{\theta}_{ji}\, a_j^* a_i$$

So the connection on $\Lambda(E^*)$ is

$$D = d + \rho(\theta) = d + \bar{\theta}_{ji}\, a_j^* a_i$$

and its curvature is

$$D^2 = \bar{\omega}_{ji}\, a_j^* a_i = \rho(\omega)$$

where ω is the curvature of E.

Now

$$L = \iota_v - e_{v^*} = z^j a_j - \bar{z}^j a_j^*$$

$$L^2 = -|v|^2 = -|z^j|^2$$

and

$$\begin{aligned}
[D, L] &= [d + \bar{\theta}_{ji}\, a_j^* a_i,\ z^j a_j - \bar{z}^i a_i^*] \\
&= dz^j a_j - d\bar{z}^i a_i^* - \bar{\theta}_{ji} z^j a_i - \bar{\theta}_{ji} \bar{z}^i a_j^* \\
&= (dz^i - \bar{\theta}_{ji} z^j)\, a_i - (d\bar{z}^j + \bar{\theta}_{ji} \bar{z}^i)\, a_j^* \\
&= (dz^i + \theta_{ij} z^j)\, a_i - \overline{(dz^i + \theta_{ij} z^j)}\, a_i^*
\end{aligned}$$

This calculation should be obvious:

$$\begin{aligned}
[d + \rho(\theta),\ \overbrace{\iota_v - e_{v^*}}^{c(v)}] &= c(dv) + c(\theta v) \\
&= c(dv + \theta v)
\end{aligned}$$

So our character form is

★ $$\operatorname{tr}_s\left\{ e^{\rho(\omega) + c(dv+\theta v) - |v|^2} \right\}$$

This form comes from a similarly written equivariant form, where this time [redacted] we have to be careful [redacted] about the status of ω. It seems that the above is perfectly defined as an entire function of ω. So we have forms on $V = \mathbb{C}^n$ depending on $\omega \in \operatorname{Lie} U(n)$ which are entire functions of ω. θ becomes zero [redacted] upon passing to equiv. forms: $[S(\mathfrak{g}^*) \otimes \Omega(V)]^G$.

Recall

$$\operatorname{tr}_s\left(e^{\rho(\omega)}\right) = \det\left(1 - e^{\bar\omega}\right) = \det\left(1 - e^{-\omega}\right)$$

Set $\quad J_i = dz^i + \theta_{ij} z^j$, so that

$$c(dv+\theta v) = Ja - \bar J a^* = Ja + a^* \bar J$$

and
$$a^*\bar\omega a + Ja + a^*\bar J = \Big(a^* + \overset{J\bar\omega^{-1}}{\frac{J}{\bar\omega}}\Big)\bar\omega\Big(a + \overset{\bar\omega^{-1}\bar J}{\frac{\bar J}{\bar\omega}}\Big) - J\frac{1}{\bar\omega}\bar J$$

use $(\omega^t)^{-1} = (\omega^{-1})^t$

$$-J\frac{1}{\bar\omega}\bar J = -J_i\left(\frac{1}{\bar\omega}\right)_{ij}\bar J_j = J_i\left(\frac{1}{\omega}\right)_{ji}\bar J_j = -\bar J\frac{1}{\omega}J$$

So the Chern character form above should be (as equiv. form)

$$\det\left(1 - e^{-\omega}\right) e^{-d\bar z \frac{1}{\omega} dz} e^{-|z|^2}$$

$$= \det\left(\frac{1-e^{-\omega}}{\omega}\right) \underbrace{(\det\omega)\, e^{-d\bar z\frac{1}{\omega}dz}\, e^{-|z|^2}}_{\text{Thom form}}$$

October 21, 1984

Chern classes of representations: Let V be a representation of G, so that we have a homomorphism

$$\rho : G \longrightarrow \mathrm{Aut}(V) = \mathcal{U}$$

Consider V as an equivariant G-bundle over a point. Then $\mathcal{U}$ is the associated principal bundle with $\mathcal{U}$ acting by right multiplication and G-acting by left multiplication thru ρ. The MC form $u^{-1}du$ on $\mathcal{U}$ is then a G-invariant connection in this principal bundle, which we want to descend after modifying to kill the momentum.

Let $X \in \mathfrak{g}$. Then we have the operators $\mathcal{L}_X$, ι_X defined on $\Omega(\mathcal{U})$. To define these we have to convert the left action of G on $\mathcal{U}$ to a right action. Thus

$$\mathcal{L}_X \omega = \frac{d}{dt}\Big|_{t=0} \left(L_{e^{-t\rho(X)}}\right)^* \omega$$

so if I want $\iota_X(u^{-1}du)$, I change u to $(1-t\rho(X))u$:

$$\delta u = -\delta t\, \rho(X)\, u$$

and then the corresponding $u^{-1}du$ is

$$-u^{-1}\rho(X)\,u.$$

$$\therefore \quad \iota_X(u^{-1}du) = -u^{-1}\rho(X)\,u$$

So the momentum is

$$\varphi_X = -\iota_X(u^{-1}du) = u^{-1}\rho(X)\,u.$$

and the "equivariant" connection form on $\mathcal{U}$ is

$$u^{-1}du + \theta^a\varphi_a = u^{-1}du + u^{-1}\rho(\theta)\,u$$

$$= u^{-1}(d + \rho(\theta))u \qquad ?$$

We first have to get connections on a vector bundle and on the associated principal bundle straight. Let E be a vector bundle over M, and let P be the associated principal $U = U(n)$ - bundle. A point of P_m is an isomorphism $p : \mathbb{C}^n \xrightarrow{\sim} E_m$. If $\pi : P \to M$ is the projection map, then one has a canonical isom

$$\pi^*(E) = P \times_M E \xleftarrow{\sim} P \times \mathbb{C}^n$$
$$(p, p(v)) \longleftarrow (p, v)$$

U acts on the right of P and on the left on $\overset{V}{\overset{\shortparallel}{\mathbb{C}^n}}$; sections of E lift back to U-invariant sections of $P \times V$, i.e.

$$s(p) = (p, f(p))$$

where
$$s(pu) = s(p) \cdot u \qquad \text{i.e.}$$
$$f(pu) = u^{-1} f(p).$$

A connection D on E is an operator on $\Omega(M, E)$ of degree 1 which is a derivation wrt d. Connections pull-back, so it induces a connection on $\pi^*(E) = P \times V$ over P. The latter can be written as an op. on $\Omega^*(P) \otimes V$:

$$D = d + \theta \qquad \theta \in \Omega^1(P) \otimes \operatorname{End} V.$$

What properties does θ have, which would tell when $d + \theta$ on $\Omega(P) \otimes V$ comes from a connection on E?

One has

$$\Omega(M, E) \xrightarrow{\sim} \big(\Omega(P) \otimes V\big)_{basic}$$

It seems clear then that we want $d + \theta$ to be

[redacted]

invariant under the U-action, which means that $\theta \in \Omega^1(P)\otimes \mathrm{End}\,V$ is U-invariant. Also we need to be able to show $\iota_X(d+\theta)\alpha = 0$ if $\alpha \in [\Omega(P)\otimes V]_{basic}$ and it seems we want

$$[\iota_X, d+\theta] = \mathcal{L}_X .$$

In effect if $d+\theta$ came from a D, then both operators are derivations wrt d and agree on $\Omega^0(M,E)$, so are equal.

Here $\mathcal{L}_X$ on $\Omega(P)\otimes V$ denotes the full [redacted], or total, action by $\mathrm{Lie}(U)$. Since $[\iota_X, d]$ = partial $\mathcal{L}_X$ on $\Omega(P)$, it follows that we must have

$$\iota_X\theta = X \quad \text{in } \Omega^0(P)\otimes \mathrm{End}(V).$$

Here the X on the right is the image of X under the [redacted] representation map $\mathrm{Lie}(U) \longrightarrow \mathrm{End}\,V$. So far we have not assumed that D is unitary, and also [redacted] P/M, U could have been any principal bundle over which E is trivialized.

Now I want to bring in the group G acting on E/M, whence G acts on P preserving the canonical isomorphism $\pi^*E = P\times V$. [redacted] So, if D is G-invariant, it is clear that $\theta \in \Omega^1(P)\otimes\mathrm{End}\,V$ is G-invariant, i.e. each matrix component of θ [redacted] is a G-invariant 1-form on P.

Supposing G acts freely on M we can inquire whether D descends to give a connection on $E/\bar{M}$ where $\bar{M} = G\backslash M$, etc. This will happens iff θ is G-basic, i.e. [redacted] each matrix component of θ is G-invariant and killed by ι_X, $X \in \mathfrak{g}$.

If we only have an invariant connection, then we can attach to $A \in \mathfrak{g}$ a Higgs field

$$-\varphi_A = \iota_A \theta \in (\Omega^0(P) \otimes \mathrm{End}\, V)^{\mathfrak{u}} = \Omega^0(M, \mathrm{End}\, E)$$

and we have

$$[\iota_A, D] = [\iota_A, d+\theta] = \mathcal{L}_A - \varphi_A$$

or

$$\mathcal{L}_A = [\iota_A, D] + \varphi_A \qquad A \in \mathfrak{g}.$$

Now supposing G acts freely on M, let us pick a connection form ω for the principal G-bundle $M \longrightarrow \bar{M}$. Then $\omega = \omega^a A_a \in \Omega^1(M) \otimes \mathfrak{g}$ and we can modify the G-invariant connection D in order to kill the Higgs field:

$$\bar{D} = D + \omega^a \varphi_a = d + \theta + \omega^a \varphi_a$$

This is an operator on $\Omega(M, E) \hookrightarrow \Omega(P) \otimes V$

Here's why I got involved with this stuff. I believe in the formulas

$$[W(\mathfrak{g}) \otimes \Omega(M,E)]_{basic} \xrightarrow{\sim} [S(\mathfrak{g}^*) \otimes \Omega(M,E)]^G$$

$$\bar{D} = D + \omega^a \varphi_a \longleftrightarrow D - \Omega^a \iota_a$$

$$\bar{D}^2 \longleftrightarrow (D - \Omega^a \iota_a)^2 = D^2 + \Omega^a \varphi_a$$

Now take $M = pt$ and the D. and ι_a are zero, so we conclude that on $[S(\mathfrak{g}^*) \otimes V]^G$ for any representation V of $\mathfrak{g}$ we should have multiplication by $\Omega^a \rho_a$ is zero. In other words given $f: \mathfrak{g} \longrightarrow V$ invariant, then $\rho(A) f(A) = 0$ for all $A \in \mathfrak{g}$. I found this surprising, but here is a simple proof. Invariance implies

$$0 = \mathcal{L}_X f(A) = f'(A;[X,A]) + \rho(X) f(A)$$

so putting $X = A$ gives $\rho(A) f(A) = 0$.

Now let us return to the Gaussian Thom form

$$U = \det(\omega)\, e^{-d\bar z \frac{1}{\omega} dz}\, e^{-|z|^2}$$

It would be nice to relate this to the transgression of the Euler form $\det(\omega)$. The idea will be to use the rescaling $z \mapsto tz$, and the fact that the character $tr_s\, e^{(D+tL)^2}$ is independent of t up to exact forms

$$\partial_t\, tr_s\, e^{(D+tL)^2} = d\, tr_s\, e^{(D+tL)^2} L .$$

Somehow the classical limit of this identity should give

$$\partial_t U = \partial_t \left\{ \det(\omega)\, e^{-t^2 d\bar z \omega^{-1} dz}\, e^{-t^2|z|^2} \right\} = \tilde d \beta_t$$

where

$$\boxed{\beta_t = \int e^{\psi\omega\bar\psi - \psi t dz - t d\bar z \bar\psi - t^2|z|^2} (-\psi z - \bar z \bar\psi)}$$

Let's evaluate for $n = 1$.

$$\beta_t = \int (+\psi t dz + t d\bar z \bar\psi)(\psi z + \bar z \bar\psi)\, e^{-t^2|z|^2}$$

$$= e^{-t^2|z|^2}(-t\bar z dz + t z d\bar z)$$

I want to compute $\tilde d \beta$, where $\tilde d = d - \omega^a \iota_a$ is the equivariant form differential on $[S(\mathfrak{g}^*) \otimes \Omega(V)]^G$. Now $\omega^a \iota_a = \iota_\omega$ is the derivation of $\Omega(V)$ of degree -1 such that

$$\iota_\omega dz^\mu = X_\omega z^\mu = (-\omega^t)_{\nu\mu} z^\nu = -\omega_{\mu\nu} z^\nu$$

so that
$$\tilde{d}\,(dz^\mu) = d(dz^\mu) - (-\omega_{\mu\nu})z^\nu$$
$$= \omega_{\mu\nu} z^\nu$$

Thus
$$\tilde{d}\beta_t = e^{-t^2|z|^2}\Big(\overbrace{-t^2(z\,d\bar z + \bar z\,dz)(-t\bar z\,dz + tz\,d\bar z)}^{-t|z|^2 d\bar z dz + t|z|^2 dz d\bar z}$$
$$+ \underbrace{\tilde{d}(-t\bar z\,dz + tz\,d\bar z)}\quad\Big)$$

$$-t\,d\bar z\,dz + t\,dz\,d\bar z \quad \leftarrow -2t\,d\bar z\,dz$$
$$-t\bar z\,\omega z + tz\,\bar\omega\bar z \quad \leftarrow -2t\,\omega|z|^2$$

$$= e^{-t^2|z|^2}\left\{2t^3|z|^2 d\bar z\,dz - 2t\,\omega|z|^2 - 2t\,d\bar z\,dz\right\}$$

Next
$$\partial_t U_t = \partial_t\left\{e^{-t^2|z|^2}\left(\omega - t^2 d\bar z\,dz\right)\right\}$$
$$= e^{-t^2|z|^2}\left\{(-2t|z|^2)(\omega - t^2 d\bar z\,dz) - 2t\,d\bar z\,dz\right\}$$

so it works.

Now integrating one get
$$U_t - \omega = \tilde{d}\int_0^t dt\,\beta_t$$

which means means that upon letting $t \to \infty$ we get
$$-\omega = \tilde{d}\int_0^\infty dt\,\beta_t \qquad \text{off } 0 \text{ section.}$$

But
$$-\int_0^\infty dt\,\beta_t = \int_0^\infty dt\; e^{-t^2|z|^2}\,2t|z|^2\left(\frac{\bar z\,dz - z\,d\bar z}{2|z|^2}\right)$$
$$= \frac{\bar z\,dz - z\,d\bar z}{2|z|^2} = \frac{1}{2}\left(\frac{dz}{z} - \frac{d\bar z}{\bar z}\right)$$

and $\tilde{d}\,\frac{1}{2}\left(\frac{dz}{z} - \frac{d\bar z}{\bar z}\right) = \frac{1}{2}\left(\frac{\omega z}{z} - \frac{\bar\omega \bar z}{\bar z}\right) = \omega$, so it works

Now we evalute β_t in general but choosing our coordinates so that ω is diagonal.

October 22, 1984

Recall the formula of yesterday for the Thom form

$$U = (\det \omega)\, e^{-d\bar{z}\frac{1}{\omega}dz}\; e^{-|z|^2}$$

This is closed with respect to $d_\omega = d - \iota_\omega$. Now pull-back by the map $t, z \longrightarrow tz$ and we get

$$\tilde{U} = (\det\omega)\, e^{-(t\,d\bar{z} + \bar{z}\,dt)\frac{1}{\omega}(t\,dz + z\,dt)}\; e^{-t^2|z|^2}$$

which must be closed under $dt\,\partial_t + d_z - \iota_\omega$, so if we write $\tilde{U} = U_t + dt\,\beta_t$ we get the standard formula

$$\partial_t U_t = d_\omega \beta_t .$$

Expand the exponent

$$(t\,d\bar{z} + \bar{z}\,dt)\frac{1}{\omega}(t\,dz + z\,dt) = t^2\, d\bar{z}\frac{1}{\omega}dz + t\,d\bar{z}\frac{1}{\omega}z\,dt + \bar{z}\,dt\frac{1}{\omega}t\,dz$$

$$= t^2 d\bar{z}\frac{1}{\omega}dz + t\,dt\left(\bar{z}\frac{1}{\omega}dz - d\bar{z}\frac{1}{\omega}z\right)$$

Thus

$$\tilde{U} = (\det\omega)\, e^{-t^2\left(|z|^2 + d\bar{z}\frac{1}{\omega}dz\right)}\left(1 - t\,dt\left(\bar{z}\frac{1}{\omega}dz - d\bar{z}\frac{1}{\omega}z\right)\right)$$

So.

$$+\beta_t = \underbrace{(\det\omega)\, e^{-t^2\left(|z|^2 + d\bar{z}\frac{1}{\omega}dz\right)}}_{U_t}(-t)\left(\bar{z}\frac{1}{\omega}dz - d\bar{z}\frac{1}{\omega}z\right)$$

and

$$-\int_0^\infty dt\,\beta_t = \det\omega \left[\int_0^\infty 2t\,dt\; e^{-t^2(\qquad)}\right]\frac{1}{2}(\qquad)$$

$$\star \quad \boxed{-\int_0^\infty dt\,\beta_t = (\det\omega)\,\frac{1}{|z|^2 + d\bar{z}\frac{1}{\omega}dz}\;\frac{1}{2}\left(\bar{z}\frac{1}{\omega}dz - d\bar{z}\frac{1}{\omega}z\right)}$$

Check: $\quad n = 1. \qquad \frac{1}{|z|^2}\,\frac{1}{2}\left(\bar{z}\cdot dz - d\bar{z}\cdot z\right) = \frac{1}{2}\left(\frac{dz}{z} - \frac{d\bar{z}}{\bar{z}}\right)$

Check that the above form when d_ω is applied gives the Euler class $\det\omega$. But

$$d_\omega\,\frac{1}{2}\left(\bar{z}\frac{1}{\omega}dz - d\bar{z}\frac{1}{\omega}z\right) = \frac{1}{2}\Big(d\bar{z}\frac{1}{\omega}dz + \bar{z}\frac{1}{\omega}\omega z$$
$$+ d\bar{z}\frac{1}{\omega}dz - (\bar{\omega}\bar{z})^t\frac{1}{\omega}z\Big)$$

$$= |z|^2 + d\bar{z}\frac{1}{\omega}dz$$

Thus the form above is

$$(\det\omega)\,\frac{\gamma}{d_\omega\gamma}$$

and so

$$d_\omega\left\{(\det\omega)\frac{\gamma}{d_\omega\gamma}\right\} = \det\omega\cdot\frac{d_\omega\gamma}{d_\omega\gamma} = \det\omega$$

~~In fact it looks like we really want to look at~~

$$\cancel{\alpha = (\det\omega)\,\frac{1}{2}\left(\bar{z}\frac{1}{\omega}dz - d\bar{z}\frac{1}{\omega}z\right).} \quad ?$$

Question: Is the form above ($\star$) basic relative to the $\mathbb{R}_+$-action?

$$\iota_X dz = z \qquad\qquad \iota_X\left(\bar{z}\frac{1}{\omega}dz - d\bar{z}\frac{1}{\omega}z\right) = \bar{z}\frac{1}{\omega}z - \bar{z}\frac{1}{\omega}z = 0.$$

$$\left(X = z\frac{\partial}{\partial z} + \bar{z}\frac{\partial}{\partial\bar{z}}\right)$$

Also

$$\iota_X\,\frac{1}{|z|^2 + d\bar{z}\frac{1}{\omega}dz} = -\frac{1}{(\quad)}\underbrace{\left(\bar{z}\frac{1}{\omega}dz - d\bar{z}\frac{1}{\omega}z\right)}_{\alpha}\frac{1}{(\quad)}$$

so $\quad \iota_X(\star) = 0 \qquad$ as $\qquad \alpha^2 = 0$

October 23, 1984

Let $G = U(n)$ and $\mathfrak{g} = \mathrm{Lie}\, U(n)$ = skew-hermitian matrices act on $V = \mathbb{C}^n$ via the standard representation: $(gv)_i = g_{ij} v_j$. Then there is an induced action on the algebra $\Omega(V)$ of forms, which is determined by the fact that G acts on the dual V^* via the contragredient representation. Thus $\omega \in \mathfrak{g}$ acts on functions via the vector field

$$X_\omega = (-\omega^t)_{ij}\, z^i \frac{\partial}{\partial z^j} + \text{complex conjugate}$$

$$= \bar{\omega}_{ij}\, z^i \frac{\partial}{\partial z^j} + \omega_{ij}\, \bar{z}^i \frac{\partial}{\partial \bar{z}^j}$$

Next we consider the differential algebra of equivariant forms:

$$\left[S(\mathfrak{g}^*) \otimes \Omega(V) \right]^G, \quad \text{differential } d_\omega = d - \iota_\omega .$$

Here $S(\mathfrak{g}^*)$ is viewed as the algebra of polynomial functions $f(\omega)$, where ω is a variable element of $\mathfrak{g}$, and ~~contraction with X_ω~~ $S(\mathfrak{g}^*) \otimes \Omega(V)$ is the algebra of forms on V depending in a polynomial fashion on ω. On such forms we have the operator $d_\omega = d - \iota_\omega$, where ι_ω is contraction with X_ω:

$$d_\omega z^j = dz^j$$

$$d_\omega\, dz^j = -\iota_\omega\, dz^j = -(-\omega^t)_{ij} z^i = \omega_{ji} z^i$$

or simply $\quad d_\omega z = dz \quad , \; d_\omega(dz) = \omega z$

if we let z be the column vector of the functions z^i.

(Problem: Explain why $d_\omega^2 = 0$ on the G-invariant forms depending on ω.)

Let's now recall the formula for the Thom form which is obtained from using the superconnection formalism

$$\boxed{\mathcal{U} = (\det\omega)\quad e^{-(|z|^2 + d\bar{z}\frac{1}{\omega}dz)}}$$

Here $d\bar{z}\frac{1}{\omega}dz$ is $(d\bar{z})^t\frac{1}{\omega}dz = d\bar{z}^i\left(\frac{1}{\omega}\right)_{ij}dz^j$. We can see directly that $d_\omega\mathcal{U} = 0$:

$$d_\omega\left(|z|^2 + d\bar{z}^t\frac{1}{\omega}dz\right) = d\bar{z}^t z + \bar{z}^t dz + \overline{\omega z}^t\frac{1}{\omega}dz - d\bar{z}^t\frac{1}{\omega}\omega z = 0$$

(Better notation: $dz^*\frac{1}{\omega}dz$). Recall the above formula, which at first sight makes sense only for ω invertible, is a polynomial in ω.

Pulling back via $t, z \longmapsto tz$ we get

$$\tilde{\mathcal{U}} = (\det\omega)\ e^{-t^2|z|^2 - (tdz + zdt)^*\frac{1}{\omega}(tdz+zdt)}$$

in $[S(\mathfrak{g}^*)\otimes\Omega(\mathbb{R}\times V)]^G$ which is killed by $dt\,\partial_t + d_V - \iota_\omega = d_{\mathbb{R}\times V} - \iota_\omega$.

$$(tdz + zdt)^*\frac{1}{\omega}(tdz+zdt) = t^2 dz^*\frac{1}{\omega}dz + tdt\left(z^*\frac{1}{\omega}dz - dz^*\frac{1}{\omega}z\right)$$

$$\tilde{\mathcal{U}} = \underbrace{(\det\omega)\,e^{-t^2(|z|^2 + dz^*\frac{1}{\omega}dz)}}\left(1 - tdt\left(z^*\frac{1}{\omega}dz - dz^*\frac{1}{\omega}z\right)\right)$$

$$= \mathcal{U}_t + dt\,\beta_t \qquad , \qquad \beta_t = \mathcal{U}_t(-t)\left(z^*\frac{1}{\omega}dz - dz^*\frac{1}{\omega}z\right)$$

Then as $\mathcal{U}_t + dt\,\beta_t$ is closed under $dt\,\partial_t + d_V - \iota_\omega$ we get

$$\partial_t\mathcal{U}_t = d_\omega\beta_t$$

and so

$$\mathcal{U}_t - \underbrace{\mathcal{U}_0}_{\det\omega} = d_\omega\int_0^t dt\,\beta_t$$

Now let $t \to +\infty$, and then for $z \neq 0$ we have

$$\det(\omega) = \cancel{d_\omega \int_0^\infty dt(-\beta_t)} \quad d_\omega \gamma \quad , \text{ where}$$

$$\gamma = \int_0^\infty dt(-\beta_t) = (\det\omega)\int_0^\infty 2t\,dt \; e^{-t^2\left(|z|^2 + dz^* \frac{1}{\omega} dz\right)} \frac{1}{2}\left(z^* \frac{1}{\omega} dz - dz^* \frac{1}{\omega} z\right)$$

$$\boxed{\gamma = (\det\omega) \frac{1}{|z|^2 + dz^* \frac{1}{\omega} dz} \; \frac{1}{2}\left(z^* \frac{1}{\omega} dz - dz^* \frac{1}{\omega} z\right)}$$

Notice that γ is a closed equivariant form on $V - \{0\}$ which is $\mathbb{C}^\times$-invariant and basic for the action of $(\mathbb{R}_{>0})^\times$:

$$d_\omega \frac{1}{2}\left(z^* \frac{1}{\omega} dz - dz^* \frac{1}{\omega} z\right) = \frac{1}{2}\Big(dz^* \frac{1}{\omega} dz + z^* \frac{1}{\omega} \omega z - (\omega z)^* \frac{1}{\omega} z + dz^* \frac{1}{\omega} dz\Big)$$

$$= |z|^2 + dz^* \frac{1}{\omega} dz$$

Hence if we put $\alpha = \frac{1}{2}\left(z^* \frac{1}{\omega} dz - dz^* \frac{1}{\omega} z\right)$, then

$$\gamma = \det(\omega) \frac{1}{d_\omega \alpha} \alpha$$

is clearly killed by d_ω. Also if $X = z\frac{\partial}{\partial z} + \bar{z}\frac{\partial}{\partial \bar{z}}$ is the generator for the $(\mathbb{R}_{>0})^*$-actions we have

$$\iota_X \alpha = \frac{1}{2}\left(z^* \frac{1}{\omega} z - z^* \frac{1}{\omega} z\right) = 0$$

$$\iota_X \frac{1}{|z|^2 + dz^* \frac{1}{\omega} dz} = -\frac{1}{|z|^2 + dz^* \frac{1}{\omega} dz} \underbrace{\iota_X\left(|z|^2 + dz^* \frac{1}{\omega} dz\right)}_{z^* \frac{1}{\omega} dz - dz^* \frac{1}{\omega} z \;=\; 2\alpha} \frac{1}{|z|^2 + dz^* \frac{1}{\omega} dz}$$

so that indeed $\iota_X \gamma = 0$ as $\alpha^2 = 0$.

The fact that γ is basic for the $(\mathbb{R}_{>0})^\times$-action means that it comes from a form on the sphere SV. Thus

$$\det \omega = d_\omega \gamma$$

means that γ is the transgression form for the Euler class.

October 23, 1984:

Straighten out the Pfaffian and related formulas. If ω_{ij} is a skew-symmetric matrix, then

$$Pf(\omega)\,\psi^1 \cdots \psi^n = \left(e^{\frac{1}{2}\omega_{ij}\psi_i\psi_j}\right)^{[n]}.$$

Here n is even. Thus

$$Pf\begin{pmatrix} 0 & \omega_{12} \\ -\omega_{12} & 0 \end{pmatrix} = \int e^{\omega_{12}\psi_1\psi_2} = \omega_{12}$$

So one concludes that in general

$$\det(\omega) = Pf(\omega)^2$$

Now recall that the infinitesimal version of the spin representation is $\rho : \omega_{ij} \longmapsto \omega_{ij}\frac{1}{4}\gamma^i\gamma^j$, and that one is interested in the tr_s for the spin representation. For $n=2$.

$$tr_s\left(e^{\frac{1}{4}\omega_{ij}\gamma^i\gamma^j}\right) = tr_s\left(e^{\frac{1}{2}\omega_{12}\gamma^1\gamma^2}\right)$$

$$= tr_s\left(e^{\frac{1}{2}\omega_{12}\,i\varepsilon}\right) = e^{\frac{i}{2}\omega_{12}} - e^{-\frac{i}{2}\omega_{12}} \sim i\omega_{12}$$

as $\omega \longrightarrow 0$. Thus in general

$$tr_s\left(e^{\frac{1}{4}\gamma^t\omega\gamma}\right) \sim i^{n/2}\,Pf(\omega) \qquad \text{as } \omega \longrightarrow 0$$

As a check note that we expect

$$tr_s\left(e^{\frac{1}{4}\gamma^t\omega\gamma}\right) \sim (2i)^m \int e^{\frac{1}{4}\psi^t\omega\psi} = \frac{(2i)^m}{2^m}\int e^{\frac{1}{2}\psi^t\omega\psi}$$

$$= i^m\,Pf(\omega).$$

Now let us run through the ~~[illegible]~~ computation of $ch(L,1)$. Again we work with equivariant forms:

$$\left[S(\mathfrak{g}^*) \otimes \Omega(\mathbb{R}^n)\right]^G \qquad G = Spin(n).$$

This time the standard representation on $\mathbb{R}^4$ is isomorphic to the contragredient representation, so

$$X_\omega = \omega_{ij}\, x^i \partial_j$$

and
$$d_\omega(dx^j) = -X_\omega x^j = -\omega_{ij} x^i = \omega_{ji} x^i$$

or $\quad d_\omega(dx) = \omega x \quad$ as before. Then as before we take the connection $D = d_\omega$ on the equivariant bundle given by the spinor representation S. One has

$$D^2 = (d_\omega)^2 = -\mathcal{L}_\omega = \rho(\omega) \quad \text{on} \quad \left(S(\mathfrak{g}^*)\otimes S\right)^G$$

$$\rho(\omega) = \omega_{ij}\tfrac{1}{4}\gamma^i\gamma^j .$$

$$L = i x^j \gamma^j \qquad\qquad L^2 = -|x|^2$$

$$[D, L] = i\, dx^j \gamma^j$$

So the character is

$$\star \quad \operatorname{tr}_s\left(e^{\frac{1}{4}\gamma^t\omega\gamma + i\,dx^t\gamma - |x|^2}\right)$$

To evaluate we complete the square

$$\tfrac{1}{4}\left(\gamma - \tfrac{2i}{\omega}dx\right)^t \omega\left(\gamma - \tfrac{2i}{\omega}dx\right)$$

$$= \tfrac{1}{4}\left\{\gamma^t\omega\gamma - \gamma^t\omega\tfrac{2i}{\omega}dx - 2i\, dx^t\left(\tfrac{1}{\omega}\right)^t\omega\left(\gamma - \tfrac{2i}{\omega}dx\right)\right\}$$

$$= \tfrac{1}{4}\left\{\gamma^t\omega\gamma - 2i\gamma^t dx + 2i\, dx^t\left(\gamma - \tfrac{2i}{\omega}dx\right)\right\}$$

$$= \tfrac{1}{4}\left\{\gamma^t\omega\gamma - 2i\gamma^t dx + 2i\,dx^t\gamma + 4\,dx^t\tfrac{1}{\omega}dx\right\}$$

$$= \tfrac{1}{4}\gamma^t\omega\gamma + i\,dx^t\gamma + dx^t\tfrac{1}{\omega}dx .$$

Thus
$$\star = \operatorname{tr}_s\left(e^{\frac{1}{4}\left(\gamma - \frac{2i}{\omega}dx\right)^t\omega\left(\gamma - \frac{2i}{\omega}dx\right)}\right) e^{-\left(dx^t\frac{1}{\omega}dx + |x|^2\right)}$$

As usual, we expect that the first term will be the same as

$$tr_s\left(e^{\frac{1}{4}\gamma^t\omega\gamma}\right)$$

so we will get

$$ch(\iota, 1) = tr_s\left(e^{\frac{1}{4}\gamma^t\omega\gamma}\right) e^{-\left(|x|^2 + dx^t \frac{1}{\omega} dx\right)}$$

Question: Can the first term be written as a Pfaffian?

Claim: $\boxed{tr_s\left(e^{\frac{1}{4}\gamma^t\omega\gamma}\right) = (i)^{n/2}\det\left(\frac{\sinh(\omega/2)}{\omega/2}\right)^{1/2} Pf(\omega)}$

where $\det\left(\frac{\sin \omega/2}{\omega/2}\right)^{1/2}$ is the unique square root which is entire in ω and has the value 1 at $\omega = 0$. Enough to check for $n = 2$:

$$e^{\frac{i}{2}\omega_{12}} - e^{-\frac{i}{2}\omega_{12}} = tr_s\left(e^{\frac{1}{4}\gamma^t\omega\gamma}\right)$$

$$\omega_{12} = Pf(\omega).$$

Now

$$\det\left(\frac{\sinh(\omega/2)}{\omega/2}\right) = \left(\frac{\sinh(i\omega_{12}/2)}{i\omega_{12}/2}\right)^2$$

because the eigenvalues of $\begin{pmatrix} 0 & \omega_{12} \\ -\omega_{12} & 0 \end{pmatrix}$ are $\pm i\omega_{12}$. Also

$$\frac{\sinh(i\omega_{12}/2)}{i\omega_{12}/2} = \frac{e^{\frac{i\omega_{12}}{2}} - e^{-i\frac{\omega_{12}}{2}}}{\cancel{2} i\,\omega_{12}/\cancel{2}} \qquad \text{etc.}$$

October 26, 1984

I want to consider again the problem of constructing heat operators e^{tH} where H is a Laplacean type operator. The main point I am concerned with is to show how to obtain e^{tH} starting from an approximate kernel $L(t)$ with "tangent vector H" at $t=0$. The random walk analysis leads to

$$(1) \qquad e^{tH} = \lim_{N\to\infty} L\left(\tfrac{t}{N}\right)^N$$

It occurred to me that one should think of $L(t)$ as a rough parametrix for the operator $D = \partial_t - H$, in the same way that for an elliptic operator D one can find a rough parametrix P_0 by taking any ΨDO with symbol inverse to the symbol of D. Then one has

$$DP_0 = I - K$$

where K is of order -1, so

$$D\{P_0(I + K + \cdots + K^{n-1})\} = (I-K)(I + \cdots + K^{n-1}) = I - K^n$$

where K^n has order $-n$.

My hope is now to interpret $L(t-t', x, x')$ as a rough parametrix P_0 for $D = \partial_t - H$. If I can define a concept of order such that $K = I - DP_0$ is of order -1 and such that order $\ll m \Rightarrow$ the kernel is C^m, then because the kernels are supported in $t \geq t'$, the series

$$I + K + \cdots = (I + \cdots + K^{n-1})(I + K^n + K^{2n} + \cdots)$$

will converge, as K^n is a Volterra kernel.

There is a discrepancy between the process (1) and the process

$$(2) \qquad e^{tH} = L(t) + (L*K)(t) + (L*K*K)(t) + \cdots$$

which goes as follows. In the case where H is a finite dimensional operator, if $L(t) = 1 + L'(0)t + \cdots$, then

$$L\left(\tfrac{t}{N}\right)^N \longrightarrow e^{L'(0)t}$$

But [redacted] (2) will always work, e.g. if $L(t) = 1$, [redacted] then $P_0(t) = \theta(t)$, $(\partial_t - H)P_0 = 1 - H$, so $K \overset{(t)}{=} H$ and (2) is just the exponential series for e^{tH}.

Because of this discrepancy it is not clear that the [redacted] Neumann series uses all the "geometry" which goes into L.

Review of Michael's lecture on mixed volumes and isoperimetric inequalities. Let G_c be the complex 2-torus $(\mathbb{C}^\times)^2$, and identify its character group with the lattice points in the plane $\mathbb{Z}^2 \subset \mathbb{R}^2$. Let $S \subset \mathbb{Z}^2$, whence we get a representation of G_c on $V_S = \bigoplus_{x \in S} \mathbb{C}_x$ and on the associated projective space [redacted] P_S. Let X be the closure of a generic G_c-orbit in P_S. It is an algebraic surface which we suppose is nonsingular, just to simplify the sequel.

There is a canonical Kähler form ω on P_S invariant under the compact torus $G \subset G_c$, and which represents $c_1(\mathcal{O}(1))$. Strictly speaking $c_1(\mathcal{O}(1)) \in H^2(P_S, \mathbb{Z})$ goes into [redacted] the class represented by the form $\frac{\omega}{2\pi}$ in DR cohomology. Hence $\int_X \left(\frac{\omega}{2\pi}\right)^2$ is an integer which is in fact the degree of the surface X.

$$\deg(X) = \int_X \left(\frac{\omega}{2\pi}\right)^2$$

Now one can do this integral using the moment map for the G-action. In general this should be a map

$$X \xrightarrow{\mu} (\mathrm{Lie}\, G)^* \qquad \text{(real dual)}$$

which sends $x \in X$ to [redacted]$(v \longmapsto \varphi_v(x))$, φ_v denoting the Hamiltonian of the element $v \in \mathfrak{g}$. I believe we have the compatibility:

$$\begin{tikzcd}
X \arrow[r] \arrow[d, "\mu"] & \mathbb{P}_S \arrow[d, "\tilde{\mu}"] \\
(\mathrm{Lie}\, G)^* & (\mathrm{Lie}\ U(S))^* \arrow[l]
\end{tikzcd}$$

In any case we [redacted] should have that [redacted] over the open $G_{\mathbb{C}}$-orbit in X, μ is a quotient map for the G-action. The symplectic volume ω^2 should be compatible with the volume on G and the volume in $(\mathrm{Lie}\, G)^* = \mathbb{R}^2$. Thus

$$\int_X \frac{1}{2!}\left(\frac{\omega}{2\pi}\right)^2 = \text{area } \mu(X) \underset{\text{convexity of moment map thm.}}{=} \text{convex hull of } S$$

Let's check the constants as follows. Let $\mathbb{C}^{n-1}$ act[redacted] on $\mathbb{P}^{n-1} = \mathbb{P}(\mathbb{C}^n)$ by muliplying $v \in \mathbb{C}^n$ as follows $z = (z_1, \dots, z_{n-1}) \in \mathbb{C}^n$ acts on $v = (z_0, \dots, z_{n-1})$ as $(v_0, z_1 v_1, \dots, z_n v_n)$. Then the characters appearing are the points $0, e_1, \dots, e_{n-1} \in \mathbb{Z}^{n-1}$. This has volume $\frac{1}{(n[redacted]-1)!}$.

Alternative interpretation of the degree. Think of the torus G as consisting of $(z_1, z_2) \in (\mathbb{C}^\times)^2$ and as a lattice point as a character $(z_1, z_2) \longmapsto z_1^m z_2^n$. A point of V_S is then a sequence of numbers c_{mn} $(m,n) \in S$ and its transform under G consists of the numbers $\{ c_{mn} z_1^m z_2^n \mid (m,n) \in S \}$. A point of $\mathbb{P}_S$ means we worry about the $\{c_{mn}\}$ up to homothety, and a typical hyperplane section would be where a linear combination of these numbers is zero. From the viewpoint of the kind of variety in $\mathbb{P}_S$ obtained one might as well take the orbit of the point where all $c_{mn} = 1$. Then two generic hyperplane sections [redacted] intersect at those (z_1, z_2) satisfying

$$\sum_{(m,n)\in S} a_{mn} z_1^m z_2^n = 0, \quad \sum_{(m,n)\in S} b_{mn} z_1^m z_2^2 = 0$$

So the degree count the number of common zeros of two generic polynomials with support S.

Next go onto mixed volumes. This time one picks two sets $S, T \subset \mathbb{Z}^2$ and takes a generic orbit $X \subset \mathbb{P}_S \times \mathbb{P}_T$. One now [redacted] has the line bundles $pr_1^* \mathcal{O}(m) \otimes pr_2^* \mathcal{O}(n)$, which are positive for $m,n > 0$.

October 27, 1984

S is a finite set of characters of the complex torus G_c which we map into $\mathbb{P}_S = \mathbb{P}\{\bigoplus_{\chi\in S} \mathbb{C}_\chi\}$ by assigning to $u \in G_c$ the line $\mathbb{C}\cdot(\chi(u))_{\chi\in S}$. We now use that the moment map for $(S^1)^n$ acting on $\mathbb{P}^{n-1}$ can be identified with the map

$$\begin{aligned} \mathbb{P}^{n-1} &\longrightarrow \Delta(n-1) \\ \mathbb{C}(z^j) &\longmapsto \left(\frac{|z^j|^2}{\sum |z^j|^2}\right) \end{aligned}$$

So it seems that the moment map for G acting on G_c with the symplectic form induced from $\mathbb{P}_S$ is

$$u \longmapsto \sum_{\chi\in S} \frac{|\chi(u)|^2}{\left|\sum_{\chi\in S} \chi(u)\right|^2}\, \chi$$

Here is a direct way to see this. We have the map $G_c \longrightarrow V_S = \bigoplus_{\chi\in S} \mathbb{C}_\chi$, $u \longmapsto (\chi(u))_{\chi\in S}$, i.e. $z_\chi = \chi$. The symplectic form on $\mathbb{P}_S$ lifts to d of the form

$$\alpha = \frac{\sum \bar{z}_\chi\, dz_\chi}{\sum |z_\chi|^2}$$

So the momentum φ_ξ, $\xi \in Lie(G)$, will be $\varphi_\xi = -\iota_\xi \alpha$. Now we need $X_\xi \chi$ which should be

$$(X_\xi \chi)(u) = \frac{d}{dt}\Big|_{t=0} \chi(e^{-t\xi}u) = -\dot{\chi}(\xi)\,\chi(u)$$

so

$$\varphi_\xi(u) = -(\iota_{X_\xi}\alpha)(u) = \frac{\sum |\chi(u)|^2\, \dot{\chi}(\xi)}{\sum |\chi(u)|^2}$$

Let's see now what it means for the image of the moment map to be convex. Replace G_c by its universal covering, so now u will belong to [redacted] $\mathbb{C}^n$, $\chi(u) = e^{iku}$, $k \in \mathbb{R}^n$. So we have a finite subset $S \subset \mathbb{R}^n$ and the image of the moment map [redacted] consists of all linear combinations

$$\frac{\sum_{k\in S} e^{-kv} k}{\sum_{k \in S} e^{-kv}} \qquad v \in \mathbb{R}^n.$$

The result is that as v ranges over $\mathbb{R}^n$ we get all the points in the interior of the convex hull of the set S.

Let's set this up a bit cleaner. Let S be a finite subset of a real vector space V and then let's consider the function on V^* given by

$$Z(\lambda) = \sum_{v \in S} e^{\text{[redacted]}\lambda(v)}$$

Then
$$\frac{\delta Z(\lambda)}{Z(\lambda)} = \frac{\sum e^{\lambda(v)} \, \delta\lambda(v)}{\sum_{v\in S} e^{\lambda(v)}} = \left\langle \delta\lambda, \frac{\sum e^{\lambda(v)} v}{\sum e^{\lambda(v)}} \right\rangle$$

So we are looking at the map from V^* to V which sends λ to the derivative of $\log Z(\lambda)$ at λ which we identify with a linear fnl on V^*, i.e. an element of V. The result seems to be that this map $V^* \longrightarrow V$ has image the **interior** of the convex hull of S.

So we should replace S by a positive measure $d\mu$ with compact support. Then

$$Z(\lambda) = \int d\mu \, e^{\lambda(v)}$$

is essentially the Fourier transform of $d\mu$, [redacted] when it is viewed as an entire function of $\lambda \in V^* \otimes_{\mathbb{R}} \mathbb{C}$. Such an entire function satisfies a Paley-Wiener type growth condition, and the PW theorem says that an entire fn. satisfying this growth condition is the F.T. of a distribution supported in the associated convex subset of V.

What is the relation between the Fourier transform and the moment map considered before? First notice that our map $G_c \longrightarrow \mathbb{P}_S$ which gives the orbit of the line $\mathbb{C}(1, \cdots, 1)$ should be replaced by the orbit of a [redacted] line $\mathbb{C}(\rho_x)_{x \in S}$ with $\rho_x > 0$ for all x. Any line can be transformed into such a line under the action of the torus $(S^1)^S$ provided the line is not contained in [redacted] $\mathbb{P}_{S'}$ for $S' < S$. Also $(S^1)^S$ preserves the 2-form and commutes with G_c. So it is natural to put in such a $\rho = (\rho_x)$. Also this will correspond to a measure supported on S.

Now $\quad z_\chi = \rho_x \chi \quad , \qquad X_\xi z_\chi = -\rho_x \dot{\chi}(\xi) \chi(u)$

and

$$\varphi_\xi(u) = \frac{\sum \rho_x^2 |\chi(u)|^2 \dot{\chi}(\xi)}{\sum \rho_x^2 |\chi(u)|^2}$$

It might be better to write this with (ρ_x) replaced by a vector v, and the G-action denoted $\rho(u)$. Then

$$\varphi_\xi(u) = \frac{\langle \rho(u) v, \dot{\rho}(\xi) \rho(u) v \rangle}{|\rho(u) v|^2}$$

So when we pass to the universal covering of G_c and put $\chi(u) = e^{iku}$, then we get

$$\varphi(u) = \frac{\sum_{k\in S} |e^{iku}\rho_k|^2 k}{\sum_{k\in S} |e^{iku}\rho_k|^2}$$

$$= \frac{\sum_{k\in S} e^{-2k\,\mathrm{Im}\,u} |\rho_k|^2 k}{\sum_{k\in S} e^{-2k\,\mathrm{Im}\,u} |\rho_k|^2}$$

It appears that because of the 2 factor, [redacted] and the way $\sum e^{-2k\,\mathrm{Im}\,u} |\rho_k|^2$ was obtained, one should not think of this sum as part of the F.T.

$$\sum_{k\in S} e^{iku} |\rho_k|^2$$

Next lets go on to discuss the case of mixed volumes. In this case one starts with two subsets S, T of $\hat{G}$ and considers a G_c orbit in $\mathbb{P}_S \times \mathbb{P}_T$.

$$\begin{tikzcd} G_c \arrow[r] \arrow[d] & \mathbb{P}_S \times \mathbb{P}_T \arrow[d] \\ \mathbb{R} \otimes \hat{G} & \Delta_S * \Delta_T \arrow[l] \end{tikzcd}$$

The first thing to find is the moment map for $(S^1)^S \times (S^1)^T$ acting on $\mathbb{P}_S \times \mathbb{P}_T$ with the bundle $\mathcal{O}(m) \boxtimes \mathcal{O}(n)$. Segre embedding

$$\mathbb{P}_S \times \mathbb{P}_T \hookrightarrow \mathbb{P}_{S\times T}$$

would work for $m = n = 1$.

Clearly the moment maps is linear in the 2-form ω, so we must take the appropriate linear combination so the new moment map is

$$m \frac{\sum_{k\in S} e^{-kv}|\rho_k|^2 k}{\sum_{k\in S} e^{-kv}|\rho_k|^2} + n \frac{\sum_{T} e^{-kv}|\rho_k|^2 k}{\sum_{k\in T} e^{-kv}|\rho_k|^2}$$

which is minus the derivative of $m\log\left(\sum_{k\in S} e^{-kv}|\rho_k|^2\right) + n\log\left(\sum_{k\in T} e^{-kv}|\rho_k|^2\right)$

~~Let's start again with the map $G_c \to V$ $u \mapsto \rho(u)v$ in the above~~

Let's start again. We have a representation ρ of G_c on the complex vector space V and take a G_c-orbit in $\mathbb{P}V$, specifically the one generated by the line $\mathbb{C}\cdot v$ where $v \neq 0$. Then we pull-back the Kahler form ω on $\mathbb{P}V$ to the closure X of the G_c-orbit, and integrate $\frac{\omega^n}{(2\pi)^n}$ over X. Now we have been assuming more or less that $G_c \simeq$ an open dense orbit in X. This is the case if the set of characters S occurring in V are such that the differences $\{\chi - \chi' \mid \chi, \chi' \in S\}$ span the character group of G_c. So it would seem that the integral we are after can be done directly on G_c, rather than on X.

So let's consider more generally the map $G_c \to V$, $u \mapsto \rho(u)v$ where v is a vector $\neq 0$ in V. The form ω is up to an i factor the differential of the connection form $\theta = d'\log|z|^2 = \frac{\sum \bar{z}^\mu dz^\mu}{\sum |z^\mu|^2}$

And when I pull this back to G_c I get

$$z^\mu = \chi^\mu(u)\,\sigma^\mu$$

$$dz^\mu = \chi^\mu(u)\,\sigma^\mu \; \dot{\chi}^\mu(du)$$

[redacted]

$$\Theta = \frac{\sum |\chi^\mu(u)\,\sigma^\mu|^2\, \dot{\chi}^\mu(du)}{\sum |\chi^\mu(u)\,\sigma^\mu|^2} .$$

But what really is happening is that I have a holomorphic section $s(u) = \rho(u)\sigma$ of the [redacted] pull-back of $\mathcal{O}(-1)$ over $\mathbb{P}V$ and I am taking

$$\Theta = d' \log |s|^2$$

We have therefore got some sort of symplectic structure on G_c [redacted] and we want to integrate $\frac{\omega^n}{(2\pi)^n}$. Because $G_c \simeq G \times \mathrm{Lie}\, G$ and things are invariant in the G direction, this integral becomes an integral over $\mathrm{Lie}\, G$, which for some reason I am trying to find, is the volume of the convex hull of the support of ρ in $(\mathrm{Lie}\, G)^*$. This is somehow the mystery. Clearly it is a Jacobian type argument.

What is probably involved is the moment map

$$\begin{array}{ccccc} G_c & \xrightarrow{\pi} & G_c/G & \xrightarrow{j} & \mathrm{Lie}(G)^* \\ & & \shortparallel & & \\ & & \mathrm{Lie}(G) & & \end{array}$$

and calculating [redacted] that $\pi_*\left(\frac{\omega}{2\pi}\right)^n = j^*(\text{volume on } \mathrm{Lie}(G)^*)$. This calculation should be straightforward. The [redacted] crux of the argument has to be [redacted] to show that j is a diffeomorphism of [redacted] $\mathfrak{g}$ with the interior of the convex hull of the support of ρ in $\mathfrak{g}^*$.

October 28, 1984

I've been going over the relations between convex bodies, toral varieties, and the moment map explained by Atiyah. A concrete question is what is behind the proof by this theory of the fact to $\mathrm{vol}(\lambda A + \mu B)$ is a polynomial in λ, μ. My feeling is that this should be elementary, especially since Atiyah's integrals over a toral variety can be done as an integral over the Lie algebra of the torus.

Yesterday I isolated the key point, which is where the convexity of the moment map is used. Let V be a real vector space (it is the part of the complex torus orthogonal to the compact torus), and let $d\rho$ be a positive measure with compact support in V^*. I assume that the support of ρ is not contained in an affine hyperplane, or equivalently that the convex hull of the support of ρ has non-empty interior.

We now consider the function on V given by

$$Z(v) = \int_{V^*} e^{-kv} d\rho(k)$$

and look at the Legendre transform process defined by the function $\log Z$. This means that at each point $v \in V$, the differential $d \log Z|_v$ is identified with a linear functional on V^*:

$$\delta \log Z(v) = \langle \delta v, \partial_v \log Z(v) \rangle$$

i.e. we get a map

$$\Phi : V \longrightarrow V^*$$
$$v \longmapsto \partial_v \log Z(v)$$

or

$$k_\mu = \frac{\partial}{\partial v^\mu} \log Z$$

What seems to be true is that Φ is a diffeomorphism with image the interior of the convex hull of Supp(ρ). The integral of $\left(\frac{\omega}{2\pi}\right)^n$ over the toral variety turns out to be

$$\int_V d^n v \det\left(\partial^2_{\mu\nu} \log Z\right) = \int_{\Phi(V)} d^n k = \operatorname{vol}(\Phi(V))$$

So the key point is to see Φ is a diffeomorphism with the correct image.

The first thing to verify is that the Jacobian of Φ

$$\partial_\mu k_\nu = \partial^2_{\mu\nu} \log Z = \frac{\partial_{\mu\nu} Z}{Z} - \frac{\partial_\mu Z}{Z}\frac{\partial_\nu Z}{Z}$$

is non-singular. But

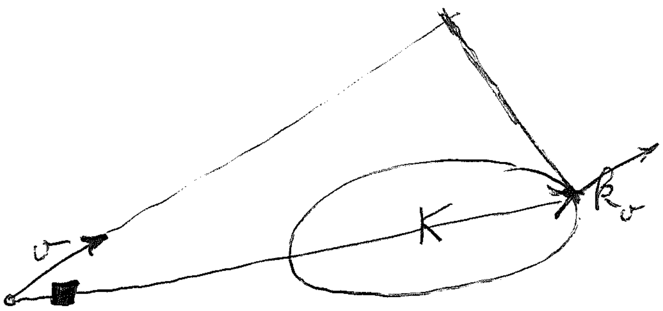
$$Z = \int d\rho(k)\, e^{-kv} \qquad -\frac{\partial_\mu Z}{Z} = \frac{1}{Z}\int d\rho(k) e^{-kv} k_\mu = \langle k_\mu \rangle$$

$$\frac{\partial^2_{\mu\nu} Z}{Z} = \frac{1}{Z}\int d\rho(k)\, e^{-kv}\, k_\mu k_\nu = \langle k_\mu k_\nu \rangle$$

and by standard stuff $\partial^2_{\mu\nu} \log Z = \langle k_\mu k_\nu \rangle - \langle k_\mu \rangle \langle k_\nu \rangle$ is at least positive semi-definite: If $k_x = x^\mu k_\mu$, then $\langle k_x^2 \rangle - \langle k_x \rangle^2 = \langle (k_x - \langle k_x \rangle)^2 \rangle \geq 0$. In fact it will be a strictly positive definite matrix since along any (affine) line $\mathbb{R} \subset V$, the restriction of Z to this line will be the Laplace transform of the measure on $\mathbb{R}^*$ obtained by pushing $d\rho$ forward under the map $V^* \to \mathbb{R}^*$ and this is not concentrated at a single point by our assumption. (Better: k_x is not constant on Supp ρ for $x \neq 0$, hence $\langle (k_x - \langle k_x \rangle)^2 \rangle > 0$.)

Next let's consider a [redacted] ray tv and evaluate $\log Z$ on this ray as $t \to +\infty$. Change the sign in the definition of Z. I'm also going to suppose that ρ is a smooth measure on a convex body K whose boundary ∂K is C^∞ and strictly convex. Also I'll use an inner product in V in order to identify V and V^*.

Now
$$Z(tv) = \int d\rho(k)\, e^{tv\cdot k}$$



As $t \to \infty$ the probability measure
$$\frac{d\rho(k)e^{tv\cdot k}}{\int d\rho(k)\, e^{tv\cdot k}} \quad \text{on } K$$
gets concentrated near the point k_v on ∂K where the outward normal is in the same direction as v. Thus as $t \to \infty$

$$\Phi(tv) \,[redacted]\, = \,[redacted]\, \frac{\int d\rho(k)\, e^{tv\cdot k}\, k}{\int d\rho(k) e^{tv\cdot k}} \longrightarrow k_v$$

It is more or less clear now that [redacted] in this smooth case we can extend Φ from V to $V \cup$ sphere at infinity and we get a smooth map $V \cup SV \longrightarrow K$ which maps SV diffeomorphically onto ∂K. Then we would have a 1-1 etale map which would be [redacted] a diffeomorphism by 1-connectedness.

I think I now see how to handle the case of finite support:

It clear that given $\varepsilon > 0$ we should be able to choose an R such that for any v with $|v| > R$ the measure $\dfrac{d\rho(k)\, e^{kv}}{\int d\rho(k)\, e^{kv}}$ will be so concentrated in the v-direction that when k is averaged with respect to this measure we end up within ε of the boundary of K. This said it follows that Φ carries $\{v, |v| > R\}$ into the ε nbd of ∂K. Thus as $R \to \infty$, $B_R = \{v, |v| \leq R\}$ exhausts V, and $\Phi(B_R)$ exhausts interior of K, and since $\Phi : B_R \simeq \Phi(B_R)$ we win.

Now that we understand the case of the volume we should be able to handle the mixed volumes.

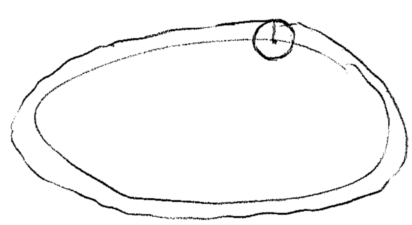
First do the isoperimetric folklore: Let K be a convex body in the plane with area A and perimeter S. The isometric inequality says

$$S^2 \geq 4\pi A \qquad \text{equality} \iff K = \text{a disk}.$$

Suppose to simplify that K has smooth boundary. Let D be the unit disk. Then

$$A(t) = \text{area of } K + tD$$

is quadratic in t, hence we can work out its coefficients using that



$$\frac{d}{dt} A(t) = S(t).$$

$$\text{and} \quad A(t) \sim \pi t^2$$

We find then that

$$A(t) = \underbrace{A_0}_{\text{area } K} + t \underbrace{S_0}_{\text{circum } K.} + \pi t^2$$

So in particular $S(t) = S_0 + 2\pi t$.

Then

$$\frac{d}{dt}\log\left(\frac{\sqrt{A}}{S}\right) = \frac{1}{2}\,\frac{1}{A}\,\overbrace{\frac{dA}{dt}}^{S} - \frac{1}{S}\,\overbrace{\frac{dS}{dt}}^{2\pi}$$

$$= \frac{1}{2AS}\left(S^2 - 4\pi A\right)$$

which is ≥ 0 by the isoperimetric inequality. Thus by increasing t we can make $\frac{\sqrt{A}}{S}$ increase or better $\frac{S^2}{A}$ decrease and get as close to the limiting value 4π as we want.

The general inequality here is that the quadratic form

$$\mathrm{vol}(\lambda K + \mu L) = \lambda^2 \mathrm{vol}(K) + \lambda\mu\, \mathrm{vol}(K, L) + \mu^2 \mathrm{vol}(L)$$

has negative discriminant.

Mixed volumes. This time suppose we have two measures $d\rho_i(k)$ with corresponding transforms $Z_i(\sigma)$. Then given integers $m, n > 0$ we can form $Z_1^m Z_2^n$. Now I know that product of L.T.'s corresponds to convolution and it seems clear that convex hull is compatible with sum:

$$CH(A+B) = CH(A) + CH(B)$$

Check this: $\subset$ clear. Given $\sum t_i a_i \in CH(A)$, $\sum t_j' b_j \in CH(B)$, then

$$\sum t_i a_i = \sum t_i t_j' a_i$$
$$\sum t_j' b_j = \sum t_i t_j' b_j$$

$$\sum t_i a_i + \sum t_j' b_j = \sum t_i t_j' (a_i + b_j)$$, so it is clear that $\supset$ also holds.

Thus $Z_1^m Z_2^n$ belongs to a measure whose convex hull of its support is $mK_1 + nK_2$, where K_j = convex hull of Supp $d\rho_i$. Then we can calculate the volume of $mK_1 + nK_2$ by integrating $\det\{\partial^2_{\mu\nu} \log (Z_1^m Z_2^n)\}$ over V. Finally $\det(mT_1 + nT_2)$ is a polynomial of degree r in m, n where $r = \dim V$.

October 29, 1984

Stroock tells me some things about Stratonovich versus Itô DE's. First one looks at the stochastic integral

$$\int_0^1 \theta(t)\, d\beta(t) \qquad \beta(t) = \text{Brownian motion process.}$$

where $\theta(t)$ is "adapted" in the sense that it is a random variable depending only on $\beta(t')$ for $t' \le t$, hence independent from $\beta(t')-\beta(t'')$ for $t' > t'' > t$. Then one can define the integral as a limit of Riemann sums

$$\sum_k \theta\left(\tfrac{k}{n}\right) \beta\left(\tfrac{k+1}{n} - \tfrac{k}{n}\right)$$

Notice these Riemann sums are special in the sense that θ is evaluated at the left point of the interval. The above defn. is due to Itô and is modeled on [redacted] earlier work of Paley & Wiener.

Stratonovich idea is to mollify $\beta(t)$ to a smooth path process $\beta^\varepsilon(t)$, do the integral and [redacted] let $\varepsilon \to 0$. The two approaches are different as we shall see. Itô's method is not coordinate invariant, but is suited for calculation and estimates. The problem with Strat. integral is that depends on the $\frac{1}{2}$-norm of θ in some sense, +as not of zeroth order. (In spirit: one converts a DE to an integral equation so as to render first order diffn. a zeroth order process, but it's no good if the integral is $\frac{1}{2}$ order.)

Example:

$$dx_t = \lambda x_t \, d\beta_t$$

Considered in the sense of Straton. the solution is clearly

$$x_t = e^{\lambda \beta_t}$$

However as an Itô equation the solution is

$$x_t = e^{\lambda \beta_t - \lambda^2 t/2}$$

To see this use general Itô calculus

$$d f(x_t) = f'(x_t)\, dx_t + \tfrac{1}{2} f''(x_t)(dx_t)^2$$

$$= f'(x_t)(\lambda x_t)\, d\beta_t + \tfrac{1}{2} f''(x_t)(\lambda x_t)^2 \underbrace{(d\beta_t)^2}_{=dt}$$

with $f(x) = \log(x)$ Then

$$d\log(x_t) = \lambda\, d\beta_t + \tfrac{1}{2}\left(-\tfrac{1}{x_t^2}\right)(\lambda x_t)^2\, dt$$

$$= \lambda\, d\beta_t - \tfrac{1}{2}\lambda^2 dt$$

which can be directly integrated.

Another thing Stroock said is that the Itô procedure involves a normal ordering. In the above example we get instead of $\frac{\beta_t^n}{n!}$ for the coeff of λ^n, the Hermite poly $\frac{1}{n!} H_n(\beta_t)$.

October 31, 1984

Let's go back to the problem of constructing $e^{t\Delta}$, where $\Delta = g^{ij}\partial_i\partial_j$ + lower terms. I decided various things about this problem which I now review.

There are really two different methods, which I shall call the path integral and parametrix methods. In both cases one starts with an approximation $L(t)$ which one refines to the operator $e^{t\Delta}$ by a process of successive modifications and passing to the limit. In the path integral approach one uses

$$e^{t\Delta} = \lim_{N\to\infty} L(t/N)^N$$

and then it is necessary to require that $L(t) = I + t\Delta + \cdots$ in some sense. The tangent vector to L at the identity must be Δ, and this means the moments of $L(t)$ have to have the correct first order dependence on t so as to give all the coefficients of Δ.

In the parametrix method one requires only that

$$(\partial_t - \Delta)L = 1 - K,$$

where K is of order -1 in some sense. This is a condition that L has to be correct only ~~up to the top~~ on the symbol level. In fact Seeley uses

$$L(t,x,x') = \int \frac{d^n\xi}{(2\pi)^n}\, e^{i\xi(x-x') - t g^{ij}(x')\xi_i\xi_j}$$

$$= \frac{1}{(4\pi t)^{n/2}}\left(\det g_{ij}(x')\right)^{1/2} e^{-\frac{1}{4t} g_{ij}(x')(x-x')^i(x-x')^j}$$

Once one has chosen such an L, the operator $e^{t\Delta}$ is found by the Neumann series

$$e^{t\Delta} = L(I + K + K^2 + \cdots)$$

Now my goal will be to carry out this process and see that it works. The foremost problem is to specify the class of kernels $K(t,x,x')$ where one is to work, to define order so that order of a composition adds, to see that order $-\infty$ is the same as smooth, etc. Set $x'=0$, and let $K(t,x) = K(t,x,0)$

Properties: $K(t,x)$ is smooth outside $(0,0)$ and is identically zero for $t<0$. In a sense I have to make precise I want the singularity at $(0,0)$ to be described by an asymptotic expansion of terms

$$e^{\frac{-S(x)}{t}}\, t^k a_k(x)$$

where $S(x) = \frac{1}{4} g_{ij}(0)\, x^i x^j + O(x^3)$.

Let's see if I can now pin down the singularity. I choose a coord system so that $S(x) = x^2$. I have to understand the meaning of an expansion

$$K(t,x) \sim e^{-\frac{x^2}{t}} \sum_k t^{k} a_k(x)$$

or better

$$K(t,x) \sim e^{-\frac{x^2}{t}} \sum_{k,\alpha} t^{k} x^\alpha \cdot a_{k,\alpha} .$$

The key idea is what Melrose told me, namely, that

$$\star \qquad e^{-\frac{x^2}{t}}\, t^k x^\alpha = e^{-\left(\frac{x}{\sqrt{t}}\right)^2} \left(\frac{x}{\sqrt{t}}\right)^\alpha t^{k+\frac{|\alpha|}{2}}$$

is of order $k + \frac{|\alpha|}{2}$ in some sense.

Let's concentrate then on understanding the singularity of the function $\star$. There is an obvious way [redacted] to resolve the singularity, namely we blowup $\mathbb{R} \times \mathbb{R}^n = \{(h, x)\}$ at $h = 0$, $x = 0$, and map the blowup to (t, x)-space by setting

$$t = h^2$$
$$x = x.$$

On the blowup there are smooth functions y^μ extending $\frac{x^\mu}{\sqrt{t}}$; we [redacted] write

$$y^\mu = \frac{x^\mu}{\sqrt{t}}.$$

[redacted] Then our function $\star$ when pulled back to the blowup extends to a smooth function of h, y:

$$e^{-\frac{x^2}{t}}\, t^{k+|\alpha|/2} \left(\frac{x}{\sqrt{t}}\right)^\alpha = e^{-y^2} y^\alpha h^{2k+|\alpha|}$$

What is the blowup? It consists of all pairs (l, v) where l is a line in $\mathbb{R} \times \mathbb{R}^n$ thru $(0,0)$ and where v is a point on this line. For the moment we leave out the lines lying in the fibre $h = 0$. Then l has a unique generator $(1, y^\mu)$, and $v = (h, x^\mu)$ where, as $v \in l$ we have

$$x^\mu = h y^\mu \qquad \text{or} \qquad y^\mu = \frac{x^\mu}{h}$$

So strictly speaking the y^μ are not defined on the closed set $Z = \{(\ell, v) \mid \ell \subset 0 \times \mathbb{R}^n\}$. The [redacted] vector [redacted] y^μ becomes infinite as one approaches a point of Z. Hence $e^{-y^2} y^\alpha h^b$ does extend smoothly by zero over Z.

So far we have constructed a resolution:

$$(h, y)\text{-space} \cup Z \longrightarrow (t, x)\text{-space}$$

and the function $e^{-x^2/t} t^k x^\alpha$ for $2k + |\alpha| > 0$ when lifted back extends to a smooth function $e^{-y^2} y^\alpha h^{2k+|\alpha|}$ <u>vanishing on Z.</u>

Change of variables: $\quad x = [redacted]\, hy \qquad y = \frac{x}{\sqrt{t}}$
$\qquad\qquad\qquad\qquad\quad t = h^2 \qquad\quad h = \sqrt{t}$

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial y} \underbrace{\frac{\partial y}{\partial x}}_{=\frac{1}{\sqrt{t}}} + \frac{\partial u}{\partial h} \underbrace{\frac{\partial h}{\partial x}}_{=0} = \frac{1}{h} \frac{\partial u}{\partial y}$$

$$\frac{\partial u}{\partial t} = \frac{\partial u}{\partial y} \frac{\partial y}{\partial t} + \frac{\partial u}{\partial h} \frac{\partial h}{\partial t} = -\frac{1}{2} \frac{y}{h^2} \frac{\partial u}{\partial y} + \frac{1}{2h} \frac{\partial u}{\partial h}$$

$$\frac{\partial y}{\partial t} = -\frac{1}{2} \frac{x}{t^{3/2}} = -\frac{1}{2} \frac{y}{h^2} \qquad \frac{\partial h}{\partial t} = \frac{1}{2h}$$

So the heat equation $\left[\partial_t - a(x) \partial_x^2 - b(x) \partial_x - c(x)\right] u = 0$ becomes

$$\left[-\frac{1}{2} \frac{y}{h^2} \frac{\partial u}{\partial y} + \frac{1}{2h} \frac{\partial u}{\partial h} - a(hy) \frac{1}{h^2} \partial_y^2 - b(hy) \frac{1}{h} \partial_y - c(hy)\right] u = 0$$

Take $a = 1$, $b = c = 0$ and we get

$$\left[\frac{1}{2h} \frac{\partial}{\partial h} - \frac{1}{h^2} \partial_y^2 - \frac{1}{2h^2} y\, \partial_y\right] u = 0$$

not regular in <u>h, so it's not parabolic</u> in standard sense.

Let's review the situation. I am trying to decide whether the fundamental solution for $\partial_t - \Delta$ with pole at $(0,0)$, when lifted to the blowup, extends smoothly over the exceptional divisor. If this is true then along the exceptional divisor we can expand in powers of t (or h). This expansion does depend on an identification of [redacted] a nbd of [redacted] $0 \in M$ with the tangent space at this point, but one might hope that the coefficients of the powers of h, which are functions on the tangent space are of the form

$$e^{-y^2} \cdot \text{polynomial in } y.$$

We ought to be able to make this work formally. So let's return to what we did in May, where we were able to proceed with $e^{-tp^2}\cdots$, but got stuck when it came to the x-picture. This time around I want to lift to the blowup. I won't use h, but rather work with $\sqrt{t}$. Thus we want the change of variable
$$\begin{cases} x = \sqrt{\tau}\, y \\ t = \tau \end{cases} \quad \text{or} \quad \begin{matrix} y = \dfrac{x}{\sqrt{t}} \\ \tau = t \end{matrix}$$

Then
$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial y}\frac{\partial y}{\partial x} + \frac{\partial u}{\partial \tau}\frac{\partial \tau}{\partial x} = \frac{1}{\sqrt{\tau}}\frac{\partial u}{\partial y} \qquad \frac{\partial y}{\partial t} = -\frac{1}{2}\frac{x}{t^{3/2}}$$
$$\frac{\partial u}{\partial t} = \frac{\partial u}{\partial y}\frac{\partial y}{\partial t} + \frac{\partial u}{\partial \tau}\frac{\partial \tau}{\partial t} = \frac{\partial u}{\partial \tau} + \frac{\partial u}{\partial y}\left(-\frac{1}{2}\frac{1}{\tau}y\right)$$

so our operator becomes
$$\frac{\partial u}{\partial \tau} - \frac{1}{2}\frac{y}{\tau}\frac{\partial u}{\partial y} - a(\sqrt{\tau}y)\frac{1}{\tau}\frac{\partial^2 u}{\partial y^2} - b(\sqrt{\tau}y)\frac{1}{\sqrt{\tau}}\frac{\partial u}{\partial y} - c(\sqrt{\tau}y)u$$

Since we are expanding in powers of τ probably we should multiply by τ in order to get

$$\tau \frac{\partial}{\partial \tau} - a(\sqrt{\tau} y)\, \partial_y^2 - \tfrac{1}{2} y \partial_y - b(\sqrt{\tau} y) \sqrt{\tau}\, \partial_y - c(\sqrt{\tau} y)\tau$$

Now the leading term in τ is clearly

$$\tau \partial_\tau - a(0)\, \partial_y^2 - \tfrac{1}{2} y \partial_y$$

Suppose $a(0) = 1$, and let's check that $\dfrac{e^{-y^2/4}}{t^{n/2}}$ is a solution:

$$\left(\tau \partial_\tau - \tfrac{n}{2}\right) - \underbrace{\left(\partial_{y^r} - \tfrac{y^r}{2}\right)^2}_{\partial_y^2 - y\partial_y + \frac{y^2}{4} - \frac{n}{2}} - \tfrac{1}{2} y\left(\partial_y - \tfrac{y}{2}\right)$$

$$= \tau \partial_\tau - \partial_y^2 + \tfrac{1}{2} y \partial_y$$

which is roughly what was encountered before. (see p. 83 $\star$ and p. 85). But now I know more, namely that it is completely OK to have polynomials in the y's occurring. They are intrinsic because of the linear structure on the exceptional fibre.

But the operator to be inverted is going to be

$$r - \partial_y^2 + \tfrac{1}{2} y \partial_y$$

where r is an integer or half-integer > 0. Maybe this will be invertible on the Schwartz space; then I can avoid having to mention polynomials and how the degrees grow.

But

$$e^{\frac{y^2}{8}}\left(+\partial_y^2+\frac{1}{2}y\partial_y\right)e^{-\frac{y^2}{8}} = \left(\partial_y-\frac{y}{4}\right)^2+\frac{1}{2}y\left(\partial_y-\frac{y}{4}\right)$$

$$= \partial_y^2 - \cancel{\frac{1}{2}y\partial_y} + \frac{y^2}{16} - \frac{n}{4} + \cancel{\frac{1}{2}y\partial_y} - \frac{y^2}{8}$$

$$= \partial_y^2 - \frac{y^2}{16} - \frac{n}{4} = -\Big(\underbrace{-\partial_y^2 + \frac{y^2}{16}}_{\text{harmonic oscillator } \geq \frac{n}{4}} + \frac{n}{4}\Big)$$

This is not enough.

What I want to prove is that $\partial_y^2 + \frac{1}{2}y\partial_y$ is invertible on the Schwarz space.

November 2, 1984

Program: In order to construct heat operators $e^{t\Delta}$ geometrically I have to prove that the class of kernels with the proper singularity is closed under composition. The kernels are $K(t,x,x')$ and composition means convolution in the t variable. This creates complexities with the blowups which have to be analyzed. So I propose first to do the composition for kernels $K(h,x,x')$ on Connes tangent groupoid. Here the h variable multiplies under composition.

I seem to get hung up on the convolution in the t variable, so maybe it would be worthwhile exploring the idea of doing the Laplace transform in t. This converts the singularity at $t=0$ into some kind of growth condition in the transform variable λ. Let's take a typical term

$$e^{-\frac{x^2}{t}}\, x^\alpha t^k$$

and take the Laplace transform.

$$\int_0^\infty dt\, e^{-\lambda t}\, e^{-\frac{x^2}{t}}\, x^\alpha t^k.$$

This involves us with imaginary Bessel functions such as

$$K_s(\lambda) = \int_0^\infty e^{-\frac{\lambda}{2}(t+t^{-1})}\, t^s \frac{dt}{t}$$

which we recall can be nicely evaluated when $s \in \frac{1}{2}+\mathbb{Z}$ in terms of an exponential:

$$\int_0^\infty dt\, e^{-\lambda t - \frac{u}{t}}\, t^{-1/2} = \sqrt{\frac{\pi}{\lambda}}\, e^{-2\sqrt{\lambda u}}$$

However what I will need is the growth information as $\lambda \longrightarrow \infty$, and this should come out of simple asymptotics.

$$\int_0^\infty \frac{dt}{t}\, e^{-\lambda t - \frac{u}{t}}\, t^s$$

$$= \int_0^\infty \frac{dt}{t}\, e^{-\lambda a t - \frac{u}{a}\frac{1}{t}}\, a^s t^s \qquad \lambda a = \frac{u}{a} \qquad a = \sqrt{\frac{u}{\lambda}}$$

$$= \left(\sqrt{\frac{u}{\lambda}}\right)^s \int_0^\infty \frac{dt}{t}\, e^{-2\sqrt{\lambda u}\left(\frac{t+t^{-1}}{2}\right)}\, t^s$$

$f(t) = \frac{1}{2}(t+t^{-1}) = 1$ at $t=1$

$f'(t) = \frac{1}{2}(1-t^{-2}) = 0$ at $t=1$

$f''(t) = \frac{1}{t^3} = 1$ at $t=1$

$$\sim \left(\sqrt{\frac{u}{\lambda}}\right)^s e^{-2\sqrt{\lambda u}} \int_{-\infty}^{\infty} du\, e^{-\frac{1}{2}(2\sqrt{\lambda u})u^2}$$

$$= \left(\sqrt{\frac{u}{\lambda}}\right)^s e^{-2\sqrt{\lambda u}} \frac{1}{\left(\pi\sqrt{\lambda u}\right)^{1/2}} = \frac{1}{\pi^{1/2}} \frac{u^{\frac{s}{2}-\frac{1}{4}}}{\lambda^{s/2+\frac{1}{4}}}\, e^{-2\sqrt{\lambda u}}$$

Thus
$$\int_0^\infty \frac{dt}{t}\, e^{-\lambda t - \frac{x^2}{t}}\, t^{k+1} \sim \frac{1}{\sqrt{\pi}} e^{-2\sqrt{\lambda}|x|} \frac{|x|^{k+\frac{1}{2}}}{\lambda^{k/2+3/4}}$$

Seems this is going to be messy.

Equivariant cohomology can be defined using entire functions on the Lie algebra. Does this provide any insight into the Baum-Connes problem?

I recall this deals with the periodic cyclic homology of the crossed product

$$C^\infty(G) \tilde{\rtimes} C^\infty(M)$$

where $C^\infty(G)$ is the convolution algebra. Take the simplest case where $M = pt$. Then we have the convolution algebra whose finitely generated projective modules would appear to be finite dimensional representations of G. So the K-theory is the repn. ring $R(G)$.

I guess an invariant transversally elliptic operator on M defines a trace on the convolution algebra.

November 3, 1984

A key point in the construction of a heat operator $e^{t\Delta}$ is the formal calculation in powers of t. This is not the same as the formal series

$$1 + t\Delta + \frac{t^2}{2!}\Delta^2 + \cdots .$$

There appears to be a general philosophy of asymptotic expansions, where there is an exponential factor containing the geometry times the formal series.

It is this geometry which I have to concentrate on. [redacted] I want to consider [redacted] the way the exponential factor behaves under composition. Could it be possible that there is a way to incorporate the fermions?

Let's set up the composition for kernels $K(h, x, x')$ including also the case of fermions eventually. First I want to look at the constant coefficient, or translation-invariant, case: $K(h, x-x')$. Also I begin with operators on functions. Now the typical kernel one has in mind is

$$K(h, x-x') = \int \frac{d^n p}{(2\pi)^n h^n} e^{i\frac{p}{h}(x-x')} \hat{K}(p)$$

where $\hat{K}(p)$ is smooth rapidly decreasing. Notice that this can be written in terms of $v = \frac{x-x'}{h}$ as

$$\int \frac{d^n p}{(2\pi)^n h^n} e^{ipv} \hat{K}(p)$$

and [redacted] this **is** smooth and rapidly decreasing in v.

I want to consider $K(h, x)$ such that upon changing variables to h, v, $v = \frac{x}{h}$:

$$h^n K(h, hv)$$

one gets a smooth function of h, v vanishing as $v \to \infty$, and also which extends to $h=0$.

The good way geometrically to say this should be to introduce $\widetilde{\mathbb{R}\times V}$ = bloup up of $\mathbb{R}\times V$ at $(0,0)$. Let's review the description.

Given a vector space W, we can form $\mathbb{P}W$ and the line bundle $\mathcal{O}(-1)$ over $\mathbb{P}W$. Then

$$\mathcal{O}(-1) \subset \mathbb{P}W \times W$$

consists of pairs (l, w) where $l \subset W$ is a line and $w \in l$. Set $\tilde{W} = \mathcal{O}(-1)$ with the map $\tilde{W} \longrightarrow W$, $(l,w) \longmapsto w$. Clearly the fibre of $\tilde{W}$ over 0 is $\mathbb{P}W$ and $\tilde{W} - \mathbb{P}W \xrightarrow{\sim} W - 0$.

~~In order to describe~~ $\cancel{\tilde{W}}$ ~~as a manifold we have to introduce local coordinates. Let~~ $\cancel{W = \mathbb{R}^n}$ ~~with coordinate functions~~ $\cancel{x^\mu}$. ~~Let~~ $\cancel{U_\mu \subset \mathbb{P}W}$ ~~be the open set of~~ $\cancel{l}$ ~~with~~ $\cancel{x^\mu | l \neq 0}$.

In order to describe $\tilde{W}$ we need local coordinates. Let's choose a decomposition $W = \mathbb{R}e \oplus V$, where V is a hyperplane in W and $e \in V - W$. Then any l in $\mathbb{P}W - \mathbb{P}V$ has a unique generator $e + v$, so we get an isom. $\mathbb{P}W - \mathbb{P}V \simeq V$. The a point (l, w) of $\mathcal{O}(-1)$ over l determines an $h \in \mathbb{R}$ by $w = h(e+v)$. Thus

$$\tilde{W} - \tilde{V} \xrightarrow{\sim} \mathbb{R} \times V$$

and so we have coordinatized the open set

$\tilde{W} - \tilde{V}$ by choosing an $e \in W - V$.

Next do this more concretely so that we can get the transition functions. Suppose $W = \mathbb{R}^n$ and let V be the μ-th coordinate hyperplane: $x^\mu = 0$, and set $e = e_\mu$. Then $l = \mathbb{R}(x^1, \ldots, \underset{\uparrow}{1}, \ldots x^n)$, giving coords
μth position

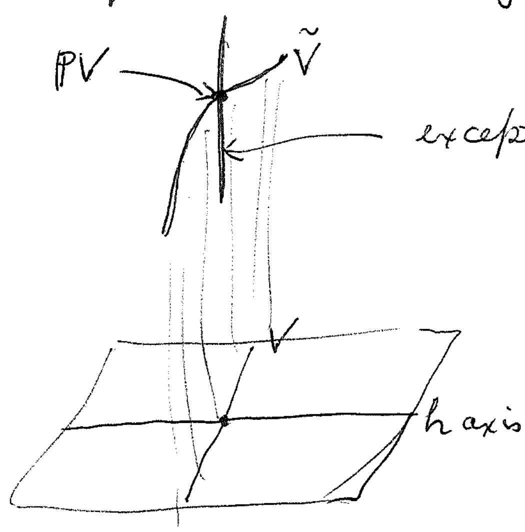
$x^\nu, \nu \neq \mu$ for $\mathbb{P}W - \mathbb{P}V$. And if $y = (y^\mu) \in l$ we have $y^\nu = h x^\nu$ where $h = y^\mu$.

Thus on $\tilde{W} - \tilde{V}$ we have the coordinates h, x^ν $x^\mu = 1$ and the map to W is

$$h, x^\nu \longmapsto h x^\nu.$$

(not too clear).

Let's instead consider $W = \mathbb{R} \oplus V$, $V = \mathbb{R}^n$ with coordinates x^μ, and let h be the first coordinate of W. Then $w = (h, x^\mu)$ will be coordinates for W. A line $l \subset W$, but not in V, has a unique generator $(1, y^\mu)$. and then if $w = (h, x^\mu) \in l$ we have $x^\mu = h y^\mu$. Thus on $\tilde{W} - \tilde{V}$ we have the coords (h, y^μ) and the map to W is given by $x^\mu = h y^\mu$.



exceptional fibre $= \mathbb{P}(\mathbb{R} \oplus V) = V \cup \mathbb{P}V$

So we have

exceptional fibre $= \mathbb{P}(\mathbb{R} \oplus V)$

$= \tilde{V} \cup \mathbb{P}V$

inverse image of $h = 0$

$= \tilde{V} \cup \mathbb{P}(\mathbb{R} \oplus V)$

$\tilde{V} \cap \mathbb{P}(\mathbb{R} \oplus V) = \mathbb{P}V.$

Now that I see the geometry of $\widetilde{\mathbb{R}\oplus V}$ I can go back to

$$K(h,x) = \int \frac{d^n p}{(2\pi h)^n}\, e^{i\frac{p}{h}x} F(p) . \qquad h\neq 0$$

If we then pull this back via $(h,y^\mu) \longmapsto (h,hy^\mu)$ we get

$$h^n K(h,hy) = \int \frac{d^n p}{(2\pi)^n}\, e^{ipy} F(p)$$

Notice that this is a smooth function of h,y defined for all h including $h=0$. Put a better way, K is only defined for $h\neq 0$, but $h^n K(h,hy)$ extends to all of $\tilde{W}-\tilde{V}$.

The next point is that as $y \longrightarrow \infty$, the transform on the left goes to zero. Better we assume F is a Schwartz function, ~~so~~ $h^n K(h,hy) = \hat{F}(y)$ will be a Schwartz function on V. This means $\hat{F}$ can be identified with a smooth function on $\mathbb{P}(\mathbb{R}\oplus V)$ vanishing to infinite order on $\mathbb{P}V$.

$$\begin{tikzcd} \widetilde{\mathbb{R}\oplus V} \arrow[r] \arrow[d] & \mathbb{P}(\mathbb{R}\oplus V) \\ \mathbb{R}\oplus V & \end{tikzcd}$$

Now one should maybe look at the converse, namely, take a smooth function on $\widetilde{\mathbb{R}\oplus V}$ vanishing to infinite order on $\tilde{V}$, and then one should identify such a function with a $K(h,x)$.

November 4, 1984

I am now involved with the program of making precise the algebra of smooth kernels on the tangent groupoid to M. The tangent groupoid is obtained by blowing up $0 \times (\Delta M) \subset \mathbb{R} \times M \times M$. So a kernel on it is a function $K(h, x, x')$ with some kind of singularity along $h=0$, $x = x'$. Actually function should be section of some bundle.

I want to begin by working out the constant coefficient case which means M is a real vector space V, or possibly, a torus V/Γ, and then $K(h, x, x') = K(h, x-x')$, and the singularity of $K(h,x)$ is at $h=0, x=0$. My example is

$$K(h,x) = \int \frac{d^n p}{(2\pi h)^n} \, e^{i\frac{p}{h}x} \, F(h,p)$$

where $F(h,p)$ is a family of Schwartz functions of p depending smoothly in h.

Schwartz functions are functions $f(p)$, $p \in \mathbb{R}^n$ which are smooth and satisfy estimates

$$\left| p^\alpha \frac{\partial^\beta f}{\partial p^\beta} \right| \leq C_{\alpha,\beta}$$

on all of $\mathbb{R}^n$. These estimates mean that f and each of its derivatives $\partial^\beta f$ decay faster than $\frac{1}{\text{poly}}$ for any poly. A basic fact is that the Fourier transform is defined on the space of Schwartz functions. Usually this is proved as follows.

First one shows that $\hat{f}$ is bounded. Let's 'ft to standard x, ξ notation:

This implies (take same formula with $f(x)$ replaced by $f(x)-f(y)$) that $H_s \subset C^0$ for $s > n/2$. So for $\mathbb{R}$ one needs $> \frac{1}{2}$ derivatives.

Once one bounds $|\hat{f}|_{L^\infty}$ in terms of $|f|_{L^1}$ the rest is clear since the F.T. interchange derivatives and polynomials.

Next let's go on to try to give a geometric proof of this fact about F.T. being defined on ~~the~~ Schwartz space S. I propose to use the idea that a Schwartz function on V is the same thing as a function on $\mathbb{P}(\mathbb{R}\oplus V) = V \cup \mathbb{P}V$ which vanishes to infinite order along $\mathbb{P}V$. One can also use the 1-point compactification $S(\mathbb{R}\oplus V) = V \cup \infty$.

We have V and V^* and form $V \times V^*$ on which we have the function $e^{ix\xi}$. Now enlarge to $\bar{V} \times \bar{V^*}$, where $\bar{V} = \mathbb{P}(\mathbb{R}\oplus V) = V \cup \mathbb{P}V$. Or maybe go to $\overline{V \times V^*}$? What's important is that we will have this function $f(x)$ on $\bar{V} \times \bar{V^*}$ which we multiply by $d^n x\, e^{ix\xi}$, which has singularities as either x or ξ go to ∞, and then integrate over the fibre to get $\hat{f}(\xi)$ on $\bar{V}^*$. Geometrically, $\hat{f}(\xi)$ is defined and smooth for $\xi \in V^*$ — this is nothing but the fact that f vanishing to infinite order swamps the singularities of $d^n x\, e^{ix\xi}$ as $x \to \infty$. What isn't clear geometrically, at least so far is why $\hat{f}$ decays as $\xi \to \infty$. This ultimately rests on integration by parts.

$$\hat{f}(\xi) = \int d^n x \; e^{-i\xi x} f(x)$$

$$f(x) = \int \frac{d^n\xi}{(2\pi)^n} e^{i\xi x} \hat{f}(\xi)$$

which in physics notation means

$$\langle \xi | f \rangle = \int d^n x \; \langle \xi | x \rangle \langle x | f \rangle$$

$$\langle x | f \rangle = \int \frac{d^n\xi}{(2\pi)^n} \langle x | \xi \rangle \langle \xi | f \rangle .$$

Then to bound $\hat{f}$ we use

$$\left| \hat{f}(\xi) \right| \leq \int d^n x \; |f(x)| = L^1\text{-norm of } f.$$

But more is true

$$\hat{f}(\xi) - \hat{f}(\eta) = \int d^n x \underbrace{\left(e^{i\xi x} - e^{i\eta x} \right)}_{\to 0 \text{ as } \xi \to \eta} f(x)$$

so that by the dominated convergence thm. $\hat{f}$ is continuous and bounded for $f \in L^1$.

This has the standard consequence that if

$$(1 + |\xi|^2)^{s/2} \left| \hat{f}(\xi) \right| \leq C$$

then

$$(2\pi)^{-n} \int d^n\xi \; |\hat{f}| \leq (2\pi)^{-n} \int d^n\xi \frac{C}{(1+|\xi|^2)^{s/2}} \qquad \text{finite for } s > n$$

so f will be continuous. What are the Sobolev things?

$$f(x) = \int \frac{d^n\xi}{(2\pi)^n} \frac{e^{i\xi x}}{(1+|\xi|^2)^{s/2}} (1+|\xi|^2)^{s/2} \hat{f}(\xi)$$

$$|f(x)|^2 \leq \underbrace{\int \frac{d^n\xi}{(2\pi)^n} \frac{1}{(1+|\xi|^2)^s}}_{\text{finite } s > n/2} \; \underbrace{\int \frac{d^n\xi}{(2\pi)^n} (1+|\xi|^2)^s |\hat{f}(\xi)|^2}_{\text{Sobolev [redacted] } H_s\text{-norm.}}$$

Q: What makes an oscillatory integral

$$\int d^n x \; e^{it\varphi(x)} \, g(x)$$

a Schwartz function of t, when $d\varphi \neq 0$ on Supp g ?

By means of a partition of 1 one can make Supp g (supposed compact) small. Then there is a vector field X with $X\varphi = 1$, and one can suppose $d^n x$ preserved by X ([redacted] change g). Then

$$X e^{it\varphi} = e^{it\varphi} \, it$$

so

$$\int d^n x \, e^{it\varphi} g = \frac{1}{(it)^m} \int d^n x \; X^m\left(e^{it\varphi}\right) g$$

$$= \frac{1}{(it)^m} \int d^n x \; e^{it\varphi} (-X)^m g = O\left(\frac{1}{t^n}\right)$$

for any m.

Q: What makes DH setup work?

In this case the vector field X one works with, namely, the generator of the circle action, preserves φ. One supposes that $e^{it\varphi(x)} g(x) \, d^n x$ = an n-form ω_n [redacted] which is part of an equivariant form ω [redacted] which is closed $d\omega = \iota_X \omega$. Suppose that one can find a "connection" form α [redacted] such that [redacted] $d\alpha = 0$. Connection form means $\mathcal{L}_X \alpha = 0$ and $\iota_X \alpha = 1$. Then we have that

$$(d - \iota_X)(\alpha\omega) = -\omega$$

in particular $d(\alpha\omega_{n-1}) - \overbrace{\cancel{\iota_X(\alpha\omega_n)}}^{=0 \text{ if } n = \dim M} = -\omega_n$, and so one has $$\int_M \omega_n = \int_M -d(\alpha\omega_{n-1}) = -\int_{\partial M} \alpha\omega_{n-1} .$$

So it would appear that the DH setup is based on completely different principles than stationary phase.

What I would really like is a unification of stationary phase, steepest descent, possibly in the complex domain where conjugate variables would enter, and where one could see the DH phenomena happening.

Actually the best way to think seems to be via the level surfaces of the Hamiltonian, i.e. via the Archimedes theorem.

So next let's go back to the original program of smooth kernels on the tangent groupoid. I start by looking at the translation invariant kernels $K(h, x-x')$ where

$$K(h,x) = \int \frac{d^np}{(2\pi h)^n}\, e^{\frac{ipx}{h}} f(p)$$

with f a Schwarz function. This K is defined for $h \neq 0$, however if we make the change of variable $y = \frac{x}{h}$, then

$$h^n K(h, hy) = \int \frac{d^np}{(2\pi)^n}\, e^{ipy} f(p) = \check{f}(y)$$

is a Schwartz function of y.

More generally I want to look at

$$K(h,x) = \int \frac{d^n p}{(2\pi h)^n} e^{i\frac{p}{h}x} f(h,p)$$

where $h \longmapsto f(h,p)$ is a family of Schwartz functions of p depending smoothly in h. Then

$$h^n K(h, hy) = \int \frac{d^n p}{(2\pi)^n} e^{ipy} f(h,p) = \tilde{f}(h,y)$$

is a smooth family in h of Schwartz functions of y. This should be the same as a smooth function on $\mathbb{R} \times (V \cup \mathbb{P}V)$ vanishing to infinite order on $\mathbb{R} \times \mathbb{P}V$.

Now let's relate this to $\widetilde{\mathbb{R} \times V} \subset (\mathbb{R} \times V) \times \mathbb{P}(\mathbb{R} \times V)$. I want to compare complements:

$$\widetilde{\mathbb{R} \times V} - \tilde{V} = \{(\omega, \ell) \mid \ell \subset \mathbb{R} \times V,\ \ell \not\subset V,\ \omega \in \ell$$

$$\simeq \mathbb{R} \times V \quad \text{with coords } h, y^\mu$$

where $\ell = \mathbb{C}(1,y)$, $\omega = (h, hy^\mu) = (h, x^\mu)$. Also

$$\{\mathbb{R} \times (V \cup \mathbb{P}V)\} - \{\mathbb{R} \times \mathbb{P}V\} = \mathbb{R} \times V.$$

Therefore there should be a map

$$\begin{array}{ccc} \mathbb{R} \times (V \cup \mathbb{P}V) & \longrightarrow & \widetilde{\mathbb{R} \times V} \\ \cup & & \cup \\ \mathbb{R} \times \mathbb{P}V & \longrightarrow & \tilde{V} \end{array} \qquad ?$$

All this is very confusing. I am trying to compare two different ideas. One is a smooth function on $\widetilde{\mathbb{R} \times V}$ vanishing to infinite order on $\tilde{V}$. The other is a family of Schwartz functions $\tilde{f}(h,y)$ on V depending smoothly on h. There should be no difference in this case, but the point is that $\widetilde{\mathbb{R} \times V}, \tilde{V}$ makes sense for V replaced by M, e.g. $M = V/\Gamma$.

I am now beginning to think there is a condition to be satisfied. Actually I probably have the above map going the wrong way:

$$\begin{array}{ccc} \widetilde{\mathbb{R}\times V} & \longrightarrow & \mathbb{R}\times(V\cup \mathbb{P}V) \\ \cup & & \cup \\ \tilde{V} & \longrightarrow & \mathbb{R}\times\mathbb{P}V \end{array}$$

such a map is obtained by

$$\widetilde{\mathbb{R}\times V} \subset (\mathbb{R}\times V)\times \mathbb{P}(\mathbb{R}\times V) \twoheadrightarrow \mathbb{R}\times\mathbb{P}(\mathbb{R}\times V)$$

so what happens is we have

$$\underset{(h,x)}{\overset{\parallel}{w}} \in l \subset \mathbb{R}\oplus V \quad \text{goes to} \quad (h,l)\in \mathbb{R}\times\mathbb{P}(\mathbb{R}\oplus V)$$

Over $h=0$, two cases occur: $\left| \begin{array}{l} l\subset V \implies x \text{ arb. in } l \\ l\not\subset V \implies x = 0. \end{array}\right.$

So the map is neither injective nor surjective. On the complements we have $\mathbb{R}\times V$ in both cases and the map is an isomorphism. Therefore what we have is two different compactifications (proper over the h-line) of $\mathbb{R}\times V$.

~~Now I think we can see how to handle V/Γ. When we take $K(h,x)$, where $x\in V/\Gamma$ and $h\neq 0$ now and lift it to $h^n K(h,hy)$, then we get a smooth function on $\mathbb{R}\times V$ vanishing to infinite order at $\mathbb{R}\times\infty$, but it is not an arbitrary function with this property because of the periodicity conditions.~~ NO NO.

Let's see what happens in the case of $M = V/\Gamma$. Let U be an open nbd of 0 in M covered by a disk around 0 in V, say the unit disk. Then $\tilde{M}$ = the blow-up of M along 0 is a union

$$\tilde{M} = (M - \{0\}) \underset{U-\{0\}}{\cup} (\tilde{V}/U).$$

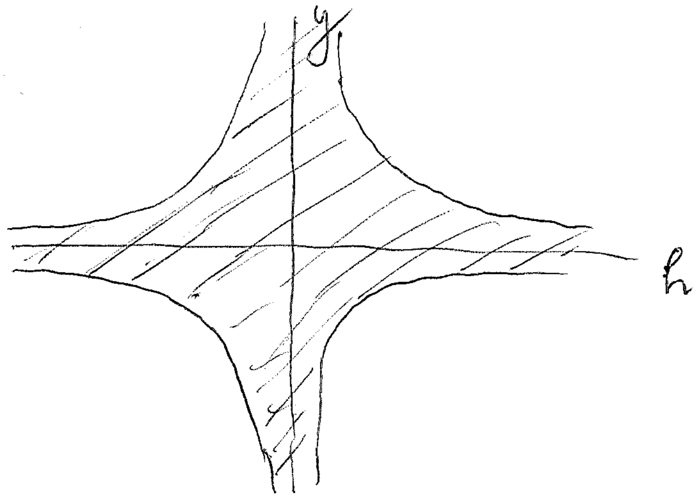
The map $\pi : \tilde{M} \longrightarrow M$ is a diffeomorphism off 0 and over U we can identify $\tilde{M}/U$ with $\tilde{V}/U = \{(v, \ell) \mid \ell \in \mathbb{P}V,\ v \in \ell,\ |v| \leq 1\}$. Similarly $\widetilde{\mathbb{R}\times M} \xrightarrow{\pi} \mathbb{R}\times M$ is a diffeomorphism off $(0,0)$, and over $\mathbb{R}\times U$ we can identify $\widetilde{\mathbb{R}\times M}$ with $\widetilde{\mathbb{R}\times V}/\mathbb{R}\times U = \{(w, \ell) \mid \ell \subset \mathbb{R}\times V,\ w \in \ell$, and if $w = (h, v)$, then $|v| \leq 1\}$.

So now suppose we have a $K(h,x)$ on $(\mathbb{R}-0)\times M$ which extends smoothly over $\widetilde{\mathbb{R}\times M}$ and vanishes to infinite order on $\tilde{M}$. Take a point of $\widetilde{\mathbb{R}\times M}$ lying in $V \subset \mathbb{P}(\mathbb{R}\times V)$ = exceptional fibre. Such a point is a pair $(0, \ell_0)$, where $\ell_0 = \mathbb{C}(1, y_0)$, $y_0 \in V$. A nearby point is (w, ℓ), where $\ell = \mathbb{C}(1, y)$ and $w = (h, hy)$, and where we assume $|hy| \leq 1$ so that we are in the open set lying over $\mathbb{R}\times U$. Thus a coordinate nbd of $(0, \ell_0) \in V$ is described by

$$|h| \leq \varepsilon, \qquad |y| < \frac{1}{\varepsilon}$$

or maybe better just by the condition $|hy| < 1$.

Picture:



By assumption $K(h, hy)$ will have a smooth extension to this region. Clearly that gives an expansion

$$K(h, hy) \sim \sum_{n \geq 0} h^n k_n(y) \quad \text{as } h \to 0$$

which is at least uniform on compact subsets of V.

But now we also want to look at the case of a point of $\mathbb{P}V$

November 5, 1984

I am trying to find equivalent descriptions of a singularity. Better, I want to define a class of smooth functions $K(h,x)$ on $\mathbb{R}\times U - (0,0)$, where U is an open subset of $V = \mathbb{R}^n$ containing 0. My first description is that if K is lifted to the blowup $\widetilde{\mathbb{R}\times U}$ [redacted], then it has a smooth extension which vanishes to infinite order along $\tilde{U}$.

Another possibility goes as follows. First we require $K(h,x)$ vanishes to infinite order along $\times(U-(0)) \subset (\mathbb{R}\times U)-(0,0)$. Then we require an asymptotic expansion as $h \longrightarrow 0$

$$K(h,hy) \sim \sum h^n k_n(y)$$

where $k_n \in$ Schwartz space of V. What should asymptotic mean in this case? Note that $K(h,hy)$ is defined for $|hy| < 1$ say. Set

$$K_N(h,hy) = K(h,hy) - \sum_{n\le N} h^n k_n(y)$$

Then maybe we want to require that on $|hy| \le \frac{1}{2}$, $|h| \le K$.

$$\left| \frac{\partial}{\partial h^j}\, y^\alpha \frac{\partial}{\partial y^\beta} K_N(h,hy) \right| \le C\, h^{N+1-j}$$

(This estimate comes from the fact that the Schwartz space is defined by the norms $\sup_y \left| y^\alpha \frac{\partial}{\partial y^\beta} f \right|$. So what I have written down is the inequality satisfied by the next term in the asymptotic expansion.)

A third possibety goes as follows. Choose $\rho \in C_0^\infty(V)$ with support in U and with $\rho \equiv 1$ for $|x| \leq \frac{1}{2}$. Then $\rho(x)K(h,x)$ is now defined on $R\times V-(0,0)$ and it has the same singularity at $(0,0)$. Now we can require that ρK vanishes to infinite order on $0\times(V-0)$ and that

$$\rho(hy)\,K(h,hy) \qquad \text{on} \quad R\times V$$

is a family of Schwartz functions on V depending smoothly on h. Actually it seems that if $\rho(hy)K(h,hy)$ is a smooth family of Schwartz functions, then necessarily $\rho(x)K(h,x)$ vanishes to infinite order on $h=0$, $x\neq 0$.

What does smooth family of Schwartz functions mean? Want $f(h,y)$ to be smooth in both variables and then each derivative with respect to h to be a Schwartz fn: For all y, $|h|\leq K$

$$\left|\frac{\partial}{\partial h^j}\, y^\alpha \frac{\partial}{\partial y^\beta} f\right| \leq C$$

Now I would like to show that these are equivalent.

It seems that the basic geometric idea is as follows. You start with $K(h,x)$ defined on $(\mathbb{R}-0)\times M$, then you lift back to $\widetilde{\mathbb{R}\times M}$ and assume K has a smooth extension vanishing to infinite order on $\widetilde{M}$. Next you let V be the tangent space to M at O, and identify an open nbd U of O in M with the unit disk of V. Then choose $\rho(x)\in C_0^\infty(U)$ such that $\rho\equiv 1$ near O.

But before choosing ρ, one notes that if $\pi: \widetilde{\mathbb{R}\times M}\to\mathbb{R}\times M$

then $\pi^{-1}(\mathbb{R}\times U)$ can be viewed as the open subset of $\widetilde{\mathbb{R}\times V}$ sitting over $\mathbb{R}\times U$. Thus our original function K upstairs gives a smooth function on [redacted] $\pi^{-1}(\mathbb{R}\times U) \subset \widetilde{\mathbb{R}\times V}$. So everything I need to know [redacted] about K in a nbd of the exceptional divisor is given.

Start again: Let's begin with K on $\widetilde{\mathbb{R}\times M}$ vanishing to infinite order on $\tilde{M}$. Next choose $V \supset U \subset M$ and $\rho \in C_0^\infty(U)$. Then we have

$$\begin{tikzcd}
\widetilde{\mathbb{R}\times V} \arrow[d] & \widetilde{\mathbb{R}\times U} \arrow[l, phantom, "\supset"] \arrow[r, phantom, "\subset"] \arrow[d] & \widetilde{\mathbb{R}\times M} \arrow[d] \\
\mathbb{R}\times V & \mathbb{R}\times U \arrow[l, phantom, "\supset"] \arrow[r, phantom, "\subset"] & \mathbb{R}\times M
\end{tikzcd}$$

and ρK on $\widetilde{\mathbb{R}\times M}$ has support in $\widetilde{\mathbb{R}\times U}$, and so can be transferred to $\widetilde{\mathbb{R}\times V}$. As K vanishes to infinite order on $\tilde{M}$, ρK does also, and so [redacted] ρK on $\widetilde{\mathbb{R}\times V}$ vanishes to infinite order on $\tilde{V}$. But now use the map

$$\begin{tikzcd}
\widetilde{\mathbb{R}\times V} \arrow[r] & \mathbb{R}\times(V\cup PV) \arrow[r] & \mathbb{R}\times(V\cup\infty) \\
\tilde{V} \arrow[u, phantom, "\cup" rotate=90] \arrow[r] & 0\times PV \arrow[u, phantom, "\cup"] \arrow[r] & 0\times(\infty) \arrow[u, phantom, "\cup"]
\end{tikzcd}$$

to [redacted] transfer ρK to $\mathbb{R}\times(V\cup PV)$ or $\mathbb{R}\times(V\cup\infty)$. Then you get a family of Schwartz functions.

Geometrically what I am doing is to take $K(h, hy)$ which is defined for $h\neq 0$, $|hy|<1$ and then multiply to get $\rho(hy)K(h,hy)$ which now is a 1-parameter family of functions with compact support defined for $h \neq 0$. I claim that this family extends to a smooth family of Schwartz functions defined for all h.

Now so far I have used the fact that when I have a birational proper map of algebraic varieties, then smooth functions vanishing to infinite order on the exceptional locus correspond in one-one fashion.

November 6, 1984

Let's go back to the Thom forms and try to relate this to Bott's residue calculations. The Thom form is

$$\mathcal{U} = \det(\omega)\, e^{-\left(|z|^2 + dz^* \frac{1}{\omega} dz\right)} \qquad \in \left[S(\mathfrak{g}^*) \otimes \Omega(V)\right]^G$$

where $G = U(n)$ acting on $V = \mathbb{C}^n$. Recall $z = \begin{pmatrix} z^1 \\ \vdots \\ z^n \end{pmatrix}$ and

$$d_\omega\,(dz) = \omega z\,, \qquad d_\omega z = dz$$

$$d_\omega\,(dz^*) = (\omega z)^*, \qquad d_\omega z^* = dz^*$$

The exponent in $\mathcal{U}$ can be written:

$$d_\omega \Big[\underbrace{\tfrac{1}{2}\left(z^* \tfrac{1}{\omega} dz - dz^* \tfrac{1}{\omega} z \right)}_{\alpha'} \Big] = |z|^2 + dz^* \tfrac{1}{\omega} dz$$

which shows that $\mathcal{U}$ is killed by d_ω.

Let's now compute the transgression using α'. Let $\tilde{\mathcal{U}}$ be the pull-back under the map $t, z \mapsto tz$.

$$\tilde{\alpha}' = t^2 \alpha' + \tfrac{1}{2}\left(t z^* \tfrac{1}{\omega} z\, dt - dt\, z^* \tfrac{1}{\omega} t z \right)$$

$$= t^2 \alpha'$$

so

$$\tilde{\mathcal{U}} = \det(\omega)\, e^{\left(-(dt\,\partial_t + d_\omega)(t^2\alpha')\right)}$$

$$= \det(\omega)\, e^{\left(-t^2 d_\omega \alpha' - 2t\,dt\,\alpha'\right)}$$

$$= \underbrace{\det(\omega)\, e^{-t^2 d_\omega \alpha'}}_{\mathcal{U}_t} + dt \cdot \underbrace{\det(\omega)\, e^{-t^2 d_\omega \alpha'} (-2t)\, \alpha'}_{V_t}$$

is closed under $dt\,\partial_t + d_\omega$, which implies

$$\partial_t \mathcal{U}_t = d_\omega V_t\,, \quad \text{hence}$$

$$\mathcal{U}_t - \underbrace{\mathcal{U}_0}_{\det \omega} = d_\omega \left\{ \det(\omega) \int_0^t (-2t)\,dt\; e^{-t^2 d_\omega \alpha'}\, \alpha' \right\}$$

$$\mathcal{U}_t = \det(\omega) + d_\omega\left\{\det(\omega)\left(\frac{e^{-t^2 d_\omega \alpha'} - 1}{d_\omega \alpha'}\right)\alpha'\right\}$$

so letting $t \to \infty$ we get simply

$$\det(\omega) = d_\omega\left\{\det(\omega)\,\frac{\alpha'}{d_\omega \alpha'}\right\}$$

where inside the braces is the transgression form.

Now the problem at hand is to look at the circle action on a complex vector bundle E/M. We know the localized equivariant cohomology of E − zero section is trivial, and in a specific way. Let's take the equivariant $G = U(n)$ - model for the forms on E, namely $[S(\mathfrak{g}^*) \otimes \Omega(V)]^G$. The circle group can be identified with the scalars inside G. Let u denote a typical scalar matrix in $\mathfrak{g}$, so that $u \in i\mathbb{R}\cdot I$. Then

$$X_u = -u\left(z^j \frac{\partial}{\partial z^j} - \bar{z}^j \frac{\partial}{\partial \bar{z}^j}\right)$$

and the differential we will be using is

$$d_{\omega,u} = d_\omega - \iota_u = d - \iota_\omega - \iota_u .$$

The way we see the localized equivariant cohomology for the circle action is trivial is to use the 1-form

$$\alpha = \tfrac{1}{2}\left(z^* dz - dz^* z\right)$$

which is clearly invariant.

In order to get to the bottom of this stuff with the Thom class and the circle action it seems desirable to go back to the relation in the projective bundle. Let $\xi = c_1 \mathcal{O}(1)$. Over PV one has the exact sequences

$$\cancel{0 \to \mathcal{O} \to \pi^*V \otimes \mathcal{O}(1) \to Q \otimes \mathcal{O}(1) \to 0}$$

$$0 \to \mathcal{O}(-1) \to \pi^*V \to Q \to 0$$

$$0 \to \mathcal{O} \to \pi^*V \otimes \mathcal{O}(1) \to Q \otimes \mathcal{O}(1) \to 0$$

hence if rank $V = n$, we have

$$c_n(\pi^*V \otimes \mathcal{O}(1)) = c_{n-1}(Q \otimes \mathcal{O}(1))\, \cancel{c_1(\mathcal{O})} = 0$$

$$\parallel$$

$$\xi^n + c_1(V)\,\xi^{n-1} + \cdots + c_n(V)$$

Lifting this relation to $V - 0$, or SV, we get a reason for $c_n(V)$ to vanish, which should furnish the transgression class.

In other words what we have to do is to prove on the differential form level the Chern product formula for the exact sequence

$$0 \longrightarrow \mathcal{O} \longrightarrow \pi^* V \longrightarrow Q \longrightarrow 0$$

of holomorphic bundles over $V - 0$.

Notice that we ought to be able to do the holomorphic theory equivariantly, using as group the unitary group. This group preserves the metrics and holom. structures hence we get invariant connections. Hence the above is an exact sequence of equivariant holomorphic vector bundles, and we have an invariant metric on $\pi^* V$, so we have invariant connections.

At this point I need to know what the Bott-Chern theory says about this exact sequence. Review their theory. In general given a holomorphic vector bundle with metric one has a unique connection preserving the metric and extending the $\bar\partial$-operator. The curvature of such a connection will be of type (1,1), so the Chern forms, etc., are of type (p,p). As we vary the metric the Chern forms change by something in the image of $d''d'$. Recall how this goes. Choose a holom. framing, whence a section is a vector function f and

$$D = d + \theta \qquad \theta \text{ matrix of } (1,0) \text{ forms.}$$

The metric is given by $|f|^2 = f^* N f$ where N is a positive definite matrix fn. Then

$$d|f|^2 = (df)^* N f + f^* dN f + f^* N (df)$$

is supposed to equal

$$\langle Df, f\rangle + \langle f, Df\rangle = (df + \theta f)^* N f + f^* N (df + \theta f)$$

whence

$$dN = \theta^* N + N\theta \quad \Longrightarrow \quad \boxed{\theta = N^{-1} d' N}$$

Given a family of metrics depending on a parameter we set $L = N^{-1}\dot N$, whence $\dot N = NL$

$$\underbrace{\dot\theta}_{\dot D'} = (d' + \theta)^{\cdot} = (N^{-1}(d')N)^{\cdot} = [\underbrace{N^{-1}(d')N}_{D'}, L]$$

$$\boxed{\dot D' = [D', L]}$$

So now

$$\partial_t \operatorname{tr}(e^{D^2}) = \operatorname{tr}(e^{D^2}[D, \dot D]) = d \operatorname{tr}(e^{D^2}\dot D)$$

and $\dot D = \dot D' = [D', L]$, also $0 = [D, D^2] = [\overset{2,1}{D', D^2}] + [\overset{1,2}{D'', D^2}]$

so $[D', D^2] = [D'', D^2] = 0$. Thus

$$\operatorname{tr}(e^{D^2}\dot D) = \operatorname{tr}(e^{D^2}[D', L]) = \operatorname{tr}([D', e^{D^2}L]) = d' \operatorname{tr}(e^{D^2}L)$$

and so $\quad \partial_t \operatorname{tr}(e^{D^2}) = d''d' \operatorname{tr}(e^{D^2}L)$.

Notice in the above proof that what is used is that D'' stays fixed, D' changes up to inner autos. (so that $(D')^2 = 0$, $(D'')^2 = 0$ and the curvature is of type $(1,1)$.)

Now one considers an exact sequence of holomorphic bundles

$$0 \to E_I \longrightarrow E \longrightarrow E_{II} \longrightarrow 0$$

with a metric on the bundle E. Use the metric to split the sequence, the $\bar\partial$-operator on E will have the form

$$D'' = \begin{pmatrix} D''_I & W \\ 0 & D''_{II} \end{pmatrix}$$

and the D' operator will be

$$D' = \begin{pmatrix} D'_I & 0 \\ W^* & D'_{II} \end{pmatrix}.$$

Now one considers the family

$$D'_t = \begin{pmatrix} D'_I & 0 \\ e^{-t}W^* & D'_{II} \end{pmatrix}$$

Then $\left[D'_t, \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix}\right] = \left[\begin{pmatrix} 0 & 0 \\ e^{-t}W^* & 0 \end{pmatrix}, \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix}\right] = \begin{pmatrix} 0 & 0 \\ e^{-t}W & 0 \end{pmatrix} = -\dot{D}'_t$

so that we have $\quad \dot{D}'_t = [D'_t, L] \quad$ with $L = \begin{pmatrix} -I & 0 \\ 0 & 0 \end{pmatrix}$.
This gives then the Bott-Chern deformation.

Now I want to apply this to [redacted] $E = \pi^* V$ over $V - 0$, where $V = \mathbb{C}^n$. E is given the trivial

Connection $\overset{D=}{d} = d' + d''$ and we to put this connection in a family $D_t = D_t' + d''$ where only the D_t' varies.

Earlier work Jan-Mar 1983 p 611

<u>November 7, 1984</u>

I now want to carry out the Bott Chern deformation in the case of the exact sequence

$$0 \longrightarrow \mathcal{O} \longrightarrow \pi^* V \longrightarrow Q \longrightarrow 0$$

over $V-(0)$, $V = \mathbb{C}^n$, using the standard metric on V. The projection on the subbundle $\mathcal{O}$ is

$$e = \frac{z z^*}{|z|^2} \qquad z = \begin{pmatrix} z^1 \\ \vdots \\ z^n \end{pmatrix} \text{ coords on } V.$$

The connection in $\pi^* V$ is $D = d = d' + d''$ relative to the trivialization, so relative to the orthogonal splitting we have

$$D' = d' = \begin{pmatrix} e \\ 1-e \end{pmatrix} d' \begin{pmatrix} e & 1-e \end{pmatrix} = \begin{pmatrix} e\,d'e & e\,d'(1-e) \\ (1-e)d'e & (1-e)d'(1-e) \end{pmatrix}$$

where if things check we have $e\,d'(1-e) = 0$.

$$d'e = d' \frac{z z^*}{|z|^2} = \frac{dz\, z^*}{|z|^2} - \frac{1}{|z|^4} (z^* dz)\, z z^*$$

$$e\,d'e = \frac{z z^* dz\, z^*}{|z|^4} - \frac{(z^* dz)\, z z^*}{|z|^4} = 0.$$

So now our new connection will deform D' but leave $D'' = d''$ alone; in fact we want to go from

$$d' = \begin{pmatrix} e\,d'e & 0 \\ d'e & (1-e)\cdot d' \cdot (1-e) \end{pmatrix} \quad \text{to} \quad \begin{pmatrix} e\,d'e & 0 \\ 0 & (1-e)d'(1-e) \end{pmatrix}$$

so the simplest way to do this is

$$D'_t = d' - t d'e \qquad 0 \leq t \leq 1 .$$

Next let's [redacted] write this family as $N_t^{-1} \cdot d' \cdot N_t$ for a family of metrics. Put

$$N_t = \text{[redacted]} \, (1-t)e + (1-e) = 1 - te$$

$$d' N_t = -t d'e$$

$$N_t^{-1} d' N_t = \left(\frac{1}{1-t} e + (1-e) \right) \left(-t d'e \right) = -t d'e$$

Next we write this in the form

$$\partial_t D'_t = [D'_t , L_t]$$

where we [redacted] recall

$$L_t = N_t^{-1} \partial_t N_t = \left(\frac{1}{1-t} e + (1-e) \right) (-e) = \frac{-1}{1-t} e$$

Check: $$[D'_t, L_t] = \left[d' - t d'e , -\frac{1}{1-t} e \right] = \frac{-1}{1-t} d'e + \frac{t}{1-t} [d'e, e]$$

$$[d'e, e] = (d'e)e - \cancel{e d'e} = d'(e^2) - \cancel{e d'e} = d'e$$

$$\therefore \; [D'_t, L_t] = \frac{-1+t}{1-t} d'e = -d'e = \partial_t D'_t .$$

November 8, 1984

Feynman's formula for simplifying denominators

$$\frac{1}{ab} = \int_0^1 \frac{dt}{((1-t)a + tb)^2}$$

Standard proof is

$$\frac{d}{dt}\,\frac{1}{(1-t)a+tb} = \frac{+a-b}{((1-t)a+tb)^2}$$

so
$$\int_0^1 \frac{dt}{((1-t)a+tb)^2} = \frac{1}{a-b}\left[\frac{1}{(1-t)a+tb}\right]_0^1 = \frac{1}{a-b}\left[\frac{1}{b}-\frac{1}{a}\right]$$
$$= \frac{1}{ab}$$

Feynman says he found this trick from Gaussian manipulations of Schwinger. [redacted] Here is perhaps what is meant:

$$\frac{1}{ab} = \int_0^\infty dx\, e^{-xa} \int_0^\infty dy\, e^{-yb} = \int_0^\infty\!\!\int_0^\infty dx\,dy\, e^{-(xa+yb)}$$

Now change variables

$$xa+yb = \underbrace{(x+y)}_{r}\left(\underbrace{\frac{x}{(x+y)}}_{1-t}a + \underbrace{\frac{y}{(x+y)}}_{t}b\right)$$

so that
$$x = (1-t)r \qquad\Longrightarrow\qquad dx = (1-t)dr - r\,dt$$
$$y = tr \qquad\qquad\qquad dy = r\,dt + t\,dr$$

$$dx\,dy = ((1-t)r + rt)\,dr\,dt = r\,dr\,dt$$

Then
$$\frac{1}{ab} = \int_0^1 dt \int_0^\infty r\,dr\, e^{-r((1-t)a+tb)} = \int_0^1 dt\,\frac{1}{((1-t)a+tb)^2}$$

It appears to me that Gaussian methods are more powerful possibly than the existing Bott methods. So let me do as much as possible from the Gaussian side.

Recall that we had

$$\alpha' = \frac{1}{2}\left(z^* \frac{1}{\omega} dz - dz^* \frac{1}{\omega} z \right)$$

$$d_\omega \alpha' = |z|^2 + dz^* \frac{1}{\omega} dz$$

and that the Thom form is

$$U = \det(\omega)\, e^{-d_\omega \alpha'} = \det(\omega)\, e^{-|z|^2 - dz^* \frac{1}{\omega} dz}$$

Integration over the fibre $\mathbb{C}^n$ gives

$$\det(\omega) \int_{\mathbb{C}^n} e^{-|z|^2} (-1)^n \frac{\left(dz^* \frac{1}{\omega} dz\right)^n}{n!}$$

To evaluate this I can suppose ω diagonal whence

$$\frac{\left(dz^* \frac{1}{\omega} dz\right)^n}{n!} = \frac{\left(\sum \frac{d\bar{z}^j dz^j}{\omega_j} \right)^n}{n!} = \prod_{j=1}^{n} \frac{d\bar{z}^j dz^j}{\omega_j}$$

Need

$$\int_{\mathbb{C}} e^{-|z|^2} \underbrace{\overset{(dx - i dy)(dx + i dy)}{d\bar{z}\, dz}}_{2i\, dx\, dy} = 2i\pi .$$

Thus

$$\boxed{\int_{\mathbb{C}^n} \det(\omega)\, e^{-|z|^2 - dz^* \frac{1}{\omega} dz} = \left(\frac{2\pi}{i}\right)^n}$$

Now I want to see what happens with the transgression form. Recall the formulas — denote the pull-back under $t, z \mapsto tz$ by $\sim$. Then

$$\tilde{\alpha}' = t^2 \alpha' \qquad \widetilde{d_\omega \alpha'} = t^2 d_\omega \alpha' + 2t\,dt\, \alpha'$$

So what we have is

$$\partial_t \Big\{ \underbrace{\det(\omega)\, e^{-t^2 d_\omega \alpha'}}_{U_t} \Big\} = d_\omega \Big\{ \det(\omega)\, e^{-t^2 d_\omega \alpha'} (-2t)\alpha' \Big\}$$

$$U_t - \det(\omega) = d_\omega \Big\{ \det(\omega) \int_0^t (-2t)\,dt \;\; e^{-t^2 d_\omega \alpha'} \alpha' \Big\}$$

$$= d_\omega \Big\{ \det(\omega) \; \frac{e^{-t^2 d_\omega \alpha'} - 1}{d_\omega \alpha'} \; \alpha' \Big\}$$

What I want to do is to evaluate the integral of $\det(\omega) \dfrac{\alpha'}{d_\omega \alpha'}$ over S^{2n-1} using what I know about $\int_{\mathbb{C}^n} U_t = \left(\dfrac{2\pi}{i}\right)^n$. By Stokes's formula

$$\int_{|z| \leq R} d_\omega \big(\det(\omega) \cdots\cdots \big) = \int_{|z| = R} \det(\omega) \; \frac{e^{-t^2 d_\omega \alpha'} - 1}{d_\omega \alpha'} \, \alpha'$$

so letting $R \longrightarrow \infty$ we get

$$\int U_t = -\int_{S^{2n-1}} \det(\omega) \frac{\alpha'}{d_\omega \alpha'}$$

since I know the form $\det(\omega) \dfrac{\alpha'}{d_\omega \alpha'}$ is basic for the $\mathbb{R}_{>0}$-action.

Next I want to consider

$$\alpha = \tfrac{1}{2}\left(z^* dz - dz^* z \right)$$

$$d_\omega \alpha = z^* \omega z + dz^* dz$$

Let's check $\dfrac{\alpha}{d_\omega \alpha}$ is basic for the $\mathbb{R}_{>0}$-action.

$$X = z^j \frac{\partial}{\partial z^j} + \bar{z}^j \frac{\partial}{\partial \bar{z}^j} \;, \quad i_X \alpha = \tfrac{1}{2}(z^* z - z^* z) = 0$$

$$\iota_X\, d_\omega \alpha = z^* dz - dz^* \cancel{z} = 2\alpha.$$

Thus

$$\iota_X\left(\frac{\alpha}{d_\omega \alpha}\right) = -\frac{1}{d_\omega\alpha}\underbrace{\iota_X(d_\omega\alpha)}_{2\alpha}\frac{1}{d_\omega\alpha}\,\alpha = 0$$

as $\alpha^2 = 0$. Thus $\dfrac{\alpha}{d_\omega \alpha}$ is basic so

$$\int_{S^{2n-1}} \frac{\alpha}{d_\omega \alpha}$$

makes sense, ~~and~~ NO see below in fact we can replace S^{2n-1} by any homologous cycle in $V - 0$.

I forgot to require ω to be invertible, otherwise I can divide by $d_\omega \alpha$.

Claim that

$$\int_{S^{2n-1}} \frac{\alpha}{d_\omega \alpha} = \frac{1}{\det(\omega)}\left(-\left(\frac{2\pi}{i}\right)^n\right)$$

Note: There is a problem with this for

$$d_\omega \alpha = z^*\omega z + dz^* dz$$

won't be invertible even if ω is invertible. The point is that unless the eigenvalues of ω are all in $i\mathbb{R}$ or all in $-i\mathbb{R}$, there will be a "light cone" where $z^*\omega z = 0$.

I seem to be running into the same phenomenon as when I tried to derive a fundamental solution of the wave operator $\partial_t^2 - \partial_x^2$ by Gaussian methods. Recall the derivation in the elliptic case:

$$-\langle x|\Delta^{-1}|0\rangle = \int \frac{d^n\xi}{(2\pi)^n} e^{i\xi x}\frac{1}{\xi^2} = \int \frac{d^n\xi}{(2\pi)^n} e^{i\xi x}\int_0^\infty e^{-t\xi^2}dt$$

$$= \int_0^\infty dt\, \frac{e^{-\frac{x^2}{4t}}}{(4\pi t)^{n/2}} = \int_0^\infty \frac{dt}{t}\, t^{1-\frac{n}{2}} \frac{e^{-\frac{x^2}{4t}}}{(4\pi)^{n/2}} = \int_0^\infty \frac{dt}{t}\, t^{\frac{n}{2}-1}\frac{e^{-\frac{x^2}{4}t}}{(4\pi)^{n/2}}$$

$$= \frac{\Gamma\left(\frac{n}{2}-1\right)}{(4\pi)^{n/2}} \frac{1}{\left(\frac{x^2}{4}\right)^{\frac{n}{2}-1}} = \frac{\Gamma\left(\frac{n}{2}-1\right)}{4\,\pi^{n/2}} \frac{1}{|x|^{n-2}}$$

So if $n=3$, we get $\frac{\sqrt{\pi}}{4\pi^{3/2}}\frac{1}{|x|} = \frac{1}{4\pi|x|}$ which checks.

November 9, 1984

Yesterday I learned that there are singular transgression forms in some sense. This I find intriguing in connection with the fundamental solution for the wave operator $\partial_t - \Delta$. Why? Somehow because one is using the differential forms things are [redacted] very regular - one always has

$$\left(\frac{i}{2\pi}\right)^n \int_{S^{2n-1}} \frac{\frac{1}{2}(z^*dz - dz^*z)}{z^*\omega z + dz^*dz} = -[redacted]\ \frac{1}{\det(\omega)}$$

just as long as ω is invertible, but independent of ω being definite, so that $z^*\omega z$ is invertible as a function on S^{2n-1}.

Notice that we have

$$z^*\omega z = \underbrace{z^* \frac{\omega+\omega^*}{2} z}_{Re(z^*\omega z)} + i\,\underbrace{z^* \frac{\omega-\omega^*}{2i} z}_{Im(z^*\omega z)}$$

Thus if I start with a skew-adjoint ω and add a small [redacted] positive-definite or negative perturbation, say $\varepsilon I + \omega$, then

$$z^*(\varepsilon I + \omega) z = \varepsilon |z|^2 + z^*\omega z$$

doesn't vanish on S^{2n-1} for $\varepsilon \in \mathbb{R}$, $\varepsilon \neq 0$.

So I guess the way to proceed [redacted] is as follows. One considers

$$\left(\frac{i}{2\pi}\right)^n \int_{S^{2n-1}} \frac{\frac{1}{2}(z^*dz - dz^*z)}{z^*\omega z + dz^*dz}$$

over the open set of invertible $n \times n$ complex matrices such that $z^*\omega z$ doesn't vanish. Unfortunately I don't know this open set is either dense or connected, however, the above integral is clearly holomorphic on this open set. Also this open set contains those ω with positive-definite real part $\frac{\omega+\omega^*}{2}$ which is connected. Since we know that the above integral equals $-\frac{1}{\det(\omega)}$ for ω positive-definite, it's clear that by a limiting process we can define the integral for ω skew-adjoint invertible and it gives $-\frac{1}{\det(\omega)}$. We get the same answer if we use ω with negative-definite real part.

Go back to the problem of the relation in PE. I recall this says that the n-th Chern class of $\mathcal{O}(1)\otimes\pi^*E$ is zero:

$$c_n(\mathcal{O}(1)\otimes\pi^*E) = \sum_{k=0}^{n} \xi^k c_k(E) = 0 \qquad \xi = c_1(\mathcal{O}(1)).$$

Also it results from that fact that $\mathcal{O}(1)\otimes\pi^*E$ has a non-vanishing section:

$$0 \longrightarrow \mathcal{O}(-1) \longrightarrow \pi^*E \longrightarrow Q \longrightarrow 0$$

November 10, 1984

Program: To prove the relation

$$c_n(\mathcal{O}(1)\otimes\pi^*E) = \sum_{k=0}^{n} c_1(\mathcal{O}(1))^{n-k}\pi^* c_k(E) = 0$$

in $H^*(PE)$ using explicitly the differential forms.

The idea will be to use the fact that $\mathcal{O}(1)\otimes\pi^*E$ has a canonical non-vanishing section. Hence one will have a section of the associated sphere bundle, and up there we have the nth Chern class written as d of the transgression form.

A possible point to this exercise is to better understand the diagonal map $V \to V\times V$ which is what gives the section of $\mathcal{O}(1)\otimes\pi^*E$.

We suppose E given a connection D which preserves an inner product, and we work with a local trivialization of the principal bundle P of E for $G = U(n)$. Thus $E = M\times V$ with $D = d+A$. I should also keep in mind the Weil algebra approach

$$\begin{tikzcd} & \Omega(P) \arrow[dl] & W(\mathfrak{g}) \arrow[l] \\ \Omega(M) & \Omega(M) \arrow[u, hook] & S(\mathfrak{g}^*)^G \arrow[l] \arrow[u, hook] \end{tikzcd}$$

etc.

Now I want to discuss $\mathcal{O}(1)\otimes\bar{\pi}^*E$ over PE. Think of PE as SE/S^1; then $\mathcal{O}(1)\otimes\bar{\pi}^*E$ is the bundle induced by the equivariant bundle $\pi^*E = SE\times_M E$ with S^1 acting diagonally. Here $\pi: SE\to M$, $\bar{\pi}: PE\to M$. We need to get a connection in $\mathcal{O}(1)\otimes\bar{\pi}^*E$ starting from our connection D in E.

D induces a connection in π^*E obviously invariant under S^1. What is the inclination? It's the same as for the S^1 action on E/M, hence is

$$\varphi_u = u\cdot Id_E \qquad u\in Lie(S^1) = i\mathbb{R}.$$

Then to get a connection $\bar D$ which descends to $\mathbb{P}E$ we need a connection form $\theta \in \Omega^1(SE)$ for the S^1-action, whence we put

$$\bar D = D + \theta\varphi = D + \theta\cdot Id_E .$$

How do we construct the connection form θ? We start with a G-invariant connection for
$$\begin{array}{c} SV \\ \downarrow S^1 \\ \mathbb{P}V \end{array}$$
and then modify it by the connection in P to get a connection in
$$\begin{array}{c} P\times^G SV \\ \downarrow \\ P\times^G \mathbb{P}V \end{array} = \begin{array}{c} SE \\ \downarrow \\ \mathbb{P}E \end{array}.$$
So what this means is we make the invariant connection in $SV/\mathbb{P}V$ into an equivariant one, whence it can be applied universally given any principal G-bundle.

So we start with the circle action on SV:

$$X_u = -u\left(z^j\frac{\partial}{\partial z^j} - \bar z^j\frac{\partial}{\partial \bar z^j}\right)$$

and the 1-form $\alpha = \frac{1}{2}(z^*dz - dz^*z)$ which is G-invariant and which satisfies

$$\iota_{X_u}(\alpha) = -u|z|^2 = -u \quad \text{on } SV.$$

Thus the invariant connection we want is

$$\theta = -\alpha$$

since $\iota_{X_u}\theta = u$.

Next in order to mix α over P we really want to lift α relative to the isom.

$$[W(\mathfrak{g}) \otimes \Omega(SV)]_{basic} \xrightarrow{\sim} [S(\mathfrak{g}^*) \otimes \Omega(SV)]^G$$

and this means taking α in to

$$\prod_a (1 - \theta^a \iota_a)\alpha = \alpha - \theta^a \iota_a \alpha$$

Now if $\omega \in \mathfrak{g}$, then

$$\iota_\omega \alpha = \iota_\omega \tfrac{1}{2}(z^* dz - dz^* z) = \tfrac{1}{2}[z^*(-\omega z) - (-\omega z)^* z]$$
$$= - z^* \omega z .$$

So $-\theta^a \iota_a \alpha$ is just the element of $W^1(\mathfrak{g}) \otimes \Omega^0(SV)$ given by the matrix function

$$-\theta^a \iota_a \alpha = z^* \theta z .$$

Hence when applied to P, we get from α

$$\alpha + z^* A z \in \Omega^1(P \times SV).$$

So we conclude that an S^1 connection form in $SE/\mathbb{P}E$ is given by the 1-form

$$\boxed{\theta = -(\alpha + z^* A z) = -(\tfrac{1}{2}(z^* dz - dz^* z) + z^* A z)}$$

Then the curvature is the negative of

$$d\left\{\tfrac{1}{2}(z^* dz - dz^* z) + z^* A z\right\} = dz^* dz + dz^* A z - z^* A dz + z^* dA z$$
$$= (dz^* - z^* A)(dz + Az) + z^* A^2 z + z^* dA z$$
$$= \boxed{z^*(dA + A^2) z + (dz + Az)^*(dz + Az)}$$

represents $c_1 \mathcal{O}(-1)$ on $\mathbb{P}E$

Let's recapitulate:

We are constructing a connection in $\mathcal{O}(1)\otimes\pi^*(E)$ starting from a unitary connection in E. Then we will use the transgression form on $\mathcal{O}(1)\otimes\pi^*(E)$ − zero section together with the canonical non-vanishing section to write the form representing $c_n(\mathcal{O}(1)\otimes\pi^*E)$ as d of something.

~~So far I have constructed $c_1\,\mathcal{O}(1)$ as a form on RE using the fact that $\mathcal{O}(1)$ is obtained from the identity character of S^1, i.e. one has S^1 acting on $SE\times\mathbb{C}$, not clear.~~

I have been thinking of $\mathcal{O}(1)\otimes\pi^*(E)$ as obtained from the S^1-equivariant bundle $\pi^*E = SE\times_M E$ with S^1 acting diagonally. It has the invariant connection D and the inclination $\varphi_u = u\, id_E$, so if $\theta\in\Omega^1(SE)$ is an S^1-connection then

$$\bar{D} = D + \theta\, id_E$$

descends, and the curvature is

$$\bar{D}^2 = D^2 + d\theta\, id_E = \text{curv. of } \pi^*E + \text{curv. of } \mathcal{O}(1).$$

So I have to calculate θ and $d\theta$.

The simplest way to find θ is to use equivariant forms to reduce E to V with the G-action. Thus the connection form and curvature in $[S(\mathfrak{g}^*)\otimes\Omega(SV)]^G$ are

$$\theta = \alpha \qquad \alpha = \tfrac{1}{2}(z^*dz - dz^*z)$$

$$d_\omega\theta = -d_\omega\alpha \qquad +d_\omega\alpha = z^*\omega z + dz^*dz$$

When brought to SE we obtain

$$\boxed{\begin{aligned} \alpha &= \tfrac{1}{2}\big(z^*(dz+Az) - (dz+Az)^*z\big) = \tfrac{1}{2}(z^*dz - dz^*z) + z^*Az \\ d_\omega\alpha &= (dz+Az)^*(dz+Az) + z^*Fz \end{aligned}}$$

So the form representing $c_n(\mathcal{O}(1)\otimes\pi^*E)$ is

$$\det\left(F - z^*Fz - (dz+Az)^*(dz+Az)\right).$$

This is what we want to write as d of something.

The next step will be to bring in the sphere bundle, or complement of the zero section, of $\mathcal{O}(1)\otimes\pi^*E$. At this point I am thinking of $\mathbb{P}E$, or SE, as the bases.

Let's try treating this equivariantly. I consider then $\pi^*V = SV\times V$ over SV with the connection

$$\bar{D} = d_\omega + \theta = d_\omega - \alpha. \text{ on } \left[S(\mathfrak{g}^*)\otimes\Omega(SV)\otimes V\right]^G$$

It's the trivial bundle with fiber V with connection modified so that it can be descended to $\mathbb{P}V$. The curvature is

$$\bar{D}^2 = \omega - d_\omega\alpha \quad \in \left[S(\mathfrak{g}^*)\otimes\Omega(SV)\otimes \operatorname{End}V\right]^G$$

Now we want to consider the sphere bundle

$$S(\pi^*V) = SV\times SV$$

which introduces a new set of coordinates ξ^α. The transgression form on this sphere bundle has to be computed relative to the trivialization, the connection form $-\alpha$, the curvature $\omega - d_\omega\alpha$.

So let's consider this problem directly. I already established the equivariant form

$$\det(\omega)\frac{\alpha'}{d_\omega\alpha'} \qquad \alpha' = \tfrac{1}{2}\left(z^*\tfrac{1}{\omega}dz - dz^*\tfrac{1}{\omega}z\right)$$

which lives in $[S(\mathfrak{g}^*)\otimes\Omega(SV)]^G$. So we have to trace this thru

$$\Omega(SE) = \Omega(P\times^G SV) \longleftarrow [W(\mathfrak{g})\otimes\Omega(SV)]_{basic} \xrightarrow{\sim} [S(\mathfrak{g}^*)\otimes\Omega(SV)]^G$$

So α' lifts to $\alpha' - \theta^a\iota_a\alpha'$ and

$$-\iota_\xi\alpha' = \frac{1}{2}\left(z^*\frac{1}{\omega}\xi z - (\xi z)^*\frac{1}{\omega}z\right)$$

$$= \frac{1}{2}\; z^*\left(\frac{1}{\omega}\xi + \xi\frac{1}{\omega}\right)z$$

So

$$\alpha' - \theta^a\iota_a\alpha' = \frac{1}{2}\left[z^*\frac{1}{F}(dz+Az) - (dz+Az)^*\frac{1}{F}z\right]$$

and $d_\omega\alpha' = |z|^2 + dz^*\frac{1}{\omega}dz$ lifts to

$$|z|^2 + (dz+Az)^*\frac{1}{F}(dz+Az)$$

General formula for the transgression form is therefore

$$\boxed{\det(F)\;\frac{\frac{1}{2}\left[z^*\frac{1}{F}(dz+Az) - (dz+Az)^*\frac{1}{F}z\right]}{|z|^2+(dz+Az)^*\frac{1}{F}(dz+Az)}}$$

So now what I ought to do is to plug in A, F which are $A = -\alpha$, $F = \omega - d_\omega\alpha$ and get

$$\det(\omega-d_\omega\alpha)\;\frac{\frac{1}{2}\left[\zeta^*\frac{1}{\omega-d_\omega\alpha}(d\zeta-\alpha\zeta) - (d\zeta-\alpha\zeta)^*\frac{1}{\omega-d_\omega\alpha}\zeta\right]}{|\zeta|^2+(d\zeta-\alpha\zeta)^*\frac{1}{\omega-d_\omega\alpha}(d\zeta-\alpha\zeta)}$$

which is too messy to fathom.

I can put $\zeta = z$ in this - this amounts to pulling back via $\Delta: SV \longrightarrow SV\times SV$

New approach: I want to start with the Thom class on E/M in the equivariant cohomology for the circle action. This is the fundamental geometric gadget. If I forget the fact it has compact support I get the equivariant Euler class on M whose restriction to SE will give the class $c_n(\mathcal{O}(1)\otimes\pi^*E)$ on PE. (I am using that the equivariant c_n of π^*E on SE can be identified with $c_n(\mathcal{O}(1)\otimes\pi^*E)$ on PE.) The actual deformation of the equivariant Euler class to the equivariant Thom class is [redacted] essentially the transgression form. And since the Thom form pulls back to zero, over SE we get the relation on the projective bundle.

So it seems that I want to do the following: Let [redacted] me take the Thom form [redacted]

$$U \in \left[S(i\mathbb{R}^*)\otimes\Omega(E)\right]^{S^1}$$

and then consider the 1-parameter family of maps

$$\begin{aligned} \mathbb{R}\times SE &\longrightarrow E \\ t\;,\; z &\longmapsto tz \end{aligned}$$

and pull-back obtaining the usual [redacted]

$$U_t + dt\, V_t \qquad\qquad \partial_t U_t = d V_t$$

Then $U_0 =$ Euler class, $U_\infty = 0$ so $\int_0^\infty dt\, V_t$ should give the desired relation. But this amounts to restricting the equivariant transgression form to SE.

Seems to lead to the same mess as on the previous page.

November 11, 1984

The program is to understand the physicists integration process for differential forms on the loop space $\mathcal{L}M$. This is denoted

$$\int \mathcal{D}x\, \mathcal{D}\psi \;\; e^{-S} \,(\cdots\cdots) \qquad S = \int \tfrac{1}{4}(\dot{x}^2 + \psi\dot{\psi})\,dt$$

(the flat case)

and when it is applied to the Bismut form

$$T\left\{ e^{\int [-\dot{x}^\mu A_\mu(x) + \frac{1}{2}\psi^\mu \psi^\nu F_{\mu\nu}(x)]\,dt} \right\}$$

yields the index of the Dirac operator. In the case of a line bundle things separate somewhat, and the ψ-integral is Gaussian with exponent

$$\int \left[\tfrac{1}{4}\psi\dot{\psi} - \tfrac{1}{2}\psi^\mu\psi^\nu F_{\mu\nu}\right] dt .$$

This will give the Pfaffian of the operator $\frac{d}{dt} - 2F(x_t)$.

What I propose to do now is to take the case of a constant curvature connection on a line bundle on a torus $\mathbb{R}^n/\Gamma$. In this case the integral

$$\int \mathcal{D}x\,\mathcal{D}\psi \;\; e^{-\int [\frac{1}{4}(\dot{x}^2 + \psi\dot{\psi}) + \dot{x}^\mu A_\mu - \frac{1}{2}\psi^\mu\psi^\nu F_{\mu\nu}]\,dt}$$

should split into the product of two Gaussian integrals which I should be able to evaluate. The fermion integral is

$$\int \mathcal{D}\psi \;\; e^{-\int [\frac{1}{4}\psi\dot{\psi} - \frac{1}{2}\psi^\mu\psi^\nu F_{\mu\nu}]\,dt}$$

and the boson integral is

$$\int \mathcal{D}x \;\; e^{-\int [\frac{1}{4}\dot{x}^2 + \dot{x}^\mu A_\mu(x)]\,dt}$$

and this will break up as a sum over lattice points, i.e. homotopy classes of paths. Presumably according to Bismut only the trivial homotopy class contributes.

Let us next look at the operators corresponding to these integrals. One has

$$\not{D}^2 = D_\mu^2 + \tfrac{1}{2}\gamma^\mu\gamma^\nu F_{\mu\nu}$$

acting on sections of spinors $\otimes L$ over $M = \mathbb{R}^4/\Gamma$. Now the spinor bundle over M is the trivial bundle with fibre the spinor module S, so we can write

$$\not{D}^2 = 1 \otimes D_\mu^2 + \tfrac{1}{2}\gamma^\mu\gamma^\nu \otimes F_{\mu\nu}$$

where D_μ, $F_{\mu\nu}$ act on $\Gamma(M, L)$. Since $F_{\mu\nu}$ is constant it commutes with D_μ^2. Actually I can write

$$\not{D}^2 = 1 \otimes D_\mu^2 + \tfrac{1}{2}\gamma^\mu\gamma^\nu F_{\mu\nu} \otimes 1 \qquad \text{on } S \otimes \Gamma(M,L)$$

so

$$e^{t\not{D}^2} = e^{t\frac{1}{2}\gamma\gamma F} \otimes e^{tD_\mu^2}$$

and so

$$\boxed{\operatorname{tr}_s e^{t\not{D}^2} = \operatorname{tr}_s\left(e^{t\frac{1}{2}\gamma\gamma F}\right) \cdot \operatorname{tr}\left(e^{tD_\mu^2}\right)}$$

which is the precise version of the splitting of the functional integral into fermionic and bosonic parts.

Let's recall now how to evaluate

$$\operatorname{tr}_s\left(e^{\frac{1}{4}\omega_{ij}\gamma^i\gamma^j}\right)$$

which is the difference of the characters of ~~the~~ the Spin representations applied to $\exp(\omega) \in Spin(n)$. Do for $n=2$

$$e^{\frac{1}{4}(\omega_{12}\gamma^1\gamma^2 + \omega_{21}\gamma^2\gamma^1)} = e^{\frac{1}{2}\omega_{12}\gamma^1\gamma^2} = e^{\frac{i}{2}\omega_{12}\varepsilon}$$

so
$$\mathrm{tr}_s\left(e^{\frac{1}{4}\gamma^t\omega\gamma}\right) = \frac{e^{\frac{i}{2}\omega_{12}} - e^{-\frac{i}{2}\omega_{12}}}{2i\,\frac{\omega_{12}}{2}} \cdot i\omega_{12}$$

$$\boxed{\mathrm{tr}_s\left(e^{\frac{1}{4}\gamma^t\omega\gamma}\right) = \det\left(\frac{\sinh \omega/2}{\omega/2}\right)^{1/2} i^{n/2}\,\mathrm{Pf}(\omega)}$$

Thus
$$\mathrm{tr}_s\left(e^{t\frac{1}{2}\gamma\gamma F}\right) = \det\left(\frac{\sinh tF}{tF}\right)^{1/2} i^{n/2}(2t)^{n/2}\,\mathrm{Pf}(F)$$

Next we need to determine the partition function $\mathrm{tr}_s\left(e^{tD_\mu^2}\right)$. Take $n=2$ with $F = i\omega\, dx dy$, $\omega < 0$.

$$A = \frac{i\omega}{2}(x dy - y dx)$$

$$D_x = \partial_x - \frac{i\omega}{2}y \ ; \quad D_y = \partial_y + \frac{i\omega}{2}x$$

$$[D_x, D_y] = i\omega$$

To find the spectrum look for creation and annihilation operators
$$[D_x + iD_y, -D_x + iD_y] = +i\,2i\omega = -2\omega$$

so put
$$a = \frac{1}{\sqrt{-2\omega}}(D_x + iD_y), \quad a^* = \frac{1}{\sqrt{-2\omega}}(-D_x + iD_y)$$

and
$$(-2\omega)\,a^*a = (-D_x + iD_y)(D_x + iD_y)$$
$$= -(D_x^2 + D_y^2)\underbrace{-i(i\omega)}_{+\omega}$$

or
$$-\left(D_x^2 + D_y\right)^2 = (-2\omega)\left(a^*a + \tfrac{1}{2}\right)$$

so the only question is the multiplicity of the ground state. Look at kernel of a.

$$(D_x + iD_y)\psi = \left(\partial_x + i\partial_y - \frac{i\omega}{2}y - \frac{\omega}{2}x\right)\psi$$

$$= \left(2\partial_{\bar z} - \frac{\omega}{2}z\right)\psi = 0$$

$$\Longrightarrow \quad \psi = e^{-\frac{(-\omega)}{4}|z|^2}\left(\text{holom. fn.}\right)$$

The holomorphic function will then have to satisfy some kind of periodicity conditions relative to the lattice. Here is where the curvature $i\omega\, dx\, dy$ will have to be integral, and of course the index will be given by the [redacted] R-R thm. on the elliptic curve. $\therefore$ [redacted] Multiplicity of the ground state is index. [redacted] So

$$tr\left(e^{tD_\mu^2}\right) = \text{index} \cdot \sum_{n\geq 0} e^{-t(n+\frac{1}{2})(-2\omega)}$$

$$= \text{index} \cdot \frac{1}{e^{-t\omega} - e^{t\omega}}$$

So now let's correlate these two approaches.

$$F = \begin{pmatrix} 0 & i\omega \\ -i\omega & 0 \end{pmatrix} \sim \begin{pmatrix} \omega & \\ & -\omega \end{pmatrix} \qquad \text{so that}$$

$$\det\left(\frac{\sinh tF}{tF}\right)^{1/2} = \frac{\sinh t\omega}{t\omega} = \frac{e^{t\omega} - e^{-t\omega}}{2t\omega}$$

So $$tr_s\left(e^{\frac{1}{4}\gamma\omega\gamma}\right) = \frac{e^{t\omega} - e^{-t\omega}}{2t\omega} \cdot i\underbrace{\left(\tfrac{2}{i}t\omega\right)}_{Pf(tF)} = \frac{\left(e^{t\omega} - e^{-t\omega}\right)(-1)}{\text{[redacted]}}$$

so it checks out.

I propose now to calculate $tr(e^{tD_\mu^2})$ over the torus using the explicit construction I have given for the kernel of this operator on Euclidean space. This will give, maybe some kind of strange version of the Poisson summation formula used for the fuel equation of Θ functions.

Recall that we think of the D_μ as generating a Weyl algebra with the relations

$$[D_\mu, D_\nu] = F_{\mu\nu}$$

$$e^{uD} e^{vD} = e^{\frac{1}{2}uFv} e^{(u+v)D}$$

so that if we form the convolution algebra of

$$\int dv\, f(v) e^{vD}$$

we have the convolution

$$(f * g)(v) = \int dw\, f(v-w) g(w)\, e^{\frac{1}{2}vFw}$$

Thus $$(\delta_u * g)(v) = \int dw\, \delta(v-w-u)\, g(w)\, e^{\frac{1}{2}vFw}$$

$$= g(v-u)\, e^{\frac{1}{2}vF(v-u)} = g(v-u) e^{\frac{1}{2}uFv}$$

(On the other hand if we took the representation

★ $$D_\mu = \partial_\mu - \frac{1}{2} F_{\mu\nu} x^\nu$$

then $$e^{vD} = e^{v\partial - \frac{1}{2}vFx}$$

$$= e^{\cancel{\frac{1}{2}[v\partial, -\frac{1}{2}vFx]}} e^{-\frac{1}{2}vFx} e^{v\partial} \qquad -\tfrac{1}{4}vFv = 0$$

so $$\left(e^{u(\partial - \frac{1}{2}Fx)} g\right)(x) = e^{-\frac{1}{2}uFx} g(x+u).$$

The point is that we can't identify $\delta_u * g$ with $e^{uD} g$

in the representation A.)

We know from our earlier work that

$$e^{tD_\mu^2} = \int d\sigma^n \frac{1}{(4\pi t)^{n/2}} \det\left(\frac{tF}{\sinh tF}\right)^{1/2} e^{-\frac{1}{4}\sigma \frac{\cosh tF}{\sinh tF}\sigma} e^{\sigma D}$$

for any representation of the Weyl algebra. Now we want to take the representation on sections of a line bundle L over $\mathbb{R}^n/\Gamma$ with constant curvature $\frac{1}{2}F_{\mu\nu}dx^\mu dx^\nu$, and then take the trace.

It should be enough to know the distribution $\operatorname{tr}(e^{\sigma D})$, $\sigma \in \mathbb{R}^n$. This is supported on Γ, because $e^{\sigma D}$ is parallel translation in the direction σ. It's reasonable to expect $\operatorname{tr}(e^{\sigma D})$ to be a sum of [redacted] δ-functions located at the points of Γ. The thing we need to know is the actual [redacted] effect of $e^{\sigma D}$ on the line bundle L. What probably happens is that for $\sigma \in \Gamma$, $e^{\sigma D}$ is an endomorphism of L, essentially the character associated by F to σ, and then one has to integrate this over the torus. This should kill the contributions for $\sigma \in \Gamma$, $\sigma \neq 0$, in accordance with Bismut's theorem.

November 12, 1984

Let's return to the problem of constructing heat operators. My idea is to describe a class of kernels $K(t,x,x')$ in terms of the blowup of $\mathbb{R}\times M\times M$ along $0\times \Delta M$. A central problem is to prove these kernels can be composed.

I propose now to just look at the translation invariant case: $K(t,x-x')$, where $M=\mathbb{R}^n$. Then composition is given by convolution:

$$(K*K')(t,x) = \int_0^t dt_1 \int dx_1 \quad K(t-t_1, x-x_1)K'(t_1,x_1)$$

The class of $K(t,x)$ I consider are smooth functions on $\mathbb{R}\times M$ outside $(0,0)$ which are zero for $t<0$. Further, if we set $h^2=t$, so that K becomes a smooth fn. on h-line $\times M$ defined for $(h,x)\neq(0,0)$, then we want K to extend smooth over the blowup of $\mathbb{R}\times M$ at $(0,0)$. This means that if we make the change of variables

$$\begin{cases} t = h^2 \\ x = hy \end{cases}$$

then $K(h^2,hy)$ should be a smooth function on $\widetilde{\mathbb{R}\times M}$ which vanishes to infinite order along $\tilde{M}$.

The simplest examples should then be of the form

$$K(t,x) = f\left(\frac{x}{\sqrt{t}}\right) = f(y)$$

where $f\in \mathcal{S}(\mathbb{R}^n)$. Our first problem is to see how the convolution of such kernels looks.

Let's first get the volume factors straight. The typical kernel of interest is a Gaussian

$$K(t,x) = \int \frac{d^n\xi}{(2\pi)^n}\, e^{i\xi x}\, e^{-t\frac{1}{2}\xi^t A \xi}$$

$$= e^{-\frac{1}{2t} x^t A^{-1} x}\, \frac{1}{(2\pi t)^{n/2}}\, \frac{1}{(\det A)^{1/2}}$$

$$= t^{-n/2} f(y) \qquad f(y) = \int \frac{d^n\xi}{(2\pi)^n}\, e^{i\xi y}\, e^{-\frac{1}{2}\xi^t A \xi}$$

So let's now consider

$$K(t,x) = t^{-n/2} f\left(\frac{x}{\sqrt{t}}\right) \qquad K'(t,x) = t^{-n/2} g\left(\frac{x}{\sqrt{t}}\right)$$

where $f, g \in \mathcal{S}(\mathbb{R}^n)$, and look at

$$(K*K')(t,x) = \int_0^t dt_1 \int dx_1\, (t-t_1)^{-\frac{n}{2}} f\left(\frac{x-x_1}{\sqrt{t-t_1}}\right) t_1^{-\frac{n}{2}} g\left(\frac{x_1}{\sqrt{t_1}}\right)$$

To analyze this we use the F.T.

$$f(y) = \int \frac{d^n\xi}{(2\pi)^n}\, e^{i\xi y}\, \hat{f}(\xi)$$

$$t^{-n/2} f\left(\frac{x}{\sqrt{t}}\right) = \int \frac{d^n\xi}{(2\pi)^n}\, t^{-n/2}\, e^{i\xi\frac{x}{\sqrt{t}}}\, \hat{f}(\xi)$$

$$= \int \frac{d^n\xi}{(2\pi)^n}\, e^{i\xi x}\, \hat{f}(\sqrt{t}\,\xi)$$

$$\therefore\ (K*K')(t,x) = \int_0^t dt_1 \int d^n x_1 \int \frac{d^n\xi}{(2\pi)^n}\, e^{i\xi(x-x_1)}\, \hat{f}(\sqrt{t-t_1}\,\xi) \int \frac{d^n\eta}{(2\pi)^n}\, e^{i\eta x_1}\, \hat{g}(\sqrt{t_1}\,\eta)$$

$$\int d^n x_1\, e^{i(-\xi+\eta)x_1} = (2\pi)^n\, \delta(-\xi+\eta)$$

$$(K*K')(t,x) = \int_0^t dt_1 \int \frac{d^n\xi}{(2\pi)^n\, t^{n/2}}\, e^{i\xi\frac{x}{\sqrt{t}}}\, \hat f\left(\sqrt{t-t_1}\,\tfrac{\xi}{\sqrt{t}}\right) \hat g\left(\sqrt{t_1}\,\tfrac{\xi}{\sqrt{t}}\right)$$

$$= t^{-n/2} \int \frac{d^n\xi}{(2\pi)^n}\, e^{i\xi\frac{x}{\sqrt{t}}} \underbrace{\int_0^t dt_1\; \hat f\left(\sqrt{1-\tfrac{t_1}{t}}\;\xi\right) \hat g\left(\sqrt{\tfrac{t_1}{t}}\;\xi\right)}$$

$$t\int_0^1 du\; \hat f\left(\sqrt{1-u}\;\xi\right)\hat g\left(\sqrt{u}\;\xi\right) = t\int_0^{\pi/2} d(\sin^2\theta)\; \hat f\big((\cos\theta)\xi\big)\,\hat g\big((\sin\theta)\xi\big)$$

So what one has done is to take $\hat f(\xi)\,\hat g(\eta)$ which is a Schwartz function on $\mathbb{R}^n \times \mathbb{R}^n$, then restrict to the graph of angle θ and average.

What does this mean geometrically? It means we do something relative to the map

$$\sqrt{t-t_1}\,,\ \sqrt{t_1} \longmapsto \sqrt{t}$$

$$h_1\,,\ h_2 \longmapsto \sqrt{h_1^2+h_2^2} \qquad ?$$

Here seems to be an interesting point because the map $h_1, h_2 \longrightarrow \sqrt{h_1^2+h_2^2}$ has a singularity at $(0,0)$. Does the function extend smoothly over the blowup? No, because on a line through O we have the function $|x|$. There is an un-ramified double covering of the blown up (h_1,h_2)-plane on which the function extends smoothly. To see this consider the pairs $(u,t) \in S^1 \times \mathbb{R}$ and map to the plane by $(u,t) \longmapsto tu$. Then a given $v \in \mathbb{R}^2$ can be represented as tu with $|u|=1$ in two ways if $v \neq 0$.

Let's set this up carefully and in general starting from a vector space V. The blowup $\tilde V$ consists of pairs (ℓ, v) with $v \in \ell \in PV$. It is covered by the space of pairs (u,v), $u \in SV$, $v \in \mathbb{R}u$, which can be identified with $SV \times \mathbb{R}$. Thus we have

$$\begin{tikzcd}[column sep=small]
 & \underset{SV \times \mathbb{R}}{(u,t)} \arrow[dl, "\{\pm 1\}"'] \arrow[dr] & (u,t) \mapsto tu \\
\underset{\tilde{V}}{(\mathbb{R}u, tu)} \arrow[rr] & & V
\end{tikzcd}$$

$$(\ell, v) \longmapsto v$$

Similarly in the complex case we have

$$\begin{tikzcd}
 & SV \times \mathbb{C} \arrow[dl, "S^1"'] \arrow[dr] & \\
\tilde{V} \arrow[rr] & & V.
\end{tikzcd}$$

So now we should try to formulate a composition theorem. It appears that I might be dealing with two different kinds of composition on the same kind of kernels. Two situations

$$K(t,x) = t^{-\frac{n}{2}} f\left(\frac{x}{\sqrt{t}}\right) + \cdots\cdots$$

$$K(h,x) = h^{-n} f\left(\frac{x}{h}\right) + \cdots\cdots$$

The former occurs with heat operators and t gets convolved. The latter occurs with the classical limit + tangent groupoid; a typical example is

$$K(h,x) = \int \frac{d^n p}{(2\pi h)^n} e^{i\frac{p}{h}x} \hat{f}(p) = h^{-n} f\left(\frac{x}{h}\right)$$

I think all we have to do now is to set up the composition geometrically.

Let's first do the composition in the tangent groupoid situation supposing our manifold is $V = \mathbb{R}^n$. Then we have two kernels $K(h,x)$, $K'(h,x)$ which we wish to convolve in x

$$(K * K')(h,x) = \int d^n x_1 \; K(h, x - x_1)\, K'(h, x_1)$$

The assumption is that $h^{+n} K(h, hy)$, $h^n K'(h, hy)$ are smooth families of Schwartz functions in y with parameter h.

$$h^n (K*K')(h, hy) = h^n \int d^n(x_1) \;\; K(h, x - x_1)\, K'(h, x_1)$$

$$= \int d^n y_1 \;\; h^n K(h, hy - hy_1)\, h^n K'(h, hy_1)$$

so since convolution should be smooth it should work.

So we should ask what this means geometrically. Why from a geometric viewpoint should convolution be defined on $\mathcal{S}$? One starts with $V \times V \xrightarrow{+} V$ and one wishes to compactify. [redacted] The map is not proper so it doesn't induce a map of 1-point compactifications

Consider a surjection of vector spaces $V \xrightarrow{p} W$. It doesn't induce a map $\mathbb{P}V \longrightarrow \mathbb{P}W$ because lines in the kernel don't go to lines in W. Let $K = \mathrm{Ker}\{V \xrightarrow{p} W\}$. Then we have a map $\mathbb{P}V - \mathbb{P}K \longrightarrow \mathbb{P}W$ and the fibre over $l' \subset W$ is the space of $l \subset p^{-1}l'$ which are complementary to K; this is an affine space for the vector space $\mathrm{Hom}(l', K)$.

Let $\widetilde{\mathbb{P}V}$ be the space of pairs $(l, l') \in \mathbb{P}V \times \mathbb{P}W$ with $l \subset p^{-1}(l')$. Then [redacted] $\widetilde{\mathbb{P}V}$ is the projective bundle over $\mathbb{P}W$ associated to $p^{-1}(\mathcal{O}(-1)) \subset \pi^* V$. We have a map $\widetilde{\mathbb{P}V} \longrightarrow \mathbb{P}V$ which is an isomorphism over $\mathbb{P}V - \mathbb{P}K$.

It would seem that $\widetilde{\mathbb{P}V}$ is the blowup of $\mathbb{P}V$ along $\mathbb{P}K$. The inverse image in $\widetilde{\mathbb{P}V}$ of $\mathbb{P}K$ consists of pairs (ℓ, ℓ'), where $\ell \in \mathbb{P}K$, $\ell' \in \mathbb{P}W$. What is the normal bundle to $\mathbb{P}K \subset \mathbb{P}V$. At a line $\ell \in \mathbb{P}K$ the tangent space to $\mathbb{P}V$ is $\mathrm{Hom}(\ell, V/\ell)$, the subspace tangent to $\mathbb{P}K$ is $\mathrm{Hom}(\ell, K/\ell)$, hence the normal space is $\mathrm{Hom}(\ell, W)$. The normal bundle is thus $\mathcal{O}(1) \otimes W$ and its projective bundle is $\mathbb{P}K \times \mathbb{P}W$. Assume it's OK.

So now we prove things about Schwartz functions. Let's first identify Schwartz functions on V with smooth functions on $\mathbb{P}(\mathbb{R} \oplus V)$ vanishing to infinite order $\mathbb{P}(V)$. Then we want to see that given a surjection $V \to W$ we get an integration map $\mathcal{S}(V) \longrightarrow \mathcal{S}(W)$, say volumes on V, W being given so that functions can be identified with forms of highest degree. Then we look at the induced surjection $\mathbb{R} \oplus V \longrightarrow \mathbb{R} \oplus W$ and form the correspondence

$$\mathbb{P}(\mathbb{R} \oplus V) \longleftarrow \widetilde{\mathbb{P}(\mathbb{R} \oplus V)} \longrightarrow \mathbb{P}(\mathbb{R} \oplus W)$$

We have to begin by understanding how to integrate Schwartz functions : $\int_V dx : \mathcal{S}(V) \longrightarrow \mathbb{C}$. The idea is that we really integrate Schwartz densities, and that a Schwartz density on V is the same thing as a smooth density on $V \cup \{\infty\}$ vanishing to infinite order at ∞. So it can be integrted.

A key technical lemma will be that if M is a compact manifold and $\tilde{M}$ is the blowup of M along a closed submanifold Z, then the smooth densities or forms, or currents on M vanishing to infinite order on Z are the same as those on $\tilde{M}$ vanishing to infinite order on $\tilde{Z}$.

This lemma will enable me to go from $V \cup \infty$ to $\mathbb{P}(\mathbb{R}\times V)$, since blowing up the point ∞ yields $\mathbb{P}(\mathbb{R}\times V)$. Unfortunately blowing up yields non-orientable manifolds (blowing up a point in S^2 yields $\mathbb{R}P^2$), so one has to worry about densities.

Further ideas: We can now integrate Schwartz densities relative to a surjection $f: V \twoheadrightarrow W$. In particular we can define convolution of Schwartz densities. Hence we can probably also do this for smooth families of Schwartz densities depending on h. Recall that when $M = V$, we identified $\widetilde{\mathbb{R}\times M} - \tilde{M}$ with $\mathbb{R}\times V$. (Because a point of $\widetilde{\mathbb{R}\times V}$ is a line l in $\mathbb{R}\times V$ with a point $v=(h,x)$ on this line. If $(l,v) \notin \tilde{V}$, then l contains a generator $(1,y)$ and we have

$$h(1,y) = (h,x)$$

so that $\widetilde{\mathbb{R}\times V} - \tilde{V} = \{(h,y)\} = \mathbb{R}\times V$.

Now it would appear one has managed to prove composition is defined for $K(h,x)$ on $\mathbb{R}\times V - (0)$, so it should be a simple step to do the general case of $K(h,x,x')\,d^n x'$. Recall

$$K(h,x-x')|d^n x'| = \int \frac{|d^n p|}{(2\pi h)^n} e^{i\frac{p}{h}(x-x')} \hat{f}(p)\,|d^n x'|$$

so if we put $hy = x - x'$ and treat x as constant we get

$$K(h,hy)\,h^n|d^n y| = \int \frac{|d^n p|}{(2\pi)^n} e^{ipy}\,\hat{f}(p)\,|d^n y|$$

November 13, 1984

Reflection Positivity: Osterwalder-Schrader found axioms for the Schwinger fns. (Euclidean Green's functions) enabling reconstruction of the quantum field theory. Consider the case of a real scalar field $\varphi(x)$ with 0 space dimensions. This is just the quantum mechanics of a particle on the line. The Schwinger functions are

$$S_n(t_1,\dots,t_n) = \langle 0 | T[x(t_1)\dots x(t_n)] | 0 \rangle$$

$$= \int \mathcal{D}x \; e^{-\int S} \; x(t_1)\cdots x(t_n).$$

I shall take a Gaussian situation, i.e. where S_n comes from S_2 by the Wick rules. So it will be enough to look at the 2-point function. I guess I assume $S_1(t) = 0$.

So the axioms are that these are distributions, translation invariance which implies $S_2(t,t')$ depends only on $t-t'$: $S_2(t,t') = G(t-t')$, also permutation invariance so $S_2(t,t') = G(|t-t'|)$. Finally comes the assumption of reflection positivity, which says for $f(t)$ smooth supported in $t \geq 0$ that

$$\iint \overline{f(-t)}\, G(|t-t'|)\, f(t')\, dt\,dt' \;\geq 0$$

To understand what this all means we look first at the quantum example.

$$S(t_1,t_2) = \langle 0 | x(t_1)x(t_2) | 0 \rangle \qquad t_1 > t_2$$

$$= \langle 0 | e^{+t_1 H} x\, e^{-t_1 H}\, e^{+t_2 H} x\, e^{-t_2 H} | 0 \rangle$$

$$= e^{(t_1-t_2)E_0} \langle 0 | x\, e^{-(t_1-t_2)H} x | 0 \rangle$$

$$= \langle 0 | x\, e^{-(t_1-t_2)(H-E_0)}\, x | 0 \rangle$$

So we may suppose $E_0 = 0$. (Don't assume $|0\rangle$ is the ground state yet.)

Thus

$$G(t) = \langle 0 | x\, e^{-|t|H}\, x | 0 \rangle$$

and $\displaystyle\iint \overline{f(-t')}\, G(t-t')\, f(+t)\, dt\, dt'$

$$= \int_{-\infty}^{0} dt' \int_0^\infty dt \quad \overline{f(-t')}\, G(t-t')\, f(t)$$

$$= \int_0^\infty dt' \int_0^\infty dt \quad \overline{f(t')}\, G(t+t')\, f(t)$$

$$= \int_0^\infty dt' \int_0^\infty dt \quad \overline{f(t')}\, \langle 0 | x e^{-t'H}\; e^{-tH} x | 0 \rangle f(t)$$

$$= \left\langle \int_0^\infty dt' f(t') e^{-t'H} x|0\rangle ,\ \int_0^\infty dt\, f(t)\, e^{-tH} x|0\rangle \right\rangle$$

$$= \left\| \int_0^\infty dt\, f(t)\, e^{-tH} x|0\rangle \right\|^2$$

Conversely suppose one gives a $G(t)$ for $t \geq 0$ such that for any $t_1, \dots, t_n \geq 0$ we have that the matrix $G(t_i + t_j)$ is ≥ 0. Then how can we reconstruct the Hilbert space.

It's clear we get a Hilbert space with a path x_t $t \geq 0$ such that

$$\langle x_{t'}, x_t \rangle = G(t+t').$$

I guess what one does is to take functions $f(t)$, $t \geq 0$

with $\qquad \|f(t)\|^2 = \int_0^\infty dt' \int_0^\infty dt \; \overline{f(t')}\, G(t-t') f(t)$

Then x_{t_0} corresponds to the function $\delta(t-t_0)$. It's not clear to me how to reconstruct the rest of the Hilbert space from the data given so far.

Now what seems to be needed is a criterion for $G(t)$, $t \geq 0$ to be the Laplace transform of a probability measure:

$$G(t) = \int_0^\infty e^{-t\lambda}\, d\mu(\lambda)$$

I recall Widder defining completely positive somehow so this would work.

Let's consider the example of Brownian motion where we have x_t starting at $x=0$ at $t=0$. Then x_t is defined for $t \geq 0$ and we have

$$\langle x_t, x_{t'} \rangle = \min\{t, t'\}.$$

Not a function of $t-t'$?

Maybe it should be true that assuming $G(t_i + t_j) \geq 0$ for any subset $t_1, \ldots, t_n$ is enough.

The proof might go as follows. The function $G(t)$ gives an inner product on the linear combinations of exponential functions $e^{-t\lambda}$, $t \geq 0$. But these can be used to approximate rapidly decreasing smooth functions $f(\lambda)$, so it should be clear that we get a linear functional on these $f(\lambda)$ which is positive (if $f \geq 0$, approx $f^{1/2}$). Thus we get a measure on the λ-line.

Example: Take the measure

$$\rho(\lambda) = e^{-e^{-\lambda}} e^{-a\lambda} \, d\lambda \qquad a > 0$$

on the λ- line. As $\lambda \to -\infty$, $e^{-\lambda} \to +\infty$ fast so $\rho(\lambda)$ decays more rapidly than any exponential as $\lambda \to -\infty$. As $\lambda \to +\infty$, $e^{-e^{-\lambda}} \to 1$, so ρ decays exponentially.

Then

$$G(t) = \int_{-\infty}^{\infty} e^{-t\lambda} \, e^{-e^{-\lambda}} e^{-a\lambda} \, d\lambda$$

$$= \int_{-\infty}^{\infty} e^{-e^{-\lambda}} \left(e^{-\lambda}\right)^{t+a} d\lambda \qquad \text{set } e^{-\lambda} = u, \quad -d\lambda = \frac{du}{u}$$

$$= \int_0^{\infty} e^{-u} \, u^{t+a} \, \frac{du}{u} = \Gamma(t+a)$$

Since ρ is supported on the whole λ- line it follows that when we take the corresponding H and Hilbert space, the Hamiltonian will not have spectrum bounded below.

Could there $\exists$ a model of emission + absorption based on this example?