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Problem: Relate character forms defined by Grassmannian graph and the superconnection formalism

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Problem: Compare equivariant coh. for the map

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June 10, 1984

Yesterday I decided that the forms representing the class ch_k

$$\frac{1}{2}\frac{1}{k!}\, tr_s \left(\frac{\varepsilon}{\varepsilon^2 - L^2}\, dL \right)^{2k}$$

which were obtained via the Grassmannian graph approach, are really natural and very basic. They make sense for unbounded operators and also closely related to the resolvent used in analysis. Thus I should solve the transgression problem in this context, namely, take the character forms on the Grassmannian and somehow transgress them to the group which acts on the Grassmannian.

This leads me back ~~to~~ the following problem. I have a Lie group G acting on M and an invariant closed form ω on M. Then I want to 'transgress' ω to a form of degree one less on the group G using the map $G \longrightarrow M$ associated to a point of M. I ~~Suppose that the form is~~ have to give two reasons for the restriction of ω to G to be cohomologous to zero. One reason is given by assuming that $\omega \in \Omega(M)$ is a coboundary. The other reason should come from the fact that $G/G = pt$, and the way to handle this is to assume ω comes from, or is part of, an equivariant closed form.

Thus we start with an equivariant closed form, i.e. a cocycle in

$$[W(\mathfrak{g}) \otimes \Omega(M)]_{basic} \xrightarrow{\sim} S(\mathfrak{g}^*) \otimes \Omega(M)^G$$

which we write as $\tilde{\omega} = \sum \Omega^\alpha \omega_\alpha$ a polynomial in the generators Ω^a of $S(\mathfrak{g}^*)$ with coefficients in $\Omega(M)$. The coefficient ω_0 is our invariant form ω on M. Now we want to use $\tilde{\omega}$ to see why ω becomes cohomologous to zero in $\Omega(G)$. I want to use that $G/G = pt$, and so might as well consider the case where G acts freely on M first. One has

$$\begin{tikzcd} \Omega(M) & & W(\mathfrak{g}) \otimes \Omega(M) \arrow[ll, "\text{defined via}"', "\substack{\text{choosing a}\\ \text{connection in}\\ M \to M/G}"] \\ \cup & & \cup \\ \Omega(M/G) = \Omega(M)_{basic} & & {[W(\mathfrak{g}) \otimes \Omega(M)]_{basic}} \arrow[ll] \arrow[r, "\sim"] & {\underset{\tilde{\omega}}{[S(\mathfrak{g}^*) \otimes \Omega(M)]^G}} \end{tikzcd}$$

Now $\tilde{\omega}$ lifts to

$$\bar{\omega} = \sum \Omega^\alpha \prod_a (1 - \theta^a i_a)\, \omega_\alpha \qquad \text{in } [W(\mathfrak{g}) \otimes \Omega(M)]_{basic}$$

(Recall $\prod_a (1 - \theta^a i_a)$ is the projector onto the horizontal space). When $\bar{\omega}$ is viewed in $W(\mathfrak{g}) \otimes \Omega(M)$ it augments to ω_0. Since $W(\mathfrak{g})$ is a contractible complex, this means that $\bar{\omega} - \omega_0$ is a coboundary in $W(\mathfrak{g}) \otimes \Omega(M)$, hence on $\Omega(M)$ ω_0 will be cohomologous to a form coming from $\Omega(M/G)$. Thus when $M = G$, and ω_0 has positive degree, we see that the restriction of ω_0 to G is cohomologous to zero.

In practice the contractibility of $W(\mathfrak{g})$ is effected by deforming the connection form θ on M over M/G to zero via $t\theta$. This is the Chern-Simons business.

Example: Suppose we consider an equivariant 2-form $\tilde{\omega} = \Omega^a f_a + \omega \in [S(\mathfrak{g}^*) \otimes \Omega(M)]^G_{(2)} =$ $[\mathfrak{g}^* \otimes \Omega^0(M)]^G \oplus \Omega^2(M)^G$. For this to be closed means

$$0 = (d_M - \Omega^a \iota_a)(\Omega^a f_a + \omega)$$

$$= \Omega^a df_a + d\omega - \Omega^a \iota_a \omega$$

$$\Longleftrightarrow \quad d\omega = 0 \quad \text{and} \quad \iota_a \omega = df_a .$$

Then

$$\bar{\omega} = \Omega^a f_a + \prod_a (1 - \theta^a \iota_a)\, \omega$$

Now consider the 1-form $\theta^a f_a$. One has

$$d(\theta^a f_a) = d\theta^a f_a - \theta^a df_a$$

$$\bar{\omega} - \omega = \underbrace{\Omega^a}_{d\theta^a + \frac{1}{2} f^a_{bc} \theta^b \theta^c} f_a - \theta^a \iota_a \omega + \frac{1}{2} \theta^c \theta^b \underbrace{\iota_b \iota_c \omega}$$

$$X_b f_c = f_{[X_b, X_c]} = f^a_{bc} f_a$$

so $\quad \bar{\omega} - \omega = d(\theta^a f_a)$.

The next project will be to consider the character forms on the Grassmannian and find the sort of group G acting on the Grassmannian to which the above applies. We want the character forms to come from equivariant forms, so we want the subbundle on Grass to have an invariant connection. Hence it is clear that the whole unitary group should work.

Let us now consider the Grassmannian Grass consisting of all projectors e of a given rank with the natural action of the unitary group G. We consider the map $G \longrightarrow \text{Grass}$, $g \mapsto g e_0 g^{-1}$. and want to write the pull back of the form to $e(de)^{2k}$ as a coboundary.

The pull-back E of the subbundle on Grass is the bundle whose fibre at g is the subspace $\operatorname{Im} g e_0 g^{-1} = gV^0$ where $V^0 = \operatorname{Im} e_0$. This bundle is trivial as a section $s \in \Gamma(G, E)$ is of the form $s(g) = g \cdot \psi(g)$, where $\psi \in \operatorname{Map}(G, V^0)$. The Grassmannian connection is

$$
\begin{aligned}
(Ds)(g) &= e(g)\,(ds)(g) \\
&= g e_0 g^{-1}\left[dg\,\psi(g) + g(d\psi)(g)\right] \\
&= g\left[d\psi + e_0 g^{-1} dg\,\psi\right](g)
\end{aligned}
$$

so we can identify D on E with $d + \bar{\theta}$ on the trivial bundle over G with fibre V^0, where

$$\bar{\theta} = e_0 (g^{-1} dg) e_0$$

The curvature of this connection is

$$
\begin{aligned}
\Omega = d\bar{\theta} + \bar{\theta}^2 &= e_0(-g^{-1}dg\, g^{-1}dg)e_0 + e_0(g^{-1}dg)e_0(g^{-1}dg)e_0 \\
&= -e_0 (g^{-1}dg)(1-e_0)(g^{-1}dg)\, e_0
\end{aligned}
$$

As a check lets compute $e(de)(de) = (de)(1-e)(de)$ where $e = g e_0 g^{-1}$ so $de = dg\, g^{-1} e + e\, g\, dg^{-1}$.

$$(de)(1-e)(de) = e\,(g\,dg^{-1})(1-e)(dg\, g^{-1})\,e$$

$$= + g e_0 (-g^{-1}dg)(1-e_0)(g^{-1}dg) e_0 g^{-1}$$

which is conjugate to $d\bar{\theta} + \bar{\theta}^2$.

Now to write $tr(\Omega^k)$ as a coboundary we use the deformation of Chern-Simons: $d + t\bar{\theta}$ which gives

$\Omega_t = (d+t\bar{\theta})^2$

$$tr(\Omega^k) = d\int_0^1 dt \cdot k \cdot tr(\Omega_t^{k-1}\bar{\theta})$$

where
$$\Omega_t = (d + t\bar{\theta})^2 = t\,d\bar{\theta} + t^2\bar{\theta}^2$$
$$= -t e_0 (g^{-1}dg)^2 e_0 + t^2 e_0 (g^{-1}dg) e_0 (g^{-1}dg) e_0$$

Start again: We a group G acting by unitary transformations on $\mathcal{H}$ and are considering the map $G \longrightarrow Grass$, $g \longmapsto g e_0 g^{-1}$. The point is to see that the character forms ch_k on Grass pull-back to explicit coboundaries on G. I want to do this in the simplest way possible.

Does it help to take $G = U(N)$ and then regard Grass as $U(N)/U(n)\times U(N-n)$? If I use the principal $U(n)$-bundle $U(N)/U(N-n)$ over Grass, then we have $\pi^* ch_k = d\cdot$(Chern-Simons)

One thing that bothers me is that I seem to be showing Connes odd cocycles are trivial cohomologically. I recall he looks at $g \longmapsto g F_0 g^{-1}$ where $F_0 = 2e_0 - 1$ is an involution, and pulls back the forms

$$\frac{1}{2^{2k+1}k!} tr\left(F(dF)^{2k}\right) = \frac{1}{k!} tr\left(e(de)^{2k}\right)$$

to get his odd cocycles. I seem to be writing these forms as coboundaries.

This seems to be important enough to get straight, especially since we know in the loop group case that the cocycle is non-trivial. So let's review carefully.

I consider the map $e : g \longmapsto g e_0 g^{-1}$ from G into the Grassmannian regarded as idempotents. Then

$$de = dg\, e_0 g^{-1} + g e_0 \underbrace{d(g^{-1})}_{-g^{-1}dg\,g^{-1}}$$

$$= [dg.g^{-1}, e]$$

and so the curvature is

$$e\,de\,de = e[dg.g^{-1}, e][dg.g^{-1}, e]$$

$$= g e_0 [g^{-1}dg, e_0][g^{-1}dg, e_0] g^{-1}$$

which gives rise to left-invariant forms on G. Now

$$tr\, \bar{\theta} = tr\,(e_0 g^{-1} dg\, e_0)$$

is a left-invariant form and

$$\cancel{d\, tr\, \bar{\theta} = tr\,(e_0\, dg^{-1} dg\, e_0)}$$

$$tr\,(e(de)^2) = tr\,(d\bar{\theta} + \bar{\theta}^2) = tr\,(d\bar{\theta}) = d\, tr(\bar{\theta}).$$

In the loop group case, $tr\,\bar{\theta}$ doesn't exist since if $e_0 =$ projection on holom. fns. then

$$tr\,(e_0 f) = \sum_{n \geq 0} \langle z^n | f | z^n \rangle = \sum_{n\geq 0}{}' \int_{S^1} f(z) \frac{d\theta}{2\pi}$$

which is infinite in general.

Let's return to the Dirac operator setup. Here one has a $\mathbb{Z}_2$-graded $\mathcal{H} = \mathcal{H}^0 \oplus \mathcal{H}^1$ and a group of gauge transformation $\mathcal{G}$ acting on $\mathcal{H}$ unitarily, and a family $\mathcal{A}$ of odd degree operators L on $\mathcal{H}$ stable under $\mathcal{G}$. Next one has the Chern character

$$tr_s\left(e^{u(L^2+[D,L]+D^2)}\right)$$

which is an equivariant form on $\mathcal{A}$ for the $\mathcal{G}$-action. Now the point is that once we have the equivariant form, then the part of it obtained by ignoring the $\mathcal{G}$-action, namely,

$$tr_s\left\{e^{u(L^2+dL)}\right\}$$

becomes a coboundary upon pulling back via a $\mathcal{G}$-map $\mathcal{G} \longrightarrow \mathcal{A}$. This should happen in a relatively specific way using the family $d+t\theta$.

Now I have to produce another reason for the high degree components of $tr_s\left\{e^{u(L^2+dL)}\right\}$ to be cohomologous to zero.

June 11, 1984

Let's discuss character forms a bit more abstractly. Let's consider the super connection situation: $\tilde{D} = D + L$ on $E = E^0 \oplus E^1$. Then we can lift up to the principal bundle P of E and we obtain the following. The bundle E becomes a trivial bundle with fibre $V = V^0 \oplus V^1$ and the map L becomes an equivariant map of P into $\mathrm{End}^1(V)$. The connection D on P becomes $d + \theta$ on $\Omega(P) \otimes V$, where θ is the connection form. The curvature becomes

$$\tilde{D}^2 = (D+L)^2 = L^2 + [d+\theta, L] + (d+\theta)^2$$

and the Chern character form $tr_s\left(e^{\tilde{D}^2}\right)$ is constructed out of L, θ.

The point is that the Chern character form is really an equivariant form for the action of $\mathrm{Aut}^0(V)$ on $\mathrm{End}^1(V)$, which then is pulled back to P and descended to M using the connection D.

June 24, 1984

I want to try to reconstruct the few ideas and thoughts of the past twelve days. The basic program concerns the relation of Chern character forms defined by the super connection formalism on one hand and Grassmannian graph on the other. Analytically one wants to find the precise link between the heat operator approach and the parametrix approach to the index. Classically it is the link between θ-functions and Eisenstein series.

The basic setup: $V = V^0 \oplus V^1$, G = a group of unitary degree zero autos. of V, M = a space of odd degree skew-adjoint operators $i(T+T^*)$ of V which is stable under G.

One is interested in equivariant forms for the G-action on M. Such forms are universal for the following. Suppose we are given a superconnection $\tilde{D} = D + L$ on $E = E^0 \oplus E^1$ over Y. Then the principal bundle of E is a principal G-bundle P, where $G = \mathrm{Aut}^{(0)}(V)$, over Y. We get a connection form in P/Y and an equivariant map $P \to M$ from D and L respectively. Thus equivariant forms for (G, M) give rise to forms on Y.

Thus we have the idea that the character forms constructed via the superconnection formalism are nothing but equivariant forms for $G \subset \mathrm{Aut}^0(V)$ acting on $M \subset \mathrm{End}^1_{sk}(V)$.

The next idea is that one can

look at the transgression problem in the context of (G,M), where it looks exactly like the stuff I did for $(\mathcal{G}, \mathcal{a})$.

Recall the transgression process: One starts with an equivariant form α for (G,M). One will give two reasons for the ~~restriction~~ pull-back of α ~~by~~ a G-map $G \longrightarrow M$ to be cohomologically trivial. The difference of these reasons is a cocycle on G. One reason comes from the fact that α is an equivariant form; let's describe this by writing $\alpha = \alpha(\theta)$, where θ is a potential connection form. Thus given a P/Y, a principal G-bundle with a connection form θ, and a G-map $u : P \to M$, I get a form $(u^*\alpha)(\theta)$ on P which is basic and so can be regarded as a form on Y. If I take u to be a G-map $u : G \longrightarrow M$, and θ to be $g\,dg^{-1}$, then $u^*\alpha(\theta)$ comes from a point, and so it is zero provided degree $(\alpha) > 0$. On the other hand forgetting the G-action contribution to α is achieved by setting $\theta = 0$ to obtain a form $\alpha(0)$ on M. This restricts to $u^*\alpha(0)$ on G. Using a path $t\theta$ of connection forms expresses $u^*\alpha(0) - \underbrace{u^*\alpha(\theta)}_{0}$ as a coboundary $d\beta$, and this gives the first reason $u^*\alpha(0)$ is cohomologically trivial.

The second reason comes from the contractibility of M, i.e. one uses a path sT to shrink

T to zero.

Let us now carry out the transgression process assuming α comes from the super connection formalism. I will be working over G with the trivial super vector bundle with fibre V, and I have the G-map

$$G \xrightarrow{L} M \qquad\qquad L(g) = gL_0g^{-1}.$$

The super-connection which descends to $G|G$ is

$$d + g\,dg^{-1} + gL_0g^{-1} = g\,(d+L_0)\,g^{-1}$$

and belongs to $\theta = g\,dg^{-1}$ over G. The superconnection belonging to $\theta = 0$ is

$$d + gL_0g^{-1} = g.(d + g^{-1}dg + L_0).g^{-1}$$

Thus if we make the gauge transformation given by g we have the two connections

$$\begin{cases} d + L_0 \\ d + g^{-1}dg + L_0 \end{cases}$$

which we can join by the path

$$\tilde{D}_t = d + t\omega + L_0 \qquad\qquad \omega = g^{-1}dg$$

Let's now take α to be the total Chern character form

$$tr_s\left(e^{u\tilde{D}^2}\right) \qquad\qquad \begin{array}{l} u = \text{parameter} \\ \tilde{D} = d + \theta + L \end{array}$$

interpreted as an equivariant form for (G,M). This is pulled back, or specialized to, the above

$\tilde{D}_t$. The curvature is

$$(d + t\omega + L_0)^2 = L_0^2 + t[L_0, \omega] + (t^2 - t)\omega^2 .$$

Thus we get the formula

$$\boxed{tr_s\left(e^{u(L_0^2 + [L_0,\omega])}\right) - tr_s\left(e^{uL_0^2}\right) = d\int_0^1 dt\; tr_s\left(e^{u(L_0^2 + t[L_0,\omega] + (t^2-t)\omega^2)}\, u\,\omega\right)}$$

Note that at $t = 1$ we have

$$d + L = g \cdot (d + \omega + L_0) \cdot g^{-1}$$

with curvature

$$L^2 + dL = g\left(L_0^2 + [L_0, \omega]\right)g^{-1}.$$

so except for the 0-diml form $tr_s(e^{uL_0^2})$, the above formula expresses $tr_s\left(e^{u(L^2+dL)}\right)$, pulled back via $L : G \to M$, $L(g) = gL_0g^{-1}$, as a coboundary.

Next one uses the deformation hL_0 $0 \le h \le 1$ and the formula

$$tr_s\left(e^{u(h^2L_0^2 + h[L_0,\omega])}\right) - tr_s(e^0) = d\int_0^1 dh\; tr_s\left(e^{u(h^2L_0^2 + h[L_0,\omega])}\, uL_0\right)$$

for the contractibility of M.

Let's review the construction of cyclic cocycles on $\Omega^0(M, \mathrm{End}\, E)$ given a connection D on E. We consider $\mathcal{G}$ = gauge tranformation group of $E^{\oplus n}$ whose complexified Lie algebra is

$$\tilde{g} = gl_n \otimes \Omega^0(M, \mathrm{End}\, E),$$

and we want Lie algebra cochains on $\tilde{g}$, i.e.

left-invariant forms on $\mathcal{G}$. So on $\mathcal{G}\times M$ we consider the bundle $pr_2^* E$ with the family of connections $\qquad \omega = g^{-1}dg$

$$d_{\mathcal{G}} + t\omega + D_M \;=\; D + t\omega \qquad D = pr_2^*(D)$$

and we obtain

$$\operatorname{tr}\left(e^{u(D^2+[D,\omega])}\right) - \operatorname{tr}\left(e^{uD^2}\right) = d\int_0^1 dt \, \operatorname{tr}\left(e^{u(D^2+t[D,\omega]+(t^2-t)\omega^2)}\,\omega\right)$$

This formula can be integrate over cycles in M to obtain left-invariant forms on $\mathcal{G}$.

One hopes that the boxed formula on p. 12 in the case of a Dirac operator L_0 would yield the above formula in the classical limit.

An important point: $D^2 + [D,\omega]$ is of type $(1,1)+(0,2)$ on $\mathcal{G}\times M$, hence its k-th power integrates to zero if $k > \dim M$. I need a substitute (for the) Bott type thm. in the setting of the boxed formula on p.12 with Dirac operators. Here there is a Fredholm condition on L_0 which is not satisfied by hL_0 when $h=0$.

Discussion: In constructing cyclic cocycles using differential forms I have this place where Botts thm. is used. I want ultimately to assign cyclic cocycles to a general L_0, so I need an abstract argument to replace the Bott thm. It should be analogous to the existence of traces as used by Connes. Here seems to be a candidate.

From the Grassmannian graph business we know that the classes ch_k are represented by the forms

$$\frac{1}{2}\,\frac{1}{k!}\,tr_s\left(\frac{\lambda^{1/2}}{\lambda-L^2}\,dL\right)^{2k}$$

for any λ. (Note the rescaling $L\mapsto\lambda^{1/2}L$ reduced to $\lambda=1$.) We now show how for a Dirac operator with $k>\dim M$ these forms go to zero as $\lambda\to\infty$. We use the estimate

$$tr_s\left(\frac{\lambda^{1/2}}{\lambda-L^2}\,dL\right)^{2k}\sim\int\left(\frac{\lambda^{1/2}}{\lambda+\xi^2}\right)^{2k}\frac{d^n\xi}{(2\pi)^n}$$

$$\sim\int\left(\frac{\lambda^{1/2}}{\lambda+\lambda\xi^2}\right)^{2k}\frac{\lambda^{\frac{n}{2}}d^n\xi}{(2\pi)^n}=C\cdot\lambda^{\frac{n}{2}-k}$$

This goes to zero for $k>\frac{n}{2}=\frac{1}{2}\dim M$, which is better than <u>$k>\dim M$, so</u> I am confused. ?

Further ideas: (these are in the scratchbook left home)

1) In order to handle the Grassmannian graph version of $\tilde{D}=D+L$ use equivariant forms

2) Witten's approach to the determinant line bundle based on the massive Dirac operator in the large M limit.

3) Link up determinant line bundle with the transgression problem.

By reviewing how one obtains sections of the determinant line bundle, I saw the

Plücker embedding of the Grassmannian, which reinforces the Grassmannian viewpoint. It should be possible to compute curvature forms for the determinant line bundle and completely analyze the transgression problem.

Summary of the transgression problem: The process starts with an equivariant form $\alpha(\Theta)$ for (G, M) of pos. degree and uses that $\alpha(0)$ is a boundary because of the conical contractibility of M. The equivariant form comes from a bundle over M with invariant connection.

Let's consider the finite-diml. situation. This means $V = \mathbb{C}^n \oplus \mathbb{C}^m$ and

$$G = \mathrm{Aut}^{(0)} V = U(n) \times U(m),$$
$$M \simeq \mathrm{Hom}(\mathbb{C}^n, \mathbb{C}^m) = \{T : V^0 \to V^1\}.$$

Then we have the Graph embedding

$$M \longrightarrow \mathrm{Grass}_n(V) \cong U(n+m) / U(n) \times U(m)$$

which is equivariant for the embedding

$$G = U(n) \times U(m) \subset U(n+m).$$

<u>Question</u>: What is the effect on cohomology of the map

$$(G, M) \longrightarrow (U(n+m), \mathrm{Grass}) \;?$$

The classifying space of (G, M) represents a pair of v.b. E^0, E^1 together with a map $T : E^0 \to E^1$. The classifying space of $(U(n+m), \mathrm{Grass})$ represents a vector bundle E of rank $n+m$ together with a projector of rank n. Both classifying spaces are clearly homotopy equivalent to $BU(n) \times BU(m)$.

<u>Question</u>: What do the equivariant forms on $(U(n+m), \text{Grass})$ look like?

In general the equivariant forms on G/H are elements of

$$\{W(\mathfrak{g}) \otimes \Omega(G/H)\}_{\text{basic}} \simeq \{S(\mathfrak{g}^*) \otimes \Omega(G/H)\}^G$$

$$\simeq \{S(\mathfrak{g}^*) \otimes \Lambda(\mathfrak{g}/\mathfrak{h})^*\}^H$$

For the Grassmannian $\mathfrak{h} = \mathrm{End}(V) \oplus \mathrm{End}(W)$, $\mathfrak{g}/\mathfrak{h} = W \otimes V^* \oplus W^* \otimes V$ so things are very big.

Actually I see now that the above can be written

$$(*) \quad \{S(\mathfrak{g}^*) \otimes \Lambda(\mathfrak{g}/\mathfrak{h})^*\}^H \simeq \{S(\mathfrak{h}^*) \otimes S((\mathfrak{g}/\mathfrak{h})^*) \otimes \Lambda(\mathfrak{g}/\mathfrak{h}^*)\}^H$$

which looks exactly the same as the H-equivariant forms on $\mathfrak{g}/\mathfrak{h}$ which are polynomials on $\mathfrak{g}/\mathfrak{h}$. It would seem therefore that

$$(U(n) \times U(m), M) \subset (U(n+m), \text{Grass})$$

induces an injection of equivariant forms such that the image are polynomial on M. But this can't be correct, because the degrees don't match. The $S(\mathfrak{g}^*)$ on the left of $(*)$ has degree $(\mathfrak{g}^*) = 2$, whereas the $S((\mathfrak{g}/\mathfrak{h})^*)$ on the right is of degree 0. However I bet there is a birational map somewhere.

Anyway let's work on some formulas derived in the other scratchbook. I want to compute the character forms of the subbundle on the Grassmannian as equivariant forms. Think of the Grass as consisting of idempotents e on V and G as the unitary group of V. The Grassmannian connection is G-invariant and is to be modified by the Higgs field in order to become equivariant.

In general if D is an invariant connection, and $X \in \mathfrak{g}$, then the Higgs field φ_X is a bundle endomorphism defined by

$$\mathcal{L}_X = [\iota_X, D] \pm \varphi_X$$

To determine the sign, we recall for a line bundle that φ is the "moment" map essentially, which satisfies

$$\mathcal{L}_X \Omega = d\varphi_X$$

If $\quad \mathcal{L}_X = \nabla_X + \varphi_X \quad$ on $\Gamma^{\circ}(E)$, then invariance of ∇ ~~means~~ implies

$$[\mathcal{L}_X, \nabla_Y] = \nabla_{[X,Y]}$$

$$\parallel$$

$$[\nabla_X, \nabla_Y] + [\varphi_X, \nabla_Y]$$

or $$Y\varphi_X = [\nabla_X, \nabla_Y] - \nabla_{[X,Y]} = i(Y)i(X)\Omega.$$

Thus the correct formula is

$$\boxed{\mathcal{L}_X = [-\iota_X, D] + \varphi_X}$$

Hence if we want to modify D so that it descends, we use

$$\tilde{D} = D + \theta\varphi = D + \theta^a \varphi_a$$

where θ is a connection form, or the universal connection form relative to the Weil algebra. On

$$W(\mathfrak{g}) \otimes \Omega(M, \mathrm{End}\, E)$$

D includes the differential on $W(\mathfrak{g})$. If we use the isomorphism

$$[W(\mathfrak{g}) \otimes \Omega(M, \mathrm{End}\, E)]_{basic} \xrightarrow{\ \sim\ } [S(\mathfrak{g}^*) \otimes \Omega(M, \mathrm{End}\, E)]^G$$

$$D \longmapsto D - \Omega^a \iota_a$$

we see the curvature is on the right

$$(D - \Omega^a \iota_a)^2 = D^2 - \Omega^a(\underbrace{\iota_a D + D\iota_a}_{\mathcal{L}_a - \varphi_a})$$

$$= \boxed{D^2 + \Omega^a \varphi_a}$$

This is also consistent with

$$\tilde{D}^2 = (D + \theta\varphi)^2 = D^2 + \underbrace{[D,\theta]}_{d\theta = \Omega - \theta^2}\varphi - \theta[D,\varphi] + \theta\varphi\theta\varphi$$

$$\xmapsto{\ \theta \to 0\ } D^2 + \Omega\varphi .$$

Now consider the Grassmannian connection on the subbundle E over Grass. Let $\psi : e \longrightarrow \psi(e)$ be a section. Then $(D\psi)(e) = e(d\psi)(e)$ and for $X \in \mathfrak{g}$

$$(\iota_X D\psi) = e(\iota_X d\psi)(e) \qquad \text{really } [X,e] \text{ at } e$$

~~To compute this suppose $g = e^{tX}$, then $(X\psi)(e)$ is the derivative of $\psi(geg^{-1})$ at $t=0$. But~~

[redacted] ~~This is to be compared with~~ $\cancel{(\mathcal{L}_X\psi)(e) = X\psi(e)}$

(Think of $g = e^{tX}$; then g acts on E by $g(\psi(e)) = $ [redacted] $g \cdot \psi(e)$ and g acts on $\Gamma(E)$ by

$$(g * \psi)(e) = g\,\psi(g^{-1}eg) = g\,\psi(e + t[-X, e] + \ldots)$$

Thus $$(X * \psi)(e) = X\psi(e) - i_{[X,e]}\, d\psi(e) \qquad ?$$

Start again: G operates on V, and hence on idempotents by $g * e = geg^{-1}$. So it will act on sections ψ of E by

$$(g * \psi)(e) = g\,\psi(g^{-1}eg).$$

So $$\frac{d}{dt}\,(e^{tX} * \psi)(e)\Big|_{t=0} = X\psi - i_{[X,e]}\, d\psi$$ and this must be $-\mathcal{L}_X\psi$. It is all preserved by e, so we have

$$(\mathcal{L}_X\psi)(e) = e\, i_{[X,e]}\, d\psi(e) - eX\psi(e)$$

and so we conclude that the Higgs field is

$$? \qquad \varphi_X(e) = -eXe$$

This sign can't be correct since if we took the case $e = I$, we must have $\varphi_X = X$. Or if one wants, at a fixpt, the moment map must be a Lie homomorphism. So we use the formula

$$\boxed{\varphi_X = eXe}$$

So now we can calculate the equivariant

curvature of the subbundle. The Grass conn.

is
$$D\psi = e.d\psi$$

so
$$D^2\psi = e.d.e.d\psi = e\, de.\, d\psi$$
$$= (de)(1-e)\, d(e\psi) = (de\,(1-e)de)\psi$$
$$= (e\, de\, de)\psi$$

and the curvature is $e\, de\, de$.

Finally we see that the equivariant curvature is
$$e(de)^2 + e\theta e.$$

Now here is how I wanted to apply this. I want to assign to [redacted] a point $T: V^0 \longrightarrow V^1$ the idempotent which projects onto $\Gamma_T \subset V^0 + V^1$. This is

$$e = \begin{pmatrix} 1 \\ T \end{pmatrix} \frac{1}{1+T^*T} \begin{pmatrix} 1 & T^* \end{pmatrix}$$

and we have already computed the curvature. It is useful to identify $E = \operatorname{Im} e$ with $\begin{pmatrix} 1 \\ T \end{pmatrix} V^0$. Then a section of E is in the form

$$s[redacted](T) = \begin{pmatrix} 1 \\ T \end{pmatrix} \psi(T)$$

and
$$(Ds)(T) = e\; d\begin{pmatrix} 1 \\ T \end{pmatrix} \psi(T)$$
$$= e\left[\begin{pmatrix} 0 \\ dT \end{pmatrix}\psi + \begin{pmatrix} 1 \\ T \end{pmatrix} d\psi\right]$$
$$= \begin{pmatrix} 1 \\ T \end{pmatrix}\left\{d[redacted] + \frac{1}{1+T^*T} T^* dT\right\}\psi.$$

Hence the connection form is $\frac{1}{1+T^*T} T^* dT$ relative to this trivialization and the curvature is

$$d\left(T^*\frac{1}{1+TT^*}dT\right) + \underbrace{T^*\frac{1}{1+TT^*}dT.T^*\frac{1}{1+TT^*}dT}$$

cancel

$$= dT^*\frac{1}{1+TT^*}dT - T^*\frac{1}{1+TT^*}(dT.T^*+TdT^*)\frac{1}{1+TT^*}dT +$$

$$= \left\{1 - T^*\frac{1}{1+TT^*}T\right\}dT^*\frac{1}{1+TT^*}dT$$

$$= \frac{1}{1+T^*T}dT^*\frac{1}{1+TT^*}dT$$

To simplify things I look at just the first Chern class

$$\Omega = tr\left(\frac{1}{1+T^*T}dT^*\frac{1}{1+TT^*}dT\right)$$

The Higgs field is going to associate to $X = \begin{pmatrix} X^0 & 0 \\ 0 & X^1 \end{pmatrix}$ the endomorphism eXe of $\operatorname{Im} e = \binom{1}{T}V^0$. So we get

$$\binom{1}{T}\frac{1}{1+T^*T}(1 \quad T^*) \, X \, \binom{1}{T}\frac{1}{1+T^*T}(1 \quad T^*)$$

which pulled back to V^0 becomes

$$\frac{1}{1+T^*T}(1 \quad T^*) \, X \, \binom{1}{T} = \frac{1}{1+T^*T}\left(X^0 + T^*X^1T\right)$$

Thus we get the equivariant 2-form

$$tr(D^2+\Omega\varphi) = tr_{V^0}\left(\frac{1}{1+T^*T}dT^*\frac{1}{1+TT^*}dT\right)$$

$$+ \, tr_{V^0}\left(\frac{1}{1+T^*T}\left(\Omega + T^*\Omega T\right)\right)$$

Now the next step is to convert this to a formula which applies to a pair of vector bundles E^0, E^1 with connection D^j, and map $T: E^0 \longrightarrow E^1$. This involves inverting the isomorphism

$$[W(\mathfrak{g}) \otimes \Omega(M)]_{horiz} \xrightarrow{\sim} S(\mathfrak{g}^*) \otimes \Omega(M)$$

given by $\theta \longmapsto 0$. The inverse is $\prod_a (1 - \theta^a i_a)$ which is a projection operator onto the horizontal forms.

Let's understand this sort of projection a bit. Suppose we are given a principal G-bundle P with a connection. Then we have a ring isomorphism

$$\Omega(P) \simeq \Omega(P)_{horiz} \otimes \Lambda \mathfrak{g}^*$$

because at each point of P the tangent space splits into the vertical and horizontal subspaces.

$$0 \longrightarrow \mathfrak{g} \hookrightarrow T_P \longrightarrow \pi^* T_Y \longrightarrow 0$$

$$0 \longrightarrow \pi^* T_Y^* \longrightarrow T_P^* \longrightarrow \mathfrak{g}^* \longrightarrow 0$$

given by connection

so $\Lambda T_P^* \simeq \pi^*(\Lambda T_Y^*) \otimes \Lambda \mathfrak{g}^*$.

Thus we can project onto $\Omega(P)_{horiz}$ by using the augmentation on $\Lambda \mathfrak{g}^*$ and we get

$$\Omega(P) \simeq \Omega(P)_{horiz} \otimes \Lambda \mathfrak{g}^* \xrightarrow{\mathrm{id} \otimes \varepsilon} \Omega(P)_{horiz}$$

which is a ring homomorphism. The same should work with a bundle present, so I really need to know what happens to dT under this projector.

June 25, 1984

Let's review Witten's approach to the determinant line bundle over the space $\mathcal{A}$ of connections. He associates to each connection A the Dirac operator with mass m, and takes the limit as $m \to \infty$. The physics is given by the YM Lagrangian plus the Dirac Lagrangian. When quantized, the Dirac part is realized by a fermion Fock space, and for large m, one gets only a line in this Fock space, because the rest has too high energy. In the large m limit the fermion determinant is a Chern-Simons functional on $\mathcal{A}$. Combined with the YM Lagrangian, it quantizes to the covariant Laplacean with coefficients in a line bundle with connection.

I propose to look at the Dirac operator in two dimensions with a mass:

$$-i\gamma^\mu D_\mu + \beta m$$

where γ^μ, β anti-commute and have square 1. My feeling is that this should be very close to the Grassmannian graph embedding.

Let's analyze the spectrum of $-i\not{D} + \beta m$. Put $H = -i\not{D}$ which is self-adjoint, and such that

$$\beta H = -H\beta.$$

Then if $H\psi = \lambda\psi$ we have

$$H(\beta\psi) = -\beta(H\psi) = (-\lambda)\beta\psi$$

so the spectrum is symmetric.

If $\lambda \neq 0$, then $\psi, \beta\psi$ are independent so we can restrict $H + \beta m$ to $\mathbb{C}\psi + \mathbb{C}\beta\psi$ where it becomes

$$H + \beta m = \begin{pmatrix} \lambda & m \\ m & -\lambda \end{pmatrix}$$

The eigenvalues are $\pm\sqrt{\lambda^2 + m^2} = \pm m\left(1 + \frac{\lambda^2}{2m^2} + \cdots\right)$.

The picture is therefore as follows.

Witten's idea I believe goes as follows. We assign to the operator $-i\not{D} + \beta m$ the line in Fock space which is the ground state, i.e. it corresponds to the negative eigenspaces. Then as the connection varies we get a different line in Fock space. Now take the natural connection on the space of lines in Fock space and let $m \to \infty$.

What this amounts to is that we are assigning to $-i\not{D} + \beta m$ its negative eigenspace. This reminds me of the graph embedding. Let's write $\not{D} = i(T + T^*)$ where T goes from the $\beta = 1$ eigenspace to the $\beta = -1$ eigenspace. Thus

$$-i\not{D} + \beta m = \begin{pmatrix} m & T^* \\ T & -m \end{pmatrix} \qquad \text{on } V = V^0 \oplus V^1.$$

Notice

$$\begin{pmatrix} m & T^* \\ T & -m \end{pmatrix} \begin{pmatrix} m + \sqrt{m^2 + T^*T} \\ T \end{pmatrix}$$

$$= \begin{pmatrix} m^2 + T^*T + m\sqrt{m^2+T^*T} \\ \cancel{Tm} + T\sqrt{m^2+T^*T} \cancel{-mT} \end{pmatrix} = \begin{pmatrix} m + \sqrt{m^2+T^*T} \\ T \end{pmatrix} \sqrt{m^2 + T^*T}$$

hence the map $v \in V^0$ goes to $\begin{pmatrix} m + \sqrt{m^2 + T^*T} \\ T \end{pmatrix} v$

identifies V^0 with a subspace stable under $\begin{pmatrix} m & T^* \\ T & -m \end{pmatrix}$ such that this operator corresponds to the positive operator $\sqrt{m^2+T^*T}$ on V^0. Similarly

$$\begin{pmatrix} m & T^* \\ T & -m \end{pmatrix}\begin{pmatrix} -T^* \\ m+\sqrt{m^2+TT^*} \end{pmatrix} = \begin{pmatrix} \cancel{-mT^*} \cancel{+T^*m} + T^*\sqrt{m^2+TT^*} \\ -TT^* - m^2 - m\sqrt{m^2+TT^*} \end{pmatrix}$$

$$= \begin{pmatrix} -T^* \\ m+\sqrt{m^2+TT^*} \end{pmatrix}\left(-\sqrt{m^2+TT^*}\right)$$

So that $v \longrightarrow \begin{pmatrix} -T^* \\ m+\sqrt{m^2+TT^*} \end{pmatrix} v$ identifies V' with a subspace stable under the operator, such that the operator corresponds to the negative operator $-\sqrt{m^2+TT^*}$.

Therefore we see that we have something not identical with, but closely related to, the graph embedding, which involves the orthogonal subspaces

$$\begin{pmatrix} 1 \\ T \end{pmatrix} V^0 \qquad \begin{pmatrix} -T^* \\ 1 \end{pmatrix} V'$$

However it is clear that this embedding in the Grassmannian is more complicated than the graph embedding. Also it is not holomorphic in T, so that the connection form probably has both dT and dT^*.

Notice that

$$\begin{pmatrix} m+\sqrt{m^2+T^*T} \\ T \end{pmatrix} V^0 = \begin{pmatrix} 1+\sqrt{1+(\frac{1}{m}T^*)(\frac{1}{m}T)} \\ \frac{1}{m}T \end{pmatrix} V^0$$

so that the $m \to \infty$ limit coincides with the "classical": hT, $h \to 0$ limit.

At this point I more or less understand the significance of Witten's approach based on the Dirac operator with mass term:

$$-i\not{\partial} + \beta m = \begin{pmatrix} m & T^* \\ T & -m \end{pmatrix}$$

and letting $m \to \infty$. To this operator one associates the line in Fock space belonging to the negative or positive eigenvalues; the positive eigenspace is

$$\begin{pmatrix} m + \sqrt{m^2 + T^*T} \\ T \end{pmatrix} V^0$$

and is close to the graph of T.

I guess one thing I don't really see is what this has to do with the determinant line bundle which is normally constructed with the zero modes

June 26, 1984

Problem of Baum + Connes: To define the good notion of equivariant cohomology such that the Chern character from equivariant K-theory $\otimes\mathbb{C}$ is an isomorphism. The good equivariant cohomology should be isomorphic to the localized cyclic homology of the crossed product of $C^\infty(M)$ with G.

Example: $K_G(pt) = R(G)$. The group algebra: $C^\infty(G)$ under convolution, is in some sense a product of matrix rings, one for each irreducible representation. The cyclic homology of a matrix ring is $\mathbb{C}$ in even degrees and 0 in odd degrees. Thus the cyclic homology of the group algebra is roughly a direct sum of copies of $\mathbb{C}$, one for each elt of $\hat{G}$.

Let's make this more precise. Take G to be the circle, whence $\hat{G} = \mathbb{Z}$. The convolution algebra $A = C^\infty(S^1)$ is by Fourier isomorphic to the rapidly-decreasing functions on $\hat{G}$. Thus $HC_0(A) = A$ is not exactly the direct sum of $\mathbb{C}$ for each irreducible representation.

The cyclic cohomology $HC^0(A) = A^*$ is the space of distributions on S^1, or tempered functions on $\mathbb{Z}$. It seems clear that I should be able to construct tempered higher cocycles by using the homomorphisms $\chi_n : C^\infty(S^1) \longrightarrow \mathbb{C}$.

Thus we see that cyclic homology of $C^\infty(G)$ is necessarily a completion of $R(G)\otimes\mathbb{C}$.

The usual equivariant DR cohomology of (G, M) is given by the complex

$$\{W(\mathfrak{g}) \otimes \Omega(M)\}_{\text{basic}} \xrightarrow{\sim} \{S(\mathfrak{g}^*) \otimes \Omega(M)\}^G$$

If we take $M = pt$, then we get $S(\mathfrak{g}^*)^G$. What is the Chern character map

$$R(G) \longrightarrow \hat{S}(\mathfrak{g}^*)^G \quad ?$$

More generally given a G-manifold M and an equivariant bundle E over M, we define the Chern character in the equivariant DR complex as follows. We choose an invariant connection D and let φ be the associated Higgs field. The equivariant curvature is then

$D^2 +$ [redacted] $\Omega\varphi \in \{S(\mathfrak{g}^*) \otimes \Omega(M, \operatorname{End} E)\}^G$

and the Chern character is

$$\operatorname{tr}\left(e^{D^2 + \Omega\varphi}\right) \in \hat{S}(\mathfrak{g}^*)^G$$

When M is a point and E is a representation of G in a vector space V, then the Higgs field is the induced Lie algebra representation

$$\rho : \mathfrak{g} \longrightarrow \operatorname{End} V$$

hence $\Omega\varphi \in \mathfrak{g}^* \otimes \operatorname{End} V$ is another version of ρ and

$$\operatorname{tr}(e^{\Omega\varphi}) \in \hat{S}(\mathfrak{g}^*)^G$$

should be just the formal character of the representation, i.e. the [redacted] restriction of $\operatorname{tr}\rho$ on G to a formal nbd. of the identity.

Now what I am looking for might be a variant of the Weil algebra $W(\mathfrak{g}) = S(\mathfrak{g}^*) \otimes \Lambda(\mathfrak{g}^*)$ in which $S(\mathfrak{g}^*)$ is replaced by something like the smooth functions on the group. This suggests the De Rham complex $\Omega(G) \cong \Omega^0(G) \otimes \Lambda \mathfrak{g}^*$. I think the formal completion of the DR complex is $\hat{S}(\mathfrak{g}^*) \otimes \Lambda \mathfrak{g}^*$ and it is dual [redacted] canonically to the standard resolution $U(\mathfrak{g}) \otimes \Lambda \mathfrak{g}$ used in Lie algebra cohomology.

Thus maybe we want some super mixture of $W(\mathfrak{g})$ and $\Omega(G)$.

Look at circle actions. $R(S^1) = \mathbb{Z}[T, T^{-1}]$

$R(S^1) \otimes \mathbb{C} = \mathbb{C}[T, T^{-1}]$. This has maximal ideals for each $\zeta \in \mathbb{C}^\times$, and the 0 ideal is prime.

June 28, 1984

Another way we might go from $T: V^0 \to V^1$ to a point in the Grassmannian is to use the map

$$\mathfrak{g}/\mathfrak{h} \longrightarrow G/H \qquad \begin{array}{l} G = U(n+m) \\ H = U(n) \times U(m) \end{array}$$

which one gets from the exponential map in G. Here we identify $\mathfrak{g}/\mathfrak{h}$ with ~~[illegible]~~ skew-adjoint matrices of the form $\begin{pmatrix} 0 & -T^* \\ T & 0 \end{pmatrix}$. One has

$$\exp\begin{pmatrix} 0 & -T^* \\ T & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 0 & -T^* \\ T & 0 \end{pmatrix} + \frac{1}{2!}\begin{pmatrix} -T^*T & 0 \\ 0 & -TT^* \end{pmatrix}$$

$$+ \frac{1}{3!}\begin{pmatrix} 0 & T^*TT^* \\ -TT^*T & 0 \end{pmatrix} + \cdots$$

$$= \begin{pmatrix} 1 - \frac{1}{2!}T^*T + \frac{1}{4!}(T^*T)^2 - \frac{1}{6!}(T^*T)^3 \cdots & -T^*\left(1 - \frac{1}{3!}(TT^*) + \cdots\right) \\ T\left(1 - \frac{1}{3!}(T^*T) + \frac{1}{5!}(T^*T)^2 - \cdots\right) & 1 - \frac{1}{2!}TT^* + \frac{1}{4!}(TT^*)^2 - \cdots \end{pmatrix}$$

$$= \begin{pmatrix} \cos(T^*T)^{1/2} & -T^*\dfrac{\sin(TT^*)^{1/2}}{(TT^*)^{1/2}} \\ T\dfrac{\sin(T^*T)^{1/2}}{(T^*T)^{1/2}} & \cos(TT^*)^{1/2} \end{pmatrix}$$

Now take this unitary matrix and move V^0 by it to obtain the subspace

$$\begin{pmatrix} \cos(T^*T)^{1/2} \\ T\dfrac{\sin(T^*T)^{1/2}}{(T^*T)^{1/2}} \end{pmatrix} V^0 \qquad \text{instead of} \qquad \begin{pmatrix} m + \sqrt{m^2 + T^*T} \\ T \end{pmatrix} V^0$$

(Here $m = \frac{1}{2}$ so that as $T \to 0$ we are asymptotic to the graph $\left(\frac{1}{T}\right)V^0.$)

At this point I should begin to review what I would like to do with the Grassmannian. Especially since Gelfand will be here next week, I should probably think in terms of Pontryagin, or Chern forms, and how they restrict to orbits under the multiplicative group. Idea here: The rescaling: $T \mapsto hT$, which is one of the things I play with, comes from a $\mathbb{G}_m$ of the sort occurring in various ways.

[redacted]

Let's review the trangression ~~situation~~. I have G acting on $V^0 \oplus V^1$ preserving the grading and an odd $L = i(T+T^*)$, $T: V^0 \longrightarrow V^1$. Then I can consider the family of superconnections on the trivial bundle over G with fibre V.

$$d + t\omega + L \qquad\qquad \omega = g^{-1}dg$$

with curvature

$$L^2 + t[L,\omega] + (t^2-t)\omega^2 .$$

We have

$$\operatorname{tr}_s e^{u(L^2+[L,\omega])} - \operatorname{tr}_s e^{uL^2} = d\int_0^1 dt \, \operatorname{tr}_s\left(e^{u(L^2+t[L,\omega]+(t^2-t)\omega^2)} u\omega\right)$$

Now we divide by u and take Laplace transform (Motivation comes from the formal identity

$$\int_0^\infty \operatorname{tr}\left(e^{-tA}\right)\frac{dt}{t} = -\log\det(A) \qquad)$$

Thus applying $\int_0^\infty e^{-\lambda u}\frac{du}{u}$? we get

$$[redacted]\ -\log\det\left(\frac{\lambda-L^2-[L,\omega]}{\lambda-L^2}\right) = d\int_0^1 dt\; tr_s\left(\frac{1}{\lambda-L^2-t[L,\omega]-(t^2-t)\omega^2}\,\omega\right)$$

$$\|$$

$$-tr\log\left(1-\frac{1}{\lambda-L^2}[L,\omega]\right) = \sum_{k\geq 1}\frac{1}{2k}\,tr_s\left(\frac{1}{\lambda-L^2}[L,\omega]\right)^{2k}.$$

In particular it appears that

$$\boxed{\frac{1}{2}\,tr_s\left(\frac{1}{\lambda-L^2}[L,\omega]\right)^2 = d\;tr_s\left(\frac{1}{\lambda-L^2}\,\omega\right)}$$

Check:

$$tr_s\left\{\frac{1}{\lambda-L^2}(L\omega+\omega L)\frac{1}{\lambda-L^2}(L\omega+\omega L)\right\}$$

$$= tr_s\left\{\frac{1}{\lambda-L^2}L\omega\,\frac{1}{\lambda-L^2}L\omega + \frac{1}{\lambda-L^2}\omega L\,\frac{1}{\lambda-L^2}\omega L\right\}$$ ← this is 0 as moving L around produces a sign

$$tr_s\left\{\frac{1}{\lambda-L^2}\,\omega\,\underbrace{L\frac{1}{\lambda-L^2}L}_{-1+\frac{\lambda}{\lambda-L^2}}\,\omega + \underbrace{(-L)\frac{1}{\lambda-L^2}L}_{1-\frac{\lambda}{\lambda-L^2}}\omega\,\frac{1}{\lambda-L^2}\,\omega L\right\}$$

$$= tr_s\left(-\frac{1}{\lambda-L^2}\omega^2 + \omega\frac{1}{\lambda-L^2}\omega\right) = 2\,tr_s\left(\frac{1}{\lambda-L^2}(-\omega^2)\right)$$

$$= 2\,d\,tr_s\left(\frac{1}{\lambda-L^2}\,\omega\right)$$ so it checks.

Note that the importance of this result is that it should be part of the picture for [redacted] the [redacted] determinant line bundles. It gives the equivariant trivialization.

Recall that the curvature of the det. line bundle is

$$h = \lambda^{-1/2}$$

$$\frac{1}{2}\, tr_s \left(\frac{\lambda^{1/2}}{\lambda - L^2}\, dL \right)^2 . = \frac{1}{2}\, tr_s \left(\frac{1}{1 - h^2 L^2}\, d(hL) \right)^2$$

We now want to use the contraction hL; $0 \leq h \leq 1$ to write this as a coboundary. We should have the formula

$$\frac{d}{dh}\, tr_s \left(\frac{1}{1-h^2L^2}\, d(hL) \right)^2 = d \frac{d}{dh}\, tr_s \left(\frac{h^2}{1-h^2L^2}\, L\, dL \right).$$

Actually let's do this more carefully. I know that for a family D_t of connections

$$\frac{d}{dt}\, tr\left(e^{D^2}\right) = d\, tr\left(e^{D^2} \dot{D}\right)$$

so that
$$\frac{d}{dt}\, tr\left(D^2\right) = d\, tr\left(\dot{D}\right) = d\, tr\left(\dot{\Theta}\right)$$

where $\dot{\Theta}$ is the derivative of the connection form. In particular

$$tr\left(D_1^2\right) - tr\left(D_0^2\right) = d\left\{ tr\, \Theta_1 - tr\, \Theta_0 \right\},$$

so it would seem we have

$$\frac{1}{2}\, tr_s \left(\frac{1}{1-L^2}\, dL \right)^2 = d \frac{1}{2} tr_s \left(\frac{1}{1-L^2}\, L\, dL \right).$$

This checks because if $\Theta = L \frac{1}{1-L^2} dL$, then Θ is of odd degree and so $tr_s\, \Theta^2 = 0$, thus $tr_s\, \Omega = tr_s (d\Theta + \Theta^2) = d\, tr_s(\Theta)$. With λ put in it is

$$\frac{1}{2}\, tr_s \left(\frac{\lambda^{1/2}}{\lambda - L^2}\, dL \right)^2 = d\, \frac{1}{2}\, tr_s \left(\frac{1}{\lambda - L^2}\, L\, dL \right)$$

Now put $[L, \omega] = dL$ and this becomes

$$d \, \frac{1}{2} \, tr_s \, \frac{1}{\lambda - L^2} L(L\omega + \omega L) = d \, tr_s \left(\frac{L^2}{\lambda - L^2} \omega \right)$$

So we have the two ways of writing the first character form:

$$\frac{1}{2} \, tr_s \left(\frac{\lambda^{1/2}}{\lambda - L^2} [L, \omega] \right)^2 = d \, tr_s \left(\frac{\lambda}{\lambda - L^2} \omega \right)$$

$$= d \, tr_s \left(\frac{L^2}{\lambda - L^2} \omega \right)$$

The former comes from the equivariant cohomology, the latter comes from the radial flow. So the transgression is

$$\boxed{tr_s \left(\frac{\lambda}{\lambda - L^2} \omega \right) - tr_s \left(\frac{L^2}{\lambda - L^2} \omega \right) = tr_s \, \omega}$$

which is very good.

June 29, 1984

Suppose G acts on V and we consider the induced action on the Grassmannian Grass. We know that the character forms on Grass pull back under a G-map $G \longrightarrow$ Grass to become coboundaries. Let's carry this out explicitly.

Let the G-map be $g \longmapsto gV^0$. Then the pull-back of the canonical subbundle over Grass can be identified with the trivial bundle $G \times V^0$. Better: The pull-back of the subbundle should be identified with the subbundle of $G \times V$ whose fibre at g is gV^0. A section of this pull-back is therefore of the form $g\psi(g)$, where $\psi : G \to V^0$. In this way the pull-back is trivialized, i.e. identified with $G \times V^0$.

The grass connection is formed as follows:

$$g e_0 g^{-1} d\, g\psi(g) = g e_0 d\psi(g) + g e_0 g^{-1} dg \cdot \psi(g)$$

$$= g\{d\psi(g) + e_0 g^{-1} dg\, e_0 \psi(g)\}$$

where $e_0 =$ projector on V^0. Thus we have

$$D = d + e_0 \omega e_0 \qquad \omega = g^{-1}dg$$

in terms of the trivialization.

The curvature is

$$D^2 = e_0(-\omega^2)e_0 + e_0\omega e_0 \omega e_0$$

$$= e_0\omega[e_0, \omega]e_0 = e_0\omega(1-e_0)[e_0,\omega]$$

$$= e_0[e_0,\omega][e_0,\omega]$$

We can check this as follows. Pull back $ededе$

via $e = g e_0 g^{-1}$. Then

$$de = dg\, e_0\, g^{-1} + g e_0 (-g^{-1} dg\, g^{-1})$$
$$= [g e_0 g^{-1}, g\, dg^{-1}]$$

so

$$e\,de\,de \longmapsto g e_0 g^{-1} [g e_0 g^{-1}, g\, dg^{-1}]^2$$
$$= g \{ e_0 [e_0, -g^{-1} dg]^2 \} g^{-1}$$

which agrees.

Let's review: We consider G acting on V, a subspace V^0 of V corresponding to a given point of Grass, and the map

$$f : G \longrightarrow \text{Grass} \qquad g \longmapsto g V^0.$$

This map can also be described as the map associated to the subbundle E of the trivial bundle $G \times V$ whose fibre E_g at g is $g V^0$. From the Grass connection, E gets a connection D. On the other hand E is trivial, because it is equivariant. Thus we make G act diagonally on $G \times V$, whence we get an action of G on E. The invariant sections are of the form $s(g) = gv$ with $v \in V^0$. Thus we get a G-trivialization

$$G \times V^0 \longrightarrow E$$
$$(g, v) \longmapsto gv \in E_g = g V^0$$

where G acts on $G \times V^0$ on the left.

In terms of this G-trivialization we compute D. If $e_0 =$ the projector on V^0, ~~then~~ and $s(g) = g \psi(g)$

is the section of E corresponding to the section of $G \times V^0$ given by ψ, then

$$(Ds)(g) = g e_0 g^{-1} \, d \, g\psi(g)$$

$$= g e_0 g^{-1} dg + g e_0 \, d\psi$$

so $\quad D = d + e_0 (g^{-1}dg) e_0 \quad$ acting on $\Omega^*(G) \otimes V^0$

Next change notation $e_0 \mapsto e$. We have on the trivial bundle over G with fibre V^0 the left-invariant connection $D = d + e\omega e$ having the curvature $e[\omega, e][\omega, e]$. Use the path of connection $D_t = d + te\omega e$ to join D to the connection d which descends. The curvature is

$$F_t = t\,(e[\omega,e]^2) + (t^2 - t)(e\omega e)^2 .$$

The k-th character form for D is therefore a coboundary:

$$\frac{1}{k!} \operatorname{tr} \left(e[\omega,e]^2\right)^k = d \int_0^1 dt \; \operatorname{tr}\left(\frac{1}{(k-1)!} F_t^{k-1} e\omega e \right)$$

Thus we get simply the Chern-Simons form with the curvature $F = e[\omega,e]^2$ and connection $A = e\omega e$.

The next project will be to relate this to the super-connection formalism. This time we suppose $V = V^0 \oplus V^1$ is $\mathbb{Z}_2$-graded and G is the group of degree zero unitary autos. V^0 here is not the same as the V^0 above. We now fix the projector e to be the projector on the graph of $T: V^0 \longrightarrow V^1$. Thus

$$e = \begin{pmatrix} 1 \\ T \end{pmatrix} \frac{1}{1+T^*T} \begin{pmatrix} 1 & T^* \end{pmatrix}$$

Now $\omega = \begin{pmatrix} \omega^0 & 0 \\ 0 & \omega^1 \end{pmatrix}$ so that

$$e\omega e = \begin{pmatrix} 1 \\ T \end{pmatrix} \frac{1}{1+T^*T} \left(\omega^0 + T^*\omega^1 T\right) \frac{1}{1+T^*T} \begin{pmatrix} 1 & T^* \end{pmatrix}$$

Problem: To make the connection between the super connection formalism and the Grassmannian + graph formalism.

Let us consider then $E = E^0 \oplus E^1$ with a $D + L$. We can then consider the graph embedding of E^0 in $E^0 \oplus E^1$ given by $L = i(T + T^*)$. The projector on the graph $\begin{pmatrix} 1 \\ T \end{pmatrix} E^0$ is

$$e = \begin{pmatrix} 1 \\ T \end{pmatrix} \frac{1}{1+T^*T} \begin{pmatrix} 1 & T^* \end{pmatrix}.$$

Using this projector and the connection D on E one gets an induced Grassmannian connection on $\Gamma_T = \begin{pmatrix} 1 \\ T \end{pmatrix} E^0$ which can be identified with E^0. Then we can ask how it differs from D on E^0. Let $\psi \in \Gamma(E^0)$, and associate to it $\begin{pmatrix} 1 \\ T \end{pmatrix}\psi = \begin{pmatrix} \psi \\ T\psi \end{pmatrix}$ which is a section of Γ_T. Then its covariant derivative is

$$e\, D\begin{pmatrix} \psi \\ T\psi \end{pmatrix} = e\left\{\begin{pmatrix} 1 \\ T \end{pmatrix} D\psi + \begin{pmatrix} 0 \\ [D,T] \end{pmatrix}\psi\right\}$$

$$= \begin{pmatrix} 1 \\ T \end{pmatrix}\left\{ D\psi + \frac{1}{1+T^*T} T^*[D,T]\psi \right\}$$

The new connection on E^0 is therefore

$$\tilde{D} = D + \frac{1}{1+T^*T} T^*[D,T]$$

If we take the sum of the new connections on E^0, E^1 we should get

$$\boxed{\tilde{D} = D + \frac{1}{1-L^2} L[D,L] = D + L\frac{1}{1-L^2}[D,L]}$$

sign wrong?

The new curvature is

$$\tilde{D}^2 = D^2 + [D,L]\frac{1}{1-L^2}[D,L]$$
$$- L\frac{1}{1-L^2}\Big([D,L]L - L[D,L]\Big) \qquad \frac{1}{1-L^2}[D,L]$$
$$- L\frac{1}{1-L^2}[D,[D,L]]$$
$$+ L\frac{1}{1-L^2}[D,L]\,L\frac{1}{1-L^2}[D,L]$$

$$= \left(\frac{1}{1-L^2}[D,L]\right)^2 + D^2 - \frac{1}{1-L^2}L[D^2,L]$$

$$\boxed{\tilde{D}^2 = \left(\frac{1}{1-L^2}[D,L]\right)^2 + \frac{1}{1-L^2}\left(D^2 - LD^2L\right)}$$

This agrees with what I found using equivariant forms on p. 21.

Now that I have this formula for the connection and curvature we can hope to relate the Chern character forms computed via $\tilde{D}^2$ with the one constructed from the super connection business. The problem is that the latter are not homogeneous.

$$tr_s\; e^{u(D^2 + [D,L] + L^2)}$$

This project seems almost hopeless, because $\tilde{D}^2$

is apparently not a function of $L^2, [D,L], D^2$.

$$\mathrm{tr}_s(\tilde{D}^2) = \mathrm{tr}_s\left(\frac{1}{1-L^2}[D,L]\right)^2 + \mathrm{tr}_s\left(\frac{1}{1-L^2}(D^2 - LD^2L)\right)$$

and $$\mathrm{tr}_s\left(\frac{1}{1-L^2}(-LD^2L)\right) = \mathrm{tr}_s\left(L\frac{1}{1-L^2}LD^2\right)$$

so $$\mathrm{tr}_s\left(\frac{1}{1-L^2}(D^2 - LD^2L)\right) = \mathrm{tr}_s\left(\frac{1}{1-L^2}(D^2 + L^2D^2)\right)$$

This checks with what I got on page 21:

$$\mathrm{tr}_{V^0}\left(\frac{1}{1+T^*T}(\Omega^0 + T^*\Omega^1 T)\right) = \mathrm{tr}_{V^0}\left(\frac{1}{1+T^*T}\Omega^0\right) + \mathrm{tr}_{V^1}\left(\frac{TT^*}{1+TT^*}\Omega^1\right)$$

$$\mathrm{tr}_{V^1}\left(\frac{1}{1+TT^*}(\Omega^1 + T\Omega^0 T^*)\right) = \mathrm{tr}_{V^1}\left(\frac{1}{1+TT^*}\Omega^1\right) + \mathrm{tr}_{V^0}\left(\frac{T^*T}{1+T^*T}\Omega^0\right)$$

Subtracting gives

$$\mathrm{tr}_{V^0}\left(\frac{1-T^*T}{1+T^*T}\Omega^0\right) - \mathrm{tr}_{V^1}\left(\frac{1-TT^*}{1+TT^*}\Omega^1\right)$$

but it looks <u>too</u> hard to do anything with.

$$\mathrm{tr}_s\, e^{u(L^2 + [D,L] + D^2)} - \mathrm{tr}_s\, e^{uL^2} \sim \sum_{k\geq 1} u^k ch_k$$

$$\therefore\ -\log \det_s\left(\frac{\lambda - L^2 - [D,L] - D^2}{\lambda - L^2}\right) \sim \sum_{k\geq 1} \frac{(k-1)!}{\lambda^k}\, ch_k$$

$$\parallel$$

$$-\log \det_s\left(1 - \frac{1}{\lambda - L^2}([D,L] + D^2)\right) = \sum_{k\geq 1} \frac{1}{k}\,\mathrm{tr}_s\left(\frac{1}{\lambda - L^2}([D,L] + D^2)\right)^k$$

Now take components of degree 2.

$$\frac{0!}{\lambda}\, ch_1 \sim \frac{1}{1}\, tr_s\left(\frac{1}{\lambda - L^2} D^2\right) + \frac{1}{2}\, tr_s\left(\frac{1}{\lambda - L^2}[D,L]\right)^2$$

so it seems that ch_1 is rep. by

$$\boxed{\frac{1}{2}\, tr_s\left(\frac{\lambda^{1/2}}{\lambda - L^2}[D,L]\right)^2 + tr_s\left(\frac{\lambda}{\lambda - L^2} D^2\right)}$$

If I take $\lambda = 1$ I get

$$(*) \qquad \frac{1}{2}\, tr_s\left(\frac{1}{1-L^2}[D,L]\right)^2 + tr_s\left(\frac{1}{1-L^2} D^2\right)$$

which is not quite $\frac{1}{2} tr_s(\tilde{D}^2)$. Note

$$\frac{1}{2}\, tr_s\, \tilde{D}^2 = \frac{1}{2}\, tr_s\left(\frac{1}{1-L^2}[D,L]\right)^2 + \frac{1}{2}\, tr_s\, \frac{1}{1-L^2}\underbrace{(D^2 + L^2 D^2)}_{2D^2 + (L^2-1)D^2}$$

$$= (*) - \frac{1}{2}\, tr_s\, D^2$$

This is all very interesting. Why? You know that $tr_s\, \tilde{D}^2$ is supposed to represent ch_1 essentially because of the contraction $L \to 0$. But

$$tr_s\, \tilde{D}^2 = tr_s\left(\frac{1}{1-L^2}[D,L]\right)^2 + tr_s\left(\frac{1}{1-L^2}(D^2 + L^2 D^2)\right)$$

and the $\frac{1}{2}$ factor is missing.

Take the $\mathbb{P}^1$ example where L goes to infinity

<u>June 30, 1984</u>

Let's consider a pair $E = E^0 \oplus E^1$ of vector bundles with $D = D^0 \oplus D^1$ over M and an $L = i\begin{pmatrix} 0 & T^* \\ T & 0 \end{pmatrix} : E^0 \rightleftarrows E^1$ which goes to ∞ as we go toward the boundary of M. I am interested in the K-class with compact support defined by $E^0 \xrightarrow{T} E^1$; it can be non-trivial even when E^0, E^1 are trivial.

Then we look at the Grassmannian bundle $\mathrm{Grass}(E)$ over M of subspaces of E of $\dim = \dim E^0 = \dim E^1$, and the section s obtained from the graph of T: $\Gamma_T = \begin{pmatrix} 1 \\ T \end{pmatrix} E^0 \subset E$. Γ_T carries an induced connection from D, and we are interested in the character forms belonging to this connection. Let us consider one of them $\varphi(\Gamma_T)$. Since T is going to infinity, we know that far out Γ_T is very close to E^1, hence $\varphi(\Gamma_T) - \varphi(E^1)$ goes to zero.

What should φ of the K-class be? You normally take $\varphi(E^0)$, $\varphi(E^1)$ and write $\varphi(E^0) - \varphi(E^1) = d\beta$ using a path between D^0 and $T^{-1}D^1T$. What this means is that one equips E^0 with a connection compatible with D^1 and T. Clearly the induced connection on $\Gamma_T \simeq E^0$ works.

Thus we see that $\varphi(\Gamma_T) - \varphi(E^1)$ is a differential form representing the φ-class of the K-element.

By symmetry we can consider $\Gamma_T^\perp = \begin{pmatrix} -T^* \\ 1 \end{pmatrix} E^1$ which is isomorphic to E^1 but approaches E^0 at ∞. Then $-\varphi(\Gamma_T^\perp) + \varphi(E^0)$ also represents the φ-class of the K-element. So we now average:

$$\tfrac{1}{2}\left\{\varphi(\Gamma_T) - \varphi(\Gamma_T^\perp)\right\} + \tfrac{1}{2}\left\{\varphi(E^0) - \varphi(E^1)\right\}$$

represents the φ class.

Let's check this against yesterday's formula where $\varphi = \mathrm{tr}\, D^2$. Then

$$\tfrac{1}{2}\,\underset{\substack{\uparrow \\ \tilde{D} \text{ curvature} \\ \text{of } \Gamma_T \oplus \Gamma_T^\perp}}{\mathrm{tr}_s(\tilde{D}^2)} + \tfrac{1}{2}\,\mathrm{tr}_s(D^2) = \underbrace{\tfrac{1}{2}\,\mathrm{tr}_s\left(\frac{1}{1-L^2}[D,L]\right)^2 + \mathrm{tr}_s\left(\frac{1}{1-L^2}D^2\right)}_{\text{this is what came out of super connection}}$$

At this point I should be able to work out completely the determinant line bundle theory, since I have now connected up the super-connection version with the graph.

Observation: If we use the super character $\mathrm{tr}_s\, e^{u(L^2 + [D,L] + D^2)}$, then we can define the determinant-line-bundle's curvature form even though the above formula

$$\tfrac{1}{2}\,\mathrm{tr}_s\left(\frac{\lambda^{1/2}}{\lambda - L^2}[D,L]\right)^2 + \mathrm{tr}_s\left(\frac{\lambda}{\lambda - L^2}D^2\right)$$

doesn't have a meaning because the operators are not of trace class.

What seems to be happening is that one has

$$\begin{array}{ccc} \displaystyle\int_0^\infty \mathrm{tr}_s\left(e^{u\tilde{D}^2}\right) e^{-\lambda u} u^s \frac{du}{u} & \sim & \displaystyle\int_0^\infty \sum u^k ch_k\, e^{-\lambda} u^s \frac{du}{u} \\ \| & & \| \\ \mathrm{tr}_s\left(\dfrac{\Gamma(s)}{(\lambda - \tilde{D}^2)^s}\right) & \sim & \displaystyle\sum \frac{\Gamma(s+k)}{\lambda^{s+k}}\, ch_k \end{array}$$

$\tilde{D} = D + L$

Thus $tr_s(e^{u\tilde{D}^2})$ is defined for $0<u<\infty$, but a priori it will have negative powers of u in the asymptotic expansion as $u \downarrow 0$. These have to be killed by u^s before one can integrate. The above formula shows formally that it is possible to represent ch_k by the component of degree $2k$ in

$$tr_s\left(\frac{1}{(\lambda-\tilde{D}^2)^s}\right)$$

up to a constant (which is the function $\frac{\Gamma(s+k)}{\Gamma(s)\lambda^{s+k}}$.)

Notice that we are assuming that the cohomology class of $tr_s(e^{u\tilde{D}^2})$ is $\sum u^k ch_k$, a polynomial in u.

July 1, 1984

Attempt to write paper for Gelfand. The big difficulty will be at the beginning with the construction of the determinant line bundle as a holomorphic bundle, and the consistency of this with eigenspaces.

Let's consider what we see from the eigenspace viewpoint. We work around a fixed operator T_0 and let a be a positive number not an eigenvalue of T_0^*T (or $T_0T_0^*$), then this is true in a neighborhood of T_0 and so we get a finite-dimensional subcomplex

$$\begin{array}{ccc} \Omega^0 & \xrightarrow{T} & \Omega^1 \\ \cup & & \cup \\ F^0_{a,T} & \xrightarrow{T} & F^1_{a,T} \end{array}$$

which is also a quotient complex. Note: T^* depends upon the inner products in Ω^0, Ω^1 so one can't use the Sobolev 1-inner product in Ω^1.

So it seems to me that I have to be prepared to use a smoothly varying f.d. subspace F_T of Ω^1 which is transversal to T in the sense that $\operatorname{Im} T + F_T = \Omega^1$. Then I have to show that $T^{-1}(F_T)$ is a smoothly varying subspace of Ω^0 and the exact sequence

$$0 \longrightarrow \operatorname{Ker} T \longrightarrow T^{-1}(F_T) \longrightarrow F_T \longrightarrow \operatorname{Cok} T \longrightarrow 0$$

leads to an isomorphism

$$\mathcal{L}_T = \lambda(F_T) \otimes \lambda(T^{-1}F_T)^*$$

of smooth line bundles.

Let's discuss the difficulties with doing this properly and convincingly. Fundamentally you need to give a line bundle via local trivializations and transition functions. So how does this work in the situation at hand? Let's try the Grassmannian viewpoint. We associate to T its graph $\Gamma_T \subset \Omega^0 \oplus \Omega^1$ which is a closed subspace projecting 1-1+ densely in Ω^0 and such that $\Gamma_T \longrightarrow \Omega^1$ is Fredholm.

July 2, 1984

The problem is the following. I want to consider the family S of holomorphic structures on a fixed C^∞ v.b. E over M. I claim that there is a determinant line bundle on S, but I have to explain why. The general reason this is true is as follows: One forms over $S \times M$ a holomorphic vector bundle $\mathcal{E}$, one considers the projection $\pi : S\times M \longrightarrow S$, and then appeals to the general theorem that $R\pi_*(\mathcal{E})$ is a perfect complex. This general theorem probably comes out of arguments similar to the Grothendieck proof of finite-dimensionality of cohomology of coherent sheaves, i.e. using Frechet spaces of sections over open sets, Montel stuff, etc.

What I have to do is to find arguments which are consistent with the Fredholm picture. In this picture I replace the complex of sheaves $R\pi_*(\mathcal{E})$ by a complex of infinite dimensional vector bundles over S, and where the differentials in the complex are holom. on S. Let's be specific.

Each point of S determines a $\bar\partial$ operator $T: W \longrightarrow V$ where $W = \Omega^{0,0}(M,E)$, $V = \Omega^{0,1}(M,E)$. The first thing to understand is why the kernel + cokernel of T are finite dimensional. This is the theory of elliptic operators on a compact manifold. What one does to prove it is to write down a parametrix

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What I need to know is the following. I suppose given a family T_y depending on some parameters y in a holomorphic way. Suppose also given a family F_y of f.d. subspaces of V depending holom. on y. (This means that locally we can find a basis for F_y whose members are holomorphic in y.). Assume T_y is transversal to F_y for $y = y_0$. Then for y near y_0, T_y is transversal to F_y and $T_y^{-1}(F_y)$ is a family of f.d. subspaces of W depending holom. on y.

In order to prove this sort of thing it probably enough to go thru the proof that the inverse image is OK in a transversal ~~situation~~ and set that it adapts to the Fredholm situation.

Simplest case: Suppose T_{y_0} is invertible, then we know T_y is invertible and T_y^{-1} is holomorphic. This is because the inverse is a holomorphic function by the geometric series.

Next case: Assume T_{y_0} is surjective, and take $F = 0$, whence you want to show $\mathrm{Ker}(T_y)$ is a holomorphic subbundle of the trivial bundle with fibre V. This means one must produce a trivialization. The obvious method is to choose a projection π onto $\mathrm{Ker}(T_{y_0})$; this gives

$$V \xrightarrow{(\pi, T_y)} \mathrm{Ker}(T_{y_0}) \times W$$

which is an isomorphism at y_0 hence near y_0. Thus $\pi : \mathrm{Ker}(T_y) \xrightarrow{\sim} W$ is the desired trivialization.

July 7, 1984

Having finished the note for Gelfand, there are two problems of importance.

1) How to do the calculations in Riemannian, as opposed to holomorphic, notation. The idea would be to generalize to the case where the holom. structure on the surface varies.

2) The link between the connection on the determinant line bundle and your proposed definition of Chern character forms, and also the version with the Grassmannian.

<u>July 9, 1984</u>

Problem: Link between Connes S-operator and Bott periodicity. This problem arises as follows from the theory of dilog.

Dilog has something to do with the 2nd Chern class c_2, probably c_2 restricted to rank 2 bundles. I believe it describes an element of $H^3(GL_2\mathbb{C}, \mathbb{C}^*)$ in the sense of Segal's continuous or Wigner's borelian cohomology. On the other hand dilog is related to the central extension of the loop group and the tame symbol. The central extension acts on the determinant line bundle over the Hilbert-Schmidt Lagrangian, and this is c_1 type stuff. The link between c_1 and c_2 should come from Bott periodicity.

So let's try to make this more precise. Bott periodicity sets up a homotopy equiv.

$$\Omega U \simeq \mathbb{Z} \times BU$$

One direction ($\longleftarrow$) associates to a vector bundle E/X the product $pr_1^*(E) \times [\mathcal{O}(1) - 1]$ over $X \times S^2$. The other direction ($\longrightarrow$) uses the index for a family of Toeplitz operators. The determinant line bundle over ΩU is the det. line bundle of this family.

c_2, or ch_2, comes in in the following way. One takes the ch_2 class on BU, then follows it under $U = \Omega BU$ and then $\Omega U = \Omega^2 BU$. On U it is represented by $tr(\omega^3)$ essentially, and from this 3-form one manufactures a 2-form on ΩU. This whole business should be the local index formula for the family of Dirac operators on the circle.

I should go over and get straight the local index theorem for the family of Dirac operators on S^1. This is an odd degree ~~situation~~. I should review the super [redacted] connection business in this case.

Let us start with a vector bundle $E = E^0 \oplus E^1$ which is a super C_1-module, where $C_1 = \mathbb{C} + \mathbb{C}\sigma$ is the Clifford algebra. Think of C_1 as acting to the right of E. Endos. of E as a C_1-module are endos T commuting with σ strictly, since

$$T(s\sigma) = (Ts)\sigma$$

involves no change in order. Thus σ itself is an endo

$$\mathrm{End}_{C_1}(E) \cong \mathrm{End}(E^0) \otimes C_1$$

and any endo of E as a C_1-module is of the form $u + v\sigma$, where u, v are degree zero endos. and hence can be identified with elts of $\mathrm{End}(E^0)$.

The super-trace is

$$\mathrm{End}_{C_1}(E) \longrightarrow C_1/[C_1, C_1] = \mathbb{C}\bar{\sigma}$$

$$u + v\sigma \longmapsto \mathrm{tr}(v)\bar{\sigma}$$

Now we have the formula

$$\mathrm{tr}_s\left(e^{a+b\sigma}\right) = \mathrm{tr}\left(e^{a+ib}\right)_{odd}(-i\bar{\sigma})$$

where a is even, b is odd.

I propose to apply this to the curvature

$$(D+L)^2 = D^2 + [D, L] + L^2$$

$$= \underbrace{D^2 + L^2}_{a} + \underbrace{[D, L]}_{b\sigma}$$

where $L = \begin{pmatrix} 0 & u \\ u & 0 \end{pmatrix} = u\sigma$ and [redacted]
$u^* = -u$.

We are going to want to take u to be [redacted] a family of Dirac operators over the odd manifolds. These are skew-adjoint, for example $u = \partial_x + A(x)$ with A purely imaginary. Now in order to use the above superconnection formalism we take two copies of the space on which u acts, and then we take $L = u\sigma = \begin{pmatrix} 0 & u \\ u & 0 \end{pmatrix}$. But a natural thing to do is to form something like

$$\frac{1}{i}\partial_t + u$$

or better the $\bar{\partial}$-operator $D = u + i\partial_t$ in one higher dimension. The operator ~~of~~ interest is

$$-i\begin{pmatrix} 0 & D^* \\ D & 0 \end{pmatrix} = -i\begin{pmatrix} 0 & -u+i\partial \\ u+i\partial & 0 \end{pmatrix} = \begin{pmatrix} 0 & \partial \\ \partial & 0 \end{pmatrix} + i\begin{pmatrix} 0 & u \\ u & 0 \end{pmatrix}$$

July 10, 1984

[redacted] Project: "A ~~Local~~ index formula for families of Dirac operators." It is necessary to understand what are the proper formulations. Let's go over the background including the examples.

The first idea [redacted] is that the Dirac operator on the total space of a fibre bundle is the Dirac operator on the base with coefficients in the family of Dirac operators along the fibres. In order to formulate this we need ~~to~~ define a [redacted] Dirac operator with coefficients in a super bundle equipped with super connection: $\not{D} = \gamma^\mu D_\mu + \varepsilon L$.

The next idea is to prove the index thm. for ~~such~~ a $\not{D}$. We introduce h to rescale the metric: $\not{D} = h\gamma^\mu D_\mu + \varepsilon L$. The index $\mathrm{Tr}_s(e^{\not{D}^2})$ is independent of h, so it can be evaluated in the classical limit: $h \to 0$. $\not{D}$ operates on $S \otimes E$ where S is the spinors. Now the idea is to introduce Getzler's filtration into the algebra of differential operators on $S \otimes E$. This means we introduce the graded algebra over $\mathbb{C}[h]$ generated by [redacted] $\varphi \in \Gamma(\mathrm{End}\,E)$, $h\gamma^\mu$, hD_μ and then specialize to $h=0$. $\not{D}$ does not belong to the algebra but $[\not{D}, \ ]$ is a derivation on the algebra, whose square is the inner derivation $[\not{D}^2, \ ]$.

The $h=0$ specialization or associated graded algebra is a tensor product

$$S(T) \otimes \Lambda T^* \otimes \mathrm{End}\, E$$

twisted by the curvature. ~~What the [illegible]~~ The square of $\not{D}$ in the $h=0$ limit is

$$-p^2 + \left(D^2 + [D, L] + L^2\right).$$

These two ~~terms~~ commute, and as Getzler shows

e^{-p^2} brings in the $\hat{A}$-genus because of the twisted product structure yielding

$$\mathrm{Tr}_s\, e^{\not{D}^2} = \int_M \left(\frac{i}{2\pi}\right)^m \hat{A}(M)\cdot \mathrm{tr}_s\left(e^{D^2+[D,L]+L^2}\right)$$

where $\dim M = 2m$.

From this index thm. one sees that the good character form for a superconnection is $\mathrm{tr}_s(e^{(D+L)^2})$.

Next we should consider the odd case. The problem will be to get the correct link between the two ways of thinking of an odd-style operator, that is, either ungraded, or graded over C_1.

Let's start with a Dirac operator on $\mathbb{R}^n = \mathbb{R}^{2m-1}\times\mathbb{R}$ which we can write

$$\gamma^\mu D_\mu + i\gamma^n\left(\frac{1}{i}D_n\right) \qquad \text{on } S\otimes E$$

where S is the module of spinors over C_n, $\mu<n$, and E is ungraded. Take sections over the vertical, or $\mathbb{R}$ fibres, we get an operator

$$\not{D} = \gamma^\mu D_\mu + i\gamma^n(u) \qquad \text{on } S\otimes\mathcal{H}$$

where u is self-adjoint on $\mathcal{H}$, which is ungraded. So our first idea is to regard a odd diml Dirac operator as an ungraded self-adjoint (or skew-adjoint) operator u, and then form the above $\not{D}$. Introduce h:

$$\not{D} = h\gamma^\mu D_\mu + \gamma^n(iu).$$

Idea: $S\otimes\mathcal{H} = (\mathcal{H}\otimes C_1)\otimes_{C_1} S$. S is a left super-module over $C_n = C_{n-1}\hat{\otimes}C_1$, whence it would seem this tensor product is naturally a C_{n-1}-module

There is a possibility that the last section of the superconnection paper is wrong with its convention that things strictly commute with σ. Hence we should really produce the correct formalism out of the index theorem.

So let's begin with a Dirac on $\mathbb{R}^n$, n even, and suppose $n = k + l$ where k, l are odd. Corresponding to $\mathbb{R}^n = \mathbb{R}^k \times \mathbb{R}^l$ we write the Dirac operator

$$\not{D} = \gamma^\mu D_\mu + \overbrace{\gamma^j D_j}^{L} \qquad \text{on } S \otimes E$$

where S = spinors over C_n. Rescale:

$$\not{D} = h\gamma^\mu D_\mu + L$$

$$\not{D}^2 = h^2 D_\mu^2 + \tfrac{1}{2} h^2 \gamma^\mu \gamma^\nu F_{\mu\nu} + h\gamma^\mu [D_\mu, L] + L^2$$

As usual we want to compute the index by letting $h \to 0$. This amounts to filtering

$$\mathrm{End}(S \otimes E) = C_n \otimes \mathrm{End}(E)$$

$$C_n = C_k \hat{\otimes} C_l$$

C_k generated by the γ^μ, $1 \leq \mu \leq k$

When we take the associated graded we get

$$\mathrm{gr}\, C_n = \Lambda(k) \hat{\otimes} C_l$$

$\Lambda(k)$ = ext. alg. gen. by $\omega^\mu = \overline{h\gamma^\mu}$.

The only thing you seek from the $\mathbb{R}^k$ viewpoint is an infinite-diml super bundle with the connection D_μ, the Clifford mult γ^μ and the odd operator L. This seems like a good place to begin. Thus we reach again the proper axioms for "Dirac operator".

So we start with a "Dirac" operator over R^{n-1}, $n=2m$. This is an operator of the form

$$\not{D} = \gamma^\mu D_\mu + L \qquad \text{on} \qquad E = E^0 \oplus E^1$$

where E is a C_{n-1}-module, D is a connection on E compatible with the C_{n-1}-action, and L is an odd degree operator on E compatible with C_{n-1}-action. Now E has a grading, so it is a graded C_{n-1} module, or an ungraded C_n-module, hence can be written canonically

$$E = S \otimes F$$

where F is ungraded.

Now we want to adopt the convention that $i^m \varepsilon = \gamma^1 \cdots \gamma^n$ in S defines a grading ε in S. From $\not{D}$ we get $\gamma^1, \ldots, \gamma^{n-1}, \varepsilon$ and then define γ^n by this relation.

Then γ^n anti-commutes with $\gamma^1, \ldots, \gamma^{n-1}, \varepsilon$; as L does also we see that $\gamma^n L$ commutes with $\gamma^1, \ldots, \gamma^{n-1}, \varepsilon$ hence is of the form $1 \otimes u$, where u acts on F. So we get the form

$$\not{D} = \gamma^\mu D_\mu + \gamma^n u \qquad \text{on} \qquad S \otimes F = E$$

with the grading on E induced from the grading on S.

<u>Conclusion</u>: Over R^{n-1}, $n = 2m$, the graded Dirac operators (these are the ones which can have indices) are of the form

$$\not{D} = \gamma^\mu D_\mu + \gamma^n u \qquad \text{on} \qquad S \otimes F$$

where D_μ is a connection on F, and u is a (skew-adjoint) operator on F.

Now we have to go thru the proof of the index theorem to see what differential form results, i.e. gets integrated over M^{n-1}. Let's make a guess based on

$$\not{D}^2 = h^2 D_\mu^2 + \frac{h^2}{2}\gamma\gamma F + h\gamma^\mu\gamma^n[D_\mu u] + u^2$$

The important thing is that γ^n is going to be like any old σ, but there will be some differences. So we go thru the perturbation expansion.

$$\cdots \gamma^\mu\gamma^n[D_\mu, u] \cdots \gamma^\nu\gamma^n[D_\nu, u] \cdots$$
$$= -\gamma^\mu[D_\mu, u] \cdots \gamma^\nu[D_\nu, u]$$

It seems like you get the same signs as if you worked with $\gamma^\mu[D, L]$ where $L = \gamma^n u$ and observed the sign conventions. You need someway to pick out the odd number of factors.

Maybe the important point is that the associated graded algebra is

$$\left(\Omega(M)\hat{\otimes} C_1\right) \otimes_{\Omega^0(M)} \Omega^0(\operatorname{End} F)$$

and that I can identify this with the algebra

$$\Omega(M, \operatorname{End} E)^\sigma = \Omega(M)\hat{\otimes}_{\Omega^0(M)}\left(\Omega^0(M, \operatorname{End} F)\otimes C_1\right)$$

where $E = F\otimes C_1$ considered as a right C_1-module

I want to go over the index theorem for the Dirac operator over $\mathbb{R}^{2m-1}$ which is $\mathbb{Z}_2$-graded

$$\not{D} = \gamma^\mu D_\mu + L.$$

Here $\not{D}$ acts on sections of a graded C_{2m-1}-module E. $E = E^0 + E^1$ is a graded C_{2m-1} module, D is a connection on E preserving the grading, and L is an odd degree endom. of E. Thus $[D_\mu, \gamma^\nu] = 0$, $[\gamma^\mu, L]_+ = 0$

Because E is a graded C_{2m-1}-module it is an ungraded C_{2m}-module, where C_{2m} has the generators $\gamma^1, \ldots, \gamma^{n-1}, \varepsilon$, ε being the grading of E. One defines γ^n on E so that one has the relation

$$\gamma^1 \cdots \gamma^n = i^m \varepsilon$$

Thus we know from simplicity of C_{2m} that

$$E \simeq S \otimes F \qquad F = \mathrm{Hom}_{C_{2m}}(S, E)$$

where F is ungraded vector bundle, and the grading on E comes from the grading on $S = S^0 \oplus S^1$.

Since $[D_\mu, \gamma^\nu] = 0$ it follows that D_μ comes from a connection on E. As $\gamma^n L$ [redacted] commutes with the γ^μ, ε it is induced by an endom. u of F. Note that L anti-commutes with $\gamma^1, \ldots, \gamma^{2m-1}, \varepsilon$, so it commutes with [redacted] γ^n. So we get the ~~forms~~ for the Dirac operator

$$\not{D} = \gamma^\mu D_\mu + \underbrace{\gamma^n u}_{L} \qquad \text{on } S \otimes F$$

Next we want to go through the proof of the index thm. which means we rescale the γ^μ: $\gamma^\mu \mapsto h\gamma^\mu$ and let $h \to 0$ in $\mathrm{tr}(e^{\not{D}^2})$ where

$$\not{D}^2 = h^2 D_\mu^2 + \frac{h^2}{2}\gamma^\mu \gamma^\nu F_{\mu\nu} + h\gamma^\mu [D_\mu, L] + L^2.$$

This operator lies in the algebra generated by the operators $h\gamma^\mu$, hD_μ, γ^n, $\Omega^0(\mathrm{End}\,F)$ whose associated graded algebra is generated by ω^μ, p_μ, γ^n, $\Omega^0(\mathrm{End}\,F)$ and is isom. to

$$\mathbb{C}[p_\mu] \otimes \Lambda[\omega^\mu] \hat{\otimes} C_1 \otimes \Omega^0(\mathrm{End}\,F).$$

Now I want the limit of the super-trace on this algebra. Notice that

$$\Lambda[\omega^\mu] \hat{\otimes} C_1 \otimes \Omega^0(\mathrm{End}\,F) = \Omega(\mathrm{End}\,F) \hat{\otimes} C_1$$

is generated by forms, endos of F, and γ^n. We want a trace on this with values in $\Omega(M)$ which somehow results from the natural trace on endos. of $S \otimes F$.

End S is generated by γ^μ, γ^n; we restrict to the subalgebra generated by $h\gamma^\mu, \gamma^n$. The supertrace on End $S = C_n$ see only the top

$$\mathrm{Tr}_S(h\gamma^1 \cdots h\gamma^{n-1}\gamma^n) = h^{n-1}(2i)^m$$

So if we divide by h^{n-1} we get in the $h = 0$ limit the linear functional

$$\Lambda[\omega^\mu] \hat{\otimes} C_1 \longrightarrow \mathbb{C}$$

which vanishes on $F_{n-2}\Lambda[\omega^\mu] \hat{\otimes} C_1 + \Lambda[\omega^\mu] \otimes \mathbb{C}$ and is such that

$$\omega^1 \cdots \omega^{n-1}\gamma^n \longmapsto (2i)^m.$$

Next we have to worry about the e^{-p^2} part which we expect should give the trace contribution

$$\int \frac{d^{n-1}p}{(2\pi h)^{n-1}} e^{-p^2} = \frac{(\sqrt{\pi})^{n-1}}{(2\pi)^{n-1} h^{n-1}}$$

I have decided that the evaluation of the index is compatible with what I wrote about superconnections in the odd case. The index formula should be

$$\text{Index} = \left(\frac{\sqrt{\pi}}{2\pi}\right)^{n-1} (2i)^m \int_M tr_s\left(e^{D^2 + [D,L] + L^2}\right)$$

where it is understood that

$$D^2 + \underbrace{[D,L]} + L^2 \quad \in \quad \Omega(M, \text{End}\, F) \otimes C_1$$

$$[D, \gamma^n u] = \gamma^n [D, u]$$

and the super trace is evaluated by the rule

$$tr_s\,(a + b\gamma^n) = tr_F(b) \in \Omega(M)$$

Example: $\quad \gamma^1 \partial_x + \gamma^2 i x = \begin{pmatrix} 0 & \partial_x + x \\ \partial_x - x & 0 \end{pmatrix}$

Here $\quad D = dx\,\partial_x, \quad L = \gamma^2 i x,\quad$ so that

$$[D, L] = dx\; i\gamma^2$$

$$tr_s\left(e^{D^2 + [D,L] + L^2}\right) = e^{-x^2}\, dx\, i$$

$$\text{Index} = \frac{1}{2\sqrt{\pi}}\, 2i \int e^{-x^2} dx\, i = (-1)\frac{1}{\sqrt{\pi}} \int e^{-x^2} dx = -1.$$

which is correct.

The above constant in front of $\int_M$ is

$$\left(\frac{1}{2\sqrt{\pi}}\right)^{n-1} (2i)^m = \left(\frac{i}{2\pi}\right)^m 2\sqrt{\pi}$$

July 11, 1984

Yesterday we went over the local index thm. for a odd-diml Dirac operator on $\mathbb{R}^{n-1}$, $n = 2m$

$$\not{D} = \gamma^\mu D_\mu + \text{[redacted]}\, \gamma^n u \qquad \text{on } S_n \otimes \text{[redacted]} F$$

where $u = -u^*$ is an operator on F [redacted]. We obtained

$$\text{Index} = \int_{\mathbb{R}^{n-1}} \left(\frac{i}{2\pi}\right)^m 2\sqrt{\pi}\ \ tr_\sigma\left(e^{D^2 + [D,u]\sigma + u^2}\right)$$

with the notation in my super connection paper. Thus

$$(D + u\sigma)^2 = D^2 + [D,u]\sigma + u^2 \quad \in \Omega(M, \operatorname{End} F) \hat{\otimes} C_1$$

$$tr_\sigma(a + b\sigma) = tr_F(b) \quad \in \Omega(M).$$

~~Let's consider the form in degree 1~~

Now once we have this formula we know how to define the character form for a bundle F equipped with connection D and skew-adjoint endom. u, namely

$$tr_\sigma\left(\text{[redacted]}\, e^{D^2 + [D,u]\sigma + u^2}\right).$$

Then we apply this to families of Dirac operators.

Let work out the formula in degree one. Precisely, let me consider a family of self-adjoint operators A [redacted] on a Hilbert bundle with conn. D. Put $u = iA$. The above form gives in degree 1 the form

$$tr\left(e^{-A^2}[D,A]\right) i.$$

This has to be multiplied by $\frac{i}{2\pi} 2\sqrt{\pi}$ in order to get the integral period form. So we obtain over the parameter space the 1-forms

$$-\frac{1}{\sqrt{\pi}}\ tr\left(e^{-h^2A^2}\, h\,\delta A\right)$$

where I have put in the scaling factor h.

This is to be compared with the one form belonging to the η-invariant. Recall that $A \longmapsto e^{i\pi\eta(A)}$ is a map of the parameter space to S^1. The corresponding 1-form with integral periods is

$$\delta\, \frac{1}{2\pi i} \log\left(e^{i\pi\eta(A)}\right) = \frac{1}{2}\,\delta\eta(A)$$

which one calculates in a standard way as follows.

$$\eta(s) = \sum \operatorname{sgn}(\lambda)\, |\lambda|^{-s}$$

$$\delta\eta(s) = \sum \operatorname{sgn}(\lambda)\, (-s)\, |\lambda|^{-s-1} \operatorname{sgn}(\lambda)\, \delta\lambda$$

$$= -s\ \mathrm{Tr}\ (A^2)^{-\frac{s+1}{2}}\, \delta A$$

$$= -s\ \frac{1}{\Gamma\left(\frac{s+1}{2}\right)} \int_0^\infty \mathrm{Tr}\left(e^{-tA^2}\delta A\right) t^{\frac{s+1}{2}}\, \frac{dt}{t}$$

$$\sim -s\ \frac{1}{\sqrt{\pi}}\ \frac{2}{s}\ \underbrace{\mathrm{Tr}\left(e^{-tA^2}\sqrt{t}\,\delta A\right)\Big|_{t=0}}_{\substack{\text{means coeff of } t^0 \\ \text{in the asymptotic expansion}}}$$

$$\therefore \quad \frac{1}{2}\,\delta\eta(A) = -\frac{1}{\sqrt{\pi}}\ \mathrm{Tr}\left(e^{-tA^2}\sqrt{t}\,\delta A\right)\Big|_{t=0}$$

showing things are very consistent.

Now we want to apply the formulas to the case of Dirac operators on the circle. Actually I haven't worked out the local index theorem for a family of operators yet.

We now consider a family u of Dirac operators on odd-diml manifolds. I then define character forms of odd degree on the parameter space by the formula

$$tr_\sigma\left(e^{h^2u^2 + h[D,u]\sigma + D^2}\right).$$

Now the problem is to evaluate this as $h \to 0$. The point is that the above formula gives forms whose cohomology class is independent of h.

One problem is whether the character forms have a limit as $h \to 0$, or do we have to take the 0-th coefficient in an asymptotic expansion?

I propose to work things out carefully in the case of Dirac operators on the circle

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Here's the problem I encountered. Let me consider the family of Dirac operators $D\!\!\!/_A$ on an odd diml. manifold. Then I define closed 1-forms on the space of connections $\mathcal{A}$ by

$$tr\left(e^{h^2 D\!\!\!/^2}\, h\, d D\!\!\!/\right)$$

for any h, which all have the same DR class. If the local index thm. for families works, then the above form has a limit as $h \to 0$, not just an asymptotic series in h. This limit is essentially the differential of the η-invariant.

What is surprising is the cancellation or vanishing of negative terms in the asymptotic expansion.

Let's try to compute over S^1! A Dirac operator has the form

$$\not{D} = \partial_x + A(x) \qquad x \in S^1 = \mathbb{R}/2\pi\mathbb{Z}$$

where A is a skew-adjoint matrix. Up to the action of gauge transformations the only invariant $\not{D}$ has is the conjugacy class of its monodromy:

$$T\left\{e^{-\int_0^{2\pi} A(x)\,dx}\right\}$$

In particular the η-invariant must be a central function of the monodromy.

We know that

$$\frac{1}{2}\,\delta\eta\left(\tfrac{1}{i}\not{D}\right) = -\frac{1}{\sqrt{\pi}}\, tr\left(e^{t\not{D}^2}\sqrt{t}\,\tfrac{1}{i}\,\delta A\right)\Big|_{t=0}$$

and since we are in dimension 1 we use the leading term of the heat kernel.

$$\frac{e^{-\frac{|x-x'|^2}{4t}}}{(4\pi t)^{1/2}}$$

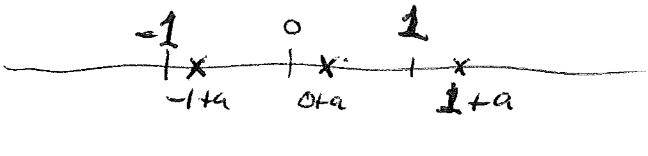
whence

$$\frac{1}{2}\,\delta\eta\left(\tfrac{1}{i}\not{D}\right) = \underbrace{-\frac{1}{\sqrt{\pi}}\,\frac{1}{\sqrt{4\pi}}\,\frac{1}{i}}_{-\frac{1}{2\pi i}}\int_0^{2\pi} tr(\delta A(x))\,dx$$

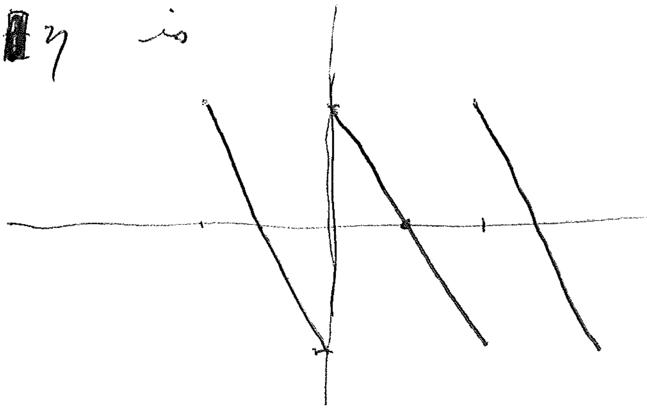
Example: $\not{D} = \partial_x + ia$ where a is constant. Here the eigenvalues are $n+a$ corresp. to the eigenfns. e^{inx} on $S^1 = \mathbb{R}/2\pi\mathbb{Z}$. Then

$$\eta(s) = \sum_{n\in\mathbb{Z}} sgn(n+a)\,|n+a|^{-s}$$

The eigenvalues appear thus say $0 < a < 1$



Clearly $\eta(s) = 0$ if $a \equiv \frac{1}{2} \mod \mathbb{Z}$.
Also if $0<a<1$ we can see roughly that $\eta(0) > 0$. As a move from 0^+ to 0^- η drops by -2, also the Weyl thm. implies η is linear in a except for the jumps at integers. Thus graph η is



$\eta = 1-2a$

$0<a<1.$

$$\frac{1}{2}\,\delta\eta = -\delta a = -\frac{1}{2\pi}\int_0^{2\pi} tr\,(\delta a)\,dx$$

and

$$e^{i\pi\eta} = e^{i\pi(1-2a)} = -e^{-2\pi i a} = -e^{-\int_0^{2\pi} A\,dx}$$

The formula should be in general for $\not{D} = \partial_x + A(x)$:

$$\boxed{\begin{aligned} e^{i\pi\eta(\frac{1}{i}\not{D})} &= (-1)^{\text{rank}} \det T\left\{e^{-\int_0^{2\pi} A(x)\,dx}\right\} \\ &= (-1)^{\text{rank}}\, e^{-\int_0^{2\pi} tr\,A(x)\,dx} \end{aligned}}$$

Check: $$\frac{1}{2\pi i}\,\delta \log e^{i\pi\eta} = \frac{1}{2\pi i}\left(-\int_0^{2\pi} tr\,\delta A(x)\,dx\right)$$