Connes (1/19/84)

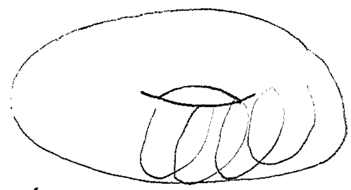
$$\begin{tikzcd} K^*(X) \arrow[r, "ch"] \arrow[d, "\text{any elliptic op gives}"'] & H^*(X) \arrow[d, "\text{any homology class}"] \\ \mathbb{Z} & \mathbb{C} \end{tikzcd}$$

rational isom.

$$\begin{tikzcd} X \arrow[dr] & \subset & S^n \arrow[dl] \\ & \mathbb{Z} & \end{tikzcd}$$

problem is that when you push forward you get something on open nbd. which then has to be compared with Bott element.

$$-\frac{d^2}{dx^2} + \underbrace{V(x)}_{\text{quasi periodic}} = D$$



T^2

D not invertible mod smooth kernel

D is invertible mod the algebra of the foliation

$$DQ - 1 = S_0 \qquad C^*(V, F)$$
$$QD - 1 = S_1$$

properties describing $C^*(V, F)$.

1) fibration: $C^*(V,F) \sim C^*(V/F) \otimes k$

2) local $W \subset V \Longrightarrow C^*(W, F_W) \subset C^*(V, F)$

3) functoriality.

Index $D \in K^0(C^*(V, F))$

Index thm. for meas. foliations

transverse measure $\longrightarrow$ tr on $C^*(V, F) \longrightarrow \mathrm{Hom}(K_0(C^*(V,F)), \mathbb{C})$

$$\langle ch\, \sigma_D\, Td_{\mathbb{C}} V, [C] \rangle$$

Ruelle-Sullivan current

two obvious question arise.

If scalar curvature of leaves > 0 $D =$ Dirac $\Rightarrow$ Index $D = 0$
$K^{\delta}(C^*(V,F))$.

leaf-wise h. eq. Signature of leaf $\in K_0(C^*(V,F))$

1) How to get info on $K(C(V,F))$
2) ~~How to pass from~~ K to cohomology.

V/F $\langle ch \times P(p_i)\, Q(c_i)$ GF classes, $[V/F]\rangle$ transv. oriented

Idea: to integrate.

Results: If $\hat{A}(V) \neq 0$ can't have foliation $R > 0$
some signature result.

G graph of foliation $\rightarrow BG$
$\Gamma, V \longrightarrow V \times_{\Gamma} E\Gamma = V_{\Gamma}$

Main conjecture

$$\begin{tikzcd}
K_{*,\tau}(BG) \arrow[r, "\mu", "\sim"'] \arrow[d, "ch"] & K(C^*(V,F)) \arrow[d, dashed, "\text{problem}"] \\
H_*(BG) \arrow[r, "\forall \omega \in H^*(BG)"'] & \mathbb{C}
\end{tikzcd}$$

ω can be pontragin or GF class $BG \longrightarrow B\Gamma_q$

Thm: says $\downarrow$ $\exists$ an

This theorem says you can integrate over the leaf space both in K-theory + cohomology

Proof involves three steps

① $C^\infty(V)$ $\quad \int f^0 df^1 \cdots df^n = \tau(f^0, \ldots, f^n)$

based on Grassmann alg. — doesn't work

So need cyclic cohomology

Lemma: $\forall \tau$ n cyclic cocycle, then $e \in \mathrm{Proj}(\mathcal{A})$

$\longmapsto \tau(e, \ldots, e)$ is an additive map $K(\mathcal{A}) \longrightarrow \mathbb{C}$.

cyclic cohomology is the linearization of algebra

$\mathcal{A} \longmapsto \mathcal{A}^\natural \xrightarrow{\mathrm{Ext}^n_\Lambda}$ Cyclic theory of $\mathcal{A}$.

derived functor of traces.

② C^* alg $A \overset{\text{dense}}{\supset} \mathcal{A}$ + cyclic cocycle τ defined on $\mathcal{A}$

problem is to show that map defined by cyclic cocycle on $\mathcal{A}$ extend to $K(A)$

1-cocycle When meaningful on A?

$$\tau(f^0, f^1) = \langle \delta(f^1), f^0 \rangle \qquad \delta : \mathcal{A} \subset A \to A^*$$

δ is a derivation

$$\langle \delta(f^1), f^0 \rangle = -\langle \delta(f^0), f^1 \rangle$$

So $\delta(f)$ is closable

basic lemma says closure has same K-theory
Banach alg; holom.

So a cyclic cocycle satisfying a weak continuity condition will pair with topology

$$\left| \tau(x^1 da^1 \, x^2 da^2 \quad x^n da^n) \right| \leq C_{a^1 \ldots a^n} \prod \|x^i\|$$

③ completely new non-comm. phenomenon

$V, \Gamma \qquad \sum f_g U_g \qquad \sum \omega_g U_g \qquad U_g \omega U_g^{-1} = g^*$

$$d \sum f_g U_g = \sum df_g U_g$$

$$\int \sum \omega_g U_g = \int \omega_e \qquad \mathcal{U}$$

$\omega^1, \ldots, \omega^n$ basis $\qquad f \omega^j = \sum \omega^i \theta_i^j(f)$

like Tomita modular theory.

turns out to be a dual action of $GL(n, \mathbb{R})$

But not a unitary repn.
no invariant Riem. metric

General idea: hyperbolic $\xrightarrow[\text{reduction}]{}$ parabolic

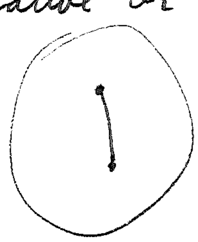
Γ, V | reduction of T_V to $\begin{bmatrix} SO(p) & 0 \\ * & SO(q) \end{bmatrix}$
| Γ-inv. reduction

Put ideas of Mackey, Zeeman, Kasparov

$V \rtimes \Gamma \qquad \begin{matrix} GL(n, \mathbb{R}) \\ SO(n, \mathbb{R}) \end{matrix}$ | 1-frame bdle $\underbrace{J_1(V)^+ / SO(n)}_{W} \xrightarrow{p} V$

fibres are symmetric spaces GL_n^+/SO_n

negative or zero curvature, so can apply Kasparov



geodesics
angle goes to zero

get $\delta \in KK_\Gamma(V, W)$ $\qquad$ action of Γ on W is distal

$$K(V/\Gamma) \longrightarrow K(W/\Gamma)$$ can integrate here

this is exactly like passing from III to II by taking cross product with the modular auto group

Where Gelf-Fuks classes come from. Consider

$$J_k(V)^+ / SO(n).$$

Generalization of Tomita Takesaki modular theory to cyclic cocycles involving a dual action of GL_n^+.

Physics

Hurder

two new theorems about cocycles over amenable groups
applications to characteristic classes for foliations.

A foliated set whose leaves are C^∞
obstruction to foliated spaces having certain measure properties.

<u>Problem</u>: Classify all C^2 foliations on a compact manifold

<u>Def</u>: Say two codim n foliations are concordant if $\exists$ fol. on $M \times \mathbb{R}$ ....

Haefliger-Thurston: concordance classes with given ν = [redacted] lifts $M \xrightarrow{} B\Gamma_n$, $M \xrightarrow{\nu} BO_n$, $B\Gamma_n \to BO_n$

Remark 1: The direction $\Leftarrow$ is a great existence thm.
2: Uncountable types

<u>Problem</u>: Find another (better) equiv. relation

(Connes: Are you interested in the longitudinal or transverse aspects.
longitudinal ↑ dynamic
transverse ↑ topology of leaf space)

Hurder wants to pass to measurable foliated space, and find equiv. relns.
What are the invariants of a foliation as a dynamical system?
So let X be a meas. subset = union of leaves. $\subset M$.
For $\mathcal{F}/X$ ergodic, we have invariants

A. von Neumann type
  I $\iff X/\mathcal{F}$ = standard
  II $\iff$ no type I component but has invariant transv. measure
  III what's left

B. Amenability   Thm. (Connes-Feldman-Weiss) F/X amenable
  $\iff F/X$ hyperfinite
  $\iff F/X$ is a limit of type I

<u>Ex</u>: Solvable Lie gp. action.

C. Growth rate of leaves.

volume in leaf of volume of ball of rad r)

D. Representations of Virtual group $(X, \mathcal{F}|X)$.

E. Properties of measure + C^* algebra.

Secondary classes of $\mathcal{F}$

$$WO(n) \longrightarrow \Delta^*(M)$$

$$\shortparallel$$

$$H^*(gl_n, O_n) \otimes \left(R[c_1, \ldots, c_n] / \deg > 2n\right)$$

$$y_I \quad \otimes \quad c_J$$

(Connes – $WO(n)$ is invariant differential forms on higher jet bundles.)

If M is oriented, then $\Delta_*(y_I c_J) \in H^p(M) = H^{n-p}(M)^*$

<u>Thm</u>: (Heitsch - Hurder) If X is saturated (and $\deg c_J = 2n$), then get fnl on $H^{n-p}(M)$, $[\phi] \longrightarrow \int_X \Delta(y_I c_J) \wedge \phi$

<u>Thm</u>: There is an operator $\chi(y_I)$ on X s.t. $\Delta_*(y_I c_J) = \chi_X(y_I)[c_J]$ and the operator $\chi_X(y_I)$ depends only on measurable coh. class of the normal GL_n-cocycle to F

The goal: These characteristic classes distinguish the different types A, B, C, D, E.