D. Quillen: Infinite determinants over Riemann surfaces.

Let M be a closed Riemann surface, let E be a C^∞ vector bundle over M, and let $\mathcal{A}$ be the space of holomorphic structures on E. To a holomorphic structure we can associate a $\bar\partial$ operator

$$D : E \longrightarrow E \otimes T^{0,1}$$

which is a first order differential operator whose symbol is the map

$$id \otimes \pi \;:\; E \otimes T^* \longrightarrow E \otimes T^{0,1}$$

where π is the projection of $T^* = T^{1,0} \oplus T^{0,1}$ on the second factor. Conversely any first order operator with this symbol is the $\bar\partial$-operator belonging to a unique holomorphic structure on E: The holomorphic sections are the sections killed by the operator. It follows that the different holomorphic structures are distinguished by the 0th order parts of their $\bar\partial$-operator. Precisely if we fix a basepoint in $\mathcal{A}$ with $\bar\partial$-operator D_0, then the other structures are described by the operators

$$D = D_0 + \alpha$$

as α runs over $\Gamma(\mathrm{Hom}(E,E) \otimes T^{0,1})$. Thus $\mathcal{A}$ is an affine space over the complex vector space $\Gamma(\mathrm{Hom}(E,E)\otimes T^{0,1})$, and hence is an infinite-dim'l complex manifold.

We also know that the $\bar\partial$-operator is a resolution of the sheaf of holomorphic sections $\mathcal{E}_D$, hence

$$0 \to H^0(M, \mathcal{E}_D) \longrightarrow \Gamma(M, E) \xrightarrow{\Gamma(D)} \Gamma(M, E \otimes T^{0,1}) \to H^1(M, \mathcal{E}_D) \to 0.$$

(Dolbeault thm.). Also we have Riemann-Roch:

$$h^0(\mathcal{E}_D) - h^1(\mathcal{E}_D) = \deg E + (\operatorname{rank} E)(1-g)$$

and this number is independent of the holom. structure.

Example: $M = \mathbb{C}/\Gamma$ elliptic curve, E = trivial line bundle. $T^{0,1}$ has base $d\bar{z}$, so the usual holom. ~~structure~~ belongs to the operator

$$D_0 = \partial_{\bar{z}} : \mathbb{1} \longrightarrow \mathbb{1} \qquad\qquad \partial_{\bar{z}} = \tfrac{1}{2}(\partial_x + i\partial_y)$$

The other holomorphic structures belong to

$$D = \partial_{\bar{z}} + \alpha$$

where $\alpha \in C^\infty(M)$. We will be particularly interested in the case where α is constant; this gives a 1-diml cx. line in $\mathcal{A}$.

Now I propose to define a holomorphic line bundle $\mathcal{L}$ over $\mathcal{A}$ with

$$\mathcal{L}_D = \lambda\left(H^1(\mathcal{E}_D)\right) \otimes \lambda\left(H^0(\mathcal{E}_D)\right)^*$$
$$= \lambda\left(\mathrm{Cok}\,\Gamma(D)\right) \otimes \lambda\left(\mathrm{Ker}\,\Gamma(D)\right)^*$$

where λ denotes the highest exterior power of a finite diml vector space. The idea is that as the holom. str. varies we get a family of elliptic operators over M parameterized by $\mathcal{A}$, hence a virtual complex bundle over $\mathcal{A}$ representing the index of this family; then we take the highest exterior power of this virtual bdle, which is a line bundle over $\mathcal{A}$.

More generally consider two Hilbert spaces V_1, V_0 and the space $\mathcal{F}$ of all linear transf.

$$V_1 \xrightarrow{T} V_0$$

which are Fredholm. Let F be a finite-diml subspace of V_0 and consider the open set of T in $\mathcal{F}$ which are transversal to F. Then we have

$$\begin{tikzcd}
0 \arrow[r] & \operatorname{Ker} T \arrow[r] & V_1 \arrow[r, "T"] & V_0 \arrow[r] & \operatorname{Cok} T \arrow[r] & 0 \\
0 \arrow[r] & \operatorname{Ker} T \arrow[r] \arrow[u, equal] & T^{-1}F \arrow[r, "T"] \arrow[u, phantom, "\cup"] & F \arrow[r] \arrow[u, phantom, "\cup"] & \operatorname{Cok} T \arrow[r] \arrow[u, equal] & 0
\end{tikzcd}$$

and hence a canonical isomorphism

$$\mathcal{L}_T^* = \lambda(\operatorname{Ker} T) \otimes \lambda(\operatorname{Cok} T)^* = \lambda(T^{-1}F) \otimes \lambda(F)^*$$

As T varies over the open set the right side is evidently a line bundle, holomorphic in the evident sense. As F increases we exhaust $\mathcal{F}$.

The dual bundle $\mathcal{L}$ is better because it has lots of holom. sections as follows. Usually one thinks of the exterior algebra $\Lambda(V)$ as a covariant functor of V. But it is also a contravariant functor for Fredholm maps in the following sense. Given $V_1 \xrightarrow{T} V_0$ as above one has

$$0 \to \operatorname{Im} T \to V_0 \to \operatorname{Cok} T \to 0$$

hence

$$\operatorname{gr} \Lambda(V_0) = \Lambda(\operatorname{Im} T) \otimes \Lambda(\operatorname{Cok} T)$$

and in particular we have an epimorphism

$$\Lambda^{q}(V_0) \twoheadrightarrow \Lambda^{q-p} \operatorname{Im} T \otimes \lambda(\operatorname{Cok} T) \qquad \text{if } \dim \operatorname{Cok} = p$$

also have

$$0 \to \operatorname{Ker} T \to V_1 \to \operatorname{Im} T \to 0$$

and this gives an injection

$$\lambda(\underbrace{\operatorname{Ker} T}_{p+d}) \otimes \Lambda^{q-p}(\operatorname{Im} T) \hookrightarrow \Lambda^{q+d}(V_1).$$

Putting these together we get

$$\Lambda_{q} V_0 \twoheadrightarrow \Lambda_{q-p} \operatorname{Im} T \otimes \lambda(\operatorname{Cok}_p T) \hookrightarrow \Lambda_{q+d} V_1 \otimes \lambda(\operatorname{Ker} T)^*_{p+d} \otimes \lambda(\operatorname{Cok} T)_p$$

i.e. a canonical embedding $\qquad d = \mathrm{Ind}(T)$

$$\mathcal{L}_T^* = \lambda(\mathrm{Ker}\, T) \otimes \lambda(\mathrm{Cok}\, T)^* \hookrightarrow \mathrm{Hom}^{(d)}(\Lambda V_0, \Lambda V_1)$$

Hence one sees that any ~~element in the dual~~ linear fnl. on the latter space gives a section of the bundle $\mathcal{L}$.

Important case: $\mathrm{Ind} = 0$. Over the open set where T is invertible we have a canonical section.