Patterson — three new results of past 2+3 years.

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should be part of a bigger theorem.

$\mathbb{H}^{N+1}$

B^{N+1} $\quad S^N$

$ds^2 = (1-\|x\|^2)^{-2}\|dx\|^2$ $\qquad Con(N) = $ gp of isoms.

$N=1: \quad Con(N) = PSL_2(\mathbb{R}) \rtimes \mathbb{Z}_2$

$N=2: \quad Con(N) \supset PSL_2(\mathbb{C})$

~~distance~~ length fn. $\quad L(\omega,\omega') = \dfrac{\|\omega-\omega'\|^2}{(1-\|\omega\|^2)(1-\|\omega'\|^2)}$

$p: \quad B^{N+1} \longrightarrow \mathbb{R}\times\mathbb{R}^{N+1}$

$x \longmapsto \frac{1}{2}\dfrac{(1+\|x\|^2)}{(1-\|x\|^2)}, \ \dfrac{x}{1-\|x\|^2}$

$g(y,\underline{x}) = y^2 - \|\underline{x}\|^2 \qquad g(p(x)) = 1$

$1 + 2L(x,x') = g(p(x),p(x'))$

Thus $Con(N)$ is (up to ~~conn.~~ coverings components) $O(1,N+1)$

Thm. of Loewe.

$$\boxed{\sinh^2 d(\omega,\omega') = L(\omega,\omega')}$$

G discrete $\subset Con(N)$

hyperbolic mans. $\qquad M = G\backslash\mathbb{H}^{N+1}$

(2) These manfs. have a kind of boundary roughly $G\backslash S^N$.

comp. of limit set $\underbrace{\Omega(G)}^{\text{ord. set}} = \{ \xi \in S^N \mid \exists\, U \ni \xi$ such that $\{g \mid gU \cap U \neq \emptyset\}$ is finite $\}$

$L(G) = S^N - \Omega(G)$

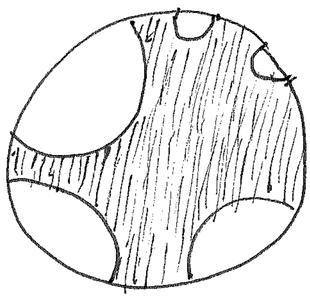
= smallest closed $\neq\emptyset$ invariant subset.

Typically $N=1$ Cantor like

$N=2$ quasi-circle

Fenchel-Nielsen

$FN(G)$ = smallest hyperbolically convex subset of B^N s.t. $\overline{FN(G)} \supset L(G)$.



G convex-cocompact $\iff \exists\, K$ compact $\subset B^{N+1}$ $\ni GK = FN(G)$.

$\Downarrow$ (not quite cocompact)

G geometrically finite (K finite volume instead)

$\Downarrow$

G finitely generated.

$\delta(G)$ = measure of convergence

$= \inf\{ s > 0 \mid \sum_{g\in G} L(\omega, g\omega)^{-s} < \infty \}$

$0 \leq \delta(G) \leq N$

$\delta(G) = N \iff G \backslash B^{N+1}$ compact.

prob measure on limit set

$$\sum L(0,g(0))^{-s}\,\delta_{g(0)} \Big/ \underbrace{\sum L(0,g(0))^{-s}}_{\text{assume } \uparrow\infty} \xrightarrow[s\to 0]{\text{subsequence}} \mu$$

Poisson kernel

$$P(w,\xi) = \frac{1-\|w\|^2}{\|w-\xi\|^2} \qquad -\Delta P(\cdot,\xi)^s = s(N-s)\,P(\cdot,\xi)^s$$

$$\int P(w,\xi)^{\delta(G)}\,d\mu = F(w)$$

satisfies
$$\begin{cases} -\Delta F = \delta(N-\delta)F \\ F(g\circ w) = F(w) \end{cases}$$

Thm: If G is convex cocompact, then
$\mu =$ Hausdorff $\delta(G)$-dim measure on $L(G)$
Hauss. dim. of $L(G) = \delta(G)$

(Also $\delta(G) =$ top. entropy of geodesic flow)

Thm. Same assumption

$$\sum L(w,gw')^{-s} = \frac{c\,F(w)F(w')}{s-\delta(G)} + \text{Fn. holom in } Re(s) > \delta(G)-\eta \quad (\eta>0)$$

Thm. Same assumptions $\exists$ const c_1, c_2

$$c_1 X^{\delta(G)} \le \#\{g\in G \mid L(w,gw') \le X\} \le c_2 X^{\delta(G)}$$

Next we deal with a bad group.
Let G be convex cocompact. Let $G_1 \lhd G$ have infinite index, G_1 infinite. $\Rightarrow L(G_1) = L(G)$

Recall concepts

G amenable if $\forall$ compact G space X, then $M(X)^G \neq 0$ $\quad$ $M(X)$ finite measures

e.g. abelian $\quad$ (not $SL_n\mathbb{Z}$)

(Brooks) $\iff$ $E_1 \subset E_2 \subset \cdots$ ~~finite~~ subsets exhausting G $\quad$ (assume G gen. by finite S)

$$\partial E_j = \{g \in E_j \mid \exists \gamma \in S \quad \gamma g \notin E_j\}$$

(Cayley diagram)

$$\frac{\# \partial E_j}{\# E_j} \longrightarrow 0$$

G transient $\quad$ Assume μ measure $\quad \mu(g^{-1}) = \mu(g)$

Supp μ generates G.

$$\mu^{(n)} = \mu * \cdots * \mu$$

$$G \text{ transient} \iff \sum_{n>0} \mu^{(n)}(\{e\}) < \infty$$

(only exceptions are known $e, \mathbb{Z}, \mathbb{Z}^2$, finite extensions of these)

Varopoulos: $G \overset{\infty}{\rhd} H_1 \overset{\infty}{\rhd} H_2 \overset{\infty}{\rhd} \{e\} \implies G$ transient

Thm (M. Rees) Suppose $A = G/G_1$ abelian. Let A_p be the subgp gen. by the parabolic elts. $N=1$

Then $\delta(G_1) = \delta(G)$

and $\displaystyle\sum_{g \in G_1} L(g\omega, g\omega')^{-\delta(G)} = \infty$ precisely when $rk(A_p) + rk(A) \leq 2$ if $\delta(G) = 1$.

+ if $rk(A_p) = 0$ + $rk(A) \leq 2$ (if $\delta(G) < 1$)

Proof uses symbolic dynamics + Bowen's theory of entropy

Theorem: (Varopoulos) Notations as above $N = 1$
A not necessarily abelian and $\delta(G) = 1$. Then

$$\sum L(\omega, g\omega')^{-\delta(G)} = \infty$$

$\Longleftrightarrow$ i) A is an extension of $1, \mathbb{Z}, \mathbb{Z}^2$ by fin. gp

ii) A is not ______________________
and every parabolic elt in G maps to an elt of finite order in A and

iii) A is not transient

Proof by random walks on groups

Thm. (Brooks) N arb. $\delta(G) > N/2$

$$\delta(G_1) = \delta(G) \Longleftrightarrow G/G_1 \text{ is amenable}$$

Proof by spectral theory. Cheeger's inequality.

exists a distribution δ $\varphi \longmapsto \sum_{\rho} M(\varphi)(\rho) \overset{\text{explicit fmla}}{=} \sum \varphi(p^k) \ldots$

where M is the Mellin transform and

ρ runs over the zeroes of $\zeta(s)$. R.H. $\Longleftrightarrow$

this distribution is ~~negative~~ positive i.e. $\varphi * \bar{\varphi}\,[\delta] \geq 0$