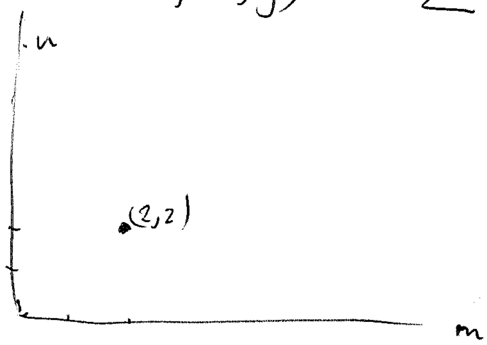
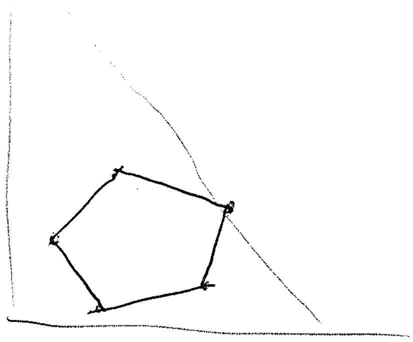
Atiyah Feb. 21, 1983 Newton Polyhedra + Alg. Geom.

Newton polygon: $f(x,y) = \sum a_{mn} x^m y^n$



plot $\{(m,n) \mid a_{mn} \neq 0\}$



plot points + take convex hull

If f, g have same Newton polygon

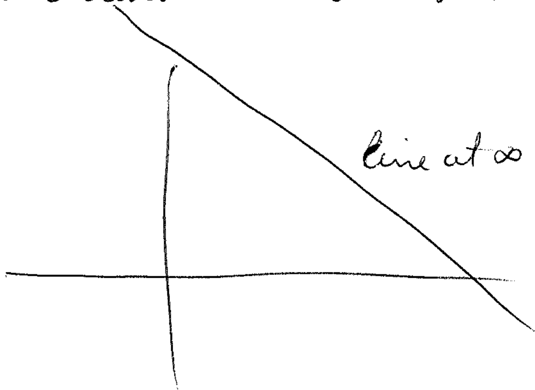
$f = 0 \qquad g = 0$

Ask for number of intersections

Assume coeff which are $\neq 0$ are general

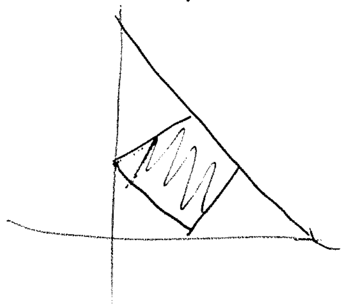
Get k^2 if all monomials are there.

Want solutions with $x \neq 0$, $y \neq 0$, finite



want to subtract off contribution due to $(x,y) = (0,0)$

Better picture



N of solns.

$= 2(\text{area of } \Delta - \text{area small } \Delta\text{'s})$

$= 2\,(\text{area of Newton polygon})$

Generalization $\quad z_1, \dots, z_n \qquad z_i \neq 0$

$$\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{Z}^n \qquad z^\alpha = z_1^{\alpha_1} \cdots z_n^{\alpha_n}$$

$$j = 1, \dots, k \qquad \sum_{\alpha \in S} c^j_\alpha z^\alpha = 0 \qquad S \subset \mathbb{Z}^n$$

the c^j_α $\alpha \in S$ are generic

$N(S) = ?$ $\qquad \hat{S}$ = convex hull.

Thm: $\quad N(S) = n!\, \mathrm{Vol}(\hat{S})$

Koushnirenko 1975

$T^n : |z| = 1$

Reformulate $\quad \alpha \in S \subset \mathbb{Z}^n$; α = 1-diml rep of T^n $\quad V_\alpha$

$z \longmapsto z^\alpha$

$$V_S = \bigoplus_{\alpha \in S} V_\alpha \quad \text{repn. of complex torus } T^n_{\mathbb{C}}$$

$$P_S = \mathbb{P}(V_S)$$

Let X be a generic orbit

$$z \cdot (1 \qquad 1) = (\qquad z^\alpha \qquad)$$

$\therefore$ X described by $\quad v_\alpha = z^\alpha \qquad z \in T^n_{\mathbb{C}}$.

Assume $\{\alpha - \beta \mid \alpha, \beta \in S\}$ generates $\mathbb{Z}^n$.

$\Longrightarrow$ generic orbit $\sim T^n_{\mathbb{C}}$

$\Longrightarrow$ $\overline{X}$ alg. subvariety of P_S of dim n.

Claim $\deg(\overline{X}) = N(S)$.

Defn. : $\deg(\overline{X})$ = no. of times it meets a generic linear subspace of codim n.

$$\sum c^j_\alpha v_\alpha = 0 \qquad j = 1, \ldots, n \qquad \text{generic linear space}$$

i.e. $$\sum c^j_\alpha z^\alpha = 0$$

Now bring in diffl geometry

$Y \subset P_{S-1}(\mathbb{C})$ — has standard closed 2-form ω

$\uparrow$ dim n.

$\Omega = \frac{\omega}{2\pi}$ integral gen. of $H^2(P)$

$$\deg Y = \int_Y \Omega^n$$

$$\operatorname{Vol} Y = \int_Y \frac{\omega^n}{n!}$$

because ω^n is the symplectic volume = Riemannian volume

$$\therefore \deg Y = \frac{n!}{(2\pi)^n} \operatorname{Vol}(Y)$$

Now the problem is to get from $\operatorname{Vol}(X)$ which is a $2n$ diml. integral to $\operatorname{Vol}(\hat{S})$ which is an n-diml integral

Moment map + Symplectic Geometry

s Harmonic oscillators

$$H = \frac{1}{2}\sum (p_j^2 + q_j^2) \qquad z_j = p_j + i q_j$$

$$\mu : \mathbb{C}^s \longrightarrow \mathbb{R}^s$$
$$z \longmapsto \frac{1}{2}\left(|z_1|^2, \dots, |z_s|^2\right)$$

conserved quantities

$H = 1$ or $\sum |z_j|^2 = 2$ stable under S^1

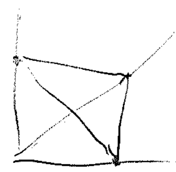
$$S^{2s-1}/S^1 = P_{s-1}(\mathbb{C})$$

$$\mu : P_{s-1} \longrightarrow \mathbb{R}^s$$

$\omega = \sum dp_j \, dq_j$ on $\mathbb{C}^s$ restricted to S^{2s-1} descends to the old ω.

$$\mu(P_{s-1}) = (s-1)\text{-simplex in } \mathbb{R}^s$$

$$\sum v_\alpha = 1$$



$T^n \qquad T^s$

$$\begin{tikzcd}
X \arrow[r, hook] \arrow[d, "\mu"] & P_s \arrow[d, "\mu"] \\
\mathbb{R}^n & \mathbb{R}^s \arrow[l] \\
\alpha & v_\alpha \arrow[l, mapsto]
\end{tikzcd}$$

naturality of moment map

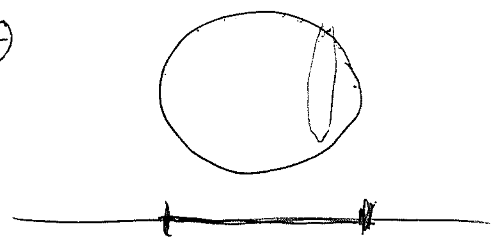
$\mu : X \longrightarrow \mu(X) \subset \mathbb{R}^n$ is a nice T^n fibration

Because this is a moment map

$$\mathrm{Vol}\ X = \mathrm{Vol}(T^n)\cdot \mathrm{Vol}(\mu X)$$

To make more familiar consider

$dx\, d\theta$



$$\therefore\ \mathrm{Vol}\, X = (2\pi)^n\, \mathrm{Vol}(\mu X)$$

So only thing needed is the convexity thm.

<u>Thm</u>: $\mu(X) = \mathrm{int}(\hat{S})$

$$\bar{X} \twoheadrightarrow \hat{S}$$

If σ is a face of $\hat{S}$ of dim i, then there is an orbit X_σ in $\bar{X}$ which is a torus bundle $X_\sigma \xrightarrow[T_\sigma]{} \sigma$ for a smaller torus.

Generalization due to D. Bernstein

$$\sum_{\alpha\in S_j} c_\alpha^j z^\alpha = 0 \qquad S_j \subset \mathbb{Z}^n$$

<u>Thm</u>: $N(S_1,\dots,S_n) = V(\hat{S}_1,\dots,\hat{S}_n)$

Minkowski mixed volume

In general if $A_1,\dots,A_n$ are convex can form $\lambda_1 A_1+\dots+\lambda A_n$. Then

$\mathrm{Vol}(\lambda_1 A_1+\dots+\lambda_n A_n)$ is a poly of degree n in $\lambda_1,\dots,\lambda_n$

$V(A_1, \dots, A_n) = $ coeff of $\lambda_1 \cdots \lambda_n$

$S_1 \cdots S_n$ $\qquad$ $X \subset \overset{\omega_1}{P_{S_1}} \times \cdots \times \overset{\omega_n}{P_{S_n}}$

$V_{S_1} \quad V_{S_n}$ $\qquad$ generic orbit

$P_{S_1} \quad P_{S_2}$ $\qquad$ Use $\omega_\lambda = \sum \lambda_i \omega_i$

If $\lambda_i \in \mathbb{N}$ get $\deg_\lambda X$

$$\deg_\lambda X = n! \operatorname{vol}(\mu_\lambda X) = n! \operatorname{vol}\left(\sum \lambda_i \mu_i(X)\right) = n! \operatorname{vol}\left(\sum \lambda_i \hat{S}_i\right)$$

$$\deg_\lambda X = \Omega_\lambda^n [X] = \left(\sum \lambda_i \Omega_i\right)^n [X]$$ poly in λ

$$\text{coeff of } \lambda_1 \cdots \lambda_n = n!\, \Omega_1 \cdots \Omega_n [X]$$

$$n!\, V(S_1, \dots, S_n)$$

This argument proves the poly property of $\operatorname{Vol}(\lambda_1 A_1 + \cdots + \lambda_n A_n)$ first for polyhedra with rational polyhedra

Arnold student $\qquad$ Hovansky