Donaldson March 7, 1983.

§1. Generalization of thm of Narasimhan-Seshadri

Y alg. curve, $\rho : \pi_1(Y) \longrightarrow SU(2)$ irred

get holomorphic v.b. on Y

Thm: This bundle is stable and all stable bundles of rank 2 and deg 0 arise from a unique repn. up to equivalence.

Def: E is stable (rank 2, $\deg E = 0$) $\Leftrightarrow$ for all L with $L \longrightarrow E$ one has $\deg L < 0$.

Prop (i) Stable bdls parametrized by a Hausdorff moduli space
(ii) Stability an open condition

For line bundles Jacobian = reps. $\pi_1(Y) \longrightarrow U(1)$.

§2. Extension to alg surfaces.

$$X \hookrightarrow \mathbb{CP}^N$$

Again have notion of stable bundle over X; same definition with degree defined by restricting to a curve which is a generic hyperplane section.

Given any holom. v.b. with metric $\exists$ unique connection: $d_A = \partial_A + \bar\partial_A$ $\qquad \bar\partial_A s = 0 \Leftrightarrow s$ holom.

Give X a Kähler metric consistent with the embedding, e.g. induced from CP^N.

Thm: A stable bundle E over X (rank 2, $\Lambda^2 E = 0$) has a unique consistent anti-self dual $SU(2)$-connection wrt ~~the~~ Kahler metric.

$\Lambda^2 T^*_X = \Lambda^2_+ \oplus \Lambda^2_-$ $\qquad$ so any connection

For Riemann M^4 $\qquad SO(4) \sim SO(3) \times SO(3)$.

$$\begin{array}{ccccc} SO(4) & \sim & SO(3) & \times & SO(3) \\ \uparrow & & \uparrow & & \parallel \\ U(2) & \sim & U(1) & \times & SO(3) \end{array}$$

Λ^2_+ splits into Kahler form $\oplus$ canonical bdle.

In Kahler case

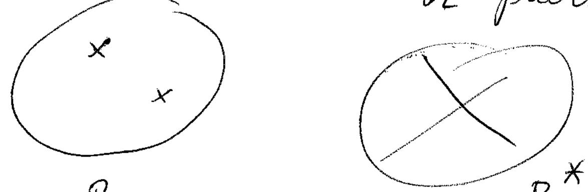
$$F_+ = \hat{F}\omega + \underbrace{\left(F^{0,2} + F^{2,0}\right)}_{\text{these vanish for a holom. bundle}}$$

Thus thm. says you can kill $\hat{F}$.
Equiv to $F \wedge \omega = 0$.

General conjecture relating stable bundles to hermitian Einstein connections. $\hat{F}.\omega$ essentially Ricci curv.

<u>Example:</u> On $\mathbb{CP}^2$, moduli space of bundles as above and $c_2 = 2$ is equivalent to space of non-singular conics in $\mathbb{P}_2^*$.

Compactify by adding pairs of lines to $\mathbb{P}_2^*$ (= degen conics) or pairs of points in $\mathbb{P}_2$



$\mathbb{P}_2$ $\qquad\qquad$ $\mathbb{P}_2^*$

§3. <u>General theory of stability</u>

$$G, G^{\mathbb{C}} \quad \text{on} \quad A \subset \mathbb{CP}^n$$

via a repn. $G^{\mathbb{C}} \longrightarrow SL_{n+1}(\mathbb{C})$.

Given $x \in \mathbb{P}^n$ choose x' over it $\in \mathbb{C}^{n+1}$.

x stable $\iff$ $G^{\mathbb{C}} \longrightarrow \mathbb{C}^{n+1}$ is proper

Criterion of Kempf + Ness: Choose norm on $\mathbb{C}^{n+1}$. Then an orbit in $\mathbb{C}^{n+1}$ is stable iff it contains a point of minimal norm, (and then this is unique up to the action of the compact gp G, assuming metric fixed under G.)

Metric gives symplectic structure on $\mathbb{C}P^n$ hence there are moment maps. The gradient of $\log|x|^2$ given by the moment map. That is, if $u \in \mathfrak{g}_{\mathbb{C}}$

$$\delta_u \log|x|^2 = (\mu, iu)$$

Hence a point of minimum norm = zero of moment map.

Thus in f.d. case $\exists$ good representative in the orbit picked out by the orbit.

§4. Application to our problem

Fixed C^∞ bdl. + metric E over X

$\mathcal{A}$ = all connections on E

Subset $\mathcal{A}^{(1,1)}$ of ones with $(1,1)$-curvature

i.e. $\ni$ $\bar{\partial}_A$ part has square zero $\Longrightarrow$ holom. structure

Gauge gp. $\mathcal{G}$ acts on $\mathcal{A}^{(1,1)}$

$\mathcal{G}^{\mathbb{C}}$ acts on holom. structures.

$$g : \bar{\partial}_A \longmapsto g\,\bar{\partial}_A\, g^{-1} \qquad \partial_A \longrightarrow (g^*)^{-1}\partial_A\, g^*$$

Two integrable connections give same holom. bdl
$\iff$ lie in same $\mathcal{G}^c$ orbit.

Now carry over the f.d. machinery.

Symplectic structure on $\mathcal{A}^{(1,1)}$. Tangent vectors are End(E) valued 1-forms

$$(a, b) = \int_X \mathrm{Tr}\,(ab) \wedge \omega$$

The moment map for action of $\mathcal{G}$ is

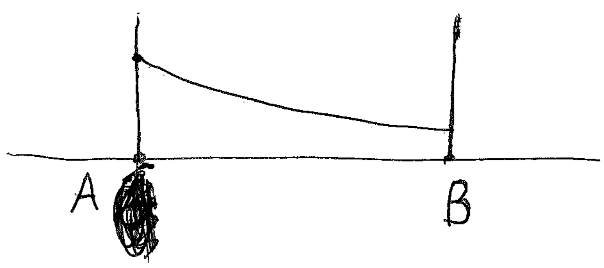
$$A \longmapsto F_A \wedge \omega$$

so that anti-self-duality condition $\iff$ this is zero.

Want analogue of $\mathbb{C}^{n+1}$ = a line bundle $\mathcal{L}$ over $\mathcal{A}^{(1,1)}$ acted on by $\mathcal{G}^c$ with metric preserved by $\mathcal{G}$. Then the connections we want lie under points in the orbits of minimal distance from 0 sections.

Note everything goes thru for ~~the bundles~~ Riemann surface, even simpler.

All we really want is the "height" functional that is, given A, B in same orbit want $M(A,B)$ such that $e^{M(A,B)}$ is the multiplier of the lengths



Familiar ~~Little~~ picture: You have a fixed holom. bundle and are varying the metric. Thus are working with space $\mathcal{G}^c/\mathcal{G}$ which parametrizes the connections up to isomorphism on a given holom. bdl.

$M(H,K)$ = functional of two metrics on same holom. bdl E.

$M(H,K)$ ~~is~~ a <u>convex</u> functional on the space of metrics, so one deduces the uniqueness of a minimum pt. if it exists.

§5. Method of proof.

To find critical points of height is to follow gradient flow. Also gives path for the Yang-Mills flow.

$$\frac{\partial M(H_t, H_0)}{\partial t} = - \|\hat{F}\|^2$$

So if M is bounded below, then $\hat{F} \to 0$.
Then use analysis of last term to get critical point.

Criterion: E over X is stable $\Longrightarrow$ $E|Y$ is semi-stable for $Y \in |nH|$.

$$M(H,K) = \underbrace{M(H|_Y, K|_Y)}_{\text{bounded by Narasimhan-Sh arguments.}} + \int_X [\mathrm{Tr}(F_H^2) - \mathrm{Tr}(F_K^2)]\psi$$

$Y = n\omega + i\partial\bar{\partial}\psi$

ψ has a mild singularity along Y.