731 – 827 April 8 – May 9, 1983

<u>April 8, 1983:</u>

There is a C^* algebra called the fermion algebra which is a direct limit of matrix rings of degrees 2^n. It has a trace whose values on idempotents belong to $\mathbb{Z}[\frac{1}{2}]$ and so gives rise to a non type I situation, in fact a $\mathrm{II}_{\text{finite}}$ factor.

I should be able to understand this C^*-alg. because it is roughly the Clifford algebra generated by creation and annihilation operators satisfying the fermion canonical commutation relations.

Let C_n be the Clifford algebra generated by $\gamma^1, \dots, \gamma^n$ which are self-adjoint, which anti-commute, and have square 1. For n even $n = 2m$, this algebra is isom. to the one generated by a_i, a_i^* $i = 1, \dots, m$ satisfying the fermion CCR. For $n = 2$ the relation is

$$\gamma^1 = a_1 + a_1^* \qquad\qquad a_1 = \tfrac{1}{2}(\gamma^1 - i\gamma^2)$$
$$\gamma^2 = ia_1 - ia_1^* \qquad\qquad a_1^* = \tfrac{1}{2}(\gamma^1 + i\gamma^2)$$

and in general $\gamma^{2j-1}, \gamma^{2j}$ are given by the same formulas in terms of a_j, a_j^*. (Here one uses that $\mathrm{End}(\Lambda W)$ is the Clifford algebra of the underlying real Euclidean space of W under $w \mapsto e(w) + i(w^*)$, hence given an orthonormal basis for W over $\mathbb{C}$: w_j we get a bunch of γ's from w_j and iw_j.)

Clearly we have natural embeddings

$$\longrightarrow C_n \longrightarrow C_{n+1} \longrightarrow \cdots$$

and if I stick to even values of n I can view this as being the natural way to include of set of a_i, a_i^* in a larger set. Also for n even

$$C_n \simeq \mathrm{End}(\Lambda \mathbb{C}^m) \qquad m = \frac{n}{2}$$

so we have an inductive system of matrix rings.

I can think of

$$C_\infty = \varinjlim C_n$$

as either the algebra of finite products of the γ^i or as the algebra of finite products of the a_i, a_i^*. We have the basis

$$\gamma^I = \gamma^{i_1} \cdots \gamma^{i_k} \qquad I = \{i_1, \ldots, i_k\} \text{ in order}$$

and the basis

$$a_I^* a_J$$

for C_{2m}, hence the same assertion in the limit.

Now the problem for me is to understand how to complete C_∞ to form a C^*-algebra.

First look at the trace. In C_{2m} the trace which most obviously presents itself is the trace as an endomorphism of ΛW. I know that this is given by

$$\operatorname{tr}_{\Lambda W}(\gamma^I) = \begin{cases} 0 & I \neq \phi \\ 2^m & I = \phi. \end{cases}$$

To get a trace on the limit I have to normalize this by dividing by 2^m. Thus the good trace on C_∞ is given by

$$\operatorname{Tr}(\gamma^I) = \begin{cases} 0 & I \neq \phi \\ 1 & I = \phi. \end{cases}$$

How does this look in the basis $a_I^* a_J$? First one must have $|I| = |J|$ for otherwise the operator won't respect the degree grading on ΛW. Actually one must have $I = J$, otherwise the part coming from a_{J-I} will commute up to sign with the rest, hence as a_{J-I} is nilpotent, so will be $a_I^* a_J$, so $a_I^* a_J$ will have trace zero. So look next at

$$a_1^* \cdots a_p^* \, a_p \cdots a_1 = (a_1^* a_1) \cdots (a_p^* a_p)$$

and note that the $a_i^* a_i$ are commuting ~~idempotents~~ decomposing ΛW into occupation number pieces. Then clearly we have

$$Tr\,(a_j^* a_j) = \frac{1}{2}$$

since half of ΛW has occupation number 1 in the jth state. Similarly

$$Tr\,(a_1^* a_1 \cdots a_p^* a_p) = \frac{1}{2^p}$$

Recall next the GNS idea that a positive linear functional on a C^*-algebra is equivalent to a Hilbert space representation with cyclic vector Φ

$$\rho(a) = \langle \Phi | a | \Phi \rangle.$$

If A is the algebra of bounded operators on $\mathcal{H}$ one of the usual ways to get such a ρ is

$$\rho(a) = Tr(\hat{\rho}\, a)$$

where ρ is a trace class operator which is hermitian ≥ 0. If we write

$$\hat{\rho} = \sum_n |n\rangle \lambda_n \langle n| \qquad \lambda_n \geq 0$$

then $\rho(a) = \sum_n \lambda_n \langle n | a | n \rangle$ and so we can see the GNS representation belonging to ρ as sitting inside a direct sum of copies of $\mathcal{H}$, one copy for each $\lambda_n > 0$.

A natural question is what is the place of the Fock space representation of C_∞ in this theory?

April 9, 1983

Yesterday I defined the algebra

$$C_\infty = \varinjlim C_n$$

whose modules are representations of the fermion CCR. C_∞ can also be described as the Clifford algebra with generators γ^i, $i = 1, 2, \ldots$ which are hermitian of square one and anti-commute. Hence C_∞ is a quotient of the group ring of the "Heisenberg" type group Γ generated by the γ^i and S^1.

The point is that because the CCR-algebra representations are the same as the reps. of the γ^i, we can describe them in analogy with reps. of the Heisenberg group.

So let us pick a maximal abelian subgroup of Γ for example the one generated by the center S^1 and the elements

$$-i\gamma^{2n-1}\gamma^{2n} = 2a_n^* a_n - 1$$

which essentially give the occupational number of the n-th state. This group is $S^1 \times \bigoplus_1^\infty \mathbb{Z}/2$. Given a representation of the CCR in a Hilbert space, we can restrict it to this subgroup, and it will decompose (as a direct integral in general) with respect to the characters of $\bigoplus_1^\infty \mathbb{Z}/2$. Let's put $W = \bigoplus_1^\infty \mathbb{Z}/2$. Then the character group is $\hat{W} = \prod_1^\infty (\mathbb{Z}/2)$

Now the actual Heisenberg group based on W is a central extension by S^1 of $W \times \hat{W}$, whereas Γ is a central extension by S^1 of $W \times W$, where $W \subset \hat{W}$ is dense. Thus we can't argue that all the characters of W occur in the repn, as we can for the full Heisenberg group. Instead the characters of W which occur will be a union of cosets for the subgroup W of $\hat{W}$.

To make this more explicit let's consider the example of Fock space where we have a vector $|0\rangle$ whose occupation numbers are 1 for n even and 0 for n odd. Thus I want $a_n|0\rangle = 0$ n even and $a_n^*|0\rangle = 0$ for n odd. This means that $|0\rangle$ is an eigenvector for W belonging to the character with value -1 for n even and $+1$ for n odd.

What we have is that the standard basis for Fock space is a basis of eigenvectors for W, and that the characters of $\hat{W}$ which occur form a single coset for W in $\hat{W}$.

So we get a description of the irreducible reprs. of the CCR for which there is an eigenvector under W. Such a thing corresponds to a partition of $\mathbb{N}-0$, i.e. character of W. Different characters give the same repn. $\Leftrightarrow$ they belong to the same W-coset.

I seem to recall that the representations of W involve measure classes on $\hat{W}$. So I should ask about interesting measure classes on $\hat{W}$ which are invariant under W.

Now $\hat{W}$ is a compact group hence has a Haar measure. This measure class is different from the Dirac one supported on a W-coset. Probably also W acts ergodically on $\hat{W}$.

An obvious thing to look at is the representation of Γ you get from the canonical representation of the Heisenberg group constructed from W. This is the space $L^2(W)$ with W and $\hat{W}$ acting in the natural way.

April 10, 1983

Yesterday I saw that the algebra $C_\infty = \varinjlim C_{2n}$ whose modules are reps. of the fermion CCR could be viewed as a "Heisenberg type" algebra. Specifically let Γ be the group generated by S^1 and the γ^μ. Let subgroups be defined by the following generators

$$W: \quad -i\gamma^1\gamma^2, \ -i\gamma^3\gamma^4, \ \ldots$$

$$W': \quad \gamma^1, \ -i\gamma^1\gamma^2\gamma^3, \ \gamma^1\gamma^2\gamma^3\gamma^4\gamma^5, \ \ldots$$

Both W, W' are elementary abelian 2 groups with the indicated bases. Moreover the commutator pairing sets up a duality between W, W' where the above bases are dual bases (e.g. $-i\gamma^1\gamma^2\gamma^3$ commutes with all basis elts for W except $-i\gamma^3\gamma^4$). However if $W = \bigoplus_1^\infty \mathbb{Z}/2$, then its dual is $\hat{W} = \prod \mathbb{Z}/2$ and $W' = \bigoplus \mathbb{Z}/2 \subset \prod \mathbb{Z}/2$.

So we see that we have an alg. semi-direct product

$$C_\infty = \mathbb{C}[W] \tilde{\otimes} \mathbb{C}[W']$$

where $W' \subset \hat{W}$, so C_∞ is a subalgebra of the Heisenberg algebra on W.

A similar situation occurs with the Kronecker foliation. There Connes uses the algebra generated by two unitaries U, V satisfying a commutation relation of the form

$$VUV^{-1} = e^{i\theta}U \qquad \theta \in \mathbb{R} - \mathbb{Q}$$

So I have again a semi-direct product

$$\mathbb{C}[\mathbb{Z}] \tilde{\otimes} \mathbb{C}[\mathbb{Z}']$$

where $\mathbb{Z}' \subset \hat{\mathbb{Z}}$ via $1 \mapsto e^{i\theta}$.

Both of these examples illustrate type II phenomena somehow. What this means I think is that one can represent these algebras on a Hilbert space in such a way that the weak closure of the algebra (= double commutant by von Neumann) is a factor (center = $\mathbb{C}$)

of type II_1 (hermitian idempotents in the algebra are classified by a trace with values in $[0,1]$.)

It might be a good idea at this point to go over the possible points of view one can adapt in order to study the Connes algebra above.

algebraic: $A =$ Laurent polynomials $\sum a_{mn} U^m V^n$

smooth: $A =$ all series $\sum a_{mn} U^m V^n$ where a_{mn} is a rapidly decreasing sequence:

$$|a_{mn}| (|m|+|n|)^k \longrightarrow 0 \qquad \text{all } k \geq 0.$$

continuous: this is some C^*-algebra completion.

measure: this should be the above-mentioned type II_1 factor.

In order to get some feeling for the C^*-completion consider first the case where $A = \mathbb{C}[U]$, where we are going to be looking at single unitary operators. Given a map

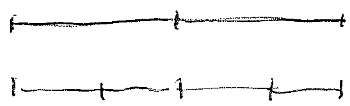
$$A = \mathbb{C}[U] \longrightarrow \mathcal{L}(\mathcal{H})$$

one gets a norm topology back on A, and so we get various C^*-algebra completions depending on what kind of unitary operator we have.

The spectral theorem for the unitary operator tells me that the norm on A is the following. By Fourier transform an element of A can be viewed as a fn. on S^1, and one takes the sup norm of this fn. on the spectrum of the unitary operator. Hence provided the operator U on $\mathcal{H}$ has all of S^1 for its spectrum we see that the completion of A in this norm is the alg. of series $\sum a_n U^n$ corresponding to continuous functions on S^1. That cannot be easily described as a condition on the coefficients a_n.

This example probably generalizes: Given a locally compact abelian group [redacted] G one can start with the convolution algebra $C_0(G)$. [redacted] The natural C^*-algebra completion of $C_0(G)$ is [redacted] probably the algebra of continuous functions on $\hat{G}$ vanishing at ∞. This ~~statement~~ is certainly correct for G discrete since Peters-Weyl says that G = characters on $\hat{G}$ generates $C(\hat{G})$.

In the case $G = W = \oplus \mathbb{Z}/2$, $\hat{G} = \prod \mathbb{Z}/2$ is a Cantor set and it is tricky to see the Fourier transform of a continuous function on $\hat{G}$. [redacted] Actually one might be able to visualize all this as a kind of dyadic subdivision process.



????

Let's go back now to the alg. Connes alg $C[U,V]$ and suppose we have a representation (* homo.)

$$\mathbb{C}[U,V] \longrightarrow \mathcal{L}(\mathcal{H})$$

Forget about V for the moment. A vector of $\mathcal{H}$ will give a measure $d\mu$ on S^1 such that the cyclic rep. of $\mathbb{C}[U]$ generated by that vector is isom. to $L^2(S^1, d\mu)$. What's important is the measure class on the circle. One can take the sup of the measure classes occurring [redacted] and this will be stable under the V action on the circle.

Now let's consider the obvious representation where $\mathcal{H} = L^2(S^1, \frac{dx}{2\pi})$, U = mult. by [redacted] e^{ix}, V = translation by θ. What is the weak closure of $\mathbb{C}[U,V]$? I know that [redacted] I can get close ~~to~~ any translation operator by taking V^n for suitable n.

Let's check this carefully. Let T be a translation operator $T_a : f(x) \longmapsto f(x+a)$. What is the norm of $T_a - I$ acting on $L^2(S^1)$? Using the F.T. we want the norm of the operator of multiplying by something like the function $e^{ina} - 1$ on $L^2(\mathbb{Z})$, so if a is irrational the norm is 2. So as $a \to 0$ we see that $T_a \not\to I$ in the operator norm topology, however $T_a \longrightarrow I$ strongly and hence weakly.

So it seems that the weak closure of $\mathbb{C}[U,V]$ in $\mathcal{L}(\mathcal{H})$ contains all multiplications and translations, and hence we get a type I factor.

In this example we have $\mathcal{H} = \ell^2$ with U acting as translation and V acting via the characters $V|n\rangle = e^{in\theta}|n\rangle$. This is formally identical to taking the measure class on the circle which is the sup of the δ-measures at the point of the orbit $\{n\theta\}$, except that U, V are interchanged. Thus we get something like repns of the CCR where $\exists$ vacuum states.

I still haven't gotten a type II representation.

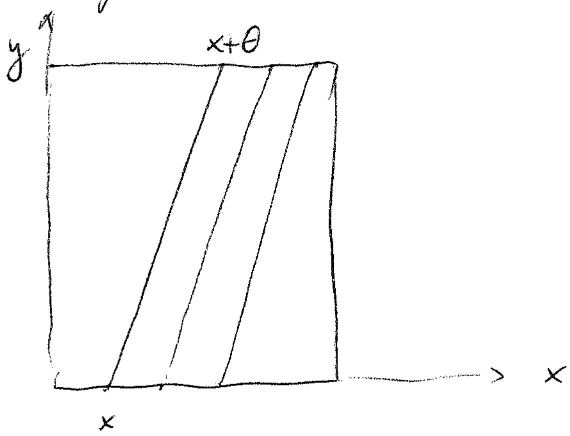
Consider the Kronecker foliation and let's try to work out Connes' reduction to the $\mathbb{C}[U,V]$-algebra. The C^*-algebra of the foliation is a crossed-product

$$C(S^1 \times S^1) \,\tilde{\otimes}\, \underbrace{C_0^\infty(\mathbb{R})}_{\text{under convolution}}$$

suitably completed. This is supposed to be Morita equalent to the $\mathbb{C}[U,V]$-algebra. This suggests that we should be looking at modules over the cross-product.

The first thing that comes to mind is the space of sections of a vector bundle over the torus $S^1 \times S^1$ equipped with a flat partial connection along the leaves. In fact since the leaves are 1-diml any partial connection along the leaves will be flat, and any vector bundle over the torus will admit such a connection because it has a (full) connection.

It is clear that an equivariant bundle over $S^1 \times S^1$ for the $\mathbb{R}$-action is determined by its restriction to a transversal S^1 and that one obtains an equivariant bdl. for the induced $\mathbb{Z}$-action. Think of the Kronecker foliation as the standard way of associating a foliation to a diffeomorphism:



So we get a vector bundle over S^1 equivariant for the action of $\mathbb{Z}$ on S^1 given by $x \longmapsto x + n\theta$, $n \in \mathbb{Z}$.

[redacted]

Something is wrong with this point of view. If you think of a vector bundle over S^1 equivariant under $\mathbb{Z}$ as being a $A_\theta = \mathbb{C}[U,V]^\wedge$-module, then clearly you are getting the analogue of a module which is f.t. projective over $\mathbb{C}[U]$. These are torsion modules. Also the kind of modules you are getting have finite-diml. rings of endos.

April 11, 1983

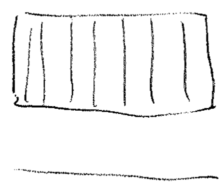
The C^*-algebra of a foliation is a direct limit of C^*-algebras associated to open subsets where we can take the ~~quotient~~. One has a cosheaf namely associated to $U \subset V$ one has $\mathcal{A}(U) \to \mathcal{A}(V)$.

If the foliation consists of points, then $\mathcal{A}(U)$ is essentially (maybe =) the continuous functions on U vanishing at ∞. [redacted] One should first understand how this varies in a covariant fashion. One has

$$\mathcal{A}(U) = \operatorname{Ker}\{C(U \cup \infty) \longrightarrow C(\infty)\}$$

and if U is an open subset of V we have an obvious map $V \cup \infty \longrightarrow U \cup \infty$ and hence a map $\mathcal{A}(U) \longrightarrow \mathcal{A}(V)$.

Now how does one proceed when the leaves are positive dimensional. In this case first consider the case where



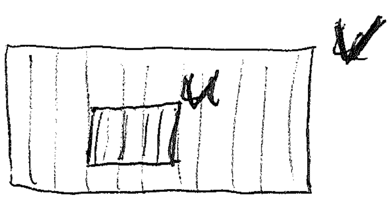
$U_0 \times \mathbb{R}^q$

U_0

$$U = U_0 \times \mathbb{R}^q$$

with which the leaves are the vertical [redacted] fibres of U_0. Then [redacted] an element of $\mathcal{A}(U)$ is a family of compact operators on $L^2(\mathbb{R}^q)$ indexed by [redacted] points of U_0 the family tending to zero as the point of U_0 goes to ∞. Such a thing can be obtained from a kernel $K(x,x')$ defined for $(x,x') \in U \times_{U_0} U$ which is smooth and of compact support.

It's clear how when we have an inclusion $U \subset V$ which looks like



then we have an obvious map $a(U) \longrightarrow a(V)$.

The next thing I would like to understand is the K-theory of this C^*-algebra. There is already some sort of problem with the K-theory of a locally compact, but not compact space defined by

$$K_c(X) = \operatorname{Ker}\{K(X \cup pt) \longrightarrow K(pt)\}.$$

The usual description is to give a complex of bundles on X which cohomology has compact support. However to define the map $K_c(U) \longrightarrow K_c(X)$ associated to an open embedding $U \longrightarrow X$, one must first replace a complex over U by a vector bundle over U trivialized near ∞.

In the Kasparov theory if one is given $E \xrightarrow{T} F$ a complex of vector bundles over U acyclic near ∞, then one just extends E, F to be a field of Hilbert spaces on X with zero fibres on $X - U$. Finally one can add H_A, $A = C(X)$, the trivial Hilbert bundle and use

$$E \oplus H_A = H_A,$$

to actually get a Fredholm map

$$H_A \cong E \oplus H_A \longrightarrow F \oplus H_A \cong H_A$$

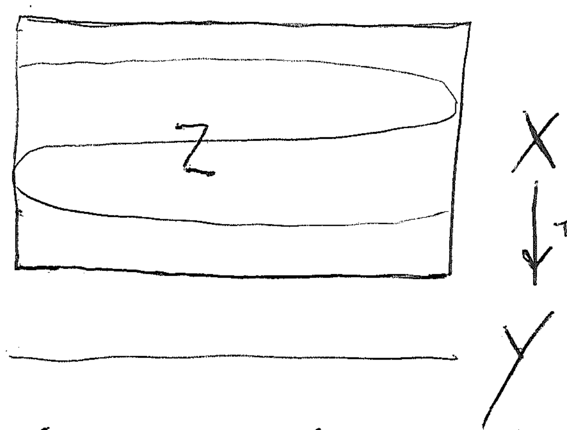
of Hilbert A-modules. What this amounts to is to do the same thing over U first: $E \oplus H_B \cong H_B$, and then use the fact that H_B comes from H_A. But then one has to use the Kuiper theorem to trivialze the map

$$H_B \cong E \oplus H_B \longrightarrow F \oplus H_B \cong H_B$$

near infinity <u>before it will extend</u> to X.

Next, let's try to understand the Kronecker foliation a bit better.

I want to understand the idea that a transversal to the foliation defines a finite type projective module over the C^* algebra of the foliation. For example in the case of a fibre bundle $X \xrightarrow{\pi} Y$ a transversal Z is a finite covering space of Y



and so if $g: Z \longrightarrow Y$, I ~~know that~~ $g_*(\mathcal{O}_Z)$ is a vector bundle over Y of rank = degree of the covering.

In this example I get a vector bundle over Y associated to the transversal. Now can I rewrite things so that I see some sort of projective module over the C^*-algebra of the foliation, which recall consists of continuous families of compact operators in the fibres?

If I take a tubular neighborhood of Z in X then we get a subalgebra of the C^*-algebra of the foliation, namely, the subalgebra consisting of kernels $K(x, x')$ supported in the tubular neighborhood. Actually we can even get something smaller namely where $K(x, x') \neq 0$ only if x, x' are in the same normal disk over Z.

So we reach the situation where we have a vector bundle V over Z and we consider it as a foliated manifold with fibres for leaves. Then our C^*-alg. consists of all continuous families of compact operators in the fibres of the vector bundle.

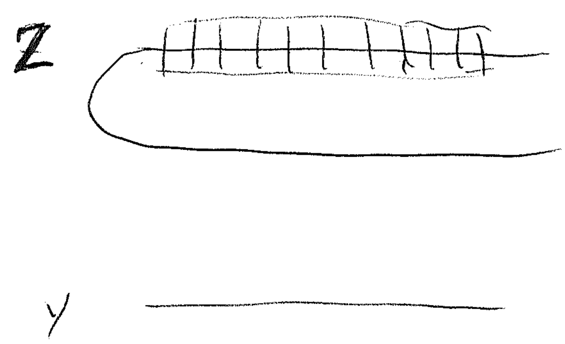
What happens when we take $Z \times \mathbb{R}$? Then the

Hilbert bundle should be the trivial bundle over Z with fibre $L^2(\mathbb{R})$, so the C^*-algebra consists of all continuous maps from Z to compact operators in $L^2(\mathbb{R})$, which is what one means by $C(Z) \otimes K$ in the sense of C^*-algebras.

A nice idempotent in this algebra is the operator which projects onto the Gaussian function - $e^{-x^2/2}$ - subspace in each $L^2(\mathbb{R})$.

Let's review the situation of a fibre bundle with a transversal inside: $Z \subset X \longrightarrow Y$. First of all we have a Hilbert space belonging to each fibre (the intrinsic one using $\frac{1}{2}$-densities). Then our C^*-algebra consists of continuous families $\{K_y\}$ of compact operators in the fibre. Such thing are given by kernels $K(x,x')$ defined on $X \times_Y X$, more or less.

Now using a tubular nbd of Z in X and [redacted] suitable Gaussian densities in the tube

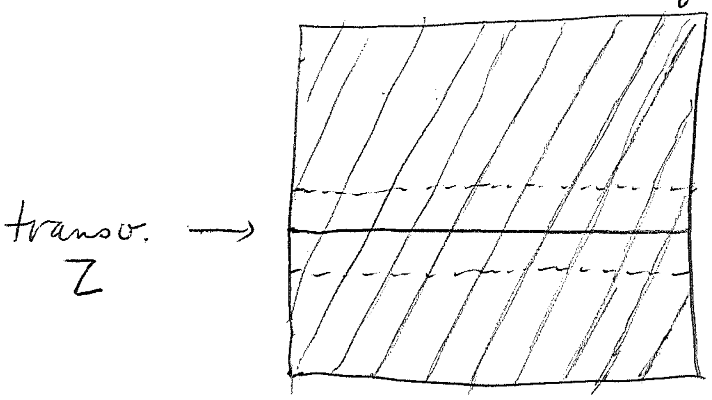


we can assign to Z a finite dimensional subspace of each Hilbert spaces. Hence we get a vector bundle over Y contained in the Hilbert bundle $\{L^2(X_y)\}$. In particular we get a projector in our C^*-algebra for the foliation. In fact projectors in the C^*-algebra correspond exactly to vector subbundles of the Hilbert bundle.

Next [redacted] up to Morita equivalence we can replace the C^* algebra by the endos. of this vector bundle.

In the case of the covering, the vector bundle comes with a natural unordered basis, and so we can see the endos. of the bundle as a family of matrix rings.

The next project will be to look at the Kronecker foliation and a transversal for it.



Again we take a [redacted] tubular nbd. and put in a Gaussian density vertically. [redacted]

[redacted] How can I describe the C^*-algebra of this foliation? The foliation has all leaves $\cong \mathbb{R}$. Each leaf has a L^2 attached so we have a family of Hilbert spaces. There is no good topology on the space of leaves, not even [redacted] a good Borel concept, so one has to be careful about how you define a family of operators in L^2(leaves). So one forms the graph of the foliation which is the set of points (x, x') with x, x' in the same leaf, topologized as $X \times \mathbb{R}$. So we will have kernels $K(x,x')$ which decay as $|x-x'| \longrightarrow \infty$. Actually one ~~starts~~ with $K(x,x')$ which are smooth and of compact supp. and complete.

Let us try to realize the transversal Z in this algebra, or really I mean that I want the projection onto the Gaussian density. Let's try to be more accurate.

In this example a fibre ([redacted] i.e. a leaf) intersects the tubular nbd of Z infinitely often, so the subspace of L^2(leaf) generated by the Gaussian densities is infinite dimensional. [redacted]

April 12, 1983

Factors are constructed via crossed products. Let a group G act on a ring A, and let M be an A-module. Consider now the abelian group

$$\tilde{M} = \mathbb{Z}[G] \otimes_{\mathbb{Z}} M \quad = \quad \bigoplus_{g\in G} g\otimes M$$

with the obvious G-action $\quad g(g_1\otimes m) = gg_1\otimes m \quad$ and the A-module structure given by

$$a(g\otimes m) = g\otimes ({}^{g^{-1}}a)m.$$

(It might look better if I put $g^{-1}ag$ instead of ${}^{g^{-1}}a$) Then $\tilde{M}$ is a module over the crossed product alg.

$$\mathbb{Z}[G] \otimes_{\mathbb{Z}} A$$

where the multiplication is defined by

$$(g_1\otimes a_1)(g_2\otimes a_2) = (g_1 g_2)\otimes ({}^{g_2^{-1}}a_1\, a_2).$$

Let's compute the center of this crossed product. A typical element of it is $\sum g\, a_g$ satisfying

$$h\left(\sum g a_g\right)h^{-1} = \sum hgh^{-1}\, {}^{h}a_g = \sum g\, a_g$$

$$\Longrightarrow \quad a_{hgh^{-1}} = {}^{h}a_g \qquad \text{for all } h, g\in G$$

and

$$\alpha\left(\sum g a_g\right) = \left(\sum g a_g\right)\alpha$$

$$\Longrightarrow \quad {}^{g^{-1}}\alpha\; a_g = a_g\,\alpha \qquad \text{for all } \alpha\in A,\ g\in G.$$

If G is commutative the first condition implies that $a_g \in A^G$ for all g. If A is commutative the second condition says that a_g must annihilate the ideal generated by $\{{}^{g^{-1}}\alpha - \alpha\}$. In the examples to be constructed $A^G = \mathbb{C}$, and $g\neq 1 \Rightarrow g$ acts non-trivially on A, hence the center will be $\mathbb{C}$.

The typical example will be where G is a [redacted] countable abelian group acting on a measure space (X,μ) preserving [redacted] the measure class of μ, and hence acting on $L^\infty(X,\mu) = A$. We assume then that the action is <u>ergodic</u> $\Longrightarrow$ $A^G = \mathbb{C}$ and <u>faithful</u> $\Rightarrow$ each $g \neq 1$ acts non-trivially on A.

The simplest example seems to be where $G = \mathbb{Z}$ and $X = \left(S^1, \frac{d\theta}{2\pi}\right)$ and G acts on the circle by an irrational rotation, [redacted] $A = L^\infty(S^1)$ acts on $M = L^2(S^1)$, and we complete $\tilde{M} = \mathbb{C}[\mathbb{Z}] \otimes M$ to get a Hilbert space

$$\mathcal{H} \; [redacted] = L^2(\mathbb{Z} \times S^1) = \ell^2 \hat{\otimes} L^2(S^1).$$

This situation we can describe as follows. Let U be the operator on $\mathcal{H}$ given by the shift on ℓ^2, i.e. $\ell^2 = \mathbb{C}[U,U^{-1}]^\wedge$. Let $V =$ multiplication by $e^{i\theta}$ on $L^2(S^1)$. Then $\mathcal{H}$ has the orthonormal basis $U^m V^n(1)$, where 1 is the constant function 1 on the circle, and the algebra we are interested in is generated by the

$$L = \mathbb{C}[U,U^{-1}] \tilde{\otimes} \mathbb{C}[V,V^{-1}] \qquad\qquad VUV^{-1} = \lambda U$$

with $\lambda \in S^1$ irrational.

The claim is that the weak closure of L in the bounded operators on $\mathcal{H}$ is a type II_1 factor. How can I see [redacted] this? It is a bit tricky because the weak closure of $\mathbb{C}[V,V^{-1}]$ acting on $L^2(S^1)$ is in fact $L^\infty(S^1)$. The same thing has to be true about $\mathbb{C}[U,U^{-1}]$, and $L^\infty(S^1)$ is not easy to see in terms of Fourier series.

We can argue as follows. What we have is a ~~situation~~ analogous to the regular representation of a group and we should be able to see that the commutant of the left regular repn. is the right regular representation.

More specifically let us consider a bounded operator T on $\mathcal{H} = \bigoplus_n \mathcal{U}^n(L^2(S^1))$ as above. Then we can consider it as a block matrix of operators $T_{mn} : \mathcal{U}^n L^2(S^1) \to \mathcal{U}^m L^2(S^1)$, the only possible problem being that it might not be easy to describe the boundedness of T in terms of the $\{T_{mn}\}$. Let us suppose T commutes with the operators $\mathcal{U}$ and the operators of multiplication by $L^\infty(S^1) = A$.

In general $T : \ell^2(G) \hat{\otimes} \underset{M}{L^2(X)} \longrightarrow \ell^2(G) \hat{\otimes} L^2(X)$ ~~has~~ components $T_{g,h} : h[redacted]M \longrightarrow gM$ and we can write

$$T_{g,h} = g \alpha_{g,h} h^{-1} \qquad \alpha_{g,h} : M \to M.$$

Since T commutes with left multiplication by $k \in G$:

$$k T_{g,h} k^{-1} = T_{kg,kh} \implies \alpha_{g,h} = \alpha_{kg,kh}.$$

Since T commutes with multiplication by $a \in A$:

$$a T_{g,h} = T_{g,h} a \qquad ??$$

These formulas are ~~too~~ confusing.

The best argument is to observe that on our Hilbert space $\mathcal{H} = \ell^2(G) \hat{\otimes} L^2(X)$ we have operators of left and right multiplication by elements of G and of $L^\infty(X)$. These commute. I should write T_{gh} using the right multiplication

$$T_{g,h} = \rho(g) \alpha_{gh} \rho(h)^{-1}.$$

Then since $\rho(h)$ commutes with ~~left~~ multiplication $\lambda(a)$ it follows that $\alpha_{gh} : L^2(X) \to L^2(X)$ commutes with mults. by A and hence is [redacted] in A.

So we see algebraically T is formally a linear comb. [redacted] of right multiplication operators $\rho(g)\rho(a)$, and clearly T is a weak limit of these linear combinations in the same way that a bounded operator when described as a matrix is the weak limit of its finite-dimensional minors.

Next we consider the trace. We know that elements in the von-Neumann algebra are described by certain formal sums $\sum_{g\in G} g\, a_g$ and the trace of this is $\int_X a_0 \, d\mu$. This is a central function when the measure $d\mu$ is invariant under the action of G. (When one takes an action preserving the measure class but not a measure in the class one gets a type III factor.)

Let's summarize what we see for $G=\mathbb{Z}$ acting as an irrational rotation on S^1. Algebraically we have an algebra generated over $\mathbb{C}$ by two invertibles U, V such that $VUV^{-1} = \lambda U$ where $\lambda \in S^1$ is irrational. This algebra consists of finite sums

$$\sum a_{mn} U^m V^n .$$

Then we have the smooth algebra consisting of series whose coefficients have rapid decrease. Also we have a C^*-algebra completion and a "von Neumann algebra of type II_1" completion, but neither of these have simple descriptions by conditions on the coefficients.

Next let's go back to the Kronecker foliation. What is the relation between the v.N. algebra of the foliation and the U,V-algebras?

The Kronecker foliation is obtained by letting $\mathbb{R}$ act on the 2-torus X in an "irrational" way. So we should find it worthwhile to treat it formally like $\mathbb{Z}$ acting irrationally on S^1. Thus I want to consider the Hilbert space

$$L^2(\mathbb{R}) \otimes L^2(X) = L^2(\mathbb{R}\times X)$$

and the operators on this given by convolution by an element of the "group ring" of $\mathbb{R}$ and by multiplying by a function on X.

It might be simpler to start out with the smooth algebra

$$\mathcal{S}(X\times\mathbb{R}) = \mathcal{S}(X)\otimes\mathcal{S}(\mathbb{R})$$

which we know how to make act on itself as well as $L^2(X\times\mathbb{R})$. The idea is that we have kernels $K(x,t)$ with composition

$$(K * K_1)(x,t) = \int_{t'+t''=t} K(x,t')K_1(x-\hat{t}',t'').$$

Maybe a simpler description is in terms of kernels $K(x,x')$ defined for x,x' ~~lying~~ in the same leaf.

The norm one uses to complete $\mathcal{S}(X\times\mathbb{R})$ is the sup over the leaves of the norm of the operator in L^2 of each leaf.

Why not ~~try to~~ do the same thing for $\mathcal{S}(S^1\times\mathbb{Z})$. Again we think of these things as kernels $K(x,x')$ when $x,x'\in S^1$ differ by λ^n. Thus we have simply a family of functions $k_n(x) = K(x,x\lambda^{-n})$ which are smooth on S^1 and which decrease rapidly as $n\to\infty$. The operator that we get on this leaf is the one with matrix elements

$$h_{mn} = K(x_0\lambda^{+m}, x_0\lambda^{+n}).$$

This matrix has rapidly decaying rows and columns but is bounded in the diagonal directions $m-n=\text{const}$, like the identity matrix.

The preceding suggests the following problem:

On one hand we know the von Neumann alg. associated to [redacted] an irrational rotation on S^1 is an algebra of operator on $L^2(S^1 \times \mathbb{Z})$. On the other hand, at least for foliations, but maybe also for this example, Connes describes the v.N. algebra as a measurable family of operators in the family of Hilbert spaces belonging to the leaves. [redacted] The problem is to relate these two descriptions. It would seem that perhaps one can decompose the Hilbert space $L^2(S^1 \times \mathbb{Z})$ as a direct integral of Hilbert spaces of the form $\ell^2 = L^2(\mathbb{Z})$.

Consider next Atiyah's [redacted] L^2-index theorem. Here one has a covering space $X \longrightarrow X/\Gamma$ with X/Γ compact, and one has an elliptic Γ-equivariant operator $D : E \longrightarrow F$ over X. Then one can consider $\mathrm{Ker}(D)$ [redacted] in $L^2(X, E)$ and define

$$\dim_\Gamma \mathrm{Ker}(D) = \int_{\substack{x \in \text{fundl. domain} \\ \text{for } \Gamma}} \mathrm{tr}\, e(x,x)$$

where [redacted] $e(x,x')$ is the kernel of the projection operator from $L^2(X,E)$ to $\mathrm{Ker}(D)$. To simplify suppose E is the trivial line bundle over X with obvious Γ-action.

Let's look carefully at this projection operator $L^2(X) \longrightarrow \mathrm{Ker}(D) \longrightarrow L^2(X)$. It is described by

$$e(x,x') = \sum_\alpha [redacted]\ u_\alpha(x)\, \overline{u_\alpha(x')}$$

where u_α is an orthonormal basis of [redacted] Ker D. This operator commutes with the Γ-action, hence

$$e(x+\gamma, x'+\gamma) = e(x,x') \qquad \forall \gamma \in \Gamma.$$

We are going to be interested in the algebra of operators on $L^2(X)$ commuting with the Γ-action. Now from the measure viewpoint $X = \Delta \times \Gamma$ where $\Delta \subset X$ is a fundamental domain, hence

$$L^2(X) = L^2(\Delta) \hat{\otimes} \ell^2(\Gamma)$$

We are interested in operators commuting with the Γ-action, say all bounded operators in $L^2(X)$ " " " ".
This is a von Neumann algebra. It is the crossed product of the bounded operators in $L^2(\Delta)$ with Γ acting trivially.

April 13, 1983

Let D be a $\bar\partial$-operator on a R.S. and δD be a first order differential operator. I want to compute the asymptotic behavior of

$$\operatorname{Tr}\left(e^{-tD^*D}\, D^{-1}\delta D\right) \qquad \text{as} \quad t \downarrow 0.$$

The idea will be to write

$$D^{-1}\delta D = P\delta D + (D^{-1}-P)\,\delta D.$$

The second term should be a smooth kernel operator and so have a trace which is the obvious candidate for the 0-th order term in the asymptotic expansion. So consider the first term. We have in general

$$P(z,z') = F(z,z')\left[-\partial_{z'} \log r(z,z')^2\right] dz' \,\frac{i}{2\pi}$$

Let's consider the flat case

$$P(z,z') = F(z,z')\,\frac{1}{z-z'} \cdot \text{const} \qquad \frac{i}{2\pi}\,dz'$$

and take δD to be $\partial_z d\bar z$. In this case $P\delta D$ is the operator

$$\left[(P\delta D)f\right](z) = \int \frac{F(z,z')}{z-z'}\,(\partial_{z'} f)(z')\,\frac{d^2 z'}{\pi}$$

and so

$$(P\delta D)(z,z') = -\partial_{z'}\,\frac{F(z,z')}{z-z'} \cdot \frac{1}{\pi}$$

$$= \left[\frac{1}{z-z'}\partial_{z'} F(z,z') + \frac{1}{(z-z')^2}F(z,z')\right] \times -\frac{1}{\pi}.$$

The heat kernel is

$$\langle z|e^{-tD^*D}|z'\rangle = \frac{e^{-\frac{|z-z'|^2}{2t}}}{2\pi t}\left\{\underset{F}{\underset{\shortparallel}{F_0(z,z')}} + t\,F_1(z,z') + \cdots\right\}$$

So I want to calculate

$$\int tr \langle 0 | e^{-tD^*D} | z \rangle \langle z | P\delta D | 0 \rangle =$$

$$\int tr \frac{e^{-\frac{|z|^2}{2t}}}{2\pi t} \left\{ F_{[redacted]}(0,z) + t F_1(0,z) + \cdots \right\} \left[\frac{1}{z} \partial_{\bar{z}} F(z,0) + \frac{1}{z^2} F(z,0) \right] \frac{-d^2 z}{\pi}$$

First notice that the term $t \, \frac{F_1(0,z) F(z,0)}{z^2}$ contributes one power of t and higher ones, so it doesn't contribute to the t^0 term. What is relevant to the answer is

$$\circledast \qquad -\frac{1}{\pi} \text{ coeff. of } z \text{ in } tr\left\{ F(0,z) \, \partial_{\bar{z}} F(z,0) \right\}$$

and this is the limit as $t \to 0$.

It should be possible to write $\circledast$ invariantly. The particular δD chosen [redacted] comes from a constant coefficient variation of the complex structure, so $\circledast$ should depend only upon the connection which defines the parallel translation F. NO.

What is confusing is the derivative w.r.t. $\partial_{\bar{z}}$ which has no meaning in [redacted] the vector bundle. So δD is <u>not</u> invariantly defined.

Let me review how one can describe the $\bar{\partial}$ operators on (M, E) where M is a smooth oriented surface and E is a smooth vector bundle over it.

A holomorphic structure on M can be identified with a sub-line-bundle $T^{1,0}$ of T^* (= bundle of cx. 1-forms) such that if ω spans $T^{1,0}$ [redacted] locally, then $i\omega \wedge \bar{\omega} \in \Omega^2(M)$ is >0 in the orientation. These line subbundles are sections of a unit disk bundle over M, namely the disk bundle obtained by taking one of the disks in $\mathbb{P}(T^*) - \mathbb{P}(T^*)_{\mathbb{R}}$. Let $\mathcal{S}(M) =$ **Holom.** ~~structures~~ on M, and $\mathcal{S}(M,E)$ the holom. structures on (M,E).

Then the fibre of the map $\mathcal{S}(M,E) \to \mathcal{S}(M)$ over a given complex ~~structure~~ represented by $T^{1,0} \subset T^*$ is an affine space modelled on $\Gamma(\mathrm{Hom}(E, E\otimes T^{0,1}))$.

If I choose a connection on E:

$$\nabla : E \longrightarrow E \otimes T^*$$

then any holom. structure on M gives rise to a $\bar\partial$-operator obtained by composing ∇ with the projection

$$E\otimes T^* \longrightarrow E \otimes T^{0,1}.$$

This gives us a section of $\mathcal{S}(M,E) \to \mathcal{S}(M)$ which is clearly holomorphic.

General ideas:

1) I want to keep track of the holomorphic concept for a family of $\bar\partial$-operators in order to be able to obtain a connection from the analytic torsion metric on the determinant line bundle.

2) The end descriptions of moduli spaces for vector bundles are purely topological (or say C^∞), but independent of the holomorphic structure on the surface. For example a connection on a bundle has a curvature which can be converted to an endomorphism of the bundle by using a volume element. Then Yang-Mills means this endom. is constant relative to the connection. The moduli spaces are described by representations of central extensions of the fundamental group.

The problem: Describe all constant coefficient $\bar\partial$-operators on the trivial line bundle over a 2-diml torus M, and the determinant line bundle over the space of these operators.

I can identify a $\bar\partial$-operator on $\mathbb{1}$ with a certain ~~quotient~~ bundle of $J_1(\mathbb{1})$, which has the form:

$$J_1(\mathbb{1}) = \mathbb{1} \oplus T^* \xrightarrow{id \oplus pr} \mathbb{1} \oplus T^{0,1} \xrightarrow{\xi + id} T^{0,1}$$

Here $T^{0,1}$ is a quotient of T^* by a line $T^{1,0}$ which is in the positive disk of $P(T^*)$ for the orientation of M. ξ is a section of $T^{1,0}$.

Thus a $\bar\partial$-operator on $\mathbb{1}$ is given by a ~~quotient~~ line bundle $T^{0,1}$ together with a section ξ of it. Hence constant coefficient $\bar\partial$-operators are described by two complex parameters the first being a point in the UHP (or unit disk) to describe the holomorphic structure and the second being a point ξ of the ~~quotient~~ line $T^{1,0}$. If I choose a fixed line $W \subset T^*$ in the opposite disk from the disk containing all the $T^{1,0}$, then $W \subset T^* \to T^{0,1}$ will be an isomorphism hence we can identify ξ with a point $w \in W$.

Therefore the set of constant coefficient $\bar\partial$-operators on $\mathbb{1}$ over M is a 2-diml complex manifold isomorphic to UHP $\times \mathbb{C}$. On this set operates the group $SL_2(\mathbb{Z})$ which acts on $M = \mathbb{R}^2/\mathbb{Z}^2$. Also we can conjugate by characters of M, hence we get an action of $\hat{M} = \mathbb{Z}^2$. Thus the semi-direct product

$$SL_2(\mathbb{Z}) \ltimes \mathbb{Z}^2$$

will act on the set $\underline{(\text{UHP} \times \mathbb{C})}$ of const. coeff. $\bar\partial$-opers.

<u>Summary:</u> Let Y = set of all constant coefficient $\bar{\partial}$-operators on $\mathbb{1}$ over an oriented 2-torus M (regarded as a compact Lie group.) Then Y is a complex manifold, a submanifold of a complex Grassmannian, isomorphic to $UHP \times \mathbb{C}$. We have a discrete group

$$G = \operatorname{Aut}(M) \ltimes \hat{M} \qquad \cong \quad SL_2(\mathbb{Z}) \ltimes \mathbb{Z}^2$$

acting on Y.

Over Y we have the determinant line bundle $\mathcal{L}$ which comes with a canonical section, and both are holomorphic. The canonical section vanishes where the $\bar{\partial}$-operator is singular. This singular set is a divisor which can be described as follows.

First of all Y has a section over the UHP which assigns to a holomorphic structure the $\bar{\partial}$-operator on $\mathbb{1}$ whose kernel is the sheaf of holomorphic functions. This section is invariant under $\operatorname{Aut}(M)$ and so translating it under the discrete group G gives to each element of $\hat{M}$ a section of Y over UHP. The divisor is the union of these sections. It is therefore a lattice bundle over UHP embedded in $Y \cong (UHP) \times \mathbb{C}$.

The next problem will be to trivialize the determinant line bundle $\mathcal{L}$. Put the standard metric on $\mathbb{1}$ and put the standard (Haar) measure on M. Then we get a metric on $\mathcal{L}$ defined by analytic torsion. Ray-Singer in their Annals paper (1973) calculate the norm of the canonical section. So it will be possible to compute the curvature.

We already know the curvature in the vertical direction, so it would be interesting to understand a bit better what happens in a horizontal direction. Also in order to [redacted] trivialize $\mathcal{L}$ I have to put a metric with the same curvature on the trivial bundle. Thus I would like there to exist a Gaussian type function on UHP. (Clearly it exists: Solve $\bar\partial\partial \log f = \omega$)

Recall that any connection on $\mathbb{1}$ gives us a holomorphic section of Y over UHP. If we take unitary connections we get a foliation of Y transverse to the fibres over UHP. Included in the leaves of this transverse foliation are the flat unitary connections. These are just the sections of Y over UHP described before which make up the Θ-divisor in Y.

Consequently if I fix a non-trivial flat unitary connection on $\mathbb{1}$, then I will get a holom. section of Y over UHP which does not meet the Θ-divisor.

Let $M = \mathbb{R}^2/2\pi\mathbb{Z}^2$. A complex structure invariant under translation will be given by $\tau \in UHP$ by requiring $T^{1,0}$ to be spanned by

$$T^{1,0}: \quad dx - \frac{1}{\tau}dy$$

$$T^{0,1}: \quad dx - \overline{\left(\frac{1}{\tau}\right)}dy.$$

We take the volume $dxdy$ on M. Then the metric on $T^{0,1}$ is given by

$$\| a(dx - \frac{1}{\bar\tau}dy)\|^2 dxdy = i\ \bar a(dx - \tau^{-1}dy)\, a(dx - \bar\tau^{-1}dy)$$

$$= |a|^2 i(\tau^{-1} - \bar\tau^{-1})\, dx\, dy$$

or $$|a(dx - \bar\tau^{-1}dy)|^2 = |a|^2\, 2\,\mathrm{Im}(-\tau^{-1})$$

Next look at the $\bar\partial$ operator

$$\mathbb{1} \longrightarrow T^* \longrightarrow T^*/(dx - \tau^{-1}dy)$$

I propose to trivialize the last quotient using the image of the section dx.

Let $M = \mathbb{R}^2 / 2\pi \mathbb{Z}^2$ with the volume form $dx\,dy$. Describe a translation-invariant complex structure on M by a point $\tau \in UHP$ by requiring

$$T^{1,0} \text{ spanned by } dx - \tau^{-1} dy.$$

Then

$$T^{0,1} = T^* / dx - \tau^{-1} dy$$

is to be trivialized. $a\,dx + b\,dy \mapsto a + b\tau$ is a good trivialization and allows us to identify the $\bar\partial$ operator with $\partial_x + \tau \partial_y$. Then the $\bar\partial$ operator has the eigenvalues $m + n\tau$, $m, n \in \mathbb{Z}$.

I also want the metric on $T^{0,1}$. One has a metric on $T^{1,0}$ defined by

$$
\begin{aligned}
|dx - \tau^{-1} dy|^2 \, dx\,dy &= i\,(dx - \tau^{-1}dy) \wedge (dx - \bar\tau^{-1} dy) \\
&= i\,(+\tau^{-1} - \bar\tau^{-1})\, dx\,dy \\
&= 2\,\mathrm{Im}(-\tau^{-1})\, dx\,dy
\end{aligned}
$$

$$\therefore\ |dx - \tau^{-1}dy|^2 = |dx - \bar\tau^{-1} dy|^2 = 2\,\mathrm{Im}\left(-\tfrac{1}{\tau}\right)$$

On the other hand, the image of $dx - \bar\tau^{-1} dy \in \overline{T^{1,0}}$ in $T^{0,1}$ when trivialized is

$$1 - \bar\tau^{-1}\tau.$$

So the generator of $\mathbb{1} \cong T^{0,1}$ corresp. to $\dfrac{dx - \bar\tau^{-1}dy}{1 - \bar\tau^{-1}\tau}$ which has norm squared

$$\left| \frac{dx - \bar\tau^{-1} dy}{1 - \bar\tau^{-1}\tau} \right|^2 = \frac{2\,\mathrm{Im}\left(-\frac{1}{\tau}\right)}{\left(1 - \frac{\tau}{\bar\tau}\right)\left(1 - \frac{\bar\tau}{\tau}\right)} = \frac{i\left(\frac{1}{\bar\tau} - \frac{1}{\tau}\right)}{(\bar\tau - \tau)(\tau - \bar\tau)} = \frac{1}{2\,\mathrm{Im}\,\tau}$$

This constant will not affect $\zeta'(0)$ because we know $\zeta(0) = 0$.

April 15, 1983

The problem: Let $Y = \bar\partial$ operators with constant coefficients on $\mathbb{1}$ over a 2-torus. Find the curvature of the determinant line bundle and describe the kind of determinant functions.

Fix an operator of interest D. This is just a point of Y. Next I need a tangent vector at this point and I have to interpret it as an operator.

Let's describe this process more carefully. A point y of Y is a quotient line $T^{0,1}$ of T^* together with a point of that ~~quotient~~ which I can lift uniquely back to $T^*_{\mathbb{R}}$ if I wish. To such data I associate a certain operator which goes from $C^\infty(M)$ to $C^\infty(M, T^{0,1})$. The second space varies with the point $y \in Y$. Suppose I trivialize the latter bundle. Then to each point of y I get an operator D_y on the same space $C^\infty(M)$. In particular it now makes sense to talk about the change δD_y in the operator corresponding to a change δy at y.

Hence I have a map $\delta y \longmapsto \delta D_y$ from the tangent space to Y at y to the operators on $C^\infty(M)$. This will be complex linear provided we use a "holom. trivialization" of $T^{0,1}$ over Y.

Let's get specific. Take $M = \mathbb{R}^2/2\pi\mathbb{Z}^2$ and describe the holom. structure by $\tau \in UHP$ using

$$T^{1,0} = \text{span of } \quad dx - \tfrac{1}{\tau}dy$$

Trivialize $T^{0,1}$ using the image of dx in T^*

$$\mathbb{1} \xrightarrow[\sim]{1 \mapsto dx} T^*/(dx - \tfrac{1}{\tau}dy) = T^{0,1}$$

$$a + \tau b \longleftarrow\!\!\shortmid\; a\,dx + b\,dy$$

Then $\bar\partial : \mathbb{1} \longrightarrow T^{0,1}$ becomes simply

$$\partial_x + \tau\,\partial_y .$$

This is a holomorphic family of operators on $C^\infty(M)$ depending on τ. Finally we must put in an element of $T^{0,1}$, better add to this an element of $T^{0,1}$ which we view as a constant operator $\mathbb{I} \longrightarrow T^{0,1}$. Using the trivialization we then get the family

$$(*) \qquad \partial_x + \tau\partial_y - i\omega \qquad \text{on } C^\infty(M)$$

depending on $(\tau, \omega) \in UHP \times \mathbb{C}$.

Notice that if we fix a connection $\nabla_x = \partial_x + a$, $\nabla_y = \partial_y + b$, then the operator

$$\nabla_x + \tau\nabla_y = \partial_x + \tau\partial_y + (a + \tau b)$$

for a, b fixed is <u>indeed</u> holomorphic in τ.

Now we can turn to the problem of regularizing the expression $\mathrm{Tr}(D^{-1}\delta D)$. The difficulty with $(*)$ is that δD involves ∂_y.

Better: I have learned from previous work the wisdom of using normal coordinates in order to do calculations involving the heat kernel. The x, y coords. above are not normal, unless $\tau = i$.

So I want to introduce a holomorphic coordinate on the torus $M = \mathbb{R}^2/2\pi\mathbb{Z}^2$ for the τ-complex structure. Clearly

$$z = \text{const}\left(x - \frac{1}{\tau}y\right).$$

In order to determine the constant most convenient for calculations, perhaps we can ask that the lattice $\mathbb{Z} + \mathbb{Z}\tau$ in the w-plane be dual to the image of the lattice $2\pi\mathbb{Z}^2$ in the z-plane. Recall that

$$\Gamma^* = \{ z \mid \bar{z}\mu - z\bar{\mu} \in 2\pi i\mathbb{Z} \quad \text{all } \mu \in \Gamma = \mathbb{Z} + \mathbb{Z}\tau \}$$
$$= \frac{\pi}{\mathrm{Im}\,\tau}\Gamma$$

Summary: I have a fixed torus $M = (\mathbb{R}/2\pi\mathbb{Z})^2$ and have decided to parametrize the constant coeff. $\bar\partial$-ops. on $\mathbb{1}$ in the form

$$\boxed{D_{\tau,\omega} = \frac{1}{i}(\partial_x + \tau\partial_y) - \omega \qquad \text{acting on } C^\infty(M)}$$

where $\tau \in UHP$, $\omega \in \mathbb{C}$. In this way gauge transformations of conjugating by a character

$$e^{i(mx+ny)}$$

correspond to translation $\omega \mapsto \omega + (m + n\tau)$, and so the Θ-divisor in $Y = UHP \times \mathbb{C}$ is the union of the lattices $\Gamma_\tau = \mathbb{Z} + \mathbb{Z}\tau$ in the fibre over τ.

To calculate the regularization of $\mathrm{Tr}(D^{-1}\delta D)$ I want to use a complex coordinate on M for the τ-structure. It seems that I want

$$\boxed{\partial_{\bar z} = \frac{1}{i}(\partial_x + \tau\partial_y)}$$

so that $D_{\tau,\omega}$ takes the familiar form $\partial_{\bar z} - \omega$.

So
$$\partial_{\bar z} = \frac{1}{i}(\partial_x + \tau\partial_y) \qquad z = c(\tau x - y)$$
$$\partial_z = -\frac{1}{i}(\partial_x + \bar\tau\partial_y) \qquad \bar z = \bar c(\bar\tau x - y)$$

$$1 = \partial_z z = i c(\tau - \bar\tau) = -2c\,\mathrm{Im}\,\tau \qquad c = \frac{-1}{2\,\mathrm{Im}\,\tau}$$

$$\boxed{z = \frac{-\tau x + y}{2\,\mathrm{Im}\,\tau}}$$

Image of $2\pi\mathbb{Z}^2$ is $2\pi \cdot \dfrac{\mathbb{Z}\tau + \mathbb{Z}}{2\,\mathrm{Im}\,\tau} = \dfrac{\pi}{\mathrm{Im}\,\tau}\Gamma = \Gamma^*$. which looks good.

$$\boxed{\partial_y = \frac{1}{2\,\mathrm{Im}\,\tau}(\partial_z + \partial_{\bar z})}$$

What is our goal? I am going through this calculation in the constant coefficient case in order to find out what happens in general. The problem is to calculate the regularization of $\mathrm{Tr}(D^{-1}\delta D)$. I would like to know if it is given by $\mathrm{Tr}\left((D^{-1}-P)\,\delta D\right)$, where P is the parametrix.

I should note that δD is a specific type of operator between the spaces associated with D, and so we don't have to worry about the actual way we lift a tangent vector to the space of $\bar\partial$-operators to an operator δD.

In the constant coefficient case $D = \partial_{\bar z} - w$ acting on the C^∞-functions on our elliptic curve. δD will be an arbitrary first order constant coeff. op

$$\delta D = a\,\partial_z + b\,\partial_{\bar z} + c.$$

The hardest part will be for $\delta D = \partial_z$.

Let's review the formulas for $\langle z|D^{-1}|0\rangle = G(z)$:

$$\sigma(z) = z \prod{}' \left(1-\frac{z}{\mu}\right) e^{\frac{z}{\mu}+\frac{z^2}{2\mu^2}}$$

$$\zeta(z) = \frac{\sigma'(z)}{\sigma(z)} = \frac{1}{z} + \sum{}' \left(\frac{1}{z-\mu} + \frac{1}{\mu} + \frac{z}{\mu^2}\right)$$

$$\zeta(z+\mu) - \zeta(z) = a_\mu = l\mu + m\bar\mu$$

$$\pi G(z) = e^{w(\bar z + \frac{l}{m} z)} \frac{\sigma(z - \frac{w}{m})}{\sigma(z)\,\sigma(-\frac{w}{m})}$$

Next I want the heat kernel - this is a sum over the lattice in the z-plane in order to make it periodic, but I need only the term for $z-z'$ small, since the terms with $z-z'-\mu$, $\mu \neq 0$ will be exponentially small. Here's how the formulas can be derived:

$$e^{w\bar z - \bar w z}\,\partial_{\bar z}\,e^{-(w\bar z - \bar w z)} = \partial_{\bar z} - w = D$$

$$\langle z | e^{-tD^*D} | z' \rangle = e^{w\bar z - \bar w z}\,\frac{e^{-\frac{|z-z'|^2}{2t}}}{2\pi t}\,e^{-w\bar z' + \bar w z'}$$

$$= \frac{e^{-\frac{|z-z'|^2}{2t}}}{2\pi t}\,\underbrace{e^{w(\bar z - \bar z') - \bar w (z - z')}}$$

this is just the parallel translation wrt the ~~unitary~~ connection from z' to z. The parametrix $P(z) = \langle z | P | 0 \rangle$ will be

$$\pi P(z) = \frac{e^{w\bar z - \bar w z}}{z}$$

Because we have constant coefficient operators we have $P\partial_z = \partial_z P$. So

$$\langle z | P\partial_z | 0 \rangle = \partial_z P(z).$$

What we want to compute is the heat regularization of $\mathrm{Tr}(P\delta D)$.

$$\int d^2 z\,\langle 0 | e^{-tD^*D} | z \rangle\,\langle z | P\partial_z | 0 \rangle$$

$$= \int d^2 z\,\frac{e^{-\frac{|z|^2}{2t}}}{2\pi t}\,e^{-(w\bar z - \bar w z)}\,\partial_z\left(\frac{e^{w\bar z - \bar w z}}{\pi z}\right)$$

$$= \int d^2 z\,\frac{e^{-\frac{|z|^2}{2t}}}{2\pi t}\,(\partial_z - \bar w)\left(\frac{1}{\pi z}\right) = 0$$

by reasons of S^1-symmetry.

So to finish off the calculation all I have to do is to determine

$$\partial_z\left[G(z) - P(z)\right]\Big|_{z=0}$$

if this isn't too much work.

I did this calculation last year (p. 706–707) in the following way. First replace ∂_z by

$$\nabla_z = \partial_z + \bar\omega$$

Then

$$\nabla_z \left[G(z) - P(z) \right] = \nabla_z \frac{e^{\omega \bar z - \bar\omega z}}{\pi} \left\{ \underbrace{\frac{e^{\left(\bar\omega + \frac{l}{m}\omega\right) z}\, \sigma(z - \frac{\omega}{m})}{\sigma(z)\, \sigma(-\frac{\omega}{m})}}_{\frac{1}{z}\left(1 + az + bz^2 + \dots\right)} - \frac{1}{z} \right\}$$

$$= \frac{e^{\omega \bar z - \bar\omega z}}{\pi}\, \partial_z \left\{ a + bz + \dots \right\} \xrightarrow[z \to 0]{} b$$

Now $\sigma(z) = z\left(1 + O(z^4)\right)$, and

$$\log\left(1 + az + bz^2 + \dots\right) = az + bz^2 - \frac{1}{2}\left(az + bz^2\right)^2$$

$$= az + \left(b - \frac{a^2}{2}\right) z^2 + \dots$$

$$= \left(\bar\omega + \frac{l}{m}\omega\right) z + \log \sigma\left(z - \frac{\omega}{m}\right) - \log \sigma\left(\frac{\omega}{m}\right)$$

Differentiate

$$a + 2\left(b - \frac{a^2}{2}\right) z + \dots = \bar\omega + \frac{l}{m}\omega + \zeta\left(z - \frac{\omega}{m}\right)$$

$$\Longrightarrow \quad a = \bar\omega + \frac{l}{m}\omega - \zeta\left(\frac{\omega}{m}\right)$$

$$2\left(b - \frac{a^2}{2}\right) = \zeta'\left(0 - \frac{\omega}{m}\right) = -\wp\left(0 + \frac{\omega}{m}\right)$$

$$\therefore \quad b = \frac{1}{2}\left[\left(\bar\omega + \frac{l}{m}\omega - \zeta\left(\frac{\omega}{m}\right)\right)^2 - \wp\left(\frac{\omega}{m}\right) \right]$$

In this formula appear the Weierstrass functions for the lattice in the z-plane which we saw is

$$\Gamma^* = \frac{\pi}{\operatorname{Im}\tau}\left(\mathbb{Z} + \mathbb{Z}\tau\right)$$

Also

$$m = \frac{\pi}{\operatorname{vol}(\mathbb{C}/\Gamma^*)} = \frac{\pi}{\left(\frac{\pi}{\operatorname{Im}\tau}\right)^2 \operatorname{vol}(\mathbb{C}/\Gamma)} = \frac{\operatorname{Im}\tau}{\pi}$$

so that

$$\Gamma^* = \frac{1}{m}\Gamma .$$

Now use the relation

$$\zeta_{t\Lambda}(tw) = \frac{1}{t}\zeta_\Lambda(w) \quad , \quad \wp_{t\Lambda}(tw) = \frac{1}{t^2}\wp(w)$$

and you get

$$\zeta_{\frac{1}{m}\Gamma}\left(\frac{w}{m}\right) = m\,\zeta_\Gamma(w) \quad , \quad \wp_{\frac{1}{m}\Gamma}\left(\frac{w}{m}\right) = m^2\wp_\Gamma(w)$$

Here $m = m_{\Gamma^*} = \frac{1}{m_\Gamma}$, hence we get

$$\nabla_{\bar z}(G-P)\Big|_{z=0} = b = \frac{1}{2m_\Gamma}\left[\left(m_\Gamma\bar w + \ell_\Gamma w - \zeta_\Gamma(w)\right)^2 - \wp_\Gamma(w)\right]$$

This is still very complicated, and it might simplify because ζ^2 has double poles where $\wp$ does.

April 16, 1983:

Let's go over the Ray-Singer calculation of the torsion.

$$\zeta(s) = \sum_\Gamma{}' \frac{1}{|w-\gamma|^{2s}} = \frac{1}{\Gamma(s)}\int_0^\infty\left(\sum e^{-t|w-\gamma|^2}\right)t^s\frac{dt}{t}$$

$D = \partial_{\bar z} - w$, $-D^* = \partial_z + \bar w$ and we have two expressions for the heat kernel:

$$\langle z|e^{-tD^*D}|z'\rangle = \sum_{\mu\in\Gamma^*}\frac{e^{-\frac{|z-z'+\mu|^2}{t} + w(\overline{z-z'+\mu}) - \bar w(z-z'+\mu)}}{\pi t}$$

$$= \sum_\gamma{}' e^{-t|\gamma-w|^2}\frac{e^{\gamma(\overline{z-z'}) - \bar\gamma(z-z')}}{\mathrm{vol}(\mathbb{C}/\Gamma^*)}$$

Since

$$\frac{\mathrm{vol}(\mathbb{C}/\Gamma^*)}{\pi} = \frac{\pi}{\mathrm{vol}(\mathbb{C}/\Gamma)}$$

we get

$$\sum_{\Gamma} e^{-t|\omega-\gamma|^2} = \frac{\pi}{t} \sum_{\Gamma^*} e^{-\frac{|\mu|^2}{t} + \omega\bar{\mu} - \bar{\omega}\mu} \frac{1}{\mathrm{vol}(\mathbb{C}/\Gamma)}$$

From this we see that this Θ fn. has asymptotic behavior $\frac{\pi}{\mathrm{vol}(\mathbb{C}/\Gamma)} \frac{1}{t}$ as $t \to 0$ to infinite order, hence the Mellin transform has a simple pole at $s=1$ with residue $\frac{\pi}{\mathrm{vol}(\mathbb{C}/\Gamma)}$. Also $\zeta(0)=0$ and $\zeta'(0) =$ value of Mellin transf. at $s=0$. So if we derive the functional equation we find

$$\zeta'(0) = \frac{\pi}{\mathrm{vol}(\mathbb{C}/\Gamma)} \left[\sum{}' \frac{e^{-\omega\bar{\mu}+\bar{\omega}\mu}}{|\mu|^{2s}} \right]_{s=1}$$

This is a periodic function of ω which satisfies

$$-\partial^2_{\omega\bar{\omega}} \zeta'(0) = \pi \sum{}' \frac{e^{\mu\bar{\omega}-\bar{\mu}\omega}}{\mathrm{vol}(\mathbb{C}/\Gamma)}$$

$$= \pi \underset{\mathbb{C}/\Gamma}{\delta(\omega)} - \frac{\pi}{\mathrm{vol}(\mathbb{C}/\Gamma)}$$

So it just's what we get if we solve Laplace's equation on the complement of the constant functions.

Let us consider the Weierstrass ζ function from the distribution viewpoint. We have

$$\partial_{\bar{\omega}} \zeta(\omega) = \pi\,\delta(\omega)$$

and the obvious periodic solution of this equation is given by the Fourier series

$$\psi = \frac{\pi}{\mathrm{vol}(\mathbb{C}/\Gamma)} \sum{}' \frac{e^{\mu\bar{\omega}-\bar{\mu}\omega}}{\mu}$$

Except $\partial_{\bar{\omega}}\psi = \pi\delta(\omega) - \frac{\pi}{\mathrm{vol}(\mathbb{C}/\Gamma)}$

Now $\zeta(\omega)$ is not periodic but $\zeta(\omega) - l\omega - m\bar\omega$ is periodic. Thus we have

$$\zeta(\omega) - l\,\omega - \underbrace{m}_{\frac{\pi}{\mathrm{vol}(\mathbb{C}/\Gamma)}}\bar\omega = \frac{\pi}{\mathrm{vol}(\mathbb{C}/\Gamma)} \sum_{\Gamma^*}{}' \frac{e^{\mu\bar\omega - \bar\mu\omega}}{\mu} + c$$

where c is a constant which can be determined if one could integrate $\zeta(\omega) - l\omega - m\bar\omega$ over $\mathbb{C}/\Gamma$.

Another possible approach which has some advantages is use $\Gamma = \{m + n\tau\}$ and first sum over m, then n in the Eisenstein way. What this means is we use

$$\begin{aligned}
\mathbb{C} &\longrightarrow \mathbb{C}^* \\
\omega &\longmapsto e^{2\pi i\omega} = t \\
\tau &\longmapsto e^{2\pi i\tau} = q .
\end{aligned}$$

Let's apply this idea to construct something like the ζ-fn. which will be a version of

$$\sum \frac{1}{\omega - m - n\tau} .$$

Use first the formula

$$\sum \frac{1}{\omega - m} = \frac{\pi\cos\pi\omega}{\sin\pi\omega} = 2\pi i\,\frac{1}{2}\,\frac{e^{2\pi i\omega}+1}{e^{2\pi i\omega}-1}$$

$$= 2\pi i\left\{\frac{1}{e^{2\pi i\omega}-1} + \frac{1}{2}\right\}$$

$$\text{or} \quad = 2\pi i\left\{\frac{1}{1-e^{-2\pi i\omega}} - \frac{1}{2}\right\}$$

Thus

$$\sum_{m\in\mathbb{Z}} \frac{1}{\omega - m - n\tau} = 2\pi i\left\{\frac{1}{e^{2\pi i(\omega - n\tau)}-1} + \frac{1}{2}\right\}$$

$$= 2\pi i\left\{\frac{1}{q^{-n}t - 1} + \frac{1}{2}\right\} = 2\pi i\left\{\frac{1}{1-q^{n}t^{-1}} - \frac{1}{2}\right\}$$

Now we want to sum this up over n, but we get something divergent because the terms approach either $\frac{1}{2}$ or $-\frac{1}{2}$. The simplest thing to do is to take

$$F(t) = 2\pi i \left\{ \sum_{n=\text{[redacted]}}^{\infty} \frac{q^n}{t-q^n} + \sum_{n=0}^{\infty} \frac{t}{t-q^{-n}} \right\}$$

Then $\quad F(t) - F(qt) = 2\pi i \left\{ \frac{t}{t-\text{[redacted]}^{1}} - \frac{1}{t-1} \right\} = 2\pi i.$

To make F periodic we can [redacted] add:

$$\tilde{F}(t) = F(t) + 2\pi i \frac{\log|t|}{\log|q|}.$$

Finally I want to integrate $\tilde{F}$ over a fundamental domain in $\mathbb{C}^*$ for the multiplication by q.

$$\int_{|q|\le|t|\le 1} \tilde{F}(t) \cdot \text{Haar}$$

I claim this integral is zero. Because suppose we integrate over the circle $|t| = r$, $|q| < r < 1$. Then

$$n\ge 1 \qquad \frac{q^n}{t-q^n} = \frac{q^n t^{-1}}{1-q^n t^{-1}} = \underset{\uparrow}{q^n t^{-1}} + \underset{\uparrow}{(q^n t^{-1})^2} + \cdots$$

integrates to give 0

$$n\ge 0 \qquad \frac{t}{t-q^{-n}} = -t\frac{q^n}{1-q^n t} = \underset{\downarrow}{(-q^n t)}\underset{\downarrow}{(1+q^n t + (q^n t)^2 + \cdots)}$$

Therefore we conclude that $\tilde{F}(t)$ realizes the Fourier series

$$\frac{\pi}{\operatorname{Im}\tau} \sum{}' \frac{e^{\mu\bar{w} - \bar{\mu}w}}{\mu}.$$

No forgotten non-anal. term

In this sense the q setup has advantages over the Weierstrass setup.

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$$F(t) = 2\pi i \left[\sum_{n=1}^{\infty} \frac{q^n}{t-q^n} + \sum_{n=0}^{\infty} \frac{t}{t-q^{-n}} \right]$$

satisfies $\quad F(t) - F(qt) = 2\pi i \left[-\frac{1}{t-1} + \frac{t}{t-1} \right] = 2\pi i$

so that

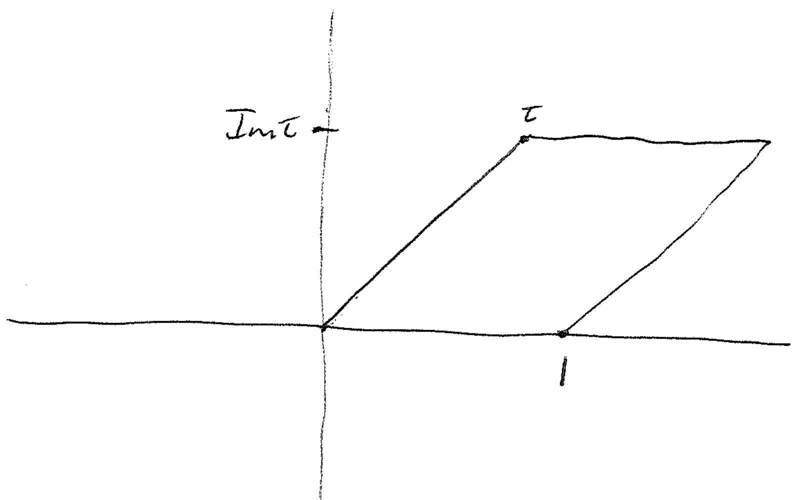
$$\tilde{F}(t) = F(t) + 2\pi i \frac{\log|t|}{\log|q|}$$

is periodic. Also we know that as a Laurent series in the annulus $|q| < |t| < 0$, $F(t)$ has no constant term.

Let us now consider $\qquad \log|t| = \operatorname{Re}(2\pi i w) = -2\pi \operatorname{Im} w$

$$\tilde{F}(e^{2\pi i w}) = F(t) + 2\pi i \frac{\operatorname{Im} w}{\operatorname{Im} \tau}$$

and compute $\displaystyle\int \tilde{F}(e^{2\pi i w})\, d^2 w$ over a period $\square$, e.g.



↑ corresponds to ↓ $|q| < |t| < 1$

Now we know that integrating F horizontally gives zero. Thus

$$\int \tilde{F}(e^{2\pi i w})\, d^2 w = \int 2\pi i \frac{\operatorname{Im} w}{\operatorname{Im} \tau}\, d^2 w = \frac{2\pi i}{\operatorname{Im} \tau} \int_0^{\operatorname{Im}\tau} y\, dy$$

$$= \pi i \operatorname{Im} \tau$$

Conclude that

$$\Phi(w) = 2\pi i \left[\sum_{n=1}^{\infty} \frac{q^n}{t-q^n} + \sum_{n=0}^{\infty} \frac{t}{t-q^{-n}} + \frac{\operatorname{Im} w}{\operatorname{Im} \tau} - \frac{1}{2} \right]$$

is the function with the Fourier series

$$\frac{\pi}{\operatorname{Im}\tau} \sum{}' \frac{e^{\mu\bar{w} - \bar{\mu} w}}{\mu} .$$

It is the solution of $\partial_{\bar{w}} \Phi = \pi\delta(w) - \frac{\pi}{\operatorname{Im}\tau}$ which is $\perp$ to the constants, i.e. the Green's fn. for $\partial_{\bar{w}}$.

Next we want to find the Green's fn. Ψ for $\partial^2_{\bar{w}w}$. This will satisfy

$$\partial_w \Psi = \Phi$$

and we know Ψ will be some way of regularizing

$$\sum \log |w - m - n\tau|^2 .$$

Let's start with

$$G(t) = \prod_{n=0}^{\infty} (1 - q^n t) \prod_{n=1}^{\infty} (1 - q^n t^{-1})$$

which is analytic in t and has simple zeroes at the right points. One has

$$\frac{G(t)}{G(qt)} = \frac{1-t}{1-t^{-1}} = -t .$$

And one has

$$\partial_w \log G = 2\pi i \; t \partial_t \log G$$

$$= 2\pi i \left[\sum_0^{\infty} \frac{-q^n t}{1-q^n t} + \sum_1^{\infty} \frac{q^n t^{-1}}{1 - q^n t^{-1}} \right] = F$$

What do I need to add to make $\log|G|^2$ periodic?

$$\log|G(t)|^2 - \log|G(qt)|^2 = \log|t|^2 = 2\,\mathrm{Re}\,\underset{\log t}{2\pi i w}$$

$$= -4\pi\,\mathrm{Im}(w)$$

We need $f(w)$ such that

$$f(w+\tau) - f(w) = 2\,\mathrm{Im}(w)$$

$$\frac{\mathrm{Im}(w+\tau)^2}{\mathrm{Im}\,\tau} - \frac{\mathrm{Im}(w)^2}{\mathrm{Im}\,\tau} = 2\,\mathrm{Im}\,w\,\cancel{\mathrm{Im}\,\tau} + (\mathrm{Im}\,\tau)^2$$

$$-\mathrm{Im}(w+\tau) + \mathrm{Im}(w) = -\mathrm{Im}\,\tau$$

$$\therefore \quad f(w) = \frac{\mathrm{Im}(w)^2}{\mathrm{Im}\,\tau} - \mathrm{Im}(w)$$

and we conclude that

$$\boxed{\log|G(t)|^2 + 2\pi\left[\frac{\mathrm{Im}(w)^2}{\mathrm{Im}\,\tau} - \mathrm{Im}\,w\right]}$$

is periodic.

<u>Check</u>: $\quad \partial_w(\mathrm{Im}\,w) = \partial_w\left(\frac{w-\bar w}{2i}\right) = \frac{1}{2i}$

which gives $\;-2\pi\left(-\frac{1}{2i}\right) = -\pi i$.

So now we know that the Green's fn. Φ we are after differs from the above expression by an additive constant. Call the above $L(w)$. We have to integrate it over a fundamental domain. Again we choose $0 < \mathrm{Im}\,w < \mathrm{Im}\,\tau$. In this case the terms in $\log G$ namely

$$\log(1-q^n t) \qquad n \geq 0$$
$$\log(1-q^n t^{-1}) \qquad n \geq 1$$

have nice series expansions, so we can conclude that the integral horizontal from ω to $\omega+1$ is zero for $\log G(t)$, hence also for its real part $\times 2$ $= \log|G(t)|^2$. So

$$\int_{\square} L\, d^2\omega = -2\pi \int_0^{\mathrm{Im}\,\tau} \left(\frac{y^2}{\mathrm{Im}\,\tau} - y\right) dy = -2\pi\left[\frac{(\mathrm{Im}\,\tau)^2}{3} - \frac{(\mathrm{Im}\,\tau)^2}{2}\right]$$

$$= \frac{\pi}{3}(\mathrm{Im}\,\tau)^2$$

Conclude that the Green's fn. we want i.e. the solution of

$$\partial^2_{\bar{\omega}\omega}\Psi = \pi\delta - \frac{\pi}{\mathrm{Im}\,\tau}$$

$\perp$ to the constants is

$$\boxed{\Psi(\omega) = \log|G(t)|^2 - 2\pi\left(\frac{(\mathrm{Im}\,\omega)^2}{\mathrm{Im}\,\tau} - \mathrm{Im}\,\omega\right) - \frac{\pi}{3}(\mathrm{Im}\,\tau)}$$

This is essentially the analytic torsion, and so now I can calculate the curvature and check the conjecture.

How do we describe elliptic curves C (= complex tori of complex dimension 1)? The Lie algebra is a complex line V and the expenential map gives an isom.

[redacted] $$V/\Gamma \xrightarrow{\sim} C$$

where Γ is a lattice in V. [redacted] Let's choose a basis for Γ with the right orientation and a generator for V. Then we get an embedding

$$\mathbb{Z}^2 \longrightarrow \mathbb{C}$$ [redacted]

that is a pair of periods (ω_1, ω_2) [redacted] with $\omega_2/\omega_1 \in UHP$. Different choices correspond to an action of $\mathbb{C}^* \times SL(2,\mathbb{Z})$ on the set of (ω_1, ω_2). [redacted] The standard thing to do is to eliminate the $\mathbb{C}^*$-action by requiring $\omega_1 = 1$. Then we can describe elliptic curves up to isom. with elements of the UHP modulo $SL_2(\mathbb{Z})$.

I have a problem of relating two descriptions of elliptic curves. One is the one just described where to a $\tau' \in$ UHP one associates the curve

$$\mathbb{C} / \mathbb{Z} + \mathbb{Z}\tau'$$

The other description occurs when I take a fixed torus, say $M = \mathbb{R}^2/2\pi\mathbb{Z}^2$, and we associate to $\tau \in$ UHP the complex structure whose holom. fns. are killed by

$$\partial_x + \tau \partial_y .$$

In order to go between these descriptions, what we need is to assign to τ a holomorphic function z which maps the lattice $2\pi\mathbb{Z}^2$ onto $\mathbb{Z} + \mathbb{Z}\tau'$, and I want the first generators + 2nd generators to correspond. Now z must be a multiple of

$$x - \frac{1}{\tau} y$$

and so we see that

$$z = \frac{1}{2\pi}\left(x - \frac{1}{\tau} y\right) \qquad \therefore \tau' = -\frac{1}{\tau}$$

Another possibility is to put

$$z = \frac{1}{2\pi}\left(-\tau x + y\right)$$

whence our family of elliptic curves is isomorphic to the family $\mathbb{C}/\mathbb{Z} + \mathbb{Z}\tau$

Let's take a different viewpoint. Suppose X/Y is a family of elliptic curves and L is a line bundle on X of degree 0 over each fibre, everything holom.

Locally we can choose a basis for the fundamental group of the fibre with the correct orientation and then take the ratio of the periods so as to get a map $Y \to UHP$ such that the family X/Y is the pull-back of the family

$$M_\tau = \mathbb{C}/\mathbb{Z}+\mathbb{Z}\tau .$$

Now comes the problem of conveniently describing line bundles of degree zero over $\mathbb{C}/\mathbb{Z}+\mathbb{Z}\tau$.

The idea will be to use constant coefficient $\bar\partial$-operators on the trivial line bundle $\mathbb{1}$. This parametrizes the line bundles of degree zero by means of the 1-diml space of invariant forms of type $(0,1)$. We also can act on these operators by conjugating with respect to the character group of the torus. Now the character group of the torus is dual, as a free abelian group, to the fundamental group. Hence corresponding to the basis $(1,\tau)$ of π_1 is a dual basis for $M_\tau^\wedge$, and we can trivialize the space of invariant $(0,1)$ forms so that the lattice inside is in a nice form.

I've already parametrized the characters of $\underbrace{\mathbb{C}/\mathbb{Z}+\mathbb{Z}\tau}_{\Gamma}$ in the form

$$e^{\mu\bar z - \bar\mu z}$$

where $\mu \in \frac{\pi}{\operatorname{Im}\tau}(\mathbb{Z}+\mathbb{Z}\tau)=\Gamma^*$, so the lattice inside the space of $\bar\partial$ operators with constant coefficients is

$$(\partial_{\bar z} - \mu)\, d\bar z \qquad \mu \in \Gamma^* .$$

Hence it would appear that the nice parametrization is

$$\boxed{w \longmapsto \left(\partial_{\bar z} - \frac{\pi}{\operatorname{Im}\tau} w\right) d\bar z}$$

But this doesn't look holomorphic in τ.

So start again. We consider the family of elliptic curves $\mathbb{C}/\mathbb{Z}+\mathbb{Z}\tau$ over the UHP, and over each curve we consider the family of const. coeff $\bar{\partial}$-ops on $\mathbf{1}$. This gives us a line bundle over the UHP which is holomorphic, since we know that the family of $\bar{\partial}$-ops. can be identified with the points in $\mathcal{O}(1)$ over $\mathbb{P}^1$ restricted to the UHP. But to see this we should describe the family of elliptic curves as the holom. structures on $\mathbb{R}^2/2\pi\mathbb{Z}^2$ of the form $\operatorname{Ker}(\partial_x+\tau\partial_y)$.

I think that I am now coming to an understanding of something that caused a great deal of confusion. The point is that if you have a holom. family of lattices Γ_τ and take the family of curves $\mathbb{C}/\Gamma_\tau$, then the family of $\bar{\partial}$-operators

$$(\partial_{\bar{z}}-w)\,d\bar{z}$$

is <u>not</u> holomorphic in τ, w. This is because $d\bar{z}$ is not a holomorphic section of $T^{0,1}=T^*/T^{1,0}$ over the UHP.

Idea: Let's consider a family of elliptic curves $X\xrightarrow{f}Y$. Over X we have the DR complex

$$\mathcal{O}_X\xrightarrow{d}\Omega^1_{X/Y}$$

and so over Y we get vector bundles $R^i f_*(\Omega^\bullet_X)$ which have Gauss-Manin connections. Complex analytically these are the complexification of the integral cohomology of the fibres. The interesting thing is the Hodge filtration on the 1-diml cohomology

$$0\to H^0(M,\Omega^1)\longrightarrow H^1(M,\mathbb{C})\longrightarrow H^1(M,\mathcal{O})\longrightarrow 0$$

The first map associates to a holom. 1-form its periods, and so assuming a given basis for $H_1(M,\mathbb{Z})$ is equivalent to the map $\tau: Y \longrightarrow UHP$. Also because of the cup-product

$$\Lambda^2 H^1(M,\mathbb{C}) \xrightarrow{\sim} H^2(M,\mathbb{C}) = \mathbb{C}$$

and so $H^0(M,\Omega^1)$ and $H^1(M,\mathcal{O})$ are dual. Actually this is true by Serre duality. On the other hand, $H^0(M,\Omega^1)$ is dual to the tangent space at the identity of the elliptic curve M, and $H^1(M,\mathcal{O})$ is the tangent space to the Jacobian, hence the tangent spaces of M and the Jacobian of line bdles of degree zero are canonically isomorphic.

So now I can easily say how to modify $d\bar{z}$ as a section of the line bundle $\tau \mapsto H^1(M_\tau,\mathcal{O})$ over the UHP so that it becomes holomorphic. We know that dz is a holom. section of $\tau \mapsto H^0(M_\tau,\Omega^1)$. Then $f(\tau)\,d\bar{z}$ will be holomorphic provided

$$dz\, f(\tau)\, d\bar{z} \quad \text{in} \quad H^2(M_\tau,\mathbb{C}) \cong \mathbb{C}$$

is holomorphic, i.e. when

$$f(\tau)\int_{M_\tau} dz\,d\bar{z} = f(\tau)\,\frac{2}{i}\,\mathrm{Im}\,\tau$$

is holomorphic. $\therefore$ $f(\tau) = \dfrac{1}{\mathrm{Im}\,\tau}$ works, which confirms the bottom of p 775.

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Consider the family of elliptic curves $\tau \mapsto M_\tau = \mathbb{C}/\mathbb{Z}+\mathbb{Z}\tau$ over the UHP. The total space of this family is a complex surface which can be described as follows. Define an action of $\mathbb{Z}\times\mathbb{Z}$ on $UHP \times \mathbb{C}$ by

$$(m,n) * (\tau, z) = (\tau, z + m + n\tau).$$

This is an action of a discrete group on a complex manifold which is free, and one can form the quotient.

Let's try to understand what modular forms are. Suppose we have a family of elliptic curves [redacted] $f: X \to Y$. Then we can define certain vector bdls. and lattice bundles over Y. For example we obtain a bundle with fibre $\mathbb{Z}^2$ by taking $R^1 f_*(\mathbb{Z})$, i.e. the bundle with fibre $H^1(f^{-1}(y), \mathbb{Z})$. And we have line bundles

$$f_* \Omega^1_{X/Y} \quad , \quad R^1 f_*(\mathcal{O}_{X/Y}) .$$

Actually [redacted] since I suppose given a zero-section for X over Y, I have a large supply of line bundles on X which I can use to get naturally-defined vector bundles over Y. Then I can apply various operations on these vector bundles to generate new vector bundles.

This amounts to the following. We have fibred [redacted] categories of elliptic curves and vector bundles. A natural way of assigning a vector bundle on the base to any family of elliptic curves is the same as a cartesian functor. [redacted] Call such cartesian functors vector bundles on the modular site. The question then becomes how to calculate these vector bundles.

The obvious candidate is [redacted] an equivariant holom. vector bundle on the UHP for the action of $SL_2(\mathbb{Z})$. This candidate is right, because given any family over Y

we can form over Y the principal $SL_2(\mathbb{Z})$-bundle $\tilde{Y}$ of bases (pos. oriented) for the homology of the fibres. Then our family is obtained from the universal family over UHP by an equivariant map $\tilde{Y} \longrightarrow UHP$ followed by descent to Y. Thus an equivariant vector bundle over UHP will pull back to an equiv. bundle over $\tilde{Y}$ which will then descend to Y.

So we want to understand equivariant holom. v.b. over the UHP. There are problems caused by the fact that $SL_2(\mathbb{Z})$ doesn't act freely. In particular the ± 1 in $SL_2(\mathbb{Z})$ doesn't act on the UHP.

Notice that the bundles

$$0 \longrightarrow f_* \Omega^1_{X/Y} \longrightarrow R^1 f_*(\mathbb{Z}) \otimes \mathbb{C} \longrightarrow R^1 f_*(\mathcal{O}_{X/Y}) \to 0$$

over the UHP have the -1 in $SL_2(\mathbb{Z})$ acting as -1.

Modular forms are global sections of certain line bundles over the modular site. Hence the -1 in $SL_2(\mathbb{Z})$ must act ~~trivially~~ on the line bundle, which is why one has <u>even</u> weights.

One way to obtain $SL_2(\mathbb{Z})$ equivariant line bundles over the UHP is to take the line bundles $\mathcal{O}(n)$ over $\mathbb{P}^1$, which are $GL_2(\mathbb{C})$ equivariant. In the above exact sequence, the middle is the dual of the standard repn of $SL_2(\mathbb{Z})$ on $\mathbb{Z}^2 \otimes \mathbb{C} = V$. $V = V^*$ over SL_2 anyway, so it should be possible to identify

$$f_* \Omega^1_{X/Y} = \mathcal{O}(-1) \qquad R^1 f_*(\mathcal{O}_{X/Y}) = \mathcal{O}(1)$$

for the universal family over the UHP. $(f_* \Omega^1)_\tau$ is generated by dz on M_τ which goes into the line

$$\left(\int_{\gamma_1} dz \, , \, \int_{\gamma_2} dz \right) \in \mathrm{Hom}(\underbrace{H_1(M_\tau)}_{\mathbb{Z}^2}, \mathbb{C})$$

where $\gamma_1 = 1$, $\gamma_2 = \tau$ in $H_1(M_\tau) = \mathbb{Z} + \mathbb{Z}\tau$.

It would seem to be true that modular forms are invariant sections of

$$\mathcal{O}(k) = \left(R^1 f_*(\mathcal{O}_{X/Y})\right)^{\otimes k}$$

over the UHP satisfying some sort of condition at ∞.

Let's consider our universal family $X \longrightarrow Y = \mathrm{UHP}$ over the UHP. I want to calculate the first Chern class of the tangent bundle (relative) $T_{X/Y}$, which is dual to the bundle $\Omega^1_{X/Y}$.

Now $\Omega^1_{X/Y}$ (or $T_{X/Y}$) has a non-vanishing holom. section dz (or ∂_z), so one has to assign a norm-squared. There are two possibilities:

1) Each of the surfaces M_τ should have the same volume. Recall the formula

$$\frac{i}{2}\, dz\, d\bar{z} = |dz|^2 \,\mathrm{vol}.$$

relating $|dz|$ and the volume 2-form. Thus

$$\mathrm{vol}(M_\tau)\,|dz|^2 = \int_{M_\tau} \frac{i}{2}\, dz\, d\bar{z} = \operatorname{Im}\tau$$

and so if we want $\mathrm{vol}(M_\tau)$ to be independent of τ, then we want

(*) $$|dz|^2 = \frac{\text{const}}{\operatorname{Im}\tau}$$

2) $|dz|^2 =$ constant independent of τ. In this case $T_{X/Y}$ and $\Omega^1_{X/Y}$ have curvature 0.

Let's compute the curvature of $\Omega^1_{X/Y}$ for the metric (*).

$$K = -\bar\partial\partial \log(\operatorname{Im}\tau) = -\bar\partial \frac{1}{\operatorname{Im}\tau}\,\frac{1}{2i}\, d\tau$$

$$= \frac{1}{(\operatorname{Im}\tau)^2}\,\frac{-1}{2i}\,\frac{1}{2i}\, d\bar\tau\, d\tau = -\frac{d\tau\, d\bar\tau}{4(\operatorname{Im}\tau)^2}$$

Thus

$$c_1(T_{X/Y}) \longleftrightarrow \frac{1}{4\pi\,(\operatorname{Im}\tau)^2}\left(\frac{i}{2}\,d\tau\, d\bar\tau\right)$$

Next I want to fix a character χ of $\mathbb{Z}^2$ which is non-trivial. This will give a flat line bundle over each of the curves M_τ. Hence we will get a holomorphic line bundle over X which has a natural metric, and so I can try to compute its curvature.

Little exercise. Define a holomorphic [redacted] line bundle over $M = \mathbb{C}/\mathbb{Z}+\mathbb{Z}\tau$ by making $\mathbb{Z}^2$ act on the trivial line bundle over $\mathbb{C}$ by the rule

$$(m,n) * (z, \xi) = (z+m+n\tau,\ \chi(m,n)\,\xi)$$

and then passing to the quotient. This action preserves the holomorphic structure on the trivial bundle and its metric, hence gives a holomorphic bundle with metric over M. The connection is flat.

Now let $\mathbb{Z}^2$ act on [redacted] the trivial line bundle over $UHP \times \mathbb{C}$ by the rule

$$(m,n) * (\tau, z, \xi) = (\tau, z+m+n\tau,\ \chi(m,n)\,\xi).$$

Again [redacted] the action is both holomorphic and metric-preserving, so the line bundle we get over

$$X = (UHP \times \mathbb{C})/\mathbb{Z}^2$$

is a holomorphic bundle with metric. Again the associated connection is flat.

[redacted]

Let's go back over the GRR conjectures. I suppose given a [redacted] holomorphic family of curves $X \to Y$ and a holom. vector bundle E over X. Say the degree of E on the fibres is ~~the~~ rank times $g-1$, and that over Y

there is no cohomology generically. Now upon choosing a metric on E and a metric on $T_{X/Y}$ I can define the analytic torsion metric on the determinant line bundle. [redacted] On the other hand I also have from these metrics explicit differential forms representing $ch(E)$ and $Todd(X/Y)$. Then the conjecture is that the curvature of the determinant line bundle is $(-1)^x$ what one expects from GRR.

I seem to have constructed a counterexample to this. Let's go over this carefully. The main point is that if one gives a non-trivial flat line bundle with metric over an elliptic curve M, and uses a multiple of the Haar volume on M, then the analytic torsion is independent of the choice of Haar volume.

Look at $\zeta(s) =$ [redacted] $\mathrm{Tr}\,(D^*D)^{-s}$. Changing the volume by a factor just changes D^*D by a factor and hence $\zeta(s)$ changes [redacted] by a^s. But

$$\frac{d}{ds}\left(a^s \zeta(s)\right) = a^s \log a\, \zeta(s) + a^s \zeta'(s)$$
$$\longrightarrow (\log a)\,\zeta(0) + \zeta'(0).$$

But for elliptic curves we know $\zeta(0)=0$. Hence the [redacted] analytic torsion is independent of the Haar volume on the elliptic curve. [redacted]

~~So I take a family of elliptic curves $X \to Y$ with simply connected bases. Then a character of the fundamental group~~

So consider a family of elliptic curves $X \to Y$ and take a character of [redacted] the fundamental group of X, supposed to be non-trivial on the fibres. Then we get a flat line bundle L with metric over X; and $ch(L)$ will be trivial as a form. On the other hand I can choose the metric on $T_{X/Y}$ so that this bundle has no curvature either. So therefore we expect from the GRR formula that the determinant line bundle has curvature zero.

But it works! Go back to the formula for $\log|s|^2$, where s is the canonical section, given on page 773. In this formula w corresponds to the flat line bundle by the rule that the $\bar\partial$-op. is

$$\left(\partial_{\bar z} - \frac{\pi}{\operatorname{Im}\tau} w\right) d\bar z \qquad \text{on } \mathbb{C}/\mathbb{Z}+\mathbb{Z}\tau.$$

The connection is

$$\left(\partial_z + \frac{\pi}{\operatorname{Im}\tau}\bar w\right) dz + \left(\partial_{\bar z} - \frac{\pi}{\operatorname{Im}\tau} w\right) d\bar z = d - \alpha$$

hence the character of the fundamental group is

$$1 \longmapsto e^{\int_0^1 \alpha} = e^{\left(\frac{\pi}{\operatorname{Im}\tau}(w-\bar w)\right)} = e^{2\pi i \frac{\operatorname{Im} w}{\operatorname{Im}\tau}}$$

$$\tau \longmapsto e^{\int_0^\tau \alpha} = e^{\left(\frac{\pi}{\operatorname{Im}\tau}(w\bar\tau - \bar w\tau)\right)} = e^{2\pi i \frac{\operatorname{Im} w\bar\tau}{\operatorname{Im}\tau}}$$

If $w = -u + \tau v$, then $\operatorname{Im} w = (\operatorname{Im}\tau) v$ and $\operatorname{Im} w\bar\tau = -\operatorname{Im}(u\bar\tau) = +\operatorname{Im}\tau\, u$. The point is that for a fixed character of $\mathbb{Z}^2$ we have

$$w = -u + \tau v \qquad u, v \text{ fixed.}$$

Hence
$$\frac{(\operatorname{Im} w)^2}{\operatorname{Im}\tau} = (\operatorname{Im}\tau) v^2. \qquad \text{Thus}$$

$$\log|s|^2 = \Phi = \log|G(e^{2\pi i w})|^2 - 2\pi\left(\frac{(\operatorname{Im} w)^2}{\operatorname{Im}\tau} - \operatorname{Im} w\right) - \frac{\pi}{3}(\operatorname{Im}\tau)$$

will be linear in $\operatorname{Im}\tau$ + harmonic $\log|G|^2$. Thus it will be harmonic in τ.

An interesting point is that the curvature of the determinant line bundle line bundle is not going to be a Kähler form on the complex manifold $UHP \times \mathbb{C}$. This is because along the 1-diml cx. submanifolds $w = -u + \tau v$ with u, v real constants the curvature is zero

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Up to now I have considered non-trivial line bundles over elliptic curves. Next I want to consider the determinant line bundle belonging to the trivial bundle $\mathcal{O}_X$ where X is the universal family of elliptic curves over the UHP. One has

$$\mathcal{L}^\vee = R^0 f_*(\mathcal{O}_X) \otimes R^1 f_*(\mathcal{O}_X)^\vee .$$

But $R^0 f_*(\mathcal{O}_X) = \mathcal{O}_Y$ and $R^1 f_*(\mathcal{O}_X)^\vee$ is $f_*(\Omega^1_{X/Y})$. $f_*(\Omega^1_{X/Y})$ for the particular family we have has a canonical trivialization given by the invariant form dz. So for the family of $M_\tau = \mathbb{C}/\mathbb{Z}+\mathbb{Z}\tau$ over the UHP we have a fairly natural holom. section of the determinant line bundle. Not perfectly canonical because dz is not invariant under $SL_2(\mathbb{Z})$.

Next we put metrics on, in this case a volume element is all that is needed. Then $\mathcal{L}$ has the analytic torsion metric and I can try to calculate the norm-squared of the above holomorphic section. The actual curvature of $\mathcal{L}$ should be zero.

We first have to compute the ζ-function determinant of the Laplacean. The eigenvalues are $|\gamma|^2$, where $\gamma \in \mathbb{Z}+\mathbb{Z}\tau$ and $\gamma \neq 0$. So

$$\zeta(s) = \sum{}' \frac{1}{|\gamma|^{2s}}$$

$$\Gamma(s)\,\zeta(s) = \int_0^\infty \underbrace{\sum{}' e^{-t|\gamma|^2}}\, t^s \frac{dt}{t}$$

$$\left(\sum e^{-t|\gamma|^2}\right) - 1 \quad = \left(\left(\frac{vol(\mathbb{C}/\Gamma)}{\pi}\right)^{-1} \sum_{\Gamma^*} \frac{e^{-\frac{|\mu|^2}{t}}}{t}\right) - 1$$

So the asymptotic expansion is $\left(\frac{vol(\mathbb{C}/\Gamma)}{\pi}\right)^{-1} \frac{1}{t} - 1$ to infinite order. Thus

$$\zeta(0) = -1$$

The functional equation is

$$\Gamma(s)\,\zeta(s) = \frac{\pi}{vol(\mathbb{C}/\Gamma)} \int_0^\infty \sum{}' e^{-\frac{|\mu|^2}{t}}\, t^{s-1} \frac{dt}{t}$$

$$= \frac{\pi}{vol(\mathbb{C}/\Gamma)}\, \Gamma(1-s) \sum_{\Gamma^*}{}' \frac{1}{|\mu|^{2(1-s)}}$$

Now what I am after is $\zeta'(0)$ which is essentially the finite part of $\Gamma(s)\zeta(s)$ as $s \to 0$. It seems to be a fairly subtle quantity which Kronecker worked out.

April 19, 1983:

The problem is to verify the GRR formula for the family of constant coefficient $\bar{\partial}$-operators on a torus.

Let's first look at the more general ~~situation~~ where we have a family of elliptic curves $f: X \to Y$ and a holomorphic line bundle L over X of relative degree 0. Recall how we can parametrize the family locally. We choose an oriented basis for $H_1(X_y, \mathbb{Z})$ varying continuously in y. Then choosing a non-zero holom. $(1,0)$-form on X_y, the ratio of its periods is a map $Y \xrightarrow{\tau} UHP$ such that the family is the pull-back of the canonical family.

We now want a nice parametrization of the line bundles of degree 0. Let's use the fact that on a line bundle of degree 0 over a curve, there is a unique metric (up to a positive factor) with zero curvature, hence a unique connection which is flat and whose holomorphy group is contained in S^1.

So given L over X, we can fix the metric over the 0 section of X over Y and then it will determined by the requirement that it is flat on each fibre. This gives us at each point y a character χ_y of $H_1(X_y, \mathbb{Z})$ which we have identified with $\mathbb{Z} \times \mathbb{Z}$. Locally this character can be lifted to real functions u, v of y such that

$$\chi_y : (m,n) \longmapsto e^{2\pi i(mu+nv)}$$

But now I need some condition on u, v equivalent to the fact L is holomorphic, something like $w = u + \tau v$ being a holomorphic function of y.

Let's go back to the fixed torus $(\mathbb{R}/2\pi\mathbb{Z})^2$ with the family of operators

$$\frac{1}{i}(\partial_x + \tau\partial_y) - w$$

for $(\tau, w) \in UHP \times \mathbb{C}$. Then we form the product $X = (\mathbb{R}/2\pi\mathbb{Z})^2 \times UHP \times \mathbb{C}$ with coords x, y, τ, w and call a function $f(x,y,\tau,w)$ holomorphic when it is killed by the operators

$$\partial_{\bar\tau}, \ \partial_{\bar w}, \ \partial_x + \tau\partial_y .$$

Clear if we put $z = \frac{1}{2\pi}(-\tau x + y)$, then τ, z, w are local holomorphic coordinates for our manifold X.

Now holomorphic sections of ~~the~~ line bundle L over X are functions $f(x,y,\tau,w)$ killed by the operators

$$\partial_{\bar\tau}, \ \partial_{\bar w}, \ \frac{1}{i}(\partial_x + \tau\partial_y) - w .$$

(Since these operators commute the integrability conditions are satisfield.) Notice that e^{ixw} is a holomorphic section (locally-defined as x, y are.), hence the $\bar\partial$ operator for L is

$$e^{+ixw}\, \bar\partial\, e^{-ixw} .$$

To compute this we use $\quad \operatorname{Im} z = \frac{-1}{2\pi}(\operatorname{Im}\tau)\, x$

or
$$x = -\frac{2\pi \operatorname{Im}(z)}{\operatorname{Im}(\tau)} \qquad \text{Then}$$

$$e^{ixw}\, \bar\partial(e^{-ixw}) = +i\, 2\pi w \left[\frac{\overbrace{\partial_{\bar z} \operatorname{Im}(z)}^{-\frac{1}{2i}}\, d\bar z}{\operatorname{Im}(\tau)} + \operatorname{Im}(z)\, \underbrace{\partial_{\bar\tau} \frac{1}{\operatorname{Im}\tau}}_{-\frac{1}{(\operatorname{Im}\tau)^2}\ -\frac{1}{2i}}\, d\bar\tau \right]$$

Thus the $\bar{\partial}$-operator defining L is

$$D = e^{ix\omega}\bar{\partial}\, e^{-ix\omega}$$

$$= \bar{\partial} \ \text{[redacted]} - \frac{\pi \,\text{[redacted]}\, \omega}{\operatorname{Im}\tau}\, d\bar{z} + \frac{\pi \,\text{[redacted]}\, \omega \operatorname{Im}(z)}{(\operatorname{Im}\tau)^2}\, d\bar{\tau}$$

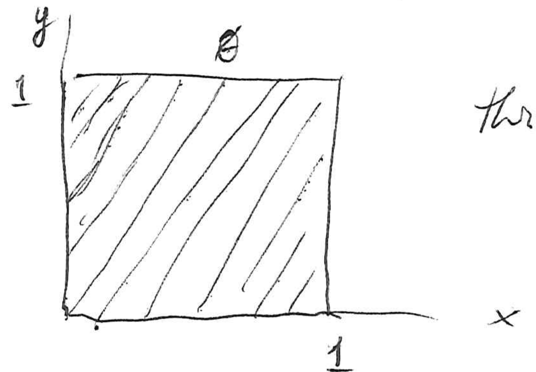
As a further check let's verify that this is periodic, that is, that it is unchanged under $(\tau, z, \omega) \longmapsto (\tau, z+m+n\tau, \omega)$. $\bar{\partial}$ is fixed under an analytic change of coords, and under $z \mapsto z+\tau$ the $(0,1)$-form changes by

$$-\frac{\pi\omega}{\operatorname{Im}\tau}\, d\bar{\tau} + \frac{\pi\omega \operatorname{Im}(\tau)}{(\operatorname{Im}\tau)^2}\, d\bar{\tau} = 0 .$$

April 20, 1983

Idea: Connes has associated a K-theory to a foliated manifold. He has some kind of notion of "vector bundle over the orbit space of the foliation", which perhaps might be important and susceptible to generalization.

Consider the Kronecker foliation as the basic examples.



corresponds to rotation through $2\pi\theta$ on S^1.

He forms a convolution algebra using smooth rapidly decreasing functions on the graph of the foliation. These are functions in

$$\mathcal{a} = \mathcal{S}(\mathbb{T}^2 \times \mathbb{R}) = \mathcal{S}(\mathbb{T}^2) \otimes \mathcal{S}(\mathbb{R})$$

with a peculiar convolution product. It seems to be useful to think of this algebra as operating in L^2 of each leaf.

A typical leaf is $x - \theta y = \alpha$ (recall that leaves are topologized with the "weak" topology, so that each leaf is $\cong \mathbb{R}$). Let's use y as a coordinate on the leaf. Then the Hilbert space of the leaf consists of $\frac{1}{2}$ densities $f(y)|dy|^{1/2}$ with $\mathcal{S}(\mathbb{R})$ acting as convolution and $F(x,y) \in \mathcal{S}(\mathbb{T}^2)$ acting by multiplying by $F(\alpha+\theta y, y)$. Thus if we use just characters $\mathcal{S}(\mathbb{T}^2)$ acts by multiplying by the functions

$$\left(e^{2\pi i y}\right)^m \left(e^{2\pi i(\alpha+\theta y)}\right)^n = e^{2\pi i(n\alpha + (m+n\theta)y)}.$$

(Looks very irreducible.)

Possible way to think of this family of Hilbert spaces. To get the von Neumann algebra, I think one forms the Hilbert space

$$\mathcal{H} = L^2(\mathbb{T}^2 \times \mathbb{R}).$$

Then the Schwarz space a of the graph under convolution acts both on the left and on the right, and the commutant of one is the weak closure of the other. I think of a acting on the left. Since a commutes with right action by $S(\mathbb{T}^2)$, the Hilbert space $\mathcal{H}$ is the direct integral of the Hilbert spaces $L^2(\mathbb{R})$ for each point of $\mathbb{T}^2$. So $\mathcal{H}$ can be recovered from the measurable family of representations of a on the [redacted] family of Hilbert spaces

$$\{ L^2(\text{leaf thru } z) \ , \ z \in \mathbb{T}^2 \}.$$

So far we have used right multiplication by functions on the torus. We have also the $\mathbb{R}$-action on the right which effectively generates the identifications among the leaves.

The fact we have $\mathcal{H}$ with structure of module over a and a^{op} [redacted] mutual commuting brings to mind the Morita equivalence. Let's review the formalism. Let $\mathcal{P}$ be an additive category closed under taking the image of projectors and having a generator. If P is a generator then we have an equivalence

$$\mathcal{P} \xrightarrow{\sim} \mathcal{P}_{A^{op}}$$

$$M \longmapsto \mathrm{Hom}(P, M)$$

where $A = \mathrm{Hom}(P, P)$. [redacted] The inverse equivalence is given by taking a right A-module R and forming

$$R \longmapsto R \otimes_A P$$

in the evident sense. So in particular if $\mathcal{P} = \mathcal{P}_B$, then the equivalence

$$\mathcal{P}_{B^{op}} \overset{\sim}{\longrightarrow} \mathcal{P}_{A^{op}}$$

is described by the functors

$$R \otimes_A P \longleftarrow\!\shortmid\; R$$

$$M \longmapsto \mathrm{Hom}_B(P, M).$$

Here P is a left A-module and right B-module. In particular we have

$$\star \qquad \mathrm{Hom}_B(P,P) = A$$

$$\mathrm{Hom}_B(P,B) \otimes_A P = B$$

Next idea is that I really should thing of $\mathcal{H}$ with its left and right $\mathcal{A}$-module structure as being some completion of $\mathcal{A}$ acting on itself. Thus $\mathcal{H} \sim \mathcal{A}$.

~~Next let e be an idempotent in a ring B and put $P = eB$. Put~~

$$\cancel{A = \mathrm{End}_B(eB)}$$

~~Now if $a \in A$, then $a: eB \to eB$ satisfies~~

$$\cancel{(aeb_1)b = a(eb_1b)};$$

~~so if we put $b_1 = e$, then~~

$$\cancel{a(eb) = (ae)b} \qquad \text{~~\& replace b by eb~~}$$

$$\cancel{\Longrightarrow a(eb) = (ae)(eb).} \quad \text{~~$\Longrightarrow$ a is mult by aee~~}$$

Start again: If we have an equivalence

$$\begin{array}{ccccc} \mathcal{P}_{B^{op}} & \sim & \mathcal{P}_{A^{op}} & & \\ B & \longmapsto & Q & & {}_BQ_A \\ P & \longleftarrow & A & & {}_AP_B \end{array}$$

then we have that the functors are given by

$$M \longmapsto \mathrm{Hom}_B(P,M) = M \otimes_B Q$$

$$N \otimes_A P = \mathrm{Hom}_A(Q,N) \longleftarrow N$$

In particular we have

$$\mathrm{Hom}_A(Q,Q) = B$$

$$\mathrm{Hom}_B(P,P) = A \,.$$

So now my idea is to just apply this to an idempotent e in A such that $eA = Q$ generates $\mathcal{P}_{A^{op}}$. Then

$$B = \mathrm{Hom}_A(Q,Q) = \mathrm{Hom}_A(eA, eA) = eAe$$

and so A can be recovered as

$$A = \mathrm{Hom}_{eAe}(P, P) = \mathrm{Hom}_{eAe}(Ae, Ae)$$

I should be able to apply this to the foliation situation in order to reduce the algebra $\mathcal{a}$.

It looks like the representation of the algebra a on any leaf Hilbert space is irreducible. Can you check this for $\mathbb{Z}$ acting irrationally on the circle. This time $a = \mathcal{S}(S^1) \otimes \mathcal{S}(\mathbb{Z})$

Let's get the notation straight. Points of the circle will be denoted $x \in \mathbb{R} \bmod \mathbb{Z}$, and we want our kernels to appear $K(x, x')$ where $x - x' \in \mathbb{Z} + \mathbb{Z}\theta$. Write $x =$ [redacted] $n\theta + x'$ to get the sense of the groupoid

$$\dot{x} \longleftarrow \dot{x}'$$

Now the algebra $\mathcal{A}$ acts faithfully on $\mathcal{S}(S^1)$. The important kernel is the one with

$$K(x,x') = 1 \qquad \text{if} \quad x - x' = \theta \quad \bmod \mathbb{Z}.$$

and its effect on a function is

$$(Vf)(x) = \text{[redacted]} \sum_{x'} K(x,x')f(x')$$

$$= f(x-\theta).$$

Hence $$(V(U\cdot 1))(x) \doteq e^{2\pi i(x - \text{[redacted]}\theta)} = e^{-2\pi i\theta}(UV1)(x)$$

or $$VUV^{-1} = \underbrace{e^{-2\pi i\theta}}_{\lambda} U.$$

Now look at a leaf Hilbert space. Thus consider the orbit $\alpha + \mathbb{Z}\theta \bmod 2\pi$. Then the Hilbert space is ℓ^2 and V will act by translation

$$V\{u_n\} = \{u_{n-1}\}$$

where U is mult. by $e^{2\pi i x}$: $\quad x = \alpha + n\theta$

$$U\{u_n\} = \{e^{2\pi i(\alpha + n\theta)}u_n\}.$$

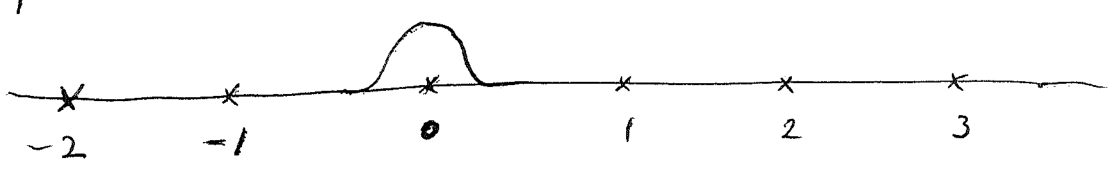
[redacted] Let's consider the algebra of operators in ℓ^2 generated by U, V. In this algebra belong the translations and the characters corresp. to $\{m\theta \in \mathbb{R}/\mathbb{Z} \mid m \in \mathbb{Z}\}$. The weak closure contains all characters, hence the representation is irreducible.

Example where the same phenomena occur. Consider representations of the fermion CCR again. Recall that these are representations of a Heisenberg type group $S^1 \times W \times G$, where $G \cong \oplus \mathbb{Z}/2$ is dense in $\hat{W} = \prod \mathbb{Z}/2$. The von Neumann algebra of interest is obtained as operators on

$$\mathcal{H} = L^2(\hat{W} \times G)$$

and again we have an $\mathcal{A} = C(\hat{W}) \otimes \mathbb{C}[G]$, which acts on the left and on the right. As in the $(S^1, \mathbb{Z})$ case we can write this as a direct integral of Hilbert spaces of the leaves, which are just the G-orbits on $\hat{W}$. Look at the operators on such a leaf Hilbert space. One has the translations by elements of G and multiplication by characters on $\hat{W}$ restricted to the orbit. Because $\hat{W}$ is dense in $\hat{G}$ the wk. closure contains all the character of G, so again we see that the leaf representation is irreducible.

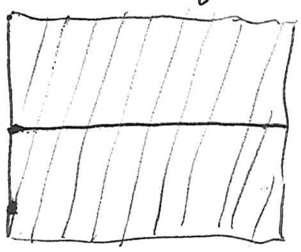
I want to take the Kronecker foliation smooth algebra and construct a projection operator inside it corresponding to a transversal. Let's first look at the problem on the level of a leaf. The transversal intersects the leaf in a $\mathbb{Z}$ orbit, and I know that I want to take the subspace of $L^2(\mathbb{R})$ spanned by a bump



and its translates under $\mathbb{Z}$. Let this function be denoted $\varphi_0(y)$ and the translations $\varphi_n(y) = \varphi_0(y-n)$. Then the projection operator on the subspace is

$$e(y,y') = \sum \frac{\varphi_n(y)\,\overline{\varphi_n(y')}}{\|\varphi\|^2}$$

The formula extends obviously for a transversal for the Kronecker foliation



namely, $e(x,x')$ will be zero unless x, x' are close to each other in the same leaf (leaf top.) and near the transversal. If φ is a bump function in the transversal direction, then

$$e(x,x') = \frac{\varphi(x)\,\overline{\varphi(x')}}{\|\varphi\|^2}$$

Question: It might be useful to know if bump functions can be replaced by Gaussians. Does the Poisson summation formula give any hints as to a nice projection operator?

On the F.T. level the subspace is $\hat{\varphi}(\xi)\cdot$periodic fns.

a_Θ is the algebra of Laurent series $\sum a_{mn} U^m V^n$ with rapidly decreasing coefficients where U, V satisfy

$$VUV^{-1} = \lambda U \qquad \lambda = e^{2\pi i \Theta}.$$

According to Connes (references to Roumanians Pincer + Voiles...) If we let a_Θ act on $\mathcal{S}(\mathbb{R})$ by the rules

$$(Vf)(x) = f(x+a)$$
$$(Uf)(x) = e^{ibx} f(x)$$

(then

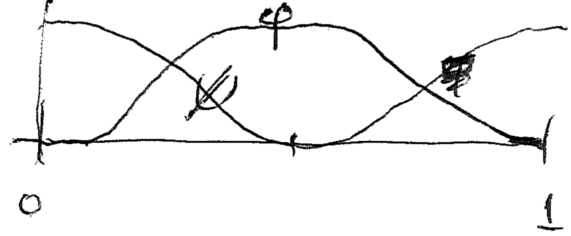
$$(VUV^{-1}f)(x) = (UV^{-1}f)(x+a) = e^{ib(x+a)} \overbrace{(V^{-1}f)(x+a)}^{f(x)}$$

so

$$= e^{iba}(Uf)(x),$$ so that we get a representation of a_Θ for $e^{iba} = \lambda$), then $\mathcal{S}(\mathbb{R})$ is a finitely generated projective module over a_Θ.

If we consider the case where $\lambda = 1$, then $a_\Theta = \mathcal{S}(\mathbb{T}^2)$, and a f.g. proj. module is the space of sections of a smooth vector bundle over $\mathbb{T}^2$. The question is which vector bundle over $\mathbb{T}^2$ do you get?

I claim that $\mathcal{S}(\mathbb{R})$ is generated by two elements. To see this suppose $a = 1$, $b = 2\pi$ (so that $\lambda = 1$). Take the function 1 on $\mathbb{R}/\mathbb{Z}$ and write it as the sum of two smooth functions $1 = \varphi(x) + \psi(x)$ such that φ vanishes near 0 and ψ vanishes near $1/2$.



Let $\varphi_0 \in C_0^\infty(\mathbb{R})$ be equal to φ in $[0,1]$ and zero outside, so that

$$\varphi(x) = \sum_n \varphi_0(x+n)$$

Given $f \in \mathcal{S}(\mathbb{R})$ we have

$$f\varphi = \sum_n \underbrace{f(x)\,\varphi_0(x+n)}_{\text{support in } (-n,-n+1)}$$

and we can find periodic function $g_n(x)$ with

$$g_n(x) = f(x-n)$$

on the support of $\varphi_0(x)$. We should define g_n to be the periodic extension of $\tilde{\varphi}_0(x) f(x-n)$ where $\tilde{\varphi}_0 \in C_0^\infty((0,1))$ and $\tilde{\varphi}_0 = 1$ on $\operatorname{Supp} \varphi_0$. This shows that $g_n \to 0$ rapidly in $\mathcal{S}(\mathbb{T})$. But now

$$f\varphi = \sum g_n(x+n)\,\varphi_0(x+n)$$

$$= \sum V^n(g_n \varphi_0)$$

and g_n is a series in U, hence $f\varphi$ will lie in the $\mathcal{Q}_\Theta$ submodule generated by φ_0. Similarly $\mathcal{S}(\mathbb{R})\psi$ will lie in the $\mathcal{Q}_\Theta$ submodule generated by ψ_0 (= piece of ψ in $[-\frac{1}{2}, \frac{1}{2}]$ extended by zero outside.)

Since $\mathcal{S}(\mathbb{R})$ is spanned by 2-elements, it is natural to expect $\mathcal{S}(\mathbb{R})$ to be the sections of a line bundle over $\mathbb{T}^2$. Since it is supposed to generate the K-theory of $\mathbb{T}^2$ along with the trivial line bundle, one expects it to be the sections of the degree 1 (or -1) line bundle.

April 23, 1983

I want to identify $\mathcal{S}(\mathbb{R})$ with the operators of translation by 1 and multiplication by $e^{2\pi i x}$ with the sections of the line bundle of degree 1 over $\mathbb{T}^2$. Let's consider the map which associates to $f \in \mathcal{S}(\mathbb{R})$, the function

$$F(x,y) = \sum_{n \in \mathbb{Z}} e^{2\pi i n y} f(x+n)$$

better to interchange x, y

Then $$F(x, y+1) = F(x,y)$$ and

$$F(x+1,y) = \sum e^{2\pi i n y} f(x+1+n)\, e^{2\pi i y - 2\pi i y}$$
$$= e^{-2\pi i y} F(x,y)$$

On the other hand recall that the Θ-function

$$G(t) = \sum q^{\frac{n(n-1)}{2}} (-t)^n$$

satisfies $$G(qt) = +\sum q^{\frac{n(n+1)}{2}} (-t)^{n+1} / -t \quad = \quad -\tfrac{1}{t}\, G(t).$$

So if we put $$t = e^{2\pi i z} \qquad q = e^{2\pi i \tau}$$

$$G(z) = \sum e^{2\pi i \tau \frac{n(n-1)}{2}} \left(-e^{2\pi i n z}\right)$$

we have

$$G(z+1) = G(z)$$
$$G(z+\tau) = -e^{-2\pi i z}\, G(z)$$

The idea is that Θ-function is the unique (up to scalar factor) holom. section of a line bundle of degree zero over the torus

$\mathbb{C}/\mathbb{Z}+\mathbb{Z}\tau$. Let's take $\tau = i$ change the sign of t and multiply [redacted]

$$G(z) = \sum e^{-\pi n(n-1)} e^{2\pi i n z}$$

by the periodic function $e^{\pm 2\pi}$ [redacted] y. Then you will get a function satisfying

$$* \quad \begin{cases} \tilde{G}(x+1, y) = \tilde{G}(x,y) \\ \tilde{G}(x, y+1) = e^{-2\pi i x}\,\tilde{G}(x,y). \end{cases}$$

Therefore the C^∞ functions of x, y satisfying $*$ can be identified with sections of the degree $+1$ line bundle over T^2. And we have a map from $\mathcal{S}(\mathbb{R})$ to the space of these sections given by

$$f \longmapsto F = \sum_{n\in\mathbb{Z}} e^{2\pi i n x} f(y+n)$$

Let's now fix our attention on this formula, and try to understand why it is an isomorphism. [redacted] Let's first figure out the operators $e^{2\pi i x}, e^{2\pi i y}$ on F in terms of f.

[redacted] $F \longmapsto e^{2\pi i x} F$ corresp. to $f \longmapsto f(y-1)$.

$F \longmapsto e^{2\pi i y} F$ corresp. to $f \longmapsto e^{2\pi i y} f(y)$

April 24, 1983

Let us compute Pic of the orbit topos of the Kronecker foliation. Because the leaves are simply connected, the line bundles on the orbit topos can be identified with line bundles over the torus $\mathbb{T}^2$ equipped with a partial (flat) connection along the leaves. Such a thing is a locally-free sheaf of rank 1 for the sheaf of smooth functions on $\mathbb{T}^2$ locally constant along the leaves. Call this sheaf $\mathcal{O}$. One has then

$$\text{Pic (orbit topos)} = H^1(\mathbb{T}^2, \mathcal{O}^*)$$

and we can compute this in analogy with the case of elliptic curves.

We first need to calculate $H^1(\mathbb{T}^2, \mathcal{O})$. We have the fine resolution

$$0 \longrightarrow \mathcal{O} \longrightarrow \underline{\mathbb{1}} \xrightarrow{\theta\partial_x - \partial_y} \underline{\mathbb{1}} \longrightarrow 0.$$

As $\theta\partial_x - \partial_y$ is a constant coefficient differential operator we can use the basis of exponential functions in $\Gamma(\mathbb{T}^2, \underline{\mathbb{1}}) = C^\infty(\mathbb{T}^2)$:

$$(\theta\partial_x - \partial_y)\, e^{2\pi i(mx+ny)} = 2\pi i(m\theta - n)\, e^{2\pi i(mx+ny)}$$

This shows that the operator is invertible except for the constant function 1, hence

$$H^0(\mathbb{T}^2, \mathcal{O}) = H^1(\mathbb{T}^2, \mathcal{O}) = \mathbb{C}$$

Next we use the exponential sequence

$$\begin{array}{ccccccccccccc} 0 \to & H^0(\mathbb{Z}) & \to & H^0(\mathcal{O}) & \to & H^0(\mathcal{O}^*) & \to & H^1(\mathbb{Z}) & \to & H^1(\mathcal{O}) & \to & H^1(\mathcal{O}^*) & \to H^2(\mathbb{Z}) \to \\ & \| & & \| & & \| & & \| & & \| & & & \| \\ & \mathbb{Z} & & \mathbb{C} & & \mathbb{C}^* & & \mathbb{Z}\oplus\mathbb{Z} & & \mathbb{C} & & & \mathbb{Z} \end{array}$$

It is fairly clearly from continuity considerations, that the map $\underset{\mathbb{Z}^2}{\overset{\parallel}{H^1(\mathbb{Z})}} \to \underset{\mathbb{C}}{\overset{\parallel}{H^1(\mathcal{O})}}$ sends the generators to the elements $1, \theta$. So we conclude that

$$[redacted]\ \mathrm{Pic}\,(\text{orbit topos}) = \mathbb{C}^*/\{e^{2\pi i n\theta}\} \times \mathbb{Z}$$

in analogy with the elliptic curve case.

One can see $H^1(\mathbb{T}^2, \mathcal{O}^*) \longrightarrow H^2(\mathbb{Z})$ is onto because any vector bundle on the [redacted] torus has a connection which when restricted to leaf directions is flat because of dimension 1.

One can realize line bundles on the orbit topos by taking the trivial bundle over the circle with the generator of $\mathbb{Z}$ acting by multiplying by $e^{2\pi i w} \in \mathbb{C}^*$ on the fibres. [redacted] Changing the trivialization of the trivial bundle by [redacted] multiplying by $e^{2\pi i x}$, changes to $e^{2\pi i w}$ to $e^{2\pi i(w+\theta)}$, so the class of the line bundle depends on $e^{2\pi i w} \in \mathbb{C}^*/\{e^{2\pi i n\theta}\}$.

Now I would like to associate to these vector bundles over the orbit topos K-elements of the reduced Connes algebra $\mathcal{A}$ whose f.t. projective modules are supposed to be the vector bundles over the "orbit space" Here we run into a problem with <u>non-unitary characters</u>.

First let's record what Roe said. One can think of a C^*-module over Connes C^*-algebra for the foliation as a being a continuous family of Hilbert spaces parametrized by points of $\mathbb{T}^2$ [redacted] together with ~~an~~ action of the holonomy. If we have a field of finite dimensional Hilbert spaces, the identity operator is compact and so one has a Kasparov K-element.

To be specific consider the $\mathcal{A}$-module we get by taking $f \in \mathcal{S}(S^1)$ with the operators

$$(Uf)(x) = e^{2\pi i x} f(x)$$

$$(Vf)(x) = e^{2\pi i \omega} f(x+\theta) \qquad VUV^{-1} = e^{2\pi i\theta}U.$$

Then $V\underline{1} = e^{2\pi i\omega}\underline{1}$, so that one might hope for an exact sequence

$$0 \longrightarrow \mathcal{A} \xrightarrow{\times(V - e^{2\pi i\omega})} \mathcal{A} \longrightarrow \mathcal{S}(\mathbb{T}) \longrightarrow 0.$$

This is certainly possible for Laurent polynomials, but here if $|e^{2\pi i\omega}| \neq 1$ we have trouble defining the map $\mathcal{A} \longrightarrow \mathcal{S}(\mathbb{T})$. In other words $\mathcal{S}(\mathbb{T})$ is not an $\mathcal{A}$-module in this case.

Review of θ-functions. Consider over $\mathbb{C}$ the unique (up to scalar factor) holomorphic line bundle + metric with the curvature form $dz\,d\bar{z}$. For example the trivial line bundle with metric given by

$$\|1\|^2 = e^{-|z|^2}$$

The Kähler form $dz\,d\bar{z}$ on $\mathbb{C}$ is translation-invariant, hence we can lift each translation on $\mathbb{C}$ to an isomorphism of the unique up to a scalar factor in S^1. Thus we get an action of a central extension of $\mathbb{C}$ by S^1 on this line bundle. The central extension then acts on the Hilbert space of L^2 holom. sections of this line bundle. The latter we can identify with $f(z)$ holom. such that

$$\|f\|^2 = \int e^{-|z|^2} |f(z)|^2 \frac{d^2z}{\pi} .$$

Given $\alpha \in \mathbb{C}$, we can define the translation operator

$$T_\alpha = e^{-\frac{|\alpha|^2}{2}} e^{\bar{\alpha} a^*} e^{-\alpha a}$$

where as usual $a = \frac{d}{dz}$, $a^* = z$ on the Hilbert space which we will denote $\mathcal{H}$. Then

$$T_\alpha T_\beta = e^{-\frac{|\alpha|^2}{2} - \frac{|\beta|^2}{2}} e^{\bar{\alpha} a^*} e^{\bar{\beta} a^*} e^{-\alpha a} e^{-\beta a} e^{[-\alpha a, \bar{\beta} a^*]}$$

$$= e^{-\frac{|\alpha|^2}{2} - \frac{|\beta|^2}{2} - \alpha\bar{\beta} + \frac{|\alpha+\beta|^2}{2}} T_{\alpha+\beta}$$

$$\therefore \quad T_\alpha T_\beta = e^{\frac{1}{2}(\bar{\alpha}\beta - \alpha\bar{\beta})} T_{\alpha+\beta}$$

Now take a lattice Γ in $\mathbb{C}$ such that the central extension splits over Γ, e.g. $\Gamma = \mathbb{Z} + \mathbb{Z}\tau$ where $\frac{1}{2}(1\tau - 1\bar{\tau}) = i \operatorname{Im}\tau \in 2\pi i \mathbb{Z}$ (or $\operatorname{Im}\tau \in 2\pi\mathbb{Z}$). We can lift the lattice back into the central extension. Then we get an action of Γ on the line bundle over $\mathbb{C}$, and as the action is free it descends to give a line bundle over the elliptic curve $\mathbb{C}/\Gamma$. Moreover the set of liftings of Γ is a torsor over the character group of Γ, hence we get a family of line bundles over $\mathbb{C}/\Gamma$ parametrized by $\hat{\Gamma}$.

The line bundle has a unique non-zero section (up to a factor), which we can get by averaging, or really summing over the lattice. For example if I use the lifting $\gamma \longmapsto \chi(\gamma) T_\gamma$, then

$$\sum_\gamma \chi(\gamma) T_\gamma \cdot 1 = \sum_\gamma \chi(\gamma) e^{-\frac{|\gamma|^2}{2}} e^{\bar{\gamma} z}$$

will correspond to the non-zero section of the line bundle over $\mathbb{C}/\Gamma$.

If I do the same to e^{hz} I get

$$\sum_\gamma \chi(\gamma)\, T_\gamma e^{hz} = \sum_\gamma \chi(\gamma) e^{-\frac{|\gamma|^2}{2} + \bar\gamma z} e^{h(z-\gamma)}$$

$$= \sum_\gamma \chi(\gamma)\, e^{-\frac{|\gamma|^2}{2} - h\gamma + (h+\bar\gamma)z}$$

All of these functions for different h should be multiples of each other, because they correspond to the unique section of the line bundle over the torus. ?

There is a mistake here which one discovers as follows. Let us compute

$$\dim_\Gamma \mathcal{H} = \int_{\mathbb{C}/\Gamma} K(z,z)$$

where K is the kernel of the projection operator on $\mathcal{H}$. An orthonormal basis for $\mathcal{H}$ is $\frac{z^n}{\sqrt{n!}}$, hence

$$K(z,z') = \sum_n \frac{z^n}{\sqrt{n!}} \frac{\bar z'^n}{\sqrt{n!}} e^{-|z'|^2} \frac{d^2z'}{\pi}$$

$$= e^{z\bar z' - |z'|^2} \frac{d^2z'}{\pi}$$

$$\int_{\mathbb{C}/\Gamma} K(z,z) = \int_{\mathbb{C}/\Gamma} \frac{d^2z}{\pi} = \frac{\operatorname{Im}\tau}{\pi} \qquad \text{if } \Gamma = \mathbb{Z} + \mathbb{Z}\tau$$

The lifting of Γ by $\gamma \mapsto T_\gamma$ will work only if $\frac{\operatorname{Im}\tau}{\pi} \in 2\mathbb{Z}$, so that we only get a line bundle of degree 2 this way.

April 25, 1983

There is supposed to be a way to see that a flat unitary bundle gives zero in $K \otimes \mathbb{R}$ using a von Neumann algebra of type II. Call this von Neumann algebra R, and suppose that $K(X) \otimes \mathbb{R}$ can be identified with some sort of thing constructed using projectors in R. I have in mind the idea that we describe a vector bundle over X by a continuous projection valued matrix e over X. We should have an embedding $\mathbb{C} \hookrightarrow R$, and be able to see that inside of R this operator e is conjugate via autos. to a constant projector.

April 26, 1983:

Consider the example of a flat line bundle over S^1 associated to a character $\chi : \mathbb{Z} \to S^1$, say $\chi(1) = q$. Then if I have a von-Neumann algebra R the image of this line bundle in the K-theory of the circle with coefficients R is going to be the bundle over the circle associated to the homom. $\chi : \mathbb{Z} \longrightarrow S^1 \subset R^*$. To trivialize this bundle one must deform this homomorphism, or better, one must find a path in R^* from q to 1. A canonical trivialization is a canonical path which doesn't seem to exist.

However if $R = C^\infty(S^1) \tilde{\otimes} \mathbb{C}[\mathbb{Z}]$, then in R^* we know q is a commutator

$$UVU^{-1}V^{-1} = q$$

where U is multiplication by e^{ix} on S^1 and V is translation in $\mathbb{Z}$. This means that the bundle over S^1 in the $\mathbb{R}$-K-theory is trivial in the algebraic K-theory, hence it should be trivial [redacted] stably. Therefore we see at least that there is a [redacted] canonical reason for the flat line bundle to become trivial in the von Neumann alg. K-theory.

Let's pursue this idea. If we work with 2×2 matrices, then we have

$$\begin{pmatrix} uvu^{-1} & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} u & 0 \\ 0 & u^{-1} \end{pmatrix}\begin{pmatrix} v & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} u & \\ & u^{-1} \end{pmatrix}^{-1}$$

and one has a standard way of writing $\begin{pmatrix} u & 0 \\ 0 & u^{-1} \end{pmatrix}$ as a product of elementary matrices. Thus we have a way to contract $\begin{pmatrix} u & 0 \\ 0 & u^{-1} \end{pmatrix}$ to the identity and hence we get a path in 2×2-matrices from g to the identity.

Next try to generalize to a representation $\Gamma \xrightarrow{g} U_n$. This time we consider a cross-product ring $R = \mathcal{S}(U_n) \otimes \mathcal{S}[\Gamma]$. [redacted] In GL_n of this ring is a canonical element g coming from $U_n \subset GL_n(\mathbb{C})$. We have an identity of the form

$$\gamma g \gamma^{-1} = g(\gamma)\, g$$

or

$$g^{-1} \gamma g = g(\gamma) \cdot \gamma$$

generalizing the above $vuv^{-1}u^{-1} = g$.

$g(\gamma)$ is a matrix with values in the center of $R = \mathcal{S}(U_n) \otimes \mathcal{S}[\Gamma]$ and hence commutes with any of the γ's.

Here's what one can do. One has the group ring of the discrete group Γ inside the von-Neuman algebra $\mathcal{R}$. [redacted]

[redacted] Recall that we have a principal Γ-bundle P over a space X and that we are looking at the bundle $P \times^{\Gamma} W$, where W is a repn. of Γ. (In fact Γ is a dense subgroup of U_n, and W is this representation). I can now for a bundle of free right $\mathcal{R}$-modules by using the left Γ-action

$$P \times^{\Gamma} \mathcal{R}.$$

I can also form $P \times^{\Gamma} (W \otimes \mathcal{R})$. Now because of the fact that we have a Γ-isom.

$$W \otimes \mathbb{C}[\Gamma] \simeq W_0 \otimes \mathbb{C}[\Gamma]$$

where Γ acts trivially on W_0, it follows that the functor

$$(*) \qquad W \longmapsto P \times^{\Gamma} (W \otimes \mathcal{R})$$

from repns. of Γ to $\mathcal{R}$-bundles over X completely forgets the Γ-action on W. Hence [redacted] I have to identify $(*)$ with the base change relative to the embedding $\mathbb{C} \subset \mathcal{R}$, at least stably.

For example, take the case $\Gamma = \mathbb{Z}$ acting on S^1 via q, so that the von Neumann algebra is

$$\mathcal{R} = \mathbb{C}[\mathbb{Z}] \tilde{\otimes} \mathcal{S}(S^1)$$

(weakly closed). Then one forms $P \times^{\Gamma} \mathcal{R}$ which will be trivialized when one deforms V in $\mathcal{R}^*$ to the identity.

April 27, 1983:

Γ is a discrete group and R is an algebra such that there exists a homomorphism $\Gamma \xrightarrow{\varphi} R^*$ and an element $g \in GL_n(R)$ such that

$$g^{-1} [redacted] \varphi(\gamma) g = \rho(\gamma) \cdot \varphi(\gamma)$$

where $\rho : \Gamma \longrightarrow GL_n(\mathbb{C})$ is a homomorphism. Then I have

$$\begin{tikzcd}
\Gamma \arrow[rr, "\varphi"] \arrow[d, "\rho"] & & R \arrow[d] \\
M_n(\mathbb{C}) \arrow[rr] & & M_n(R) = M_n(\mathbb{C}) \otimes R
\end{tikzcd}$$

where $\operatorname{Im} \varphi$ and $\operatorname{Im} \rho$ commute, and so I can form the homom. $\rho\varphi : \Gamma \longrightarrow M_n(R)$ which is conjugate to φ.

Let W be the repn. of Γ on $\mathbb{C}^n$ given by ρ. [redacted] Let P be a principal Γ-bundle over X. Then we can form fibre bundles

$$P \times^{\Gamma} W \qquad\qquad P \times^{\Gamma} (W \otimes R)$$

with fibres $\mathbb{C}$-vector spaces and right R-modules respectively. Here Γ acts [redacted] to the left ~~on~~ $W \otimes R$ by the rule

$$\gamma(w \otimes r) = \rho(\gamma) w \otimes \varphi(\gamma) r$$

and this commutes with the right action of R.

Notice that the [redacted] inclusion

$$W \subset W \otimes R$$
$$w \longmapsto w \otimes 1$$

is not compatible with the left Γ-action, so we have no obvious way to compare

$$(P\times^\Gamma W)\otimes R \qquad\qquad P\times^\Gamma (W\otimes R).$$

Nevertheless we can use the conjugacy of $\rho\otimes\varphi : \Gamma \to M_n(\mathbb{C})\otimes R = M_n(R)$ with $1\otimes\varphi$ to conclude that we have an isomorphism

$$P\times^\Gamma(W\otimes R) \cong W\otimes(P\times^\Gamma R).$$

We have two functors from representations of Γ over $\mathbb{C}$ to representations of Γ over R, namely

$$\begin{array}{lll} V \longmapsto & V\otimes R & \text{diagonal } \Gamma\text{-action} \\ \longmapsto & V\otimes R & \text{left action on just } R. \end{array}$$

which are isomorphic for the specific repn. ρ

$$\begin{array}{ccc} \mathrm{Rep}(\Gamma,\mathbb{C}) & \qquad & \mathrm{Rep}(\Gamma,R) \\ \downarrow & & \downarrow \\ K(X;\mathbb{C}) & & K(X;R) \end{array}$$

What I am missing is a way to relate K-theories.

Graeme's idea is to use a von Neumann algebra bundle over X. This time the idea is to use the isomorphism

$$\underset{\substack{\text{diagonal}\\ \text{left }\Gamma\text{ action}}}{W\otimes R} \;\sim\; \underset{\substack{1\otimes \text{ left}\\ \Gamma\text{ action}}}{W\otimes R}$$

which also commutes with right Γ-action on R, but to consider it as an isomorphism where Γ acts by conjugation on R. This gives a Γ isom. as right R-modules with Γ acting by conjugation on R.

Let Γ be a finite group and P a principal Γ-bundle over X. Then we have a functor from representations of Γ to vector bundles over X. Let $L = P \times^{\Gamma} \mathbb{C}[\Gamma]$ correspond to the regular representation. We have for any representation W a canonical isomorphism of repns.

$$\mathbb{C}[\Gamma] \otimes W \simeq \mathbb{C}[\Gamma] \otimes W_t$$
$$\gamma \otimes \gamma w \longleftarrow\!\shortmid \gamma \otimes w$$

where W_t is W with the trivial Γ action. Hence

$$L \otimes \tilde{W} \simeq L \otimes \tilde{W}_t$$

where $\tilde{W} = P \times^{\Gamma} W$.

Let $A =$ Endomorphism bundle of $L = L \otimes L^*$. This is a bundle of matrix algebras over X, an Azumaya algebra over X which is trivial in the Brauer group because it is of the form $\mathrm{End}(L)$ for some bundle L. Hence we know that the category of f.g. projective A°-modules is equivalent to the category of vector bundles over X. The equivalence is given by

$$E \longmapsto E \otimes L^* = \mathrm{Hom}(L, E) \quad \text{right } A\text{-module}$$

and the inverse equivalence takes a vector bundle M with right A-module structure into

$$M \longmapsto \mathrm{Hom}_A(L^*, M) = M \otimes_A L$$

Now consider the base extension $E \mapsto E \otimes A$ with obvious right A-module structure. This is a functor from $\mathrm{Vect}(X)$ to $\mathrm{Vect}(X, A^\circ)$ which when composed with the equivalence of the latter with $\mathrm{Vect}(X)$ gives $E \mapsto E \otimes (L \otimes L^*) \otimes_A L = E \otimes L$.

Hence we see that the map

$$K(X;\mathbb{C}) \longrightarrow K(X;A)$$

kills elements of the form $(P \times^{\Gamma} W) - (P \times^{\Gamma} W_t)$.

April 29, 1983

Consider a compact oriented surface with a fixed volume, i.e. a compact symplectic 2-manifold. Then there is an equivalence between holom. structures on M and metrics with the given volume. Call the space of these $\mathcal{H}$. We know it is a Kähler manifold. The group of volume-preserving diffeomorphisms on M acts on $\mathcal{H}$ preserving its Kähler structure, and in September (see p. 108) I computed part of the moment map. There I worked with the metrics g, but things might be simpler if I worked directly with the complex structures.

Fix coordinates x, y such that the 2-form on M giving the volume + orientation is $\omega = dx\,dy$. Then a metric will be described by

$$ds^2 = A\,dx^2 + 2B\,dx\,dy + C\,dy^2 \quad , \quad AC - B^2 = 1$$

and a cx. structure can be described by a map $\tau = \tau(x,y)$ to the UHP defined by requiring

$$T^{1,0} \text{ spanned by } dx + \tau\,dy.$$

The relation between the description is given by

$$ds^2 = A\,(dx + \tau\,dy)(dx + \bar{\tau}\,dy)$$

hence
$$\tau = \frac{B+i}{A}.$$
Since the symplectic form on the UHP is $\frac{i}{2}\frac{d\tau\,d\bar{\tau}}{(\mathrm{Im}\,\tau)^2}$ it is clear that the symplectic form on $\mathcal{H}$ will be

$$\delta_1\tau, \delta_2\tau \longmapsto \int_M \frac{i}{2}\,\frac{\delta_1\tau\,\overline{\delta_2\tau} - \overline{\delta_1\tau}\,\delta_2\tau}{(\mathrm{Im}\,\tau)^2}\,dx\,dy$$

Actually it might be better to say that the metric on $\mathcal{H}$ is given by

$$\|\delta\tau\|^2 = \int_M \frac{|\delta\tau|^2}{(\operatorname{Im}\tau)^2}\, dx\, dy$$

(up to a factor of 2).

This gets to be very messy. I need a good conceptual way of seeing things. At a given structure we have a tangent space. Deformations of the complex structure; perhaps I should think of them as C^∞ sections of $T \otimes_{\mathbb{C}} T^{0,1} = \operatorname{Hom}(T^{1,0}, T^{0,1})$. On this bundle I have a natural metric. If I have a vector field X, then it will give a section of this bundle, namely $\bar\partial X$. Now my 2 form $\tilde\Omega$ on [redacted] $C^\infty(T \otimes_{\mathbb{C}} T^{0,1})$ will be

$$\tilde\Omega(\bar\partial X, Y) = \operatorname{Im} \int [redacted] \left(\bar\partial X, \overline{Y}\right) \omega$$

Now I want a function of X and the holomorphic structure τ such that

$$(YF)(\tau) = \tilde\Omega(\bar\partial X, Y).$$

How can I analyze this? F depends upon X and the particular $\tau = T^{1,0} \subset T^*$. So [redacted] how starting with a volume preserving vector field X and a τ could you get a function? The only thing I can think of is the function

$$F_X(\tau) = \int |\bar\partial_\tau X|^2 \,\omega$$

where $\bar\partial_\tau : (T^{1,0})^* \longrightarrow (T^{1,0})^* \otimes T^{0,1}$ is the $\bar\partial$-operator belonging to the structure τ. But this isn't linear in X, so it is not correct.

Given a surface M with non-vanishing 2-form ω and a vector field X preserving ω. Then for each holomorphic structure τ on M we want a number $\varphi_X(\tau)$ depending linearly on X.

Better: The space $\mathcal{H}$ of holomorphic structures τ is a symplectic manifold with 2-form defined using ω, hence X acts as a symplectic vector field $\tilde{X}$ on $\mathcal{H}$. Then φ_X is a Hamiltonian for $\tilde{X}$.

The real problem is how can one obtain in a natural way a linear function on volume-preserving vector fields X associated to a metric on the surface.

So suppose we have 2-diml space with Riemann metric. Start off with flat 2-space and a vector field. Then this vector field has a curl and a divergence. If one identifies a vector field with a 1-form, then one can apply either d or d^* and so get 2 functions belonging to the vector field depending linearly on the vector field. Then I can integrate such a function, but of course integrating $d\eta$ gives zero.

Now we know in fact that X corresponds to a closed 1-form η by the rule $i(X)\omega = \eta$. Is $\int d^*\eta$ interesting? No because

$$\int d^*\eta = \langle 1 | d^*\eta \rangle = \langle d1 | \eta \rangle = 0.$$

The other possibility is to use the metric to convert the vector field X to a 1-form, and then take d, d^* of this 1-form. Clearly again you get functions which integrate out to zero.

The only possibilities seems as follows: Once the metric is given $T \cong T^*_{\mathbb{R}}$ and we have covariant differentiation

$$T \xrightarrow{\nabla} T \otimes_{\mathbb{R}} T^*_{\mathbb{R}} = T \otimes_{\mathbb{C}} T^* = (T \otimes T^{1,0}) \oplus (T \otimes T^{0,1})$$

where the last two isomorphisms use the complex structure on T. Now

$$T \otimes T^{1,0} = T \otimes \mathrm{Hom}_{\mathbb{C}}(T, \mathbb{C}) = \mathbb{1}$$

is canonically trivial. Hence

$$\partial : T \longrightarrow T \otimes T^{1,0} \cong \mathbb{1}$$

will give two real-valued functions associated to a vector field (or real 1-form), which must be div + curl (or d, d^*). So nothing will be obtained in this way which integrates to something non-zero.

Better analysis: I am looking for a linear fnl on symplectic v. fields X, or closed 1-forms η?, associated to a complex structure on the surface. Such a thing has to be of the form

$$\eta \longmapsto \int \eta\, \alpha$$

where α is a distributional 1-form which is determined modulo coboundaries of distributional functions. But I also know that

$$\int df \cdot \alpha = -\int f\, \Omega \qquad \forall f$$

where Ω is the curvature 2-form. But this can happen only if $d\alpha = \Omega$ which is certainly not possible when the genus of the surface $\neq 0$.

So something is definitely wrong. We have the symplectic vector fields acting on the contractible symplectic manifold $\mathcal{H}$. So certainly given X we have a function φ_X determined up to a constant on $\mathcal{H}$. So it must be the case that φ_X is not linear in X, i.e. the moment map exists on a central extension of $\mathfrak{g} =$ symplectic vector fields on M. But we know the moment map exists over the central extension (given by the functions under Poisson break) of the exact 1-forms.

This suggests that there is a natural central extension of $\mathfrak{g}$, call it $\tilde{\mathfrak{g}}$ fitting into the following diagram.

$$\begin{tikzcd}
0 \arrow[r] & C^\infty(M) \arrow[r] \arrow[d] & \tilde{\mathfrak{g}} \arrow[r] \arrow[d] & H^1(M,\mathbb{R}) \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & \left\{\begin{matrix}\text{exact}\\ \text{1-forms}\end{matrix}\right\} \arrow[r] & \underset{\text{closed 1-forms}}{\mathfrak{g}''} \arrow[r] & H^1(M,\mathbb{R}) \arrow[r] & 0
\end{tikzcd}$$

How can we attempt to describe this? Let us go back to the idea that we are after an α such that $d\alpha = \Omega$. There is an obvious candidate for α which is a connection form in a line bundle, not a 1-form. The complex structure gives us a complex line bundle T with metric and connection, hence there is a definite connection form α.

Can we associate a linear function to α on some extension of Hamiltonian vector fields?

May 1, 1983

(Conversation with George) In Russia there is a great interest in Yang-Baxter identity. This is susceptible to several formulations.

Example: Given a Lie algebra $\mathfrak{g}$ and an associated group G, then an element r in $\Lambda^2\mathfrak{g}$ can be used to define a bracket on functions on G. This bracket satisfies the Jacobi identity iff r satisfies a Y-B identity.

Example: Given $R \in \mathrm{End}(\mathfrak{g})$ one can define a new bracket on $\mathfrak{g}$ by $\{X,Y\} = [RX,Y] - [RY,X]$. This is a new Lie algebra structure $\iff$ R satisfies a YB identity.

Subexample: If $\mathfrak{g}$ is a direct sum of subalgebras $\mathfrak{a}\oplus\mathfrak{b}$ then $R = \frac{1}{2}(pr_{\mathfrak{a}} - pr_{\mathfrak{b}})$ satisfies the Y-B identity. In effect

write $x = x' + x''$, $y = y'+y''$ for the decomposition of $x, y \in \mathfrak{g}$. Then

$$\{x,y\} = [Rx,y] + [x,Ry] = \tfrac{1}{2}[x'-x'', y'+y''] + \tfrac{1}{2}[x'+x'', y'-y'']$$

$$= [x',y'] - [x'',y'']$$

is a new Lie algebra structure. (In general changing a Lie alg. bracket to $t[x,y]$ leads to an isomorphic Lie algebra for $t \neq 0$.)

Application to Kostant's thm. First note that $\mathfrak{g}^*$, the dual of $\mathfrak{g}$, has a natural Poisson bracket defined on its functions and that the subalgebra of invariant functions on $\mathfrak{g}^*$ is "in involution" (i.e. stable under Poisson bracket.) Kostants theorem says that in the $\mathfrak{g} = \mathfrak{a}\oplus\mathfrak{b}$ situation, the image of the invariant fns.

the functions on
in $\mathfrak{a}^*$ is again stable under involution. A
better theorem is that given a Y-B R $\in$ End($\mathfrak{g}$), the
$\mathfrak{g}$-invariant functions are involution for the bracket
defined using the new Lie structure. This has a
one line proof and ~~has~~ has the Kostant theorem as an
immediate consequence.

May 7, 1983

Prepare for a talk on Arakelov-Faltings and Zeta determinants. Let us start with some general considerations.

What one needs for a RR thm. on an arithmetic surface is the following: Given a holom. vector bundle E over a Riemann surface M, it has a determinant line, and one needs a metric in this line. Such a metric is provided by analytic torsion, but it depends on choosing metrics on M and E. Let's fix the metric on M and just look at the map from holom. v.b. E with metric over M to lines with metric given by analytic torsion. We want to study this map and prove enough theorems so as to be able to say we understand it.

The zeta function determinants are not 'local' i.e. integrals over M of a local expression. However the variations are given by local formulas. What does this mean? We consider a family of holom. v.b. $/M$ with metric, i.e. a holomorphic v.b. $\tilde{E}$ over $Y \times M$ with metric. Then we get a determinant line bundle over Y with metric. Take a holomorphic section (locally on Y) of the determinant line bundle which is related (in a way to be made precise) with a holom. family of Green's functions for the $\bar{\partial}$-operators over each point $y \in Y$. Then one has a formula for the covariant derivative of this section with respect to the connection on the determinant line bundle, and this formula is 'local'.

Basic problem: To understand holomorphic sections of the determinant line bundle in the context of a family.

Let us simplify and work in the context where the index is zero and the $\bar{\partial}$ operators are invertible. In this case the determinant line bundle has a canonical section and the metric 'is' just the function on Y giving the zeta determinant of the Laplacean. We know that the differential of this function is given at each y by a local formula.

$$\delta \log |s|^2_D = \delta\left[-\zeta'_{D^*D}(0)\right] = \mathrm{Tr}\left(e^{-tD^*D} D^{-1}\delta D\right)\Big|_{t=0} + c.c.$$

In using this formula one must be careful in the definition of δD. Also I should mention that one is computing the connection form corresponding to the canonical section.

Start again: suppose given a holomorphic bundle with metric, then I want the zeta determinant of the Laplacean assuming the $\bar\partial$-operator is invertible. The tool at my disposal is the variational formula, so I look for ways to deform the bundle + metric. First we can deform the metric, and what happens should be easy to describe since D^{-1} is not changing. This should result from the anomaly formula. According to Nar.-Shes. one can deform the metric to one with constant curvature at least for stable bundles. Then one has a representation of the fundamental group essentially, which one can hope to deform.

How should this work for line bundles?

May 8, 1983

I want to 'compute' analytic torsion for line bundles. Suppose given a metric on M with volume 2-form ω. We know any line bundle has a unique connection with curvature proportional to ω, hence a canonical metric up to a scalar. So the 'computation' breaks into two steps: 1) relate the analytic torsion defined by two metrics on the same holomorphic bundle 2) 'compute' the torsion defined by the canonical metric.

We want to use the fact that the variation of the torsion is given by a local formula, hence we treat these two problems by looking at suitable families. I am now going to look at the second problem. I need to find a suitable family of line bundles. (Note for future reference that there are lots of ways to parametrize line bundles:

1) divisors: $\sum n_i P_i \longmapsto \bigotimes \mathcal{O}(P_i)^{\otimes n_i}$
2) clutching function (non-algebraic) at a point ∞; this is the Baker-Akhiezer setup.
3) connections or $\bar\partial$-operators on a fixed C^∞ line bundle
4) Poincaré line bundle over $M \times \mathrm{Jac}(M)$.)

I am going to use the description 3). So I fix a holomorphic line bundle of degree d, call it L, and let $D_0 : L \longrightarrow L \otimes T^{0,1}$ be its $\bar\partial$-operator. Equip L with a metric such that the curvature is ω. Then another $\bar\partial$-operator on L is of the form

$$D = D_0 + \alpha \qquad\qquad \alpha \in C^\infty(M, T^{0,1})$$

and it has curvature $\omega \iff \partial\alpha - \bar\partial\alpha^* = 0$. If $g : M \longrightarrow U(1)$ is a gauge transformation then

$$g^{-1}(D)g = D + g^{-1}\bar\partial g$$

Included among the (0,1)-forms $g^{-1}\bar\partial g$ are the forms $i\bar\partial f$ where f is a real-valued C^∞ fn on M.

Somewhat clearer is the following. The connection is $\nabla = \nabla_0 + (\alpha - \alpha^*)$ and it has the same curvature $\iff$ $d(\alpha-\alpha^*) = \partial\alpha - \bar\partial\alpha^* = 0$. Then the form $\alpha - \alpha^* \in iC^\infty(M, T^*_{\mathbb{R}})$ changes under gauge transformations by $i\,df$. Thus we can restrict attention to $\alpha - \alpha^*$ which are harmonic, or equivalently where $\alpha \in C^0(M, T^{0,1})$ is harmonic ($\perp$ to $\operatorname{Im}\bar\partial$).

So I will consider the family of holomorphic line bundles with metric obtained as above from all $\bar\partial$-operators of the form $D = D_0 + \alpha$, with α harmonic. What I am looking at is a fixed (L_0, D_0) and tensoring with the family of degree 0 line bundles belonging to the connections on the trivial line bundle given by

$$\nabla = d + (\alpha - \alpha^*)$$

where $\alpha - \alpha^*$ is a harmonic (purely imag.) 1-form.

Falting's paper. ω is given on M and he defines a metric on a holomorphic line bundle to be <u>admissible</u> if the curvature is a constant multiple of ω. What does this look like if $L = \mathcal{O}(D)$? Enough to consider $\mathcal{O}(Q)$. Put

$$|s|^2(P) = g(P, Q).$$

Since s has a simple zero at Q we have near Q

$$|s|^2(P) = |\underbrace{z(P)}_{z} - \underbrace{z(Q)}_{z'}|^2 \, f(P) \qquad f \text{ smooth } > 0$$

and so
$$\bar\partial\partial \log|s|^2 = \pi\, \delta(z-z')\, d\bar z\, dz + \text{smooth}$$
for z near Q. Since the curvature is to be a multiple

of ω we have

$$\boxed{\bar\partial\partial \log|s|^2 = \pi\,\delta(z-z')\,d\bar z\,dz + c\,\omega}$$

where c is arranged so that the right side integrates to zero.

$$\therefore \quad 2\pi i + c\int\omega = 0.$$

(Check constants; suppose ω is the Chern form of a line bundle of degree 1. Then $\int\omega = 1$ and so $c = -2\pi i$ and so curvature of $\mathcal{O}(Q) = \frac{2\pi}{i}\,\omega$ which is correct.) Thus we see that $\log|s|^2$ has to be the Green's function for $\bar\partial\partial$ with singularity at Q, up to an additive constant.

So if $g(P,Q)$ is the Green's function we define the metric on $\mathcal{O}(Q)$ using it in the above way. Then we use multiplication to get the [redacted] metric for $\mathcal{O}(D)$. If $D = \sum_i n_i Q_i$, then one puts

$$|s|^2(P) = \prod G(P,Q_i)^{n_i}$$

where s is the canonical section of $\mathcal{O}(D)$, (merom. section).

~~the determinant line~~

Now each of the line bundles $\mathcal{O}(D)$ has a determinant line λ, and Faltings defines a metric on $\lambda(R\Gamma(\mathcal{O}(D)))$. It is defined by means of the canonical isomorphisms associated to the exact sequence

$$0 \longrightarrow \mathcal{O}(\overset{D-P}{\overset{\parallel}{D_1}}) \longrightarrow \mathcal{O}(D) \longrightarrow \mathcal{O}(D)[P] \longrightarrow 0$$

Let's introduce Faltings notation. The canonical section of $\mathcal{O}(D)$ will be denoted 1 or 1_D. Then the metric is

$$\|1_D\|^2(P) = \prod G(P,Q_i)^{n_i}$$

if $D = \sum_i n_i Q_i$ and $G = e^{g(P,Q)}$.

So next we wish to construct metrics on the determinant lines, starting from a given choice of $\lambda(\mathcal{O})$, and using the isomorphisms

$$\lambda(\mathcal{O}(D\text{[redacted]}))\ \text{[redacted]} = \lambda(\mathcal{O}(D-P)) \otimes \mathcal{O}(D)[P].$$

There is a small compatibility problem; consider $D = P+Q$. Then we have

$$(*) \quad \begin{aligned} \lambda(\mathcal{O}(P+Q)) &= \lambda(\mathcal{O}(P)) \otimes \mathcal{O}(P+Q)[Q] \\ &= \lambda(\mathcal{O}) \otimes \mathcal{O}(P)[P] \otimes \mathcal{O}(P+Q)[Q] \end{aligned}$$

and similarly an isomorphism

$$(**) \quad \lambda(\mathcal{O}(P+Q)) = \lambda(\mathcal{O}) \otimes \mathcal{O}(Q)[Q] \otimes \mathcal{O}(P+Q)[P]$$

and we want both to be isometries. Now

$$\begin{aligned} \mathcal{O}(P+Q)[Q] &\xleftarrow{\sim} \mathcal{O}(Q)[Q] \\ 1_{P+Q} &\longleftarrow\!\shortmid\ 1_Q \end{aligned}$$

but

$$|1_{P+Q}|^2(\cancel{Q}) = \infty. \quad ?$$

No: It is better to [redacted] use the section

$$\mathcal{O} \xrightarrow{1_P} \mathcal{O}(P)$$

which at the point Q multiplies the norms by $G(Q,P)$. Hence it follows that the isom.

$$\text{[redacted]} \quad \mathcal{O}(Q)[Q] \xrightarrow{1_P} \mathcal{O}(P+Q)[Q]$$

multiplies the norms by $G(Q,P)$. So the isom $(*)$ above gives an isom.

$$\lambda(\mathcal{O}(P+Q)) = \lambda(\mathcal{O}) \otimes \mathcal{O}(P)[P] \otimes \mathcal{O}(Q)[Q]$$

which is off **on** the metrics by $G(Q,P)^{-1}$. Similarly $(**)$ is off by $G(P,Q)^{-1}$. So things work because

$$G(P,Q) = G(Q,P).$$

Remark: This shows that for an exact sequence

$$0 \longrightarrow E(-\Sigma P_i) \longrightarrow E \longrightarrow \oplus E[P_i] \longrightarrow 0$$

we do not get multiplicativity for the metrics on the determinant lines. [redacted] There is a missing factor which seems to be quadratic in the divisor ΣP_i. Is this related to the non-multiplicativity for exact sequences of vector bundles?

Let's continue with Falting's work and see if the choice of metric he uses [redacted] on the surface is essential.

Consider line bundles of degree $g-1$. Here one knows the scale changing the metric on the line bundle has trivial effect on the metric on the determinant lines. Suppose then we have a family of line bundles of degree $g-1$ over M parameterized by Y. Then we can equip the bundles in the family with admissible metrics and the resulting metric on the determinant line bundle is independent of the choices. So therefore we have the following situation

$$\begin{tikzcd} \lambda(\text{family}) \arrow[r] \arrow[d] & \lambda\left(\begin{smallmatrix}\text{canonical}\\ \text{family}\end{smallmatrix}\right) \arrow[d] \\ Y \arrow[r] & \mathrm{Jac}_{g-1}(M). \end{tikzcd}$$

It would be better to argue that the family over Y is induced from the universal family over $\mathrm{Jac}_{g-1}(M)$, and that we can equip the universal family with [redacted] admissible metrics. So the determinant line bundle over Y with its metric comes from the one over $\mathrm{Jac}_{g-1}(M)$.

So it seems that Falting's construction should go through for an arbitrary volume on the surface M.

Talk on Arakelov-Faltings theory. This work extends to arithmetic surfaces the following results about divisors on algebraic surfaces:

1) intersection pairing $D_1 \cdot D_2$
2) RR thm. $\chi(\mathcal{O}(D)) = \frac{1}{2} D(D-K) + .$
3) Hodge thm. on signature of $D_1 \cdot D_2$.

curves: $\begin{array}{c} C \\ \downarrow \\ k \end{array}$ D divisor [redacted] $\deg D \in \mathbb{Z}$

$$\chi(\mathcal{O}(D)) = \dim H^0(\mathcal{O}(D)) - \dim H^1(\mathcal{O}(D))$$

RR thm. $\chi(\mathcal{O}(D)) = \deg D + 1 - g$

arithmetic curves: $[F : \mathbb{Q}] = n$, A = ring of integers.

$\operatorname{Spec}(A)$ is like an affine curve - incomplete

[redacted] $F \otimes_{\mathbb{Q}} \mathbb{R} = \mathbb{R}^{r_1} \times \mathbb{C}^{r_2}$ $\qquad r_1 + 2r_2 = n$.

$$S = \text{places of } F = \operatorname{Max}(A) \cup \underbrace{S_\infty}_{r_1 + r_2}$$

Defn. A [redacted] vector bundle over the "arithmetic curve" belonging to F is a f.g. proj A-module M together with a positive definite quadratic (resp. herm.) form on $M \otimes_A \mathbb{R}$ (resp. $M \otimes_A \mathbb{C}$) for each real place $A \to \mathbb{R}$ (resp cx. place $A \to \mathbb{C}$).

~~divisor = line bundle + $\neq 0$ meromorphic section:~~

~~$A \xrightarrow{\sim} M \otimes_{\mathbb{Q}} F$ $\qquad M \subset F$~~

~~$M =$~~

divisor = line bundle $\overset{(M,Q)}{+}$ non-zero merom. section

$M \otimes_A F = F$.

$$= \sum_{\substack{p \in S_f}}' \underset{\uparrow \in \mathbb{Z}}{n_p}\, p + \sum_{v \in S_\infty} \underset{\nwarrow \mathbb{R}}{r_v}\, v$$

RR. $\quad M \subset M \otimes_{\mathbb{Z}} \mathbb{R}$ $\qquad$ $q = \operatorname{card} k$, $\quad \operatorname{card} H^0(\mathcal{O}(D)) = q^{\dim H^0}$

analogue of this is

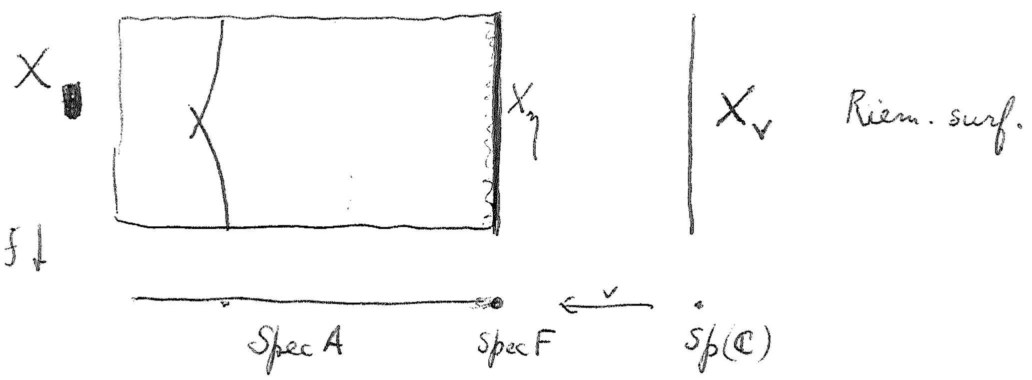
$$\sum_{m \in M} e^{-\pi Q(m)} \qquad \Theta\text{-fn.}$$

$$\sum_{m \in M}{}' e^{-\pi Q(m)} \Big/ \sum_{\mu \in M^\vee}{}' e^{-\pi Q(\mu)} = \frac{1}{\mathrm{vol}_Q (M\otimes\mathbb{R}/M)}$$

transf. formula for Θ-fn $=$ RR thm.

$$\deg(M, Q) = \log \frac{1}{\mathrm{vol}_Q (M\otimes\mathbb{R}/M)}$$

<u>arithmetic surfaces</u>



$f : X \xrightarrow{E} Y \qquad \chi(X, E) = \chi(Y, f_! E) = \deg(f_! E) + \underbrace{\operatorname{rank}(f_! E)(1-g)}_{\deg E|\text{fibre} + \cdot(1-g_F)}$

$\deg(f_! E) = \deg \lambda(f_! E)$

$$\lambda(f_! E) = \Lambda^{\max}(f_* E) \otimes \Lambda^{\max}(R^1 f_* E)^\vee$$

So if E is a v.b. over X_f, one gets a line bundle $\lambda(f_! E)$ over Spec A i.e. you get

A-lattice in $\lambda(f_! E)_\eta = \Lambda^{\max}(H^0(X_\eta, E)) \otimes \Lambda^{\max} H^1(X_\eta, E)^\vee$