648-730 March 10 - April 4, 1983

there is an extra 670-699

p.706 + 707 : ζ-values shifted by 1 for $\mathbb{Z}$
weight $p \longleftrightarrow \zeta(1-p)$

nearly all these notes are concerned with defining analytically the character of the index of a family of Dirac ops. see. formulas p. 700 + 704.

720-724. failure of analytic torsion metric to multiply for exact sequences of vector bundles

March 10, 1983 (very dizzy)

Let us consider on $\mathbb{R}^{p+q}/\mathbb{Z}^{p+q} = \underbrace{\mathbb{R}^p/\mathbb{Z}^p}_{Y} \times \underbrace{\mathbb{R}^q/\mathbb{Z}^q}_{M}$ a Dirac operator

$$\sum_{\mu=1}^{p+q} \gamma^\mu D_\mu \qquad\qquad D_\mu = \partial_\mu + A_\mu$$

acting on sections of the trivial bundle over $Y \times M$ with fibre $S \otimes E$. $E = \mathbb{C}^r$ some r. Split this operator into Y and M parts. Better, I want to think of the direct image under $pr_1 : Y \times M \to Y$, and let $(pr_1)_*(S\otimes E)$ be the infinite-diml bundle over Y with

$$\Gamma(Y, pr_{1*}(S\otimes E)) = \Gamma(Y\times M, S\otimes E)$$

or equivalently the trivial bundle with fibre $\Gamma(M, S\otimes E)$. From this point of view

$$\sum_1^{p+q} \gamma^\mu D_\mu = \sum_1^p \gamma^i D_i + \sum_{p+1}^{p+q} \gamma^\mu D_\mu$$

Let's try for a better description of the situation I have down on Y. Suppose p, q both even, whence we know that $S_{p+q} = S_p \hat{\otimes} S_q$, where S denotes the graded module of spinors. Then the Dirac operator along the fibres is $\sum_{\mu > p} \gamma^\mu D_\mu$ acting on $\Gamma(M, S_q \otimes E)$ with the coefficients A_μ varying in the Y-direction.

So what I have is a graded infinite dimensional bundle, $V = \Gamma(M, S_q\otimes E)$ are the fibres, and a self adjoint Fredholm operator $\frac{1}{i}\gamma^\mu D_\mu$, $\mu > p$ of odd degree.

Summary: Over Y I have a graded Hilbert bundle V and a self-adjoint Fredholm operator $\frac{1}{i}L$ on V of odd degree. Then over Y I can consider $S\hat{\otimes}V$ with the

operator $\sum_i \gamma^i D_i + L$ defined as follows.
Here L stands for $1 \hat{\otimes} L$ which operates with signs:

$$(1 \hat{\otimes} L)(a \hat{\otimes} b) = (-1)^{\deg a}\, a \hat{\otimes} Lb$$

Also $\gamma^i = \gamma^i \otimes 1$. Finally $D_i = \frac{\partial}{\partial y^i} + A_i$ where A_i is a degree 0 endomorphism of V.

Now the problem will be to understand the index of the operator $\not{D} + L$ on $S \hat{\otimes} V$ over Y

$$\not{D} + L = \sum_i \gamma^i \partial_i + \underbrace{\left(\sum \gamma^i A_i + L\right)}_{\text{0th order operator on } S\hat{\otimes}V.}$$

This is nice because the second term combines the connection in V, represented by the A_i, with the operators L, which is needed to make sense out of the difference $[V^+] - [V^-]$.

Let us recall what the goal is. Normally for a Dirac operator I want an expression for the index as a characteristic class involving the curvature F. This time I want the same thing for $\not{D} + L$ but it must involve L as well as the curvature of the connection D_i. The only thing it would seem one can do is to try the old proof in this slightly different situation.

$$(\not{D} + L)^2 = (\gamma^a D_a)^2 = \tfrac{1}{2}(\gamma^a\gamma^b + \gamma^b\gamma^a) D_a D_b + \tfrac{1}{2}\gamma^a\gamma^b [D_a, D_b]$$

$$= D_a^2 + \tfrac{1}{2}\gamma^a\gamma^b [D_a, D_b]$$

$$= D_i^2 + D_\mu^2 + \tfrac{1}{2}\gamma^i\gamma^j [D_i, D_j] + \gamma^i\gamma^\mu [D_i, D_\mu] + \tfrac{1}{2}\gamma^\mu\gamma^\nu [D_\mu, D_\nu]$$

$$= D_i^2 + L^2 + \gamma^i [D_i, L] + \tfrac{1}{2}\gamma^i\gamma^j F_{ij}$$

Now we need to take $\mathrm{Tr}\left(\varepsilon\, e^{+t(\not{D}+L)^2}\right)$ where ε

is the chiral γ_5 but on $S\hat{\otimes}V$. There is the possibility of taking the trace in stages.

Let's try to understand the trace of an endomorphism of $S\otimes V$. Firstly [redacted] $S\hat{\otimes}V = S\otimes V$; the only significance of that $\wedge$ is that an endo of V is [redacted] embedded as an endo of $S\otimes V$ in a strange way. [redacted] $\mathrm{End}(V)$ decomposes into eigenspaces for [redacted] conjugation by ε_V, and one embeds as follows

$$\begin{aligned} \alpha &\longmapsto 1\otimes\alpha && \text{if } \varepsilon_V\alpha\varepsilon_V^{-1}=\alpha \\ &\longmapsto \varepsilon_S\otimes\alpha && \text{if } \varepsilon_V\alpha\varepsilon_V^{-1}=-\alpha \end{aligned}$$

Put another way let $\tau: S\otimes V\to S\otimes V$ be the automorphism given by $\tau(s\otimes v) = (-1)^{\deg s\,\deg v}\, s\otimes v$. Then $\alpha\in\mathrm{End}(V)$ gets embedded by

$$\alpha \longmapsto \tau(1\otimes\alpha)\tau^{-1}$$

Finally one uses

$$\begin{tikzcd} \mathrm{End}(S\otimes V) \arrow[r, phantom, "="] \arrow[d, "\mathrm{Tr}_{S\otimes V}"] & \mathrm{End}(S)\otimes\mathrm{End}(V) \arrow[d, "\mathrm{Tr}_S\otimes\mathrm{id}"] \\ k & \mathrm{End}(V) \arrow[l, "\mathrm{Tr}_V"'] \end{tikzcd}$$

which is clear on the matrix level:

$$\mathrm{Tr}_{S\otimes V}(A_{i\mu,j\nu}) = \sum_{i,\mu} A_{i\mu,i\mu}$$

[redacted]

$$\mathrm{Tr}_V(\mathrm{Tr}_S(A_{i\mu,j\nu})) = \mathrm{Tr}_V\Big(\sum_i A_{i\mu\, i\nu}\Big) = \sum_{\mu,i} A_{i\mu,i\mu}$$

Anyway the upshot is that I can calculate traces of operators on $S\otimes V$ in either order. I must be careful to remember that $L \in End(V)^-$ becomes the operator $\varepsilon_S \otimes L$ on $S\otimes V$.

So I now have to decide about the perturbation expansion to be used for expanding the index

$$Tr\left(e^{t(\not{D}^2+L)}\varepsilon\right).$$

Since I want an index theorem for the family I should look for a formula

$$Tr_{Y\times M}\left(\varepsilon_{S\otimes V}\, e^{t(\not{D}^2+L)}\right) = Tr_Y\left(\varepsilon_S \cdots Tr_M(\varepsilon_V \cdots)\right)$$

Another ingredient is that these heat kernels are Gaussians, and a Gaussian on the product $Y\times M$ is the product of a Gaussian on Y and one on M. [redacted]

Can I make sense of the case where M is the circle? Take Y to be an odd-diml. manifold and look at a Dirac operator on $Y\times M$

$$\gamma^i D_i + \gamma^\mu D_\mu .$$

I look at the spinors on $Y\times M$. Thus we have the basic C_n- [redacted] module which is graded of dim [redacted] $2^{n/2}$ and I restrict it to C_{n-1} which is generated by the γ^i, and I know it splits into 2 irreducibles.

[redacted]

March 11, 1983

7th lecture – topics

1) curvature = Kahler form

2) analytic trivializations of $\mathcal{L}$

3) gauge transformations.

Proof of 1). The ~~curvature~~ connection form θ relative to the canonical section is

$$i(\delta D)\theta = \int tr\Big(\underbrace{F.P.\, D^{-1}(z,z)}_{J_D(z)} \cdot \delta D(z)\Big)$$

$$J_D(z) = \Big[D^{-1}(z,z') - P(z,z')\Big]_{z=z'}$$

$$P(z,z') = \underbrace{F(z,z')}_{\substack{\text{radial } \parallel \text{ transport wrt} \\ \text{the connection } \nabla \text{ chosen extending } D.}} \cdot \Big(-\partial_{z'} \log r(z,z')^2\, dz'\Big)\frac{i}{2\pi}$$

Point is that if I given a (1,0) form $\theta = a_i\, dz^i$ on a complex manifold, then $d''\theta = \sum \partial_{\bar{z}_j}(a_j)\, d\bar{z}_j dz^i$. In other words to compute $d''\theta$ I take d'' of the coefficients. In the present case I want to compute

$$\delta_1 J_D = \mathbb{C}\text{-linear in } \delta_1 D + \mathbb{C} \text{ anti linear in } \delta_1 D$$

and just take the anti-linear part.

$$\delta D^{-1} = -D^{-1}\,\delta D\, D^{-1} \qquad \mathbb{C}\text{-linear}.$$

$$F(z,z') = 1 - \beta(z')(z-z') - \alpha(z')(\overline{z-z'}) + O((z-z')^2)$$

if $$\nabla = (\partial_z + \beta)dz + \underbrace{(\partial_{\bar z} + \alpha)d\bar z}_{D}$$

Now we are taking ∇ = unique unitary connection extending D. Then if we trivialize E by an orthonormal frame for the ~~metric~~ hermitian inner product we have

$$\nabla = (\partial_z - \alpha^*)\,dz + (\partial_{\bar z} + \alpha)\,d\bar z$$

(i.e. $\nabla = d + A$, $A^* = -A$). Thus

$$\delta_1 F(z,z') = \delta_1\alpha^*(z')\boxed{(z-z')} - \delta_1\alpha(z')(\bar z - \bar z') + \cdots$$

$$\delta_1 P(z,z') = \left(\delta_1\alpha^*(z') - \delta_1\alpha(z')\,\frac{\bar z - \bar z'}{z - z'} + \cdots\right)\frac{i}{2\pi}\,dz'$$

$\therefore$ $\mathbb{C}$ anti-linear part of $\delta_1 J(z) = -\delta_1\alpha^*(z)\,\frac{i}{2\pi}\,dz$.
$= \frac{-i}{2\pi}\left(\delta_1\alpha\,d\bar z\right)^*$

Thus $d''\Theta$ is the $(1,1)$-form

$$\boxed{-\int tr\left(\delta_1\alpha^*(z)\;\delta\alpha(z)\;\frac{i}{2\pi}\,dz\,d\bar z\right)}$$

good way to write things

$$\frac{i}{2\pi}\int tr\,(\delta\alpha\,d\bar z)(\delta_1\alpha\,d\bar z)^*$$

which is just the Kahler form $\times \frac{1}{\pi}$.

Check with GRR: Over $a \times M$

$$\nabla = d_a + d_M + A$$

$$= \partial_y\,dy + \partial_{\bar y}\,d\bar y + (\partial_z + \beta)\,dz + (\partial_{\bar z} + \alpha)\,d\bar z$$

$$\nabla^2 = \partial_y\alpha\,dy\,d\bar z + \partial_{\bar y}\beta\,d\bar y\,dz + \left(\partial_z\alpha - \partial_{\bar z}\beta + [\beta,\alpha]\right)dz\,d\bar z$$

$f_*\left\{tr(e^{\nabla^2})\cdot \text{Todd}\,M\right\}$, f_* sees $dz\,d\bar z$ only

So if we want degree 2 we have 2 terms:

$$f_*\left\{\frac{1}{2}\operatorname{tr}\left(\tfrac{i}{2\pi}\nabla^2\right)^2\right\} + f_*\left(\operatorname{tr}\left(\tfrac{i}{2\pi}\nabla^2\right)\cdot \underbrace{\tfrac{1}{2}c_1(M)}_{dz\,d\bar z}\right)$$

and the second term is zero since $\operatorname{tr}(\nabla^2)$ doesn't have a $dy\,d\bar y$ term in it. Thus we get

$$\left(\frac{i}{2\pi}\right)^2 f_*\left\{\cancel{\tfrac{1}{2}}\operatorname{tr}\left(-\partial_y\alpha\,\partial_{\bar y}\beta\; dy\,d\bar y\; d\bar z\,dz\right)\right\} = \left(\frac{i}{2\pi}\right)^2\left(\int \operatorname{tr}\left(\partial_y\alpha\, d\bar z\cdot\partial_y\beta\, dz\right)\right)^{dy\,d\bar y}$$

which shows the curvature should be

$$\frac{i}{2\pi}\int \operatorname{tr}\,(\delta\alpha\, d\bar z)(\delta\alpha\, d\bar z)^* .$$

How about analytic trivializations of $\mathcal{L}$? We have to cancel the curvature.

Begin by describing the metric on the trivial bundle over $\mathcal{A}$. $\mathcal{A}$ recall is an affine space under $\Gamma(\operatorname{End}E\otimes T^{0,1}) = \mathcal{B}$.

$$\Gamma(\operatorname{End}(E)\otimes T^{0,1}) \xrightarrow{\ \sim\ } \mathcal{A}$$
$$B \longmapsto D_0 + B$$

Then I want to define an inner product $\|B\|^2$ on $\mathcal{B}$ so that if I use the metric

$$|1|^2_D = e^{-\|D-D_0\|^2}$$

then it has exactly the same curvature as $\mathcal{L}$ does.

In general $|1|^2 = e^{-z_i\bar z_i}$ leads to a curvature

$$d''d'\log|1|^2 = d''d'(-z_i\bar z_i)$$
$$= d''(-d'z_i\,\bar z_i) = d'z_i\, d''\bar z_i = dz_i\,d\bar z_i$$

Therefore I propose to define a metric on $\mathcal{B}$

by $\qquad \|\alpha\, d\bar z\|^2 = \frac{i}{2\pi}\int \operatorname{tr}\left(\alpha\, d\bar z\, (\alpha\, d\bar z)^*\right).$

Invariantly one has the inner product $\|\alpha\|^2 = \operatorname{tr}(\alpha^*\alpha)$ for $\alpha \in \Gamma(\operatorname{End}(E))$, and this extends to an inner product on $\Gamma(\operatorname{End} E \otimes T^{0,1})$

$$\|\alpha\, d\bar z\|^2 = \operatorname{tr}(\alpha^*\alpha)\, dz\, d\bar z$$

with values in $\Gamma(T^{1,1})$ which is not positive-definite until multiplied by i. Thus we do get an inner product with

$$\|\alpha\, d\bar z\|^2 = \frac{i}{2\pi}\int \operatorname{tr}\left((\alpha\, d\bar z)^*(\alpha\, d\bar z)\right)$$

$$\|B\|^2 = \frac{i}{2\pi}\int \operatorname{tr}(B^*B)$$

If I make this definition, then what is the curvature of the metric $|1|^2_{D_0+B} = e^{-\|B\|^2}$?

$$d' \log |1|^2 = -\frac{i}{2\pi}\int \operatorname{tr}(B^*\delta B)$$

$$d''d' \log |1|^2 = -\frac{i}{2\pi}\int \operatorname{tr}\left((\delta, B)^*\delta B\right)$$

so it does work, but it still isn't very clear.

Notes for 7th lecture.

1. Kähler form on $\mathcal{A}$

$\mathcal{A}$ is an affine space under $\mathcal{B} = \Gamma(\mathrm{End}(E) \otimes T^{0,1})$

$$\mathcal{B} \xrightarrow{\sim} \mathcal{A}$$
$$B \longmapsto D_0 + B$$

Inner product on $\mathcal{B}$:

$$\|B\|^2 = \frac{i}{2\pi}\int \mathrm{tr}(B^*B)$$

$$\|\alpha\, d\bar z\|^2 = \int \underbrace{\mathrm{tr}\,(\alpha^*\alpha)}_{>0}\ \underbrace{\frac{i}{2\pi} dz\, d\bar z}_{>0}$$

Kahler form on $\mathbb{C}^n$, can get as follows. Put metric on trivial bundle

$$|1|^2 = e^{-z_i \bar z_i}$$

$$\omega = \text{curvature} = d''d' \log |1|^2$$
$$= d''\left(-dz_i\, \bar z_i\right) = dz_i\, d\bar z_i$$

To define Kähler form on $\mathcal{A}$, choose a basepoint D_0 and put metric on trivial bundle

$$|1|^2_D = e^{-\|D - D_0\|^2}$$

$$d''d'\left\{-\frac{i}{2\pi}\int \mathrm{tr}(B^*B)\right\}$$

This is a form of type 1,1 so it gives a number when fed a pair of tangent vectors $B_1 = \delta_1 D$, $B = \delta D$ namely

$$-\frac{i}{2\pi}\int \mathrm{tr}\left((\delta_1 D)^*\, \delta D\right).$$

2. <u>Thm.</u> curvature of $\mathcal{L}$ = Kähler form.

To calculate $\quad d'' d' \underbrace{\log |s|^2}_{-\zeta'(0)}$

Last time saw

$$\Theta = d'(-\zeta'(0)) \; : \; \delta D \longmapsto \int \operatorname{tr}\left(FP\, D^{-1}(z,z) \cdot \delta D(z)\right)$$

where FP is constructed using the unique unitary connection ∇ extending D.

Principle: If $\Theta = \sum a_i\, dz^i$, $\quad d''\Theta = \sum \partial_{\bar z_j}(a_i)\, d\bar z_j\, dz_i$

In other words we have to calculate

$$\delta_1 \, F.P.\left(D^{-1}(z,z)\right) \qquad \text{corresp. to } \delta_1 D$$

and find the piece $\mathbb{C}$-<u>anti-linear</u> in $\delta_1 D$.

$$F.P.\, D^{-1}(z,z) = \left[D^{-1}(z,z') - P(z,z')\right]\Big|_{z=z'}$$

$$\delta_1 D^{-1} = -D^{-1}\,\delta_1 D\; D^{-1} \qquad \mathbb{C}\text{-linear in } \delta_1 D$$

$$P(z,z') = F(z,z')\left(-\partial_{z'} \log r(z,z')^2\, dz'\right)\frac{i}{2\pi}$$

$$= \left(1 - \beta(z')(z-z') - \alpha(z')\overline{(z-z')} + \dots\right)\left(\frac{-1}{z-z'} + \text{smooth}\right)\frac{i}{2\pi}dz'$$

where $\quad \nabla = (\partial_z + \beta)\, dz + \underbrace{(\partial_{\bar z} + \alpha)\, d\bar z}_{D}$

$$\delta_1 D = \delta_1\alpha\, d\bar z$$

If I use an orthonormal frame relative to given hermitian structure on E, then ∇ unitary $\Longrightarrow$
$\beta = -\alpha^*$. $\quad$ ($d + A$ unitary $\Leftrightarrow$ $A^* = -A$).

$$\delta_1 P(z,z')\Big|_{z=z'} = \delta_1\alpha(z)^* \cdot \frac{i}{2\pi}\, dz$$

$$\delta_1 \, \mathrm{F.P.}\; D^{-1}(z,z) = -\,\delta_1\alpha(z)^* \frac{i}{2\pi} dz$$

$$d''\theta : \quad -\frac{i}{2\pi}\int tr\left(\delta_1\alpha(z)^* dz \;\delta\alpha(z) d\bar{z}\right)$$

3. Construction of determinants

Thm. Given $D_0 \in \mathcal{A}$ $\exists !$ up to a constant factor analytic fn. $\det(D;D_0)$ $\ni$

$$\det_\zeta(D^*D) = e^{-\|D-D_0\|^2}\left|\det(D;D_0)\right|^2$$

Follows because $\mathcal{L}$ and $\mathbb{1}$ over $\mathcal{A}$ with the metric $|1|_D^2 = e^{-\|D-D_0\|^2}$ have same curvature, so there is a unique isom once it is given at a single point.

$$\mathbb{1} \xrightarrow{\sim} \mathcal{L}$$

$$\det(D;D_0) \longleftrightarrow s(D)$$

Consider [redacted] $Y \times M = (S)^p \times (S)^q$, $S = \mathbb{R}/L\mathbb{Z}$. and a Dirac operator over $Y \times M$ which I again write as $\gamma^i D_i + \gamma^\mu D_\mu$. One of the problems I am stuck [redacted] on is what to do when p and q are odd, say $p = q = 1$. ~~Concentrate~~ first on the case $q = 1$.

The problem is how to make sense of the break-up $Y \times M \xrightarrow{pr_1} Y \longrightarrow pt$. The problem seems to be mostly one of algebra, namely you have this module over C_{p+q} of spinors $S = S^+ \oplus S^-$ of dim $2^{(p+q)/2}$ and you want to want to separate it into some kind of tensor product of modules over C_p and C_q so that you can ~~talk~~ of the Dirac operator on Y with coefficients in $(pr_1)_*$ of something. [redacted]

What does the Dirac operator in an odd dimension look like? [redacted] There might be 2 versions related by a simple reduction.

Usually I think of the Dirac operator in dimension $2m-1$ as working on [redacted] a space of spinors of dimension 2^{m-1}; it is a self-adjoint operator. But this can [redacted] only be one of the two irreducible Clifford modules over C_{2m-1}. And it is [redacted] ungraded, because a graded C_{2m-1}-module is a ungraded C_{2m}-module.

So I am beginning to get the picture that one makes a choice of orientation to single out one of the possible Dirac operators. In even dimensions there [redacted] is one irreducible C_{2m}-module, hence one Dirac operator. However an orientation is needed to single out a grading; the orientation defines $\gamma^1 \cdots \gamma^n$ whose eigenspaces furnish the grading.

I want to start with a Dirac op. having an index. Thus I take an irreducible C_n module S with grading $S = S^+ \oplus S^-$ given by

$$\varepsilon = (i)^{\frac{n}{2}} \underbrace{\gamma^1 \cdots \gamma^n}_{\text{in the sense of the orientation}}$$

(Notice $(\gamma^1 \cdots \gamma^n)^2 = (-1)^{\frac{n(n-1)}{2}} = (-1)^{\frac{n}{2}}$, hence $\varepsilon^2 = 1$). Then if I write

$$\varepsilon = z\gamma^n \qquad z = (i)^{n/2}\gamma^1 \cdots \gamma^{n-1}$$

I know z is in the center of C_{n-1} and because γ^n and z anti-commute we must have

$$z^2 = -1$$

The eigenspaces of z give the decomposition of S into irred C_{n-1}-modules.

The first point is that the n-diml. Dirac operator will therefore give us 2 $(n-1)$-diml. Dirac operators, related so to speak by differing orientations. Thus we go from a self-adjoint anti-commuting with ε (essentially $\begin{pmatrix} 0 & D^* \\ D & 0 \end{pmatrix}$ where D is Fredholm) to two self-adjoint operators.

We have $\gamma^i D_i + \gamma^n D_n$ acting on $C^\infty(Y \times M, S)$. Think of this as $\gamma^i D_i + L$ acting on $C^\infty(Y, S \otimes C^\infty(M))$. Thus we have the infinite dimensional Dirac operator $\gamma^i D_i$; better we have this $(n-1)$-diml Dirac operator working in $S \otimes C^\infty(M)$. Normally self-adjoint operators don't give an index so what else is needed? This is a graded C_{n-1} module, the grading being given by γ^n.

I am still confused. The idea ultimately is that the operator along the fibres $\gamma^n D_n$ should give give us something in $K^1(Y)$, i.e. a map of Y to self-adjoint Fred. In addition we should have the connection D_i in the Hilbert space. Then I should have a Dirac operator on Y with coefficients in this index and then the [redacted] whole business should [redacted] have an index which is an integer.

So you need things like the [redacted] Dirac op. on Y with coefficients in a Fredholm [redacted] bundle map with connections, and its odd analogue. In the odd case you might try to take the two [redacted] C_{n-1}-modules combined with the single Hilbert space

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March 12, 1983

C_1-modules: $\quad C_1 = \mathbb{C}[\gamma^1] \cong \mathbb{C} \times \mathbb{C}$ corresp to $\gamma^1 = \pm 1$

Let n be even. Then there are two irreducible C_{n-1} distinguished by the value of $\gamma^1 \cdots \gamma^{n-1}$ (which has square $(-1)^{\frac{n}{2}-1}$). Hence there are two Dirac operators in odd dimensions.

For $n-1=1$ we have the self-adjoint operators

$$\frac{1}{i}\partial_x \qquad \text{or} \qquad -\frac{1}{i}\partial_x \qquad \text{on} \quad L^2(\mathbb{R}).$$

These are not isomorphic operators in the sense that one has an automorphism of the spinor bundle carrying one to the other. They do become isomorphic if one allows a transformation of space.

What happens if $n-1=3$. Then $\gamma^1, \gamma^2, \gamma^3$ are essentially the [redacted] Pauli spin matrices. More precisely the Pauli spin matrices are

$$\sigma_x = \begin{pmatrix} & 1 \\ 1 & \end{pmatrix} \qquad \sigma_y = \begin{pmatrix} & -i \\ i & \end{pmatrix} \qquad \sigma_z = \begin{pmatrix} 1 & \\ & -1 \end{pmatrix},$$

they anti-commute, have square 1, and satisfy

$$\sigma_x \sigma_y \sigma_z = \begin{pmatrix} i & \\ & -i \end{pmatrix}\begin{pmatrix} 1 & \\ & -1 \end{pmatrix} = \begin{pmatrix} i & \\ & i \end{pmatrix} = i.$$

So [redacted] if I take $\gamma^i = \sigma_i$ then I get the irreducible $\mathbb{C}_3$ module with $\gamma^1\gamma^2\gamma^3 = i$. The resulting Dirac operator is

$$\frac{1}{i}(\sigma_x \partial_x + \sigma_y \partial_y + \sigma_z \partial_z)$$

or $\vec{\sigma}\cdot\vec{p}$ in physical notation.

The other Dirac operator is $-$ this one. Note that changing the signs of the γ_i preserves the commutation relations, and changes the sign of $\gamma^1 \cdots \gamma^{n-1}$ if $n-1$ is odd. (For n even, changing the sign of the γ_i is the same as inner automorphism by $\gamma^1 \cdots \gamma^n$, so one has the same operator).

Let's consider the Dirac operator in $n=4$ dimensions and write $x^4 = t$.

Let us now describe the Dirac operator in n dimensions, n even, starting from the operator in $n-1$ dimensions. The picture is as follows: Given a self-adjoint operator A we consider the elliptic operator $\frac{d}{dt} + A$ in one higher dimension. To write it in Dirac form we add it to its adjoint:

$$\begin{pmatrix} 0 & -\frac{d}{dt}+A \\ \frac{d}{dt}+A & 0 \end{pmatrix} = \sigma_x A + \frac{1}{i}\sigma_y \frac{d}{dt}$$

Note that we are taking an irreducible C_{n-1} module S_{n-1} tensoring with the spin spase (an irreducible C_2 module).

$$S_n = \mathbb{C}^2 \otimes S_{n-1}$$

$$\gamma^i_n = \sigma_x \otimes \gamma^i_{n-1}$$

$$\gamma^n_n = \sigma_y \otimes 1$$

Finally $\sigma_z \otimes 1 = \varepsilon$ is the grading on S_n.

To go backwards one starts with ε and γ^n. One has

$$\varepsilon = \text{const.}\ \gamma^1 \cdots \gamma^n = -i\Big[\underbrace{\text{const}'\,(\gamma^1 \cdots \gamma^{n-1})}_{z}\gamma^n\Big]$$

where z is in the center of C_{n-1} of square $+1$. Then $z\gamma^i$ will satisfy the same commutation relations as the γ^i for $i<n$, and the $z\gamma^i$ are of ε degree 0 so

that we do get a $(n-1)$-dimensional Dirac op.

Put a different way, once we give [redacted] ε on a C_n-module, then we a C_{n+1} module structure. Furthermore we have the isomorphism

$$C_{n+1} \cong C_2 \otimes C_{n-1} \qquad \text{\underline{not} graded tensor product}$$

$$\gamma^i \longleftrightarrow \sigma_x \otimes \gamma^i \qquad i<n$$

$$\gamma^n \longleftrightarrow \sigma_y \otimes 1$$

$$\gamma^{n+1} = \varepsilon \longleftrightarrow \sigma_z \otimes 1 \qquad \text{where } \sigma_z = -i\sigma_x\sigma_y$$

[redacted] (Nice point: periodicity of Clifford algebras is due to Pauli spin matrices.)

Next point is to see if we can [redacted] describe a Dirac ~~[redacted]~~ operator on $Y \times M$, $M = S$, Y odd-dim $n-1$ in terms of a Dirac operator on Y with infinite diml coefficients and some extra structure. Now we need to give ε on S_n for $Y \times M$ because we want the index over $Y \times M$ to be defined. [redacted] So S_n has a $C_{n+1} = C_{n-1} \otimes$ [redacted] C_2 structure, and so is a tensor product

$$S_n = S_{n-1} \otimes \mathbb{C}^2.$$

Hence we will get 2 Dirac operators over Y with coefficients in the functions on $M = S^1$.

I want to be more precise. I am starting with $\gamma^i D_i + \gamma^n D_n$, where i is summed for $i<n$. [redacted] Let's now use a repn. for spin space σ_x is diagonal; in effect this means one cyclically permutes until $\sigma_x \mapsto \sigma_z$, $\sigma_y \mapsto \sigma_x$. Then

$$\frac{1}{i}\gamma^i_{(n)} D_i = \begin{pmatrix} A & \\ & -A \end{pmatrix} \qquad A = \frac{1}{i}\gamma^i_{(n-1)} D_i$$

$$\gamma^n D_n = \begin{pmatrix} & +\frac{d}{dt} + A_n \\ \frac{d}{dt} + A_n & \end{pmatrix}$$

[redacted]

Review: We consider the Dirac[redacted] operator over $Y \times M$ where M is the circle and $\dim(Y \times M) = n$ is even, so that there is an index in $\mathbb{Z}$ over $Y \times M$. Just using the Dirac op along the fibres, I want there to be an index which is in $K^1(Y)$. This is clear, namely, one gets a Hilbert bundle over Y with a self-adjoint Fredholm operator in each fibre. But then I want to be able to define a Dirac operator over Y with coefficients in this index.

To be specific the Hilbert bundle H over Y has fibres $L^2(M) = L^2(S^1)$ and the self-adjoint operator is $\frac{1}{i}\left(\partial_t + A_n(y,t)\right) = \frac{1}{i} D_n$. The other pieces of the connection $D_i = \partial/\partial y_i + A_i$ define a connection in the Hilbert bundle H. We can define the Dirac op. on Y with coefficients in H:

$$\frac{1}{i}\,\gamma^i D_i \quad \text{acting on} \quad \Gamma(Y, S_{n-1} \otimes H).$$

[redacted] Now the actual operator [redacted] which has an index is

$$\sigma_x \frac{1}{i}\gamma^i D_i + \sigma_y \cdot \frac{1}{i} D_n \quad \text{acting on} \quad \Gamma(Y, \mathbb{C}^2 \otimes S_{n-1} \otimes H).$$

[redacted] This is just the original Dirac operator over $Y \times M$.

So we reach the following construction. Suppose given a Hilbert bundle H over Y with a connection and a self-adjoint Fredholm operator in the fibres $\{A_y\}$. Then we form the operator

$$\sigma_x \frac{1}{i}\gamma^i D_i + \sigma_y A = \begin{pmatrix} 0 & B[redacted]-iA \\ B[redacted]+iA & 0 \end{pmatrix}$$

on the sections of $S \otimes H^{\oplus 2}$, where $B = \frac{1}{i}\gamma^i D_i$. The space $S \otimes H^2$ [redacted] is graded via $\sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. Thus it is clear that we are dealing with the Fredholm operator $B + iA : S \otimes H \to S \otimes H$ [redacted] on sections over Y.

Summary: Given a bundle E with connection over $Y \times M$ where $M = S^1$, $\dim(Y) = n-1$ odd, the Dirac operator with coefficients in E takes the following form. Assume a grading on the spinors over $Y \times M$ chosen - this amounts to an orientation which one assumes exists anyway. Then the Dirac operator is of the form

$$\begin{pmatrix} 0 & D^* \\ D & 0 \end{pmatrix} \quad \text{where} \quad D : \Gamma(S^+ \otimes E) \longrightarrow \Gamma(S^- \otimes E).$$

Claim that S^+, S^- can be identified with $pr_1^* S_Y$ whence

$$\Gamma(Y \times M, S^{\pm} \otimes E) = \Gamma(Y, S_Y \otimes pr_{1*}(E))$$

where $pr_{1*}E$ is the Hilbert bundle over Y, whose fibre over y is $\Gamma(M, E|y \times M)$. Then from the connection on E we get a Dirac operator, a self-adjoint operator, on the fibres of $pr_{1*}E$, and also a connection on $pr_{1*}E$. Next the connection gives rise to a Dirac op. on Y with coefficients in $pr_{1*}E$. Thus we have operators

$$D_Y = \text{Dirac op} \quad \gamma^i D_i \quad \text{using connection}$$

$$D_M = \quad " \quad " \quad \gamma^n D_n$$

on $\Gamma(Y, S_Y \otimes pr_{1*}(E))$. The claim is that

$$D = D_Y + i D_M$$

Now I should get involved with the actual index formula

March 13, 1983

Setup: We have this Dirac operator $\gamma^i D_i + \gamma^n D_n$ over $Y \times M = (S^1)^{n-1} \times (S^1)$, where $n = 2m$. I have seen that I can write this as an operator over Y. I want to make precise this form of the operator, and then carry out the analytical computation of the index in this notation.

M should be removed from the notation. We have the odd-dimensional manifold $Y = (S^1)^{n-1}$, a Hilbert bundle H over Y with a connection $D_i = \partial_i + A_i$, where the A_i are endomorphisms of the bundle, and finally a self-adjoint endomorphism L of H which is Fredholm in each fibre. Then I am interested in the operator

$\sigma_1 = \sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

$\sigma_2 = \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$

$$\sigma_1\left(\tfrac{1}{i}\gamma^i D_i\right) + \sigma_2 L = \begin{pmatrix} 0 & \frac{1}{i}\not{D} - iL \\ \frac{1}{i}\not{D} + iL & 0 \end{pmatrix}$$

on $\Gamma(Y, \mathbb{C}^2 \otimes S_{n-1} \otimes H)$.

(Notation problem: Should $\not{D}$ be $\gamma^i D_i$ or $\frac{1}{i}\gamma^i D_i$? Same problem as whether Δ should be $-\partial_i^2$ or ∂_i^2. For the moment I will stick to physicists conventions: $-\Delta = -\partial_i^2$, $\not{D} = \gamma^i D_i$.)

Start again: $Y = (S^1)^{n-1}$, $n = 2m$. $\mathcal{H}$ is a Hilbert bundle over Y with connection $D_i = \partial_i + A_i$ and s.a. Fred endom L and I consider the modified Dirac op. over Y with coefficients in $\mathcal{H}$

$$(*) \qquad \sigma_1\left(\tfrac{1}{i}\gamma^i D_i\right) + \sigma_2 L = \begin{pmatrix} 0 & \frac{1}{i}\not{D} - iL \\ \frac{1}{i}\not{D} + iL & 0 \end{pmatrix}$$

on $\Gamma(Y, S_{n-1} \otimes \mathcal{H})^2$. I am interested in calculating the index of [redacted] this operator, or more precisely of the half $\frac{1}{i}\not{D} + iL$ of it.

Let us denote the operator (*) by $\frac{1}{i}\tilde{\not{D}}$ so that

$$\tilde{\not{D}} = \sigma_1 \cdot [redacted] \not{D} + \sigma_2 \cdot (iL) = \begin{pmatrix} 0 & \not{D}+L \\ \not{D}-L & 0 \end{pmatrix}$$

and the index is given by

$$Tr\left(e^{t\tilde{\not{D}}^2}\varepsilon\right)$$

which is independent of t. (Recall L is to be given by the Dirac operator on M, so $L = \frac{1}{i}(\partial/\partial x^n + A_n)$.

$$\frac{1}{i}\tilde{\not{D}} = \sigma_1\left(\frac{1}{i}\not{D}\right) + \sigma_2 L$$

$$\parallel$$

$$\frac{1}{i}(\gamma^i D_i + \gamma^n D_n) \qquad \Longrightarrow \qquad \sigma_2 L = \frac{1}{i}\gamma^n D_n \Rightarrow L = \frac{1}{i}D_n.$$

Thus

$$\tilde{\not{D}} = \begin{pmatrix} 0 & \not{D} - iD_n \\ \not{D} + iD_n & 0 \end{pmatrix}.)$$

I need $\tilde{\not{D}}^2 = \begin{pmatrix} 0 & \not{D}+L \\ \not{D}-L & 0 \end{pmatrix}\begin{pmatrix} 0 & \not{D}+L \\ \not{D}-L & 0 \end{pmatrix}$

$$= \begin{pmatrix} (\not{D}+L)(\not{D}-L) & 0 \\ 0 & (\not{D}-L)(\not{D}+L) \end{pmatrix} = (\not{D}^2 - L^2)I + \begin{pmatrix} -[\not{D},L] & 0 \\ 0 & [\not{D},L] \end{pmatrix}$$

$$= (\not{D}^2 - L^2)\,I - [\not{D},L]\,\varepsilon, \quad \text{where } \varepsilon = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Thus

$$\boxed{-\tilde{\not{D}}^2 = (-\not{D}^2 + L^2)I + [\not{D},L]\varepsilon}$$

Now $\quad [\not{D}, L] = [\gamma^i D_i, L] = \gamma^i [D_i, L]$

because the γ^i work on spinors, and D_i, L work on $\mathcal{H}$. Now $[D_i, L]$ [redacted] is the change of L relative to the connection in $\mathcal{H}$. Thus [redacted] in the example it is a multiplication operator on the fibres.

At this point one can use the perturbation series for $e^{t\not{D}^2}$.

Where am I?

I have taken the operator $\tilde{\not{D}}$ over $Y \times M$ and replaced it by an operator over Y, a modified Dirac operator with coefficients in a Hilbert bundle.

I need an analytical index theorem for such a modified Dirac operator. Clearly this must be found within the analytical index theorem for $\tilde{\not{D}}$ over $Y \times M$ which I already have.

So let us review the index thm. for a Dirac operator over M with coefficients in a bundle E. Then replace E by a Hilbert bundle, use the modified Dirac op. and see if the theorem can be generalized.

The Dirac op. $\not{D}$ on M with coefficients E is an operator on $\Gamma(M, S \otimes E)$, where S = spinors on M. Form the operator

$$e^{t\not{D}^2}\varepsilon$$

whose trace gives the index, hence is independent of t. To compute this trace, let us think of the operator as a matrix with indices running over points m of M, and then a basis for S_m, and then a basis for E_m. So I restrict the kernel of the operator to the diagonal of $M \times M$, take the trace over $S_m \otimes E_m$ and then integrate over M. Hence I look at

$$\langle m | e^{t\not{D}^2}\varepsilon | m\rangle \in \mathrm{End}((S\otimes E)_m).$$

This has an asymptotic expansion in powers of t

Review: 1) The Dirac operator over $Y \times M$ with coefficients in the bundle E with connection can be rewritten as a modified Dirac operator over Y with coefficient in a Hilbert bundle with Fredholm endomorphism L. The formulas are

$$\begin{cases} \tilde{\not{D}} = \sigma_1 \not{D} + \sigma_2 iL & Y \text{ odd} \\ \tilde{\not{D}} = \not{D} \otimes 1 + \varepsilon \otimes iL & Y \text{ even} \end{cases}$$

and since $\sigma_2 = i\sigma_1 \varepsilon$ they are not very different

2) I am trying to obtain an analytical formula for the index of a modified Dirac operator. One starts with the operator $e^{t\tilde{\not{D}}^2}\varepsilon$ on $\Gamma(M, S \otimes H)$, where now I replace Y by M, which satisfies

$$\operatorname{Tr}\left(e^{t\tilde{\not{D}}^2}\varepsilon\right) = \text{Index}.$$

Then I want to compute the trace by looking at the kernel of $e^{t\not{D}^2}\varepsilon$ at a diagonal point (m,m), which gives me an endo. of $(S \otimes H)_m$.

The cancellation takes place when we take the spinor trace.

So lets calculate $\tilde{\not{D}}^2$.

$$\begin{aligned} \not{D}^2 &= \left(\gamma^i D_i + \varepsilon iL\right)^2 \\ &= \left(\gamma^i D_i\right)^2 + \gamma^i \varepsilon\,[D_i, iL] - L^2 \\ &= D^2 - L^2 + \tfrac{1}{2}\gamma^i\gamma^j[D_i, D_j] + \gamma^i\varepsilon[D_i, iL] \end{aligned}$$

The natural thing to do is to use the perturbation series for $\not{D}^2$ relative to $D^2 - L^2$. Then I take the spinorial trace against ε. Since an ε already appears I have to see the contributions.

In the case where Y is even the key lemma is that the trace of any non-trivial product of distinct γ-

matrices is zero, and this implies that the trace of ε against a product of distinct γ-matrices is zero unless all γ-matrices occur.

I have to take terms involving $\gamma^i\gamma^j$, $\gamma^i\varepsilon$ and take the trace against ε. When the same γ^i occurs twice one can replace it by 1. The first thing to note is that one can only eliminate all γ's in a product of these terms by having an even no of ε's and so the ε-trace is zero. So we can't ever get the empty product of γ's giving a non-zero trace. The only way is for us to get all γ's against a single ε and this we can do by using some $\gamma^i\gamma^j$ and an even no. of $\gamma^i\varepsilon$. This is indeed going to give me the sort of differential form expression that I wanted involving the variations $\delta_i L = [D_i, L]$ and the curvature $F_{ij} = [D_i, D_j]$ of the Hilbert bundle.

March 14, 1983

Kasparow cup product motivated by a simple example. Consider ψDO of order 0, P_X on X, P_Y on Y. and think of them as giving complexes of Hilbert spaces.

$$E_X^+ \xrightarrow{P_X} E_X^- \qquad\qquad E_Y^+ \xrightarrow{P_Y} E_Y^-$$

The tensor product of these complexes is ?

$$E_X^+ \otimes E_Y^+ \oplus E_X^- \otimes E_Y^- \longrightarrow E_X^- \otimes E_Y^+ \oplus E_X^+ \otimes E_Y^-$$

with the operator

$$P = \begin{pmatrix} P_X \otimes 1 & -1 \otimes P_Y^* \\ 1 \otimes P_Y & P_X^* \otimes 1 \end{pmatrix}.$$

Actually they prefer to write the complex as

$$\begin{matrix} E_X^+ \otimes E_Y^+ \\ \oplus \\ E_X^- \otimes E_Y^- \end{matrix} \longrightarrow \begin{matrix} E_X^+ \otimes E_Y^- \\ \oplus \\ E_X^- \otimes E_Y^+ \end{matrix}$$

in which case

$$P = \begin{pmatrix} 1 \otimes P_Y & P_X^* \otimes 1 \\ P_X \otimes 1 & -1 \otimes P_Y^* \end{pmatrix}$$

Now the problem is that $P_X \otimes 1$ is not pseudo-diffl of order 0. However for some reason $\Delta_X^{1/2} P_X \otimes 1$ is elliptic of order 1 (Atiyah-Singer) and so

$$\Delta_{X\times Y}^{-1/2} \left(\Delta_X^{1/2} P_X \otimes 1 \right)$$

will be elliptic of order 0. Here $\Delta_X = 1 +$ Laplacean and $\Delta_{X\times Y}$ is properly $\Delta_X \otimes 1 + 1 \otimes \Delta_Y$.

So the actual formula used for the cup product is

$$\begin{pmatrix} 1 \otimes N^{1/2} P_y & M^{1/2} P_x^* \otimes 1 \\ M^{1/2} P_x \otimes 1 & -1 \otimes N^{1/2} P_y^* \end{pmatrix}$$

where $M = \Delta_{x \times y}^{-1} \Delta_x$, $N = \Delta_{x \times y}^{-1} \Delta_y$ so $M + N = 1$.

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* It is [redacted] wrong to think of this as the tensor product of complexes, but better as a tensor product of graded modules and operators. Thus if

$$F_x = \begin{pmatrix} & P_x^* \\ P_x & \end{pmatrix} \quad \text{on} \quad \begin{matrix} E_x^+ \\ \oplus \\ E_x^- \end{matrix}$$

and similarly for F_y we have that the "good" tensor product is

$$M^{1/2}(F_x \hat{\otimes} 1) + N^{1/2}(1 \hat{\otimes} F_y)$$

So let's return to yesterday's situation where I have a modified Dirac operator

$$-i \tilde{\not{D}} = -i \gamma^i D_i + \varepsilon L$$

whose index I am trying to compute analytically. [redacted] The idea would be to scale the two terms and use the invariance of the index under deformations. [redacted] This whole business is too vague, and I think I need to become more specific.

Let me consider now $Y \times M = S^1 \times S^1$ with coordinates y, x.

Idea: The index is the trace of the identity map on a perfect complex. The identity map is represented by its kernel which is a kind of distribution on $X \times X$. To compute the trace one pulls back to the diagonal and integrates. However, pulling back distributions is only possible when transversality is satisfied, so one smooths in the normal direction first – this is where the heat kernel comes in.

Now the diagonal embedding of X factors

$$X \longrightarrow X \times_Y X \longrightarrow X \times X$$

so we can break down the smoothing process in two stages. Notice that when $X = Y \times M$, then

$$X \times_Y X = Y \times M^2 \subset Y^2 \times M = X \times X.$$

Better is to say we have the cartesian square

$$\begin{tikzcd} X \times_Y X \arrow[r] \arrow[d] & X \times X \arrow[d] \\ Y \arrow[r] & Y \times Y \end{tikzcd}$$

and so the normal bundle for $X \times_Y X$ in $X \times X$ is just the tangent bundle of Y.

The idea I had would be to smooth first in the normal direction for the embedding $X \times_Y X \subset X \times X$, then perform the pull-back ~~to~~ $X \times_Y X$. This should introduce the $\int_Y \mathrm{Todd}(Y)$ part in the index formula, and what remains should be the $ch(f_! \alpha)$.

Problem with this: One can't regularize stupidly, e.g. given an operator $D: E \to F$ where E, F happen to be isomorphic, it is necessary to regularize the identity on $\Gamma(E)$ and on $\Gamma(F)$ by Laplaceans taking into account the symbol of D.

It would seem that if I wanted to construct a partial regularization in the Y-directions that I would have to choose the symbol very carefully. Better seems to use the actual Laplacean of the operator, but with some sort of scaling (relative) of the horizontal and vertical parts.

This leads to the following problem:

Start with $\tilde{D}$ over $X = Y \times M$ and form $e^{t\not{D}^2}\varepsilon$ as an operator on $\Gamma(X, S_X \otimes E)$. Restrict the kernel of this operator to $X \times_Y X$. At a point $y \in Y$ we get an operator on $S_{Y,y} \otimes \Gamma(M_y, S_{M_y} \otimes E|M_y)$ and we can take the spinorial trace over $S_{Y,y}$. This gives an endomorphism of $\Gamma(M_y, S_{M_y} \otimes E|M_y)$, except that the endo behaves like an n-form on Y at y. That's because of what the spinor trace looks like. $n = \dim Y$

The question is to describe this endomorphism. Now if L_y is the Dirac operator in the fibre, then we expect $e^{-tL_y^2}$ to be an important part of this endomorphism, maybe even that there is an asymptotic expansion roughly of the form

$$e^{-tL_y^2}\ (\ \cdots \text{powers of } t \cdots\ )$$

except that it should be written in a non-commutative way. By this I mean that

$$\int_0^t e^{-(t-t_1)L^2} A(t_1)\, e^{-t_1 L^2}\, dt_1,$$

is a non-commutative version of

$$e^{-tL^2} \int_0^t A(t_1)\, dt_1,$$

and the last term has an obvious expansion in powers of t.

Take a Dirac operator $-i\tilde{\not{D}}$ on $Y\times M$ and look at $e^{t\tilde{\not{D}}^2}\varepsilon$ and restrict it to a fibre $\{y\}\times M = M_y$. Then we get an operator on $S_{Y,y}\otimes\Gamma(M_y, S_M\otimes E_y)$ and we can take the appropriate spinor trace over $S_{Y,y}$ to get an operator on $\Gamma(M_y, S_M\otimes E_y)$. I need to work out some examples so as to see what this operator looks like.

Let's begin with both Y, M even-dimensional.

$$-i\tilde{\not{D}} = -i(\gamma^i D_i) - i(\gamma^\mu D_\mu) = -i\not{D} + \varepsilon_Y L$$

$$+\tilde{\not{D}}^2 = +\not{D}^2 + [\not{D}, i\overset{\varepsilon_Y}{\wedge}L] - L^2$$

$$= D_i^2 + \tfrac{1}{2}\gamma^i\gamma^j[D_i, D_j] + \gamma^i\varepsilon_Y[D_i, iL] - L^2$$

So now look at

$$e^{t\tilde{\not{D}}^2}\varepsilon_Y\varepsilon_M.$$

I want to restrict this to a single fibre and take the trace over spinor indices in S_Y.

Now for the sort of thing we are doing it would look like M doesn't occur. The Y-spinor index should give something like an n-form on Y.

The problems: 1) What perturbation series do you write down? $\tilde{\not{D}}^2$ relative to $D_i^2 - L^2$, or just D_i^2.

2) Do you want to work with a 2-parameter heat kernel resulting from $-i(\gamma^i D_i)t_1 + \varepsilon_Y L t_2$?

$$\frac{1}{\lambda+\tilde{\not{D}}^2} = \frac{1}{\lambda + D_i^2 - L^2} + \frac{1}{\lambda+D_i^2-L^2}\Big[\qquad\Big]\frac{1}{\lambda+D_i^2-L^2} + \cdots$$

When I take the $\mathrm{Tr}(\varepsilon_Y\cdots)$, then the first non-zero term has $\frac{n}{2}$ interactions. In the finite dim'l case you just let $\lambda\to\infty$ and somehow argue that the D_i^2

term doesn't matter.

I have this idea of rescaling the metric on $Y \times M$ differently in the horizontal and vertical directions. Let's look at what this means for the spinors.

Start with the spinors for $\mathbb{R}^{n+k}$ with usual inner product. This is a complex vector space S with hermitian inner product equipped with self-adjoint operators γ^i $1 \leq i \leq n$, γ^μ $1 \leq \mu \leq k$ which anti-commute and have square 1. Now consider the vector space $\mathbb{R}^n \times \mathbb{R}^k$ with the new inner product $s\, x_i^2 + t\, x_\mu^2$ with $s, t > 0$. Then the spinors for this new Euclidean space can be identified with the old spinors S but equipped with the operators $\tilde{\gamma}^i = \sqrt{s}\, \gamma^i$, $\tilde{\gamma}^\mu = \sqrt{t}\, \gamma^\mu$. Thus

$$(\tilde{\gamma}^i x_i + \tilde{\gamma}^\mu x_\mu)^2 = s\, x_i^2 + t\, x_\mu^2$$

So what is happening in general is the following. We start with the Clifford algebra $C(V,Q)$ of the vector space V with quadratic form Q:

$$v^2 = Q(v)1$$

Then if (V', Q') is isomorphic to (V,Q) we get an isom. $C(V',Q') \simeq C(V,Q)$ and so the spinors for V, Q become spinors for V', Q'.

The point of the above nonsense is that any attempt to rescale the metric in $Y \times M$ will have the same effect as changing the Dirac operator from $-i\not{D} + L$ to $s(-i\not{D}) + tL$.

The real question then becomes what to make out of the heat kernels as s goes to zero before t.

Next I want to review the invariance of the index.

$$\delta\, Tr\left(e^{-tD^*D}\right) = -t\, Tr\left(e^{-tD^*D}\left(\delta D^*\cdot D + D^*\cdot\delta D\right)\right)$$
$$\delta\, Tr\left(e^{-tDD^*}\right) = -t\, Tr\left(e^{-tDD^*}\left(\delta D\; D^* + D\cdot\delta D^*\right)\right)$$

These are equal because

$$Tr\Big(\underbrace{e^{-tD^*D}}_{\text{smooth}}\,\delta D^*\cdot D\Big) = Tr\left(D\, e^{-tD^*D}\,\delta D^*\right)$$
$$= Tr\left(e^{-tDD^*} D\,\delta D^*\right).$$

Notice that D^* is not required to be the adjoint of D, only that the [redacted] heat kernels be smooth.

This means that [redacted] if $\not{D}_s = s\not{D} + iL$, then

$$Tr\left(e^{+t\not{D}_s^2}\,\varepsilon\right)$$

is independent of ~~both~~ s and t. I have seen that I can compute this trace by [redacted] integrating over [redacted] Y the trace on the fibre.

The big hope is that if I restrict [redacted] $e^{t\not{D}_s^2}\varepsilon$ to the fibre over Y, take the spinor trace over the Y spinors, and let $s \to 0$ I should get something computable.

Computation: Let's consider the case $Y \times M = S^1 \times S^1$ and the Dirac operator

$$\tilde{\not{D}}_s = s\gamma^1 D_1 + \gamma^2 D_2. \qquad \gamma^1 = \sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \gamma^2 = \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

To simplify I want to think in terms of the $\bar\partial$-operator

$$D_s = sD_1 + i[redacted]D_2$$

and its adjoint. Thus

$$\tilde{\not{D}}_s = \begin{pmatrix} 0 & +sD_2 - iD_2 \\ sD_1 + iD_2 & 0 \end{pmatrix}$$

So we have this $\bar{\partial}$-operator

$$D_s = sD_1 + iD_2 \qquad D_i = \partial_i + A_i$$

and the heat kernel will be of the form

$$\begin{pmatrix} e^{-tD_s^* D_s} & 0 \\ 0 & e^{-tD_s D_s^*} \end{pmatrix}.$$

I should be able to compute [redacted] a heat kernel such as $e^{-tD_s^* D_s}$ rather easily, [redacted] because the Riemann structure is constant, so the geodesics are straight lines.

The only general way to understand a heat operator like $e^{-tD_s^* D_s}$ $\quad D_s = sD_1 + iD_2$ is via path integrals. In any case I should be able to do an asymptotic expansion as $s \to 0$. I might as well take $t = 1$, and use paths over $[0,1]$.

$$D_s = sD_1 + iD_2$$
$$D_s^* = -sD_1 + iD_2$$

$$-D_s^* D_s = \underbrace{+s^2 D_1^2 + D_2^2}_{\substack{\text{covariant} \\ \text{Laplacean} \\ \text{for the connection}}} \underbrace{+ si[D_1, D_2]}_{\substack{\text{potential} \\ \text{energy of} \\ \text{some sort.}}}$$

changes sign for $D_s D_s^*$

$$e^{-[redacted]D_s^* D_s} = \left(e^{-\frac{1}{N} D_s^* D_s}\right)^N$$

It looks like I should get involved with a review of path integrals.

We start with a second order [redacted] differential operator $\square$ on a vector bundle E whose symbol is multiplication by a Riemann metric. Then one has

$$\square = -\nabla^2 + V$$

where ∇ is a unique connection on the bundle and V is a 0th order operator on E.

In the physics literature one supposes space is $\mathbb{R}^n$ and the bundle is trivial. Then the connection ∇ is a collection of $D_i = \partial_i + A_i$. Parallel translation with respect to the connection along a path $x(t)$ is a matrix function $P(t)$ satisfying

$$\frac{d}{dt} P(t) + A_i \frac{dx^i}{dt} P(t) = 0.$$

(i.e. if $P(t) = P(x(t))$, then $\dot{x}^i D_i P = 0$). Thus

$$P(t) = T\left\{ e^{-\int^t A_i dx^i} \right\}$$

is a way of writing the parallel translation.

Finally along a curve a connection can be changed by adding a bundle endomorphism, so we can consider

$$P(t) = T\left\{ e^{-\int^t (A_i dx^i + V dt)} \right\}$$

which satisfies

$$\left(\frac{d}{dt} + \dot{x}^i A_i + V \right) P = 0.$$

Next we need the small t expansion.

There is a principle in physics which says that all physics should be derived from a Lagrangian. Furthermore the Lagrangian is a <u>scalar</u> density on space time so that one can form $\int e^{\frac{i}{\hbar}S}$, $S = \int \mathcal{L}\, d^4x$. However path integrals, which occur for $e^{-t\square}$ where

$$\square = -\nabla^2 + V$$

is an operator on sections of a vector bundle, are not apparently of the Feynman form. Parallel translation with respect to the connection ∇ occurs and this is a sort of non-commutative integral

$$P = T\left\{ e^{-\int A\,dx} \right\}.$$

Supposedly the Dirac operator comes from a Lagrangian; this is just the aforementioned principle. Such a Lagrangian would involve fermion quantities somehow.

It seems to be very worthwhile to understand this viewpoint, both with the idea of obtaining simpler path integral expressions to work with, and for understanding more <u>physics</u>.

Let's begin with the simplest fermion situation, namely where the Hilbert space $\mathcal{H}$ is a Fock space ΛV on a finite-dimensional V. We choose an orthonormal basis for V and get corresponding operators ψ_i, ψ_i^* on $\mathcal{H}$. Then I need some sort of time evolution - which I will assume comes from a time-evolution on V.

$$H(t) = -\psi_i^* A_{ij}(t)\psi_j \qquad \text{on } \Lambda V.$$

In other words I take the trivial bundle over the time line with fibre V and put on it the connection

$$D = d/dt + A.$$

Now I have seen how the path integral

$$\int e^{-\int \psi^*(\partial_t + A)\psi \, dt} \, [D\psi][D\psi^*]$$

formally represents the determinant of $\partial_t + A$. It is necessary to put on some boundary conditions, e.g. periodic. I also know that the determinant is calculated from the parallel translation and the way it "intersects" the boundary conditions.

If I use periodic boundary conditions, then the determinants or path integral is

$$\det(1-P) = \operatorname{Tr}_{\Lambda V}(\Lambda(P)\varepsilon).$$

One mystery is how to handle the case of $\Lambda V \otimes E$. In the application I would have a vector bundle with connection over a Riemannian manifold, and I then have an induced connection over $\Lambda T^* \otimes E$ which I can then consider restricted to a loop in the manifold. Then I get a monodromy $\tilde{P}$ on $\Lambda(V)\otimes E$, assuming a basepoint given, and taking the trace over $\Lambda(V)$, I get an endo. of E. Can one write this as a path integral?

The connection on $\Lambda(T^*)\otimes E$ is the natural way of extending connections on T^* and E. So $\tilde{P}$ will be the tensor product $\Lambda(P_{T^*}) \otimes P_E$. The conclusion is that the T^* part is effectively separated from E at this level. Actually nothing interesting should occur here, for a parallelizable manifold. On the other hand the curvature introduces the term

$$\gamma^i \gamma^j F_{ij}$$

which mixes Spinors and E.

March 17, 1983

Yesterday I concluded that for a Dirac operator over a torus, there was no way to avoid the parallel translation in E occurring in the path integral. (The only alternative would be to make the E degrees of freedom into a many particle field in some way. This is not unreasonable given the Connes theory that treats the family of all E's at once.)

Let's go back to our program of computing the heat kernel $e^{+\tilde{D}_s^2}$, where (as $s \to 0$)

$$\tilde{D}_s = s\gamma^i D_i + \gamma^\mu D_\mu$$

is a Dirac operator on a product of two tori. The path integral will be taken over all paths $(x(t), y(t))$ $0 \le t \le 1$, and will involve the parallel translation in the bundle $S \otimes E$. Here E has the connection (D_i, D_μ) and S is flat. I should check that this prescription is consistent with what I get when I write $-\tilde{D}_s^2$ in the standard form $-\nabla^2 + V$. But that is clear because

$$\tilde{D}_s^2 = s^2 D_i^2 + D_\mu^2 + \tfrac{1}{2} s^2 \gamma^i\gamma^j [D_i, D_j]$$
$$+ s\gamma^i\gamma^\mu [D_i, D_\mu] + \tfrac{1}{2}\gamma^\mu\gamma^\nu [D_\mu, D_\nu]$$

and the curvature terms are of zeroth order.

Thus I see that the parallel transport term in the path integral will be

$$1_S \otimes T\left\{ e^{-\int (A_i dy^i + A_\mu dx^\mu)} \right\}.$$

The time-ordering can be removed if I suppose $\dim(E) = 1$

But this won't do me any good because I have to consider parallel transport with the potential thrown in.

It would seem to be useful to write down the path integral formula for a Dirac operator over Euclidean space. Recall the derivation

$$\Box = -\nabla^2 + V$$

$$e^{-t\Box} \quad \text{killed by} \quad \partial_t + \Box = \partial_t - \nabla^2 + V$$

$$\frac{e^{+\frac{u}{t}}}{t^{-n/2}}\left(\partial_t - \nabla^2 + V\right)\frac{e^{-\frac{u}{t}}}{t^{n/2}} = \partial_t + \frac{1}{t^2}u - \frac{n}{2}\frac{1}{t} - \left(\nabla - \frac{1}{t}\nabla u\right)^2 + V$$

$$= \frac{1}{t^2}\left(u - (\nabla u)^2\right) + \frac{1}{t}\left(2\nabla u\cdot\nabla + \nabla^2 u - \frac{n}{2}\right) + \partial_t - \nabla^2 + V$$

Then $u = (\nabla u)^2$ has soln. $u = \frac{r^2}{4}$, whence $2\nabla u\cdot\nabla = \nabla_{r\frac{d}{dr}}$

Choose $\xi(x)$ such that

$$\nabla_{r\frac{d}{dr}} \log \xi + \left(\nabla^2 u - \frac{n}{2}\right) = 0.$$

Then we know that $\xi = 1 + O(r^2)$. Also

$$\left(\frac{e^{-\frac{u}{t}}}{t^{n/2}}\xi\right)^{-1}\left(\partial_t + \nabla^2 - V\right)\frac{e^{-\frac{u}{t}}}{t^{n/2}}\xi = \partial_t + \frac{1}{t}\nabla_{r\frac{d}{dr}} - \left(\nabla + \nabla\log\xi\right)^2 + V$$

and we can begin the asymptotic expansion

$$W_0 + tW_1 + \cdots$$

One has the following transport equations

$$\nabla_{r\frac{d}{dr}} W_0 = 0$$

$$\left(1 + \nabla_{r\frac{d}{dr}}\right)W_1 + \left(-(\nabla + \nabla\log\xi)^2 + V\right)W_0 = 0$$

For the path integral we need only $W_0(x)$ through 2nd order and the value of $W_1(x)$ at $x=0$. Now if I use coordinates in E such that radial parallel transport is constant, then

$$W_0(x) = 1 \quad \text{to all orders}$$

$$W_1(0) = (\nabla \log \xi)^2_{(0)} - V(0) = -\left(V - \frac{R}{6}\right)$$

The only other point is to know what is $(\nabla \log \xi)^2(0)$. Now

$$\Delta\left(\frac{r^2}{4}\right) = \frac{n}{2} - \frac{1}{6} R_{\mu\alpha\mu\beta}\, x^\alpha x^\beta + O(x^3)$$

$$\therefore \quad \nabla_{r\frac{d}{dr}}(\log \xi) = \frac{1}{6} R_{\mu\alpha\mu\beta}\, x^\alpha x^\beta + O(x^3)$$

$$\log \xi = \frac{1}{12} R_{\mu\alpha\mu\beta}\, x^\alpha x^\beta + \cdots\cdots$$

$$\nabla^2 \log \xi = \frac{1}{6} \underbrace{R_{\mu\alpha\mu\alpha}}_{\text{scalar curvature } R} + O(x)$$

(The thing to remember is that $-\nabla^2 + \frac{R}{6}$ is effectively no potential, hence for

$$\Box = -\nabla^2 + \frac{R}{6} + V - \frac{R}{6}$$

the potential term $e^{-\int (V - \frac{R}{6})dt}$ enters in the path integral.)

~~Now let us consider~~

Now that we have the local expression for the heat kernel we can write down the path integral expression for $e^{-\Box t}$, namely

$$\langle x | e^{-\tau\Box} | x' \rangle = \int_{\substack{x(0)=x' \\ x(\tau)=x}} \mathcal{D}x(t)\; e^{-\frac{1}{4}\int_0^\tau |\dot{x}|^2 dt} \times T\left\{ e^{-\int_0^\tau A_\mu dx^\mu + \left(V - \frac{R}{6}\right) dt} \right\}$$

Next take $\Box = -\not{D}^2 = -D_\mu^2 - \frac{1}{2}\gamma^\mu\gamma^\nu F_{\mu\nu}$

whence

$$\langle x | e^{-\tau\Box} | x' \rangle = \int \mathcal{D}x(t)\, e^{-\frac{1}{4\tau}\int_0^1 |\dot{x}|^2 dt} \times T\left\{ e^{-\int_0^1 \left(A_\mu dx^\mu - \frac{\tau}{2}\gamma^\mu\gamma^\nu F_{\mu\nu}\, dt\right)} \right\}$$

Now, how can I use this to derive the index thm? One wants to set $x' = x$, apply the $\operatorname{tr} \varepsilon$ over the spinor indices, and then let $\tau \to 0$.

I need some feeling for the path integral of

$$\operatorname{tr}_{spin}\left(\varepsilon\, T\left\{ e^{-\frac{1}{4\tau}\int (\dot{x}^\mu)^2 dt - \int A_\mu \dot{x}^\mu dt - \tau\int \frac{1}{2}\gamma^\mu\gamma^\nu F_{\mu\nu}\, dt} \right\} \right)$$

In this formula the A_μ and the $F_{\mu\nu}$ are functions of x, but only the value at the point $x = x' = 0$ matters.

In this expression I can make a perturbation expansion using the last term. This should be equivalent to the old expansion

$$\frac{1}{\lambda + \not{D}^2} = \frac{1}{\lambda + D^2} + \frac{1}{\lambda + D^2}\left(-\tfrac{1}{2}\gamma^\mu\gamma^\nu F_{\mu\nu}\right)\frac{1}{\lambda + D^2} + \cdots$$

encountered before.

I propose now to look at the same expression for $\not{D}_s$. The path integral now is taken over pairs $(x(t), y(t))$ of paths and the kinetic energy of the path is now

$$\frac{1}{4}\int_0^1 \dot{x}_\mu^2\, dt + \frac{1}{4s^2}\int_0^1 \dot{y}_i^2\, dt .$$

The transport term is $\int \left(A_i \dot{y}^i + A_\mu \dot{x}^\mu \right) dt$

and the interaction term is

$$\int \left(s^2 \frac{1}{2}\gamma^i\gamma^j F_{ij} + s\,\gamma^i\gamma^\mu [D_i, D_\mu] + \frac{1}{2}\gamma^{\mu\nu} F_{\mu\nu} \right) dt$$

I fix the path $x(t)$, do the integral over all $y(t)$ joining y to y take the limit as $s \to 0$. This $y(t)$ integral is of the form

$$\int \mathcal{D}y(t)\, e^{-\frac{1}{4s^2}\int \dot{y}^2 dt \,-\, \int A_i \dot{y}^i dt \,-\, \int (s^2[\cdots] + s[\cdots] + [\cdots])\, dt}$$

The way to evaluate it is to expand around the critical point which I will assume is $y(t) \equiv 0$.

Let's try to carry out the proof when the base Y is one dimensional. In this case the operator is

$$-i\tilde{\not{D}}_s = s\sigma_1(-iD_y) + \sigma_2 L \qquad \text{or} \qquad \tilde{\not{D}}_s = s\sigma_1 D_y + \sigma_2(iL)$$

$$= \begin{pmatrix} 0 & sD_y + L \\ sD_y - L & 0 \end{pmatrix}$$

where $L = -i\not{D}$ is the Dirac operator on the fibre. This time I want the heat operator

$$\begin{pmatrix} e^{-tD_s^* D_s} & 0 \\ 0 & e^{-tD_s D_s^*} \end{pmatrix} \qquad D_s = sD_y - L$$

and we have

$$D_s^* D_s = (-sD_y - L)(sD_y - L)$$
$$= -s^2 D_y^2 + L^2 + s[D_y, L]$$
$$D_s D_s^* = -s^2 D_y^2 + L^2 - s[D_y, L].$$

Because $\dim Y = 1$, there is no curvature term from the Hilbert bundle.

Now what I want to calculate is the perturb. expansion:

$$e^{-t(H+V)} = e^{-tH} + \int_0^t dt_1\, e^{-(t-t_1)H}(-V)e^{-t_1 H} + \int_0^t dt_1 \int_0^{t_1} dt_2 \cdots$$

with $H = -s^2 D_y^2 + L^2$ and $V = \pm s[D_y, L]$. In this example the Hilbert space is 2 copies of the Hilbert space on which L acts, so that we can take the $\mathrm{Tr}\,\sigma_3$ over the Y-spinors. This gives us

$$e^{-tD_s^* D_s} - e^{-tD_s D_s^*} = -2\int_0^t dt_1\, e^{-(t-t_1)H} V e^{-t_1 H} + \cdots$$

and only the odd terms in the perturbation series survive. What do I get as $s \to 0$?

Pass to the resolvent

$$-2 \frac{1}{\lambda - H} V \frac{1}{\lambda - H} + \cdots$$

$$-2 \frac{1}{\lambda + s^2 D_y^2 - L^2}\, s[D_y, L]\, \frac{1}{\lambda + s^2 D_y^2 - L^2} + \cdots$$

To simplify the analysis, let's take the trace over the Hilbert space on which L acts. We get:

$$\mathrm{Tr}\left(e^{-tD_s^*D_s} - e^{-tD_sD_s^*}\right) = -2\int_0^t dt_1\, \mathrm{Tr}\left(e^{-tH}V\right) + \cdots$$

$$= -2t\,\mathrm{Tr}\left(e^{-tH}V\right) + \cdots$$

$$(*) \qquad = -2t\,\mathrm{Tr}\left(e^{-t(-s^2D_y^2+L^2)}s\,[D_y, L]\right) + \cdots$$

Now let's recall the formula for the derivative of $\eta_A(0)$:
$$\eta_A(s) = \mathrm{Tr}\left(\frac{A}{|A|}|A|^{-s}\right) = \mathrm{Tr}\left(A\,(A^2)^{-\left(\frac{s+1}{2}\right)}\right)$$

$$\delta\eta_A(s) = -s\,\mathrm{Tr}\left(\delta A\,(A^2)^{-\left(\frac{s+1}{2}\right)}\right)$$

$$= -s\,\frac{1}{\Gamma\left(\frac{s+1}{2}\right)}\int_0^\infty \underbrace{\mathrm{Tr}\left(\delta A\, e^{-tA^2}\right)}_{\sim\, c_{-\frac{1}{2}}t^{-1/2}} t^{\frac{s+1}{2}}\frac{dt}{t}$$

$$\sim -s\,\frac{1}{\Gamma\left(\frac{1}{2}\right)}c_{-\frac{1}{2}}\frac{2}{s} \longrightarrow -\frac{2}{\sqrt{\pi}}c_{-\frac{1}{2}}$$

where $c_{-\frac{1}{2}}$ = coeff of $t^{-1/2}$ in the asymptotic exp. for $\mathrm{Tr}\left(\delta A\, e^{-tA^2}\right)$ as $t \to 0$.

What I would like to do is to let s go to zero in $(*)$. In spite of appearances I should not get zero. This is because one first restricts the kernel to the diagonal of Y, before letting $s \to 0$.

Let's first assume $D_y = \partial/\partial y$ to simplify and ignore the non-commuting of D_y^2 and L^2 in the exponential.

$$e^{-t(-s^2D_y^2+L^2)} \sim \underbrace{e^{ts^2D_y^2}}_{\text{has diagonal value } \frac{e^0}{\sqrt{4\pi ts^2}}} e^{-tL^2}$$

and hence $(*)$ takes the approximate form

$$= -2t\,\frac{1}{\sqrt{4\pi ts^2}}\,s\,\mathrm{Tr}\left(e^{-tL^2}[D_y, L]\right) \xrightarrow[t\to 0]{} \frac{-2}{2\sqrt{\pi}}c_{-\frac{1}{2}} = \frac{1}{2}\delta\eta(0)$$

The factor $\frac{1}{2}$ is probably correct, because

$$A \mapsto e^{i\pi\eta_A} \quad \text{is the map } \mathcal{a} \to S^1$$

and the corresponding differential form is the pull-back of $\frac{d\theta}{2\pi}$, and as $\theta = \pi\eta_A$, we get $\frac{1}{2}d\eta$.

The problem then becomes to compute the diagonal part of the operator $e^{-t(-s^2D^2+L^2)}$. Here D is a connection on the Hilbert bundle on which L acts. Curious process; one can't argue that it is a perturbation in s because D^2 and L^2 don't dominate each other, since they work in different directions.

What I need is a way to compute the kernel of $e^{-t(-s^2D^2+L^2)}$ over the diagonal. Let's first consider the problem of calculating the diagonal part of e^{+tD^2}. This raises the whole mystery of why we can talk about $\mathrm{Tr}(e^{-tA})$ and not $\mathrm{Tr}\left(\frac{1}{\lambda-A}\right)$, even though to determine the asymptotic expansion of $\mathrm{Tr}(e^{-tA})$ one uses the resolvent via the formula

$$e^{-tA} = \frac{1}{2\pi i}\oint \frac{e^{-t\lambda}}{\lambda - A}\,d\lambda$$

$$\int_0^\infty e^{\lambda t}e^{-At}\,dt = \frac{-1}{\lambda - A}$$

March 18, 1983

Let's review how one constructs the diagonal values of the kernel for $e^{-t\square}$ in Seeley's work. One writes the symbol

$$\langle x|\square|\xi\rangle = e^{ix\xi}\left(p_2 + p_1 + p_0 + \cdots\right)$$

where $p_i(x,\xi)$ is homogeneous of degree i in ξ. Then formally one calculates the symbol of the resolvent

$$\langle x|\frac{1}{\lambda-\square}|\xi\rangle = e^{ix\xi}\sum_{n=0}^{\infty}\frac{b_n(x,\xi)}{(\lambda-p_2)^{n+1}}$$

Here I am assuming p_2 is a scalar, say $p_2(x,\xi) = |\xi|^2$. Then

$$\langle x|e^{-t\square}|\xi\rangle = \frac{1}{2\pi i}\oint\langle x|\frac{1}{\lambda-\square}|\xi\rangle e^{-t\lambda}\,d\lambda$$

$$= e^{ix\xi}\sum_{n=0}^{\infty} b_n(x,\xi)\underbrace{\frac{1}{2\pi i}\oint\frac{e^{-t\lambda}}{(\lambda-|\xi|^2)^{n+1}}\,d\lambda}_{e^{-t|\xi|^2}\frac{(-t)^n}{n!}}.$$

Finally

$$\langle x|e^{-t\square}|x\rangle = \int\frac{d^n\xi}{(2\pi)^n}\langle x|e^{-t\square}|\xi\rangle\overbrace{\langle\xi|x\rangle}^{e^{-i\xi x}}$$

$$= \int\frac{d^n\xi}{(2\pi)^n}\, e^{-t|\xi|^2}\sum_n b_n(x,\xi)\frac{(-t)^n}{n!}$$

and a careful analysis of the $b_n(x,\xi)$ shows that the negative powers of t they introduce when the Gaussian integral is done do not exceed the factor $(-t)^n$, and hence one does have a valid series in powers of t.

Now the question is whether I can modify the above so as to handle $\square = -\partial_y^2 + L^2$ where L^2

[redacted] is a Laplacean in some other variables.
Let me change the notation to

$$\Box = -\partial_x^2 + L^2$$

where L is something like $-i(\partial_y + A)$ and $A = A(x,y)$.

I think of L as operating on some vector space $\mathcal{H}$ [redacted] so that $\Box$ operates on $L^2(\mathbb{R}, \mathcal{H})$, functions of x with values in $\mathcal{H}$. These I can expand using a Fourier integral

$$f(x) = \int \frac{d\xi}{(2\pi)} e^{i\xi x} \hat{f}(\xi)$$

where $\hat{f}$ takes its values in $\mathcal{H}$. I guess I want to get $e^{-t\Box}$ via the resolvent.

$$\frac{1}{\lambda - \Box} = \frac{1}{\lambda + \partial_x^2 - L^2}$$

Let's first suppose that $[\partial_x, L] = 0$. Then

$$e^{-t\Box} = e^{t\partial_x^2} e^{-tL^2} = e^{-tL^2} e^{t\partial_x^2}$$

and from the kernel point of view

$$\langle x, \alpha | e^{-t\Box} | \xi, \beta \rangle = e^{ix\xi} e^{-t\xi^2} \langle \alpha | e^{-tL^2} | \beta \rangle$$

I think this ought to be the leading term of the expansion I am [redacted] after.

Notice that L commutes with multiplication by functions of x. Hence

$$\left(\frac{e^{-\frac{u}{t}}}{t^{1/2}}\right)^{-1} \left(\partial_t - \partial_x^2 + L^2\right) \frac{e^{-\frac{u}{t}}}{t^{1/2}} = \partial_t + \frac{u}{t^2} - \frac{1}{2t} - \left(\partial_x - \frac{1}{t}\partial_x u\right)^2 + L^2$$

$$= \partial_t + \frac{1}{t} x\partial_x - \partial_x^2 + L^2$$

if $u = \frac{x^2}{4}$. This maybe isn't useful.

Idea: Let us try to use ψDO techniques with the basic leading symbol $\lambda - |\xi|^2 - L^2$. This operator doesn't $\overset{(NO)}{\text{correspond}}$ to anything very simple like $e^{t\partial_x^2} e^{-tL^2}$, but certainly is the accepted gadget in the ψDO setup. Wait:

$$\int \frac{d\xi}{2\pi}\, e^{i\xi(x-x')} e^{-t(|\xi|^2 + L^2)} = \frac{e^{-\frac{(x-x')^2}{4t}}}{\sqrt{4\pi t}}\, e^{-tL_x^2}$$

really does correspond to $e^{-tL^2} e^{t\partial_x^2}$.

Thus we calculate the resolvent $\frac{1}{\lambda + \partial_x^2 - L^2}$ with respect to the operator $\frac{1}{\lambda - |\xi|^2 - L^2}$. Let

$$\langle x | \frac{1}{\lambda + \partial_x^2 - L} | \xi \rangle = e^{ix\xi}\left(g_{-2} + g_{-3} + \cdots \right)$$

$$1 = p_2 \cdot \Big(g_{-2} + g_{-3} + \cdots + \frac{1}{i}\frac{\partial p_2}{\partial \xi}\left(\frac{\partial g_{-2}}{\partial x} + \frac{\partial g_{-3}}{\partial x} + \cdots \right) \Big)$$

$$\therefore\ g_{-2} = \frac{1}{\lambda - |\xi|^2 - L^2}$$

$$(\lambda - |\xi|^2 - L^2)\, g_{-3} + \frac{1}{i}\frac{\partial p_2}{\partial \xi}\frac{\partial g_{-2}}{\partial x} = 0$$

$$g_{-3} = \frac{1}{\lambda - |\xi|^2 - L^2}(-i\xi)\,\underbrace{\frac{\partial}{\partial x}\left(\frac{1}{\lambda - |\xi|^2 - L^2} \right)}_{}$$

$$- \frac{1}{\lambda - |\xi|^2 - L^2}(-\partial_x L^2)\frac{1}{\lambda - |\xi|^2 - L^2}$$

Now we have to compute the contour integral

$$\frac{1}{2\pi i}\oint e^{-\lambda t} g_3 \, d\lambda = \frac{1}{2\pi i}\oint e^{-\lambda t} d\lambda \frac{1}{\lambda-|\xi|^2-L^2}(-i\xi)\frac{1}{\lambda-|\xi|^2-L^2}(\partial_x L^2)\frac{1}{\lambda-|\xi|^2-L^2}$$

Translate $\lambda \mapsto \lambda+|\xi|^2$ and you get

$$e^{-t|\xi|^2}\frac{1}{2\pi i}\oint e^{-\lambda t}\frac{1}{\lambda-L^2}(-i\xi)\frac{1}{\lambda-L^2}\,\partial_x L^2\,\frac{1}{\lambda-L^2}\,d\lambda$$

This looks nice because the operator L is working at the point x of interest. When one does the ξ integrals to get the diagonal value of the kernel at x one should be left with a series of contour integrals of the type

$$\frac{1}{2\pi i}\oint e^{-\lambda t}\frac{1}{\lambda-L^2}\,A\,\frac{1}{\lambda-L^2}\,B\,\frac{1}{\lambda-L^2}\,d\lambda$$

which corresponds to a convolution of heat kernels.

March 20, 1983. (In Marseille)

The basic problem I have to deal with is to understand something like

$$\langle x | e^{t(s^2\partial_x - L^2)} | x \rangle$$

or

$$\langle x | \int_0^t dt_1\, e^{-(t-t_1)(s^2\partial_x + L^2)} [\partial_x, L]\, e^{-t_1(-s^2\partial_x^2 + L^2)} | x \rangle$$

as $s \to 0$. There should be an asymptotic expansion for these quantities in powers of s.

Analogous situation: Consider a Hamiltonian

$$-\frac{\hbar^2}{2}\partial_x^2 + V \qquad\qquad \frac{p^2}{2} + V(x)$$

The Schroedinger equation is

$$i\hbar\,\partial_t \psi = \left(-\frac{\hbar^2}{2}\partial_{x^2} + V\right)\psi$$

and it has a classical limit $\hbar \to 0$, which is the classical motion based on the Hamiltonian $\frac{p^2}{2} + V(x)$. What exactly does the physics tell me? Time evolution is given by

$$\langle x | e^{-\frac{i}{\hbar}Ht} | x' \rangle = \langle x | e^{-i\left(-\frac{\hbar}{2}\partial_x^2 + \frac{1}{\hbar}V\right)} | x' \rangle$$

and as $\hbar \to 0$, this amplitude has the asymptotic expansion

$$e^{\frac{i}{\hbar}S(xt,x'0)}\,(\text{power series in } \hbar).$$

Now I don't seem to be interested in this limit because I ultimately don't want that $(1/\hbar)V$ goes off to infinity. Thus I seem to be interested in the large mass limit

$$H = \frac{p^2}{2m} + V \qquad \text{as} \quad m \to \infty.$$

Let us consider the path integral for the amplitude

$$\langle x | e^{-iH\tau} | x' \rangle = \int_{\substack{x(0)=x' \\ x(\tau)=x}} \mathcal{D}x \; e^{+i\int_0^\tau \left(\frac{m}{2}\dot{x}^2 - V(x(t))\right)dt}$$

This is an integral depending on the large parameter m and there is a standard way to get an asymptotic expansion, namely, expand around the critical point. This critical point is the straight line from x' to x in time τ. Set $x'=0$, then $x_c(t) = \frac{t}{\tau}x$, and we get the exponential factor and next term

$$e^{i\frac{mx^2}{2\tau}} \qquad e^{-i\tau V(0)}.$$

I now know that there is an asymptotic expansion, however I don't want to get involved with the actual diagrams, because they involve all kinds of derivatives of $V(x)$, and only the first few should be important.

The problem seems to be that one can't apply perturbation theory to $-\frac{1}{2m}\partial_x^2 + V$ relative to V because $-\frac{1}{2m}\partial_x^2$ is very unbounded. It seems to be another version of the fact that

$$\mathrm{Tr}\left(e^{-t\Box}\right) = \mathrm{Tr}\left(1 - t\Box + \frac{t^2}{2}\Box^2 + \cdots\right)$$

is meaningless.

Let review the index computation for the $\bar\partial$-op.

$$D = \partial_x + \frac{1}{i}\left(\partial_y + A(x,y)\right) = \partial_x + L$$

Then $$D^*D = (-\partial_x + L)(\partial_x + L) = -\partial_x^2 + L^2 - \underbrace{[\partial_x, L]}_{-i\partial_x A}$$

$$e^{-tD^*D} = e^{-t(-\partial_x^2+L)} + \int_0^t dt_1\, e^{-(t-t_1)(-\partial_x^2+L)} [\partial_x, L]\, e^{-t_1(-\partial_x^2+L^2)} + \cdots$$

$$e^{-tDD^*} = \quad '' \qquad '' \qquad -[\partial_x, L] \qquad '' \qquad + \cdots$$

Thus to first order the index is the trace of

$$2\int_0^t dt_1\, e^{-(t-t_1)(-\partial_x^2+L^2)} [\partial_x, L]\, e^{-t_1(-\partial_x^2+L^2)}$$

Let's go over the problem from the beginning. I have a $\bar\partial$ operator over the torus $S^1 \times S^1$ given by

$$D_s = s\partial_x + L = s\partial_x + \frac{1}{i}(\partial_y + A(x,y))$$

whose index I compute analytically using

$$\mathrm{Tr}\left(e^{-tD_s^*D_s}\right) - \mathrm{Tr}\left(e^{-tD_sD_s^*}\right).$$

What I want to do is to let $s \longrightarrow 0$. The index doesn't change. Now I'd like to show that the above traces have an asymptotic expansion in powers of s, and to determine the coefficients which are forms in x to be integrated in order to get the trace.

Question: Can we prove the existence of the asymptotic expansion, say by using path integral methods?

The above $\bar\partial$-operator changes

March 23, 1983

Recall that I have been looking at operators
$$\partial_x + L$$
where L is a self-adjoint operator in a Hilbert space H, and $L = L(x)$ depends on x. The operator $\partial_x + L$ acts on functions of x with values in H. My example was $L = \frac{1}{i}(\partial_y + A(x,y))$ acting on L^2 functions of y. What I wanted to do was to understand the index of this operator, especially under the deformation
$$D_s = s\partial_x + L$$
as $s \to 0$. The index doesn't change, and one might hope to understand the analysis as $s \to 0$.

Yesterday it occurred to me that one might first look at the simpler question where H is 1-diml and $L(x)$ is a function of x. I know that if I take $L(x) = x$, then $\partial_x + x$ on $L^2(\mathbb{R})$ has a non-trivial index. This situation seems to be much better from the viewpoint of periodicity.

More generally one can put in $D_x = \partial_x + A_x$ and then the operator
$$\begin{pmatrix} 0 & -D_x + L \\ D_x + L & 0 \end{pmatrix}$$
is the sort of Dirac system I used to study. Let's review the formulas. Say $D_x + L = \partial_x + ia + b$ where a, b are real functions of x. The above operator is
$$-i\sigma_2 \partial_x + \sigma_2 a + \sigma_1 b$$
and leads to the eigenvalue equation
$$\left[\sigma_2 \tfrac{1}{i}\partial_x + \sigma_2 a + \sigma_1 b - \lambda\right] u = 0$$
Now there is a matrix which conjugates the σ_i in

cyclic order. But this doesn't give the equation

$$\partial_x u = \begin{pmatrix} ik & \alpha - i\beta \\ \alpha + i\beta & -ik \end{pmatrix} u$$

I used to look at. In effect let's write this

$$\left(-\partial_x + ik\sigma_3 + \alpha\sigma_1 + \beta\sigma_2\right) u = 0$$

or

$$-\sigma_3\partial_x + ik + \alpha \underbrace{\sigma_3\sigma_1}_{i\sigma_2} + \beta \underbrace{\sigma_3\sigma_2}_{-i\sigma_1}$$

$$\left[+\sigma_3 \tfrac{1}{i}\partial_x - \sigma_2\alpha + \sigma_1\beta - k\right] u = 0$$

Now cyclically permute to get

$$\left[\sigma_2 \tfrac{1}{i}\partial_x - \sigma_1\alpha + \sigma_3\beta - k\right] \tilde{u} = 0$$

or

$$\begin{pmatrix} \beta & -\partial_x - \alpha \\ \partial_x - \alpha & -\beta \end{pmatrix} \tilde{u} = k\tilde{u}$$

This is a massive Dirac operator with mass β, so there are no possibilites of an index, (nor it seems of their being symmetry of eigenvalues $k \mapsto -k$.)

So let's return to

$$\begin{pmatrix} 0 & -D_x + L \\ D_x + L & 0 \end{pmatrix}$$

By a gauge transformation $e^{i\alpha(x)}$, α real, we can change D_x to ∂_x, and putting $L = b$ we get the eigenvector equation

$$\begin{pmatrix} 0 & -\partial_x + b \\ \partial_x + b & 0 \end{pmatrix} u = k u$$

which can be viewed as a factorization of a Schroedinger equation

$$\underbrace{(\partial_x + b)(-\partial_x + b)}_{-\partial_x^2 + (b' + b^2)} u_2 = k^2 u$$

or of the other one

$$\underbrace{(-\partial_x + b)(\partial_x + b)}_{-\partial_x^2 + (-b' + b^2)} u_1 = k^2 u_1 .$$

Factorization of $-\partial_x^2 + q$ in the form $(\partial_x + b)(-\partial_x + b)$ is only possible when one has non-vanishing solutions of $(-\partial_x^2 + q)u = 0$. Then [redacted] $b = \frac{u'}{u}$, and the spectrum of $-\partial_x^2 + q$ is $\geqslant 0$.

Next let's consider the operator $\varepsilon\partial_x + x$ on $L^2(\mathbb{R})$ which we know has index 1. Here I should be able to do all the computations algebraically, because we have the harmonic oscillator[redacted].

$$D = \varepsilon\partial_x + x \qquad D^* = -\varepsilon\partial_x + x$$

$$[D, D^*] = [\varepsilon\partial_x + x, -\varepsilon\partial_x + x] = 2\varepsilon$$

$$a = \frac{1}{\sqrt{2\varepsilon}} D \quad , \quad a^* = \frac{1}{\sqrt{2\varepsilon}} D^*$$

$$D^*D = (-\varepsilon\partial_x + x)(\varepsilon\partial_x + x) = -\varepsilon^2\partial_x^2 - \varepsilon + x^2$$
$$= (-\varepsilon^2\partial_x^2 + x^2) - \varepsilon$$

$\therefore$ harmonic oscillator hamiltonian $= -\varepsilon^2\partial_x^2 + x^2 = D^*D + \varepsilon$

$$H = 2\varepsilon\left(a^*a + \tfrac{1}{2}\right)$$

and so the eigenvalues are [redacted] $2\varepsilon\left(n + \frac{1}{2}\right)$, $n \in \mathbb{N}$.

$$D^*D = H - \varepsilon = 2\varepsilon(a^*a)$$

$$DD^* = D^*D + [D, D^*] = H - \varepsilon + 2\varepsilon = H + \varepsilon = 2\varepsilon(a^*a + 1)$$

The global index calculation will be

$$\mathrm{Tr}\left(e^{-tD^*D}\right) = \sum_{n\geq 0} e^{-(2n\varepsilon)t}$$

$$\mathrm{Tr}\left(e^{-tDD^*}\right) = \sum_{n>0} e^{-(2n\varepsilon)t}$$

and so the difference is 1.

Next I want a local calculation of the index. For this I will want $\langle x|e^{-tD^*D}|x\rangle$ mainly in the limit as $\varepsilon\to 0$. Enough to look at $\langle x|e^{-tH}|x'\rangle$. I know that because H is quadratic this kernel has the form

$$c(t)\, e^{-S_t(x,x')}$$

where S_t is quadratic in x, x'.

Start with $H = -\varepsilon^2\partial_x^2 + x^2$, then

$$e^S\left(\partial_t + H\right)e^{-S(t,x)} = \partial_t - \partial_t S - \varepsilon^2\underbrace{\left(\partial_x - \partial_x S\right)^2}_{\partial_x^2 - 2\partial_x S\,\partial_x - \partial_x^2 S + (\partial_x S)^2} + x^2$$

I want this operator to kill 1, whence

$$-\partial_t S - \varepsilon^2\left(-\partial_x^2 S + (\partial_x S)^2\right) + x^2 = 0$$

Suppose $S = ax^2 + bx + c$, where a, b, c are functions of t.

$$+\left(\dot a x^2 + \dot b x + \dot c\right) + \varepsilon^2\Big(\underbrace{(2ax+b)^2}_{4a^2x^2 + 4bax + b^2} - 2a\Big) = x^2$$

$$\dot a + 4a^2\varepsilon^2 = 1 \qquad \dot c + \varepsilon^2(b^2 - 2a) = 0$$

$$\dot b + \varepsilon^2 4ba = 0$$

$$a = -\frac{1}{4\varepsilon^2}\frac{\dot b}{b}$$

$$1 = \dot a + 4\varepsilon^2 a^2 = \left(-\frac{1}{4\varepsilon^2}\right)\left(\frac{\ddot b}{b} - \cancel{\frac{\dot b^2}{b^2}}\right) + \cancel{4\varepsilon^2 \frac{1}{(4\varepsilon^2)^2}\frac{\dot b^2}{b^2}}$$

NO good.

Start again: $H = -\varepsilon^2\partial^2 + x^2$

$$e^{-S}(\partial_t + H)e^S = [\partial_t + \partial_t S - \varepsilon^2(\partial + \partial_x S)^2 + x^2]\,1 = 0$$

$$\partial_t S - \varepsilon^2((\partial_x S)^2 + \partial_x^2 S) + x^2 = 0$$

$S = ax^2 + bx + c$ where a, b, c are fns. of t.

$$\dot a x^2 + \dot b x + \dot c - \varepsilon^2[(2ax+b)^2 + 2a] + x^2 = 0$$

$$\begin{cases} \dot a - 4\varepsilon^2 a^2 + 1 = 0 \\ \dot b - 4\varepsilon^2 ab = 0 \\ \dot c - \varepsilon^2(b^2 + 2a) = 0 \end{cases}$$

The first equation is a Riccati equation. Set

$$a = -\frac{1}{4\varepsilon^2}\alpha$$

$$-\dot a + 4\varepsilon^2 a^2 = +\frac{1}{4\varepsilon^2}\dot\alpha + 4\varepsilon^2\left(\frac{1}{4\varepsilon^2}\alpha\right)^2 = 1$$

$$\dot\alpha + \alpha^2 = 4\varepsilon^2$$

If I put $\alpha = \frac{\dot u}{u}$ then $\dot\alpha + \alpha^2 = \frac{\ddot u}{u} - \cancel{\frac{\dot u^2}{u^2}} + \cancel{\frac{\dot u^2}{u^2}} = 4\varepsilon^2$

so that $\ddot u = 4\varepsilon^2 u$ which has the solutions

$$u = c_1 e^{2\varepsilon t} + c_2 e^{-2\varepsilon t}$$

I want $a = -\frac{1}{4\varepsilon^2}\alpha = -\frac{1}{4\varepsilon^2}\frac{\dot u}{u}$ to blow up at $t=0$, hence I can take

$$u = \frac{e^{2\varepsilon t} - e^{-2\varepsilon t}}{2} = \sinh(2\varepsilon t)$$

Then $a = -\frac{1}{4\varepsilon^2}\frac{\cosh(2\varepsilon t)}{\sinh(2\varepsilon t)}\,2\varepsilon$

$\underline{i}$

$$\boxed{a = -\frac{1}{2\varepsilon}\,\frac{\cosh(2\varepsilon t)}{\sinh(2\varepsilon t)}} \sim -\frac{1}{4\varepsilon^2 t} \qquad \varepsilon t \text{ small}$$

$$\frac{\dot b}{b} = 4\varepsilon^2 a = -2\varepsilon\,\frac{\cosh(2\varepsilon t)}{\sinh(2\varepsilon t)} = -\left(\log \sinh 2\varepsilon t\right)^{\cdot}$$

$$\therefore \quad b = \frac{\text{const}}{\sinh(2\varepsilon t)}$$

Now for small t I want

$$e^{-\frac{1}{4\varepsilon^2 t}(x-x')^2 \;+\; \text{fn of } t}$$

hence

$$\boxed{b = +\frac{1}{2\varepsilon}\,\frac{2x'}{\sinh(2\varepsilon t)}}$$

Now c will be a sum of

$$c_1 = -\frac{1}{2\varepsilon}\,\frac{\cosh(2\varepsilon t)}{\sinh(2\varepsilon t)}\,x'^2 \qquad \dot c_1 = \varepsilon^2 b^2$$

and

$$c_2 = -\frac{1}{2}\log \sinh(2\varepsilon t)$$

$$\dot c_2 = -\frac{1}{2}\,\frac{\cosh(2\varepsilon t)}{\sinh(2\varepsilon t)}\,2\varepsilon \qquad +\cancel{2}\varepsilon^2 a = \cancel{+2}\varepsilon^{2}\,\frac{1}{\cancel{2}\varepsilon}\,\frac{\cosh}{\sinh} \;✓$$

$$\boxed{c = -\frac{1}{2\varepsilon}\,\frac{\cosh(2\varepsilon t)}{\sinh(2\varepsilon t)}\,x'^2 - \frac{1}{2}\log\sinh(2\varepsilon t)}$$

$$\boxed{\langle x|\,e^{-tH}\,|x'\rangle = \frac{1}{\sqrt{\frac{\sinh(2\varepsilon t)}{2\varepsilon\pi}}}\; e^{-\frac{1}{2\varepsilon}\left(\frac{\cosh(2\varepsilon t)\,x^2 - 2xx' + \cosh(2\varepsilon t)\,x'^2}{\sinh(2\varepsilon t)}\right)}}$$

The $2\varepsilon\pi$ has to be put in so that as $t \downarrow 0$ we get

$$\frac{1}{\sqrt{4\varepsilon^2\pi t}}\; e^{-\frac{1}{4\varepsilon^2 t}(x-x')^2}$$

But what happens if t is fixed and $\varepsilon \to 0$. Note the denominator contains ε^2. Put $x = x'$.

$$\langle x | e^{-tH} | x \rangle = \frac{1}{\sqrt{2\pi\varepsilon \sinh(2\varepsilon t)}} e^{-\frac{2(\cosh(2\varepsilon t)-1)}{2\varepsilon \sinh(2\varepsilon t)} x^2}$$

$$\sim \frac{1}{\sqrt{4\pi\varepsilon^2 t}} e^{-tx^2}$$

$$\langle x | e^{-tD^*D} | x \rangle = \langle x | e^{-t(H-\varepsilon)} | x \rangle \sim \frac{e^{-tx^2}}{\sqrt{4\pi\varepsilon^2 t}} e^{\varepsilon t}$$

$$\langle x | e^{-tDD^*} | x \rangle = \langle x | e^{-t(H+\varepsilon)} | x \rangle \sim \frac{e^{-tx^2}}{\sqrt{4\pi\varepsilon^2 t}} e^{-\varepsilon t}$$

Now subtract to get the local index expression:

$$\lim_{\varepsilon \to 0} \langle x | e^{-tD^*D} | x \rangle - \langle x | e^{-tDD^*} | x \rangle = \frac{e^{-tx^2}}{\sqrt{4\pi t}} \underbrace{\left[\frac{e^{\varepsilon t} - e^{-\varepsilon t}}{\varepsilon} \right]_{\varepsilon = 0}}_{2t}$$

Finally as a check do the integral over x

$$\int e^{-tx^2} dx = \int e^{-(2t)\frac{x^2}{2}} dx = \frac{\sqrt{2\pi}}{\sqrt{2t}}$$

$$\lim_{\varepsilon \to 0} = \frac{1}{\sqrt{4\pi t}} 2t \sqrt{\frac{\pi}{t}} = 1$$ which is the index.

March 24, 1983

Let's return to the general ~~situation~~ $X \xrightarrow[f]{E} Y$, say $X = Y \times M$, and where I have a Dirac operator on E. I am interested in the family of Dirac operator along the fibres, but in order to make sense of the index of this family analytically I have to use the connection on E in the horizontal direction. My idea is to [redacted] take $e^{-tD_\varepsilon^* D_\varepsilon}$ where $D_\varepsilon = \varepsilon D_{horiz} + D_{vert}$ and then let $\varepsilon \to 0$ while simultaneously restricting the kernel to the fibres.

Connes has this concept of the C^*-algebra of a foliation. This is a completion of a algebra of kernels, i.e. convolution type. Specifically one consider kernels on [redacted] $X \times_Y X$, i.e. kernels $K(x,x')$ where x, x' belong to the same fibre. One gets a C^*-algebra by completing; it looks like continuous functions on Y with values in compact operators [redacted] with the sup norm.

Now $e^{-tD_\varepsilon^* D_\varepsilon}$ is a [redacted] trace class operator over X, and as $\varepsilon \to 0$, it should give a well-defined operator in Connes algebra. No, this isn't quite right because we get negative powers in ε when we restrict to a fibre. What does seem to be possible is that the difference

$$e^{-tD_\varepsilon^* D_\varepsilon} - e^{-tD_\varepsilon D_\varepsilon^*}$$

has a limit as $\varepsilon \to 0$ which lies in Connes algebra.

Let's take a more general potential function p:

$$D = \varepsilon \partial_x + p$$

$$D^* D = (-\varepsilon\partial_x + p)(\varepsilon\partial_x + p) = -\varepsilon^2 \partial_x^2 + p^2 - \varepsilon p'$$

I know want to compute the heat kernel

$$\langle x | e^{-tD^*D} | x' \rangle$$

asymptotically in ε, as $\varepsilon \xrightarrow{\ell} 0$.. Let's assume $x' = 0$ and take away the Gaussian factor:

$$\left(\frac{e^{-\frac{x^2}{4\varepsilon^2 t}}}{\sqrt{}}\right)^{-1}(\partial_t + D^*D)\frac{e^{-\frac{x^2}{4\varepsilon^2 t}}}{\sqrt{4\pi\varepsilon^2 t}}$$

$$= \partial_t - \frac{1}{2t} + \frac{x^2}{4\varepsilon^2 t^2} + \left(-\varepsilon\left(\partial_x - \frac{x}{2\varepsilon^2 t}\right)+p\right)\left(\varepsilon\left(\partial_x - \frac{x}{2\varepsilon^2 t}\right)+p\right)$$

$$= \partial_t - \frac{1}{2t} + \frac{x^2}{4\varepsilon^2 t^2} - \varepsilon^2 \underbrace{\left(\partial_x - \frac{x}{2\varepsilon^2 t}\right)^2}_{\partial_x^2 - \frac{x\partial_x}{2t\varepsilon^2} + \frac{x^2}{(2\varepsilon^2 t)^2} - \frac{1}{2\varepsilon^2 t}} + p^2 - \varepsilon p'$$

$$= +\frac{1}{2t}x\partial_x + \partial_t + p^2 - \varepsilon p' - \varepsilon^2\partial_x^2$$

So now the asymptotic series in ε will satisfy

$$\left[\frac{1}{2t}x\partial_x + \partial_t + p^2 - \varepsilon p' - \varepsilon^2\partial_x^2\right](a_0 + a_1\varepsilon + \cdots)$$

$$\left(\frac{1}{2t}x\partial_x + \partial_t + p^2\right) a_0 = 0$$

I ought to put some conditions about the $t \to 0$ behavior for a fixed ε. Let us assume this means that a_0 has the standard t asymptotic expansion. This is confusing. All you know from the ε-expansion is recursion relations:

$$\left(\frac{1}{2t}x\partial_x + \partial_t + p^2\right) a_0 = 0$$

$$\left(\frac{1}{2t}x\partial_x + \partial_t + p^2\right) a_1 - p' a_0 = 0$$

In particular I am missing the boundary conditions necessary to specify $a_0(t,x)$.

It is likely that, because we don't want negative powers of t that a_0 is a function of x alone. Then

$$a_0(x) = +e^{-tp(x)^2} \qquad \text{NO}$$

m

Actually this seems to be a conclusion for $x=0$.

$$\left[\frac{1}{2t}\, x\partial_x + \partial_t + p(x)^2\right] a_0(t,x) = 0$$

$$\Longrightarrow \quad \left[\partial_t + p(0)^2\right] a_0(t,0) = 0 \quad \Longrightarrow \quad a_0(t,0) = e^{-tp(0)^2}$$

similarly:

$$\left[\partial_t + p(0)^2\right] a_1(t,0) - p'(0)e^{-tp(0)^2} = 0$$

$$\partial_t \left[e^{tp(0)^2} a_1\right] = p'(0)$$

$$a_1(t,0) = t\,p'(0)\,e^{-tp(0)^2} + c.\,e^{-tp(0)^2}$$

So again I don't what c is, but suppose it is 0 say by virtue of the small t asymptotics. ~~[illegible]~~

$$\cancel{\langle x | e^{-tD^*D} | 0 \rangle = \frac{1}{\sqrt{4\pi\varepsilon^2 t}}\, e^{-\frac{x^2}{4\varepsilon^2 t}}\, e^{-tp(x)^2}}$$

In fact $a_0(t,x)$ should be completely determined as a power series in t, x satisfying

$$\left[\frac{1}{2t}\, x\partial_x + \partial_t + p(x)^2\right] a_0 = 0 \qquad a_0(0,0) = 1.$$

Similarly $a_1(t,x)$ is determined by its equation and $a_1(0,0) = 0$. Thus I seem to be able to grind out the asymptotic expansion, and

$$\langle 0 | e^{-tD^*D} | 0 \rangle = \frac{1}{\sqrt{4\pi\varepsilon^2 t}}\, e^{-tp(0)^2}\left(1 + \varepsilon t\, p'(0) + \dots\right)$$

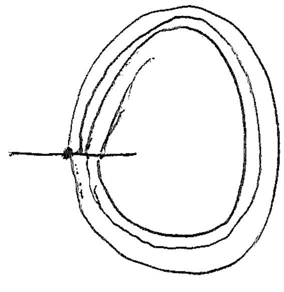
$$\langle 0 | e^{-tDD^*} | 0 \rangle = \text{———————}\left(1 - \varepsilon t\, p'(0) \qquad \right)$$

$$\langle x | e^{-tD^*D} - e^{-tDD^*} | x \rangle = \underbrace{\frac{2\varepsilon t}{\sqrt{4\pi\varepsilon^2 t}}}_{\sqrt{\frac{t}{\pi}}}\, e^{-tp(x)^2}\left(p'(x) + O(\varepsilon)\right)$$

$\therefore$ $\boxed{\text{Local trace is } \sqrt{\frac{t}{\pi}}\, e^{-tp(x)^2}\, p'(x)}$ which integrates to 1 if $p(-\infty) = -\infty$ and $p(+\infty) = +\infty$.

So now comes the generalization from $p(x)$ to an[redacted] operator $L(x)$. This should be[redacted] completely formal. [redacted].

Kronecker foliation $V = S^1 \times S^1$ with foliation $dx = \theta dy$ where θ is irrational. The first thing to do will be to find the conv. algebra of the foliation which is denoted $C_c(G)$, where G is the graph, then you complete to get $C^*(V,F)$. The graph consists of [redacted] triples (x,y,γ) where x,y lie on the same leaf and γ is a path equiv. class from x to y lying in the leaf, two such γ's being considered equivalent if they induce the same holonomy. Holonomy refers to the normal "microbundle": a closed loop produces a diffeomorphism germ on a transversal.



It's clear that [redacted] homotopic paths give the same holonomy, so that if the leaves are connected as for the Kronecker foliation, the graph is just (x,y) with x,y on the same leaf. Except that this has the wrong topology if one regards it inside $V \times V$; one wants the leaves to be unfolded. Thus in this case

$$G = S^1 \times S^1 \times \mathbb{R}. \quad [redacted]$$

In other words we should think of x,y as separated by the leaf line joining them.

Now [redacted] $C_c(G)$ will consist of functions $k(x,x')$ for $(x,x') \in G$ which are of compact support [redacted] on each leaf. A basic idea is that this kernel should act on L^2 of each leaf. Then one completes in the sup norm to get $C^*(V,F)$.

March 27, 1983

In dimension one we have the operator

$$\partial_x + x \ : \ L^2(\mathbb{R}) \longrightarrow L^2(\mathbb{R})$$

of index one. I want the generalization to $\mathbb{R}^n$. I can consider the above operator as a complex and take the n-fold tensor product. Then I get a complex

$$L^2(\mathbb{R}^n) \otimes \Lambda^{\cdot} V \qquad V = \mathbb{C}^n = \mathbb{C}v_1 + \cdots + \mathbb{C}v_n$$

v_j orthonormal

$$d = \sum (\partial_j + x^j)\, e(v_j)$$

which one converts to a single operator $d+d^*$, where

$$d^* = \sum (-\partial_j + x^j)\, i(v_j^*)$$

Then

$$d+d^* = \sum_j \frac{1}{i}\partial_j \left\{ e(iv_j) + i((iv_j)^*) \right\} + x^j\left(e(v_j) + i(v_j^*)\right).$$

Recall that ΛV is the Clifford module for the Clifford algebra $C(V)$ of V consider as a real inner product space, with $v \in V$ acting as $e(v) + i(v^*)$:

$$\left(e(v) + i(v^*)\right)^2 = e(v)i(v^*) + i(v^*)e(v) = v^*(v) = \|v\|^2$$

Hence we have

$$d+d^* = \sum_j \gamma_j' \cdot \frac{1}{i}\partial_j + \gamma_j'' x_j$$

where the γ_j', γ_j'' are generators for the Clifford algebra C_{2n}: $C_{2n} = C(\mathbb{C}^n)$ and γ_j' correspond to the standard basis of $\mathbb{R}^n \subset \mathbb{C}^n$, and the γ_j'' to the standard basis of $i\mathbb{R}^n$.

Our next goal will be to calculate analytically the index of $D = d+d^*$ using

$$\text{Index} = \operatorname{Tr}\left(e^{-tD^2}\varepsilon\right).$$

We have

$$(d+d^*)^2 = \left(\sum \gamma^\mu T_\mu\right)^2 \qquad T_\mu = \frac{1}{i}\partial_j \text{ or } x_j$$

$$= T_\mu^2 + \frac{1}{2}\gamma^\mu\gamma^\nu[T_\mu, T_\nu]$$

$$(d+d^*)^2 = -\partial_j^2 + x_j^2 + \gamma_j'\gamma_j'' \underbrace{\left[\tfrac{1}{i}\partial_j, x_j\right]}_{\frac{1}{i}}$$

$$\boxed{(d+d^*)^2 = H + \tfrac{1}{i}\gamma_j'\gamma_j'' \quad \text{where } H = -\partial_j^2 + x_j^2 \text{ is the oscillator Hamiltonian}}$$

Since H commutes with the γ's we have

$$e^{-t(d+d^*)^2} = e^{-tH}\, e^{-t\frac{1}{i}\gamma_j'\gamma_j''}.$$

If I take the spinor traces I get

$$\mathrm{tr}_{\Lambda V}\left(e^{-t(d+d^*)}\varepsilon\right) = e^{-tH}\, \mathrm{tr}_{\Lambda V}\left(e^{-t\frac{1}{i}\gamma_j'\gamma_j''}\varepsilon\right).$$

Compute the last trace:

$$\frac{1}{i}\gamma_j'\gamma_j'' = \frac{1}{i}\left[e(\iota\sigma_j) + i((\iota\sigma_j)^*)\right]\left[e(\sigma_j) + i(\sigma_j)^*\right]$$

$$= \left[e(\sigma) - i(\sigma^*)\right]\left[e(\sigma) + i(\sigma^*)\right] \qquad \sigma = \sigma_j$$

$$= 2e(\sigma)i(\sigma^*) - 1 = 2N_j - 1$$

where N_j = "occupation number" in the jth state.

$$\mathrm{tr}_{\Lambda\mathbb{C}}\left(e^{-t(2N-1)}\varepsilon\right) = e^t - e^{-t}$$

$$\therefore\quad \mathrm{tr}_{\Lambda V}\left(e^{-t\left(\frac{1}{i}\gamma_j'\gamma_j''\right)}\varepsilon\right) = \mathrm{tr}_{\Lambda V}\left(e^{-t(2N-n)}\varepsilon\right)$$

$$= \left(e^t - e^{-t}\right)^n$$

I also need

$$\mathrm{Tr}_{L^2(\mathbb{R}^n)}\left(e^{-tH}\right) = \mathrm{Tr}_{L^2(\mathbb{R})}\left(e^{-tH}\right)^n$$

$$= \left(\frac{e^{-t}}{1-e^{-2t}}\right)^n = \left(\frac{1}{e^t - e^{-t}}\right)^n.$$

Then

$$\mathrm{Tr}_{L^2(\mathbb{R}^n)\otimes\Lambda V}\left(e^{-t(d+d^*)^2}\right) = \mathrm{Tr}_{L^2(\mathbb{R}^n)}\left(e^{-tH}\right)\mathrm{tr}_{\Lambda V}\left(e^{-t\frac{1}{i}\gamma_j'\gamma_j''}\varepsilon\right) = 1$$

as it is supposed to be.

Summary: I have constructed the standard harm. osc. operator

$$d+d^* = \sum_j \gamma_j' \tfrac{1}{i}\partial_j + \gamma_j'' x_j \qquad \text{on } L^2(\mathbb{R}^n)\otimes S_{2n}$$

and I have calculated its index analytically.

The next project (following what was done in dim. 1) is to replace ∂_j by $\varepsilon\partial_j$, then let $\varepsilon \to 0$ and see what happens to $e^{-t(d+d^*)^2}$.

It might be useful before computing to ask what sort of limit one expects to obtain.

Let us review what happens in dimension 1. I have computed the actual kernel $\langle x|e^{-tH}|x'\rangle$ on p. 673 and found

$$\frac{1}{\sqrt{4\pi\varepsilon^2 t}}\, e^{-\frac{(x-x')^2}{4\varepsilon^2 t}} + F(\varepsilon^2)$$

and where the $F(\varepsilon^2)$ term is a power series in ε^2 whose coefficients are functions of x, x', t. Set $t=1$ in order to think. The point is that then I have an asymp. exp.

$$\langle x|e^{-H}|x'\rangle = \frac{1}{\sqrt{4\pi\varepsilon^2}}\, e^{-\frac{(x-x')^2}{4\varepsilon^2}}\left(F_0(x,x') + \varepsilon F_1(x,x') + \cdots\right)$$

as $\varepsilon \to 0$.

Digression: I have to understand the form of the asymptotic expansion for something like $e^{-t\square}$. There are two approaches: classical (Patodi) vs. Seeley. Patodi's formula is

$$\langle x|e^{-t\square}|x'\rangle = \frac{1}{(4\pi t)^{n/2}}\, e^{-\frac{(x-x')^2}{4t}}\left(F_0(x,x') + F_1(x,x')t + \cdots\right)$$

Seeley's formula is derived as follows:

$$\langle x|\frac{1}{\lambda-\square}|\xi\rangle = e^{ix\xi}\left\{\frac{1}{\lambda-\xi^2} + \cdots + \frac{a(x)\xi^\alpha}{(\lambda-\xi^2)^{k+1}} + \cdots\right\}$$

$$\frac{1}{2\pi i}\oint e^{-\lambda t}\frac{1}{(\lambda-\xi^2)^{k+1}}\,d\lambda = e^{-\xi^2 t}\frac{1}{2\pi i}\oint e^{-\lambda t}\frac{d\lambda}{\lambda^{k+1}}$$

$$= e^{-\xi^2 t}\frac{(-t)^k}{k!}$$

$$\therefore\ \langle x|\, e^{-t\Box}|\xi\rangle \simeq e^{ix\xi}e^{-t\xi^2}\left\{1+\cdots+a(x)\xi^\alpha\frac{(-t)^k}{k!}+\cdots\right\}$$

$$\langle x|e^{-t\Box}|x'\rangle = \int\frac{d^n\xi}{(2\pi)^n}\,e^{i(x-x')\xi-t\xi^2}\left\{1+\cdots+a(x)\xi^\alpha\frac{(-t)^k}{k!}+\cdots\right\}$$

$$= \int\frac{d^n\xi}{(2\pi)^n}\,e^{-t\left(\xi-\frac{i(x-x')}{2t}\right)^2-\frac{(x-x')^2}{4t}}\left\{\qquad\qquad\right\}$$

$$= e^{-\frac{(x-x')^2}{4t}}\int\frac{d^n\xi}{(2\pi)^n}\,e^{-t\xi^2}\left\{1+\cdots+a(x)\left[\xi+\frac{i(x-x')}{2t}\right]^\alpha\frac{(-t)^k}{k!}+\cdots\right\}$$

So the end result is an asymptotic expansion of the same form as Patodi's, but it is usually written using the F.T.

$$\langle x|e^{-t\Box}f = \int\frac{d^n\xi}{(2\pi)^n}\,e^{ix\xi}e^{-t\xi^2}\left\{1+\cdots+a_\alpha(x)\xi^\alpha\frac{(-t)^k}{k!}+\cdots\right\}\hat f(\xi)$$

$$= \cdots\frac{(-t)^k}{k!}a_\alpha(x)\left(\frac{1}{i}\partial\right)^\alpha e^{+t\partial^2}f\ \cdots$$

It therefore appears that Seeley's expansion is just the formal expansion of

$$e^{-t\Box}e^{t\Box_0}$$

in powers of t.

Ideas: In the case of a fibration $f: X\to Y$ and a Dirac operator over X, we get a heat operator $e^{-tD_E^2}$ whose limit we want to take as $h\to 0$. The kernel $\langle x|e^{-tD_E^2}|x'\rangle$ will be concentrated along $X\times_Y X$, and the idea is to restrict it to this set and we should get an asymptotic expansion in powers of ε.

Now it should be so that if we divide $\langle x|e^{-tD_\varepsilon^2}|x'\rangle$ by the Gaussian factor $\frac{1}{(4\pi\varepsilon^2 t)^{n/2}}\, e^{-(y-y')^2/4\varepsilon^2 t}$, then we obtain an asymptotic expansion valid in a nbd. of $X\times_Y X$ in $X\times X$. However this finer expansion is probably not useful for much. In particular the finer expansion of Patodi is not needed for the index thm.

Seeley's expansion seems to be something like

$$e^{-t(D_\varepsilon^2+L^2)} = e^{-tL^2}\{1+\cdots\}\,e^{-tD_\varepsilon^2}$$

and hence should be close to the [redacted] Campbell-Hausdorff formula. However the idea of putting the $e^{-tD_\varepsilon^2}$ on the right is like writing things in normal product form, and I have decided this introduces extraneous derivatives.

I now want to go back to the case of the harmonic oscillator operator, where we have seen that

$$(d+d^*)^2 = H + \underbrace{(2N-n)}_{=\frac{1}{i}\gamma_j'\gamma_j''} \qquad H = -\partial_j^2 + x_j^2$$

Now what I am supposed to do is to replace ∂_j by $\varepsilon\partial_j$, whence

$$D_\varepsilon^2 = (d+d^*)^2 = \overbrace{\left(-\varepsilon^2\partial_j^2 + x_j^2\right)}^{H_\varepsilon} + \varepsilon(2N-n).$$

[redacted] We have in this case

$$\langle x|e^{-tD_\varepsilon^2}|x\rangle = \langle x|e^{-tH_\varepsilon}|x\rangle\, e^{-t\varepsilon(2N-n)}.$$

March 28, 1983

Start with the Dirac operator on $\mathbb{R}^2$

$$-i(\sigma_1\partial_1 + \sigma_2\partial_2) = \frac{1}{i}\begin{pmatrix} 0 & \partial_x - i\partial_y \\ \partial_x + i\partial_y & 0 \end{pmatrix}$$

and tensor it with a degree 1 operator

$$L = \begin{pmatrix} 0 & P^* \\ P & 0 \end{pmatrix} \qquad \text{on } \mathcal{H} = \mathcal{H}^+ \oplus \mathcal{H}^-$$

so as to obtain the operator

$$\left(\frac{1}{i}\sigma_i\partial_i\right)\otimes 1 + \sigma_3 \otimes L \qquad \text{on} \quad \mathbb{C}^2\otimes\mathcal{H}.$$

Now

$$(\mathbb{C}^2\otimes\mathcal{H})^+ \cong \mathcal{H}^+\oplus\mathcal{H}^- \longrightarrow (\mathbb{C}^2\otimes\mathcal{H})^- \cong \mathcal{H}^-\oplus\mathcal{H}^+$$

is given by the matrix

$$\begin{bmatrix} 1\otimes P & \frac{1}{i}(\partial_x - i\partial_y)\otimes 1 \\ \frac{1}{i}(\partial_x + i\partial_y)\otimes 1 & -1\otimes P^* \end{bmatrix}$$

Pretty messy. I think that if I am going to have any success at all I have to work with the operator in the form

$$\tilde{D} = \left(\frac{1}{i}\sigma^i\partial_i\right)\otimes 1 + \sigma_3\otimes L$$

with

$$\varepsilon = \sigma_3 \otimes \varepsilon_{\mathcal{H}}$$

The program is now to replace ∂_i by $h\partial_i$ and try to understand the operator

$$e^{-t\tilde{D}^2} \qquad \text{as} \quad h \to 0.$$

I am supposed to think of L as being a potential depending on $x \in \mathbb{R}^2$. We have

$$\tilde{D} = \sigma\cdot p + \sigma_3 L$$

essentially and I am trying to understand the Hamiltonian

$$H = \tilde{D}^2 = p^2 + L^2 + \{\sigma\cdot p, \sigma_3 L\}.$$

(But this is slightly wrong as the usual physics would give $e^{-\frac{i}{\hbar}Ht}$ for time evolution, and I seem to be interested mainly in e^{-Ht}. No: you want $e^{-\beta H}$ where β is temperature (inverse). Thus the correct physics is the thermal situation in the limit as Planck's constant $\hbar \to 0$, i.e. the classical limit.)

Let's pursue the physics. I have a quantum situation at inverse temperature β and I want to understand the classical limit $\hbar \to 0$. What are some simple examples? The first example would be for a system described by

$$H = \frac{1}{2m}\left(\frac{\hbar}{i}\nabla\right)^2 + U$$

which shows that only $\hbar^2/m$ matters. We should have

$$\mathrm{Tr}\left(e^{-\beta H}\right) \qquad \text{quantum partition function}$$

being approximated as $\hbar \to 0$ by the classical partition function:

$$\int \left(\frac{dp\,dq}{2\pi\hbar}\right)^n e^{-\beta\left(\frac{p^2}{2m}+U\right)} = \left(\frac{\sqrt{2\pi}}{2\pi\hbar}\sqrt{\frac{m}{\beta}}\right)^n \int d^nq\, e^{-\beta U}$$

$$= \left(\frac{m}{2\pi\hbar^2\beta}\right)^{n/2} \int d^nq\, e^{-\beta U}$$

I am primarily interested in the correction terms.

$$H = (\tilde{D})^2 = \left(\frac{\hbar}{i}(\sigma_1\partial_1 + \sigma_2\partial_2) + \sigma_3 L\right)^2$$

$$= -\hbar^2(\partial_1^2 + \partial_2^2) + L^2 + \frac{\hbar}{i}\left(\sigma_1[\partial_1, L] + \sigma_2[\partial_2, L]\right)\sigma_3 .$$

I want the expansion as $\hbar \to 0$ of

$\langle x | \mathrm{tr}_{sp.}(e^{-\beta H}\varepsilon) | x\rangle$. Let's count to see that the expected cancellation takes place. Recall that in 2-dimension $\sigma_1 = \gamma^1$, $\sigma_2 = \gamma^2$, $\sigma_3 = \varepsilon$. Now we know that the heat kernel has the leading term

$$\langle x | e^{-\beta H} | x\rangle = \left(\frac{1}{4\pi\hbar^2\beta}\right) e^{-\beta L_x^2}$$

hence I want to see that the spinor trace will allow only $\hbar^2$ and higher. Clear since $\sigma_1\sigma_3$, $\sigma_2\sigma_3$ have trace zero.

So what is the next step?

In the more general situation where ∂_i is replaced by a connection $D_i = \partial_i + A_i$ we will have

$$(\tilde{D})^2 = \left(\frac{\hbar}{i}(\sigma_j D_j) + \sigma_3 L\right)^2$$

$$= -\hbar^2(\sigma_1 D_1 + \sigma_2 D_2)^2 + L^2 + \frac{\hbar}{i}\left(\underbrace{\sigma_1\sigma_3[D_1, L] + \sigma_2\sigma_3[D_2, L]}\right)$$

$$= -\hbar^2(D_1^2 + D_2^2) + L^2 - \hbar^2\sigma_1\sigma_2[D_1, D_2] + \frac{\hbar}{i}\left(\qquad \right)$$

Now the idea has to be to get some sort of perturbation expansion, or maybe semi-classical expansion, of $e^{-t\tilde{D}^2}$.

[redacted] Changing notation from above put

$$H = -\hbar^2(D_1^2 + D_2^2) + L^2 .$$

Then ~~the~~ leading term for its heat kernel is

$$\langle x | e^{-\beta H} | x'\rangle = \frac{1}{4\pi\hbar^2\beta} e^{-\frac{(x-x')^2}{4\hbar\beta} - \beta L_x^2} \cdot (1 + \cdots)$$

On the other hand if I expect a non-zero spinor trace against

I know I must acquire at least an $\hbar^2$, either by using the term $-\hbar^2\sigma_1\sigma_2[D_1,D_2]$ once, or both terms $\frac{\hbar}{i}\sigma_1\sigma_3[D_1,L]$, $\frac{\hbar}{i}\sigma_1\sigma_3[D_2,L]$ together; this latter is a 2nd order effect of the perturbation.

Hence it seems that the calculation takes place: The $\hbar^n$ in the denominator is cancelled against the $\hbar^n$ surviving after the spinor trace is taken.

The main problem is how to handle the algebra. Let us try the resolvant. Write

$$(\tilde{D})^2 = H + \hbar A + \hbar^2 B.$$

Then

$$\frac{1}{\lambda-(\tilde{D})^2} = \frac{1}{\lambda-H} + \frac{1}{\lambda-H}(\hbar A+\hbar^2 B)\frac{1}{\lambda-H}$$

$$+\frac{1}{\lambda-H}(\hbar A+\hbar^2B)\frac{1}{\lambda-H}(\hbar A+\hbar^2B)\frac{1}{\lambda-H}$$

$$+\cdots$$

When we take $tr(\varepsilon\,..)$ we get

$$tr\left(\frac{1}{\lambda-(\tilde{D})^2}\varepsilon\right) = \hbar^2\, tr\left(\frac{1}{\lambda-H}B\frac{1}{\lambda-H}\varepsilon\right)$$

$$+\hbar^2\, tr\left(\frac{1}{\lambda-H}A\frac{1}{\lambda-H}A\frac{1}{\lambda-H}\varepsilon\right) + O(\hbar^3)$$

Now I want to restrict to the diagonal and let $\hbar\to 0$. What I seem to need is that in dimension 2

$$\hbar^2\left\langle x\right|\frac{1}{\lambda-H}A\frac{1}{\lambda-H}A\frac{1}{\lambda-H}\left|x\right\rangle \to \frac{1}{\lambda-L_x^2}A_x\frac{1}{\lambda-L_x^2}A_x\frac{1}{\lambda-L_x^2}$$

and it might be possible to prove a general statement of this kind.

Notice that H has the form $-\hbar^2D_\mu^2+L^2$, hence is one of the Laplaceans I understand.

I can check the result I need by considering the perturbation

$$H + zA = -\hbar^2 D_\mu^2 + L^2 + zA .$$

Then because A is a multiplication operator I know that

$$\langle x | \hbar^2 e^{-\beta(H+zA)} | x \rangle \longrightarrow \frac{1}{(\sqrt{4\pi\beta})^2} e^{-\beta}$$ [redacted] $$(L_x^2 + zA_x)$$

for any z. Then we can take contour integrals in z to deduce that

$$\hbar^2 \int_0^\beta dt_1$$ [redacted] $$\int_0^{t_1} dt_2 \quad \langle x | e^{-(\beta - t_1)H} A \, e^{-(t_1 - t_2)H} A \, e^{-t_2 H} | x \rangle$$

converges to

$$\frac{1}{(\sqrt{4\pi\beta})^2} \int_0^\beta dt_1 \int_0^{t_1} dt_2 \; e^{-(\beta - t_1)L_x^2} A_x \, e^{-(t_1 - t_2)L_x^2} A_x \, e^{-t_2 L_x^2} .$$

Taking Laplace transform gives [redacted]

$$\hbar^2 \langle x | \frac{1}{\lambda - H} A \frac{1}{\lambda - H} A \frac{1}{\lambda - H} | x \rangle \longrightarrow \frac{1}{4\pi\beta} \frac{1}{\lambda - L_x^2} A_x \frac{1}{\lambda - L_x^2} A_x \frac{1}{\lambda - L_x^2}$$

?

Question: If $H = -\hbar^2 \nabla^2 + V$ does

$$\hbar^n \langle x | \frac{1}{\lambda - H} | x \rangle$$

have a limit at $\hbar \to 0$?

not well posed for $n > 1$.

Pass to the heat operator: I believe we have

$$\langle x | e^{-\beta H} | x' \rangle \approx \frac{e^{-\frac{(x-x')^2}{4\hbar^2\beta} - \beta V(x)}}{(4\pi \hbar^2 \beta)^{n/2}} \left(1 + O(h) + \cdots\right)$$

hence

$$\lim_{\hbar \to 0} \hbar^n \langle x | e^{-\beta H} | x \rangle = \frac{e^{-\beta V(x)}}{(4\pi\beta)^{n/2}}$$

hence
$$\lim_{\hbar\to 0} \hbar^n \langle x| \frac{1}{\lambda - H}|x\rangle = -\int_0^\infty e^{\beta\lambda} \frac{e^{-\beta V(x)}}{(4\pi\beta)^{n/2}} d\beta$$
$$= -\frac{\Gamma(-\frac{n}{2}+1)}{(4\pi)^{n/2}} \frac{1}{(\lambda - V(x))^{-\frac{n}{2}+1}}.$$

This is really not well-defined, except for $n=1$.

Let's review the ~~situation~~ from the beginning, paying careful attention. Consider the case of
$$\tilde{D} = \frac{\hbar}{i}\gamma^\mu \partial_\mu + \varepsilon \otimes L \quad \text{on spinors} \otimes \mathcal{H}$$

Then
$$\tilde{D}^2 = \underbrace{-\hbar^2\partial_\mu^2 + L^2}_{H} + \hbar \underbrace{\gamma^\mu \varepsilon \left[\tfrac{1}{i}\partial_\mu, L\right]}_{A} \quad \text{and}$$

I have the expansion
$$\frac{1}{\lambda - \tilde{D}^2} = \frac{1}{\lambda - H} + \frac{1}{\lambda-H}\hbar A \frac{1}{\lambda - H} + \frac{1}{\lambda - H}\hbar A \frac{1}{\lambda-H}\hbar A\frac{1}{\lambda-H} + \cdots$$

and I know that if I take the spinor trace ε only powers of $\hbar$ at least $\hbar^n$ occur. Corresponding to the above expansion is the Dyson expansion of the heat operator. Thus
$$\mathrm{tr}(\varepsilon e^{-\beta\tilde{D}^2}) = (-\hbar)^n \int_0^\beta dt_1 \int_0^{t_1} dt_2 \cdots \int_0^{t_{n-1}} dt_n \; \mathrm{tr}(\varepsilon e^{-(\beta-t_1)H} A e^{-(t_1-t_2)H} \cdots A e^{-t_n H}) + \cdots$$

Now I want to let $\hbar \to 0$ and restrict $\langle x|\mathrm{tr}(\varepsilon e^{-\beta\tilde{D}^2})|x\rangle$. So I need to know the behavior of the multiple integral. This should be the term of degree n in A in
$$\mathrm{tr}(\varepsilon e^{-\beta(H+A)}).$$

Actually $\mathrm{tr}(\varepsilon e^{-\beta(H+A)})$ begins with $(-1)^n$ the above multiple integral. On the other hand
$$\langle x| e^{-\beta(H+A)}|x\rangle = \frac{1}{(4\pi\beta)^{n/2}\hbar^n} e^{-\beta(L_x^2 + A_x)}(1 + O(\hbar))$$

So the conclusion seems to be that

$$\lim_{h\to 0} \langle x |\, \mathrm{tr}(\varepsilon e^{-\beta \tilde{D}^2}) \,| x \rangle = \frac{(-1)^n}{(4\pi\beta)^{n/2}} \int_0^\beta dt_1 \cdots \int_0^{t_{n-1}} dt_n \; \mathrm{tr}\left(\varepsilon e^{-(\beta-t_1)L_x^2} A_x \cdots A_x e^{-t_n L_x^2}\right)$$

Let's check this in the harmonic oscillator case, where

$$L = x^\mu (\varepsilon \otimes \gamma^{\mu\prime})$$

$$A = \gamma^\mu \left[\tfrac{1}{i}\partial_\mu , L\right] = -i\, \gamma^\mu \gamma^{\mu\prime}$$

March 29, 1983

Review yesterday's calculations.

$$\tilde{D} = \frac{h}{i}\gamma^\mu\partial_\mu + \varepsilon L \qquad \text{acts on} \qquad S\otimes\mathcal{H}$$

where S is the Clifford module and L operates on $\mathcal{H}$. Here S is a C_n-module where n is even and it is graded by ε; also L depends on x. Then

$$\tilde{D}^2 = -h^2\partial_\mu^2 + L^2 + h\underbrace{\gamma^\mu\varepsilon\left[\tfrac{1}{i}\partial_\mu, L\right]}_{A}$$

$$= \underbrace{-h^2\partial_\mu^2 + L^2}_{H} + hA$$

Example: $\mathcal{H} = S$ with $L = \gamma^\mu x^\mu$. Then

$$A = \gamma^\mu\varepsilon\otimes\left[\tfrac{1}{i}\partial_\mu, \gamma^\nu x^\nu\right] = \frac{1}{i}\underbrace{(\gamma^\mu\otimes 1)}_{(\gamma^\mu)'}\underbrace{(\varepsilon\otimes\gamma^\mu)}_{(\gamma^\mu)''} = (2N-n) \qquad (\text{see p. } 683)$$

$$H = -h^2\partial_\mu^2 + (x^\mu)^2$$

We use the perturbation expansion

$$e^{-tD^2} = e^{-tH} + \int_0^t dt_1\, e^{-(t-t_1)H}(-hA)e^{-t_1H} + \cdots$$

By the basic lemma on traces of γ-matrices we have

$$\operatorname{tr}\left(\varepsilon e^{-tD^2}\right) = (-h)^n\int_0^t dt_1\cdots\int_0^{t_{n-1}}dt_n\,\operatorname{tr}\left(\varepsilon e^{-(t-t_1)H}A\cdots Ae^{-t_nH}\right) + \cdots$$

The problem is to understand the limit as $h\to 0$. I can replace A by $\frac{1}{h}A$ in the above formula obtaining

$$\operatorname{tr}\left(\varepsilon e^{-t(H+A)}\right) = (-1)^n\int_0^t dt_1\cdots\int_0^{t_{n-1}}dt_n\,\operatorname{tr}\left(\varepsilon e^{-(t-t_1)H}A\cdots Ae^{-t_nH}\right) + O(h)$$

On the other hand $H+A = -h^2\partial^2 + L^2 + A$ is a standard Schroedinger operator, hence I should know that one has

$$\langle x|e^{-t(H+A)}|x'\rangle \sim \frac{e^{-\frac{(x-x')^2}{4h^2t} - t(L_x^2 + A_x)}}{(4\pi h^2 t)^{n/2}}\left(1 + O(h)\right)$$

If so, then

$$\langle x|e^{-t(H+A)}|x\rangle \sim \frac{e^{-t(L_x^2+A_x)}}{(4\pi \hbar^2 t)^{n/2}}\,(1+O(\hbar))$$

and so

$$\hbar^n \langle x|e^{-t(H+A)}|x\rangle \longrightarrow \frac{e^{-t(L_x^2+A_x)}}{(4\pi t)^{n/2}}$$

as $\hbar \to 0$. Now we compare the terms of degree n in A on both sides and we get

$$\lim_{\hbar\to 0} tr(\varepsilon\langle x|e^{-tD^2}|x\rangle) = \lim_{\hbar\to 0} \hbar^n\, tr(\varepsilon\langle x|e^{-t(H+A)}|x\rangle)_{(\deg n \text{ in } A)}$$

$$= tr\left(\varepsilon\,.\,\frac{e^{-t(L_x^2+A_x)}}{(4\pi t)^{n/2}}\right)_{\deg n \text{ in } A}$$

$$= \frac{(-1)^n}{(4\pi t)^{n/2}}\, tr\left(\varepsilon\int_0^t dt_1 \cdots \int_0^{t_{n-1}} dt_n\, e^{-(t-t_1)L_x^2} A_x \cdots A_x e^{-t_n L_x^2}\right)$$

Let's check this in the case of the oscillator where I know that H and A commute, also L^2 and A. Then the last term is

$$\frac{(-1)^n}{(4\pi t)^{n/2}}\, tr\left(\varepsilon\int_0^t dt_1 \cdots \int_0^{t_{n-1}} dt_n\, e^{-tx^2} A^n\right)$$

$$= \frac{(-1)^n}{(4\pi t)^{n/2}}\, e^{-tx^2}\, \underbrace{\frac{t^n}{n!}\, tr(\varepsilon A^n)}_{\text{term containing } t^n \text{ in } tr(\varepsilon e^{tA}) = (e^{-t}-e^{+t})^n}$$

$2N-n$

$$= \frac{1}{\pi^{n/2}}\, t^{n/2} e^{-tx^2}. \qquad \text{Since } \int d^n x\, e^{-2tx^2/2} = \frac{(\sqrt{2\pi})^n}{(\sqrt{2t})^n} = \left(\frac{\pi}{t}\right)^{n/2}$$

this integrates to one as it should.

Let's consider now

$$\tilde{D} = \frac{h}{i}\gamma^\mu D_\mu + \varepsilon L$$

where a connection is present in the bundle on which L operates.

$$\tilde{D}^2 = \underbrace{\left(-h^2 D^2 + L^2\right)}_{H} + h\,\underbrace{\gamma^\mu \varepsilon\left[\tfrac{1}{i}D_\mu, L\right]}_{A} + h^2\,\underbrace{\tfrac{-1}{2}\gamma^\mu\gamma^\nu\left[D_\mu, D_\nu\right]}_{B}$$

Again we use the perturbation expansion

$$e^{-t\tilde{D}^2} = e^{-tH} + \int_0^t dt_1\, e^{-(t-t_1)H}\left(-hA - h^2 B\right)e^{-t_1 H} + \cdots\cdots$$

By the basic lemma on the traces of γ-matrices, we know that $tr_{sp}\left(\varepsilon e^{-t\tilde{D}^2}\right)$ begins with a term of order n in h, which is in fact a sum of terms of the form

$$\int_0^t dt_1 \cdots \int_0^{t_{p-1}} dt_p \; e^{-(t-t_1)H}\begin{pmatrix}A\\ \text{or}\\ B\end{pmatrix} e^{-(t_1-t_2)H}\begin{pmatrix}A\\ \text{or}\\ B\end{pmatrix}\cdots\cdots e^{-t_p H}$$

where $n =$ numbers of A's $+ 2\cdot$number of B's. In fact if we have one of these terms, we have all permutations of it. So I should be able to understand what is happening as $h \to 0$ by looking at

$$\langle x|e^{-t(H+A+B)}|x\rangle \sim \frac{e^{-t(L_x^2 + A_x + B_x)}}{(4\pi h^2 t)^{n/2}}$$

and taking the part of a given bidegree in A, B. Since the denominator is independent of [redacted] the bidegree it would appear that

$$\lim_{h\to 0}\langle x|tr\left(\varepsilon e^{-t\tilde{D}^2}\right)|x\rangle = \text{[redacted]} \lim_{h\to 0} tr\left(\varepsilon\,\frac{e^{-t(L_x^2 + hA_x + h^2 B_x)}}{(4\pi h^2 t)^{n/2}}\right)$$

March 30, 1983

Over X we have the family L_x of self-adjoint Fredholm operators on H, and also a connection on H. (Think of L as an unbounded version of the Kasparov F.) Suppose we have the even case, i.e. L is of degree 1 relative to a grading on H. The problem is to define even differential forms on X representing the character of this family.

I begin by supposing that X is of even dimension n and defining the form of degree n. I will use the Dirac operator on X with coefficients in (L, H):

$$\tilde{D} = \frac{h}{i}\gamma^\mu D_\mu + \varepsilon L = \frac{h}{i}\not{D} + \varepsilon L$$

Here I am supposing a flat metric on X; otherwise, what I get from the analysis will be a collection of forms of degrees $n, n-2, \ldots$ corresponding to the $\hat{A}$-genus of X.

$$(\tilde{D})^2 = (-h^2 D_\mu^2 + L^2) + h \underbrace{\{\tfrac{1}{i}\not{D}, \varepsilon L\}}_{A} + h^2 \underbrace{(-\tfrac{1}{2}\gamma^\mu\gamma^\nu F_{\mu\nu})}_{B}$$

The real question is how do I get a diff'l form on X? Let's first consider the case where $L = 0$, i.e. where we have the ordinary Dirac operator over X with coefficients in a bundle (here $H^+ \oplus H^- = E \oplus 0$.) We form

$$e^{-t\tilde{D}^2} = e^{-th^2(-D_\mu^2 + B)}$$

expand in the perturbation series. First you restrict the kernel to the diagonal and take $\operatorname{tr}\varepsilon$ over the spinor indices, and then let h (or t) go to zero. This has to be an n-form on X. Why? $e^{-t\tilde{D}^2}$ is a smooth kernel operator on sections of $S \otimes E$ over X, hence its kernel is a smooth section over $X \times X$ with values:

$$\langle x | e^{-t\tilde{\partial}^2} | x' \rangle \in \mathrm{Hom}(S_{x'} \otimes E_{x'}, S_x \otimes E_x) \otimes |\Lambda^n T_{x'}|$$

Now I have to take

$\mathrm{tr}\left(\varepsilon \langle x | e^{-t\tilde{\partial}^2} | x \rangle\right)$ which I know involves $\frac{n}{2}$ factors of B. In fact I think one has

$$\mathrm{tr}\left(\varepsilon \langle x | e^{-t\tilde{\partial}^2} | x \rangle\right) = \mathrm{tr}\left(\varepsilon \frac{e^{-th^2 B_x}}{(4\pi h^2 t)^{n/2}}\right) + O(h)$$

and I probably should put $|d^n x|$ on the right since I use

$$\langle x | e^{+th^2 \partial_\mu^2} | x' \rangle = \frac{e^{-\frac{(x-x')^2}{4h^2 t}}}{(4\pi h^2 t)^{n/2}} |d^n x'|.$$

So I still have the algebraic problem of ~~seeing~~ why

$$\lim_{h \to 0} \mathrm{tr}\left(\varepsilon \frac{e^{-th^2 B_x}}{(4\pi h^2 t)^{n/2}}\right) |d^n x|$$

is an n-form at x on X.

Let us consider the $L = 0$ case carefully. All I have is a bundle over X with a connection. By considering Dirac operators on even dimensional submanifolds of X I should be able to explicitly compute the components of the character of this connection.

Let's first derive the formulas for the n-form, where n is even. I can suppose then $X = \mathbb{R}^n$ and that I am working around $x = 0$. The Dirac operator is

$$\tilde{D} = \frac{h}{i} \gamma^\mu D_\mu$$

and

$$\tilde{D}^2 = \underbrace{-h^2 D_\mu^2}_{H} \underbrace{- \frac{h^2}{2} \gamma^\mu \gamma^\nu F_{\mu\nu}}_{+h^2 B}$$

where $F_{\mu\nu} dx^\mu dx^\nu$ is an intrinsically defined $\mathrm{End}(E)$-valued 2-form namely the curvature of the connection

$$e^{-t(H+\beta B)} = e^{-tH} + \int_0^t dt_1\, e^{-(t-t_1)H}(-\beta B)e^{-t_1 H} + \cdots$$

$$\mathrm{tr}\left(\varepsilon\langle x|e^{-t(H+\beta B)}|x\rangle\right) = (-\beta)^{\frac{n}{2}}\int_0^t dt_1 \cdots \int_0^{t_{\frac{n}{2}-1}} dt_{\frac{n}{2}}\ \mathrm{tr}(\varepsilon\langle x|\, e^{-(t-t_1)H}B \cdots Be^{-t_{\frac{n}{2}}H}|x\rangle)$$

$$+ (-\beta)^{\frac{n+1}{2}}(\cdots\cdots) + \cdots$$

This is the asymptotic expansion in β. Also have the asymptotic expansion in h.

$$\mathrm{tr}\left(\varepsilon\langle x|e^{-t(H+\beta B)}|x\rangle\right) = \mathrm{tr}\left(\varepsilon\frac{1}{(4\pi h^2 t)^{n/2}}e^{-t(\beta B_x^2)}(1+O(h))\right)$$

There is some analytical point here I have to be careful about, because I am mixing two asymptotic expansions, however the expression is an entire fn. of β which should give the proper control. Steps:

$$\lim_{h\to 0} \mathrm{tr}\left(\varepsilon\langle x|e^{-t(H+h^2B)}|x\rangle\right) =$$

$$= \lim_{h\to 0} h^n\left(\text{coeff of } \beta^{n/2} \text{ in } \mathrm{tr}\left(\varepsilon\langle x|e^{-t(H+\beta B)}|x\rangle\right)\right)$$

$$= \text{coeff of } \beta^{n/2} \text{ in } \lim_{h\to 0} h^n\left(\mathrm{tr}\left(\varepsilon\langle x|e^{-t(H+\beta B)}|x\rangle\right)\right)$$

$$= \text{coeff of } \beta^{n/2} \text{ in } \frac{1}{(4\pi t)^{n/2}}\mathrm{tr}\left(\varepsilon\, e^{-t(\beta B_x^2)}\right)$$

$$= \lim_{h\to 0} \mathrm{tr}\left(\varepsilon\frac{e^{-t(h^2B_x)}}{(4\pi h^2 t)^{n/2}}\right)$$

Let's start again with a bundle E over X with connection, where $X = \mathbb{R}^n$, $n = 2m$. Then the local index is

$$\lim_{h\to 0} \mathrm{tr}\left(\varepsilon\frac{e^{-h^2B}}{(4\pi h^2)^m}\right) \qquad B = -\tfrac{1}{2}\gamma^\mu\gamma^\nu F_{\mu\nu}$$

and what I need to do is to understand why this is an n-form. So this expression is clearly

$$(*) \qquad tr\left(\varepsilon \frac{1}{(4\pi)^m}\left(\frac{1}{2}\gamma^\mu\gamma^\nu F_{\mu\nu}\right)^m \frac{1}{m!}\right).$$

Recall that ε is essentially the product of all the γ^μ:

$$(\gamma^1\cdots\gamma^n)^2 = (-1)^{\frac{n(n-1)}{2}} = (-1)^m$$

so $\qquad \varepsilon = \pm(i)^m\gamma^1\cdots\gamma^n \qquad$ depending on orientation conventions. Thus

$$tr(\varepsilon\gamma^1\cdots\gamma^n) = \mp i^m \cdot tr(1)$$
$$= \mp i^m\, 2^m$$

Now the curvature as a 2-form is

$$K = (D_\mu dx^\mu)^2 = D_\mu D_\nu\, dx^\mu dx^\nu = \frac{1}{2}[D_\mu, D_\nu]\, dx^\mu dx^\nu$$
$$= \frac{1}{2}F_{\mu\nu}\, dx^\mu dx^\nu.$$

Hence when the trace ε is taken of $\left(\frac{1}{2}\gamma^\mu\gamma^\nu F_{\mu\nu}\right)^m$ and then [redacted] multiplied by $dx^1\cdots dx^n$ we get $\mp i^m 2^m\, K^m$. Thus the above expression $*$ is essentially

$$tr\left(\left(\frac{i}{2\pi}\right)^m \frac{K^m}{m!}\right).$$

[redacted]

Let's see if we can formulate the problem nicely. The basic data that I have at a point x of X is a one-form with operator values

$$[D_\mu, L]\, dx^\mu$$

and a 2-form with operator values

$$\frac{1}{2}F_{\mu\nu}\, dx^\mu dx^\nu$$

From this data I am trying to construct various operator-valued even-diml' forms.

Next let me suppose that $L^2 = 1$. For example I can take H to be a finite dimensional graded bundle $H = E^+ \oplus E^-$ with L a unitary isomorphism between E^+ and E^-. Then I have two connections on the same bundle, so I know that the form which one gets by taking the difference of the characters will be a boundary.

It seems this case should make sense when I take a Connes situation and one makes the basic assumption of p-summability. But actually the operators should make sense even if the traces don't.

So suppose $L^2 = 1$. Then

$$\operatorname{tr}\left(\varepsilon\, \frac{e^{-t(L^2 + hA + h^2 B)}}{(4\pi h^2 t)^{n/2}}\right) = e^{-t}\operatorname{tr}\left(\varepsilon\, \frac{e^{-thA - th^2 B}}{(4\pi h^2 t)^{n/2}}\right)$$

Now expanding the exponential we get for the limit as $h \to 0$ a sum of terms

$$\frac{e^{-t}}{(4\pi t)^{n/2}} \operatorname{tr}\left(\varepsilon\, (-A) \cdots (-B) \cdots \right) t^{p+q}$$

$$\begin{matrix} p & A\text{'s} \\ q & B\text{'s} \end{matrix} \qquad p + 2q = n$$

But then $(p+q) - \frac{n}{2} = p+q - \frac{1}{2}(p+2q) = \frac{p}{2} > 0$ unless $p=0$. Hence the only thing that survives in the limit as $t \to 0$ is

$$\frac{1}{(4\pi)^m} \operatorname{tr}\left(\varepsilon\, (-B)^m \frac{1}{m!}\right)$$

which is the case studied above.

Question: I am obviously getting differential forms for each t. [redacted] Does the cohomology class stay fixed, and in particular if I [redacted] integrate the form from $t=0$ to $t=\infty$, do I get [redacted] the transgression formula for the difference of the characters?

Let's go back and try to [redacted] formulate things precisely. I want to convert the data consisting of the [redacted] operator-valued forms

$$[D_\mu, L]dx^\mu \qquad \frac{1}{2}F_{\mu\nu}\,dx^\mu dx^\nu$$

into a family of even forms. It should be a way of removing the γ's from the expression

$$\lim_{h\to 0} \operatorname{tr}\left(\varepsilon\, \frac{e^{-t(L^2+hA+h^2B)}}{(4\pi h^2 t)^{n/2}}\right) \qquad \begin{aligned} A &= \tfrac{1}{i}\gamma^\mu \varepsilon [D_\mu, L] \\ B &= -\tfrac{1}{2}\gamma^\mu\gamma^\nu F_{\mu\nu}. \end{aligned}$$

Idea: Write this in a RR form: $\langle [\sigma] \,|\, ch(\alpha)\rangle$ where the cycle σ depends on the "submanifold". [redacted]

This is a purely algebraic problem. I have a vector space T representing the tangent space to X at x and

$$[redacted] \quad L^2 \in \operatorname{End}(E), \qquad [D_\mu, L]dx^\mu \in \operatorname{End}(E)\otimes T^*$$

$$\frac{1}{2}F_{\mu\nu}\,dx^\mu dx^\nu \in \operatorname{End}(E)\otimes \Lambda^2 T^*$$

where E is $\mathbb{Z}_2$-graded and the total degree of the above three elements is zero. [redacted]

Next I suppose given an even-dimensional subspace W of T with a metric and orientation. Then I can obtain operators on $S_W \otimes E$, where S_W is the irred. $C(W)$-module with grading determined by the orientation, by using the restriction $\Lambda^\cdot T^* \to \Lambda^\cdot W^*$, and the obvious way of <u>embedding</u> W^* and $\Lambda^2 W^*$ into $C(W)$.

Recall that when I evaluate

$$\lim_{h\to 0} \operatorname{tr}\left(\varepsilon \frac{e^{-t(L^2+hA+h^2B)}}{(4\pi h^2 t)^{n/2}}\right)$$

I encounter tr ε applied to products of $h\gamma^\mu\varepsilon$, $h^2\gamma^\mu\gamma^\nu$. I know that the only way to get something non-zero is to at least n-γ factors present, in which case I have h^n.

Thus to have something non-zero present after $h\to 0$, we must use only products of $h\gamma^\mu\varepsilon$, $h^2\gamma^\mu\gamma^\nu$ where each γ appears exactly once. So what I seem to be doing is to take the differential form

$$e^{-t\left(L^2+\frac{1}{i}[D_\mu, L]\,dx^\mu - \frac{1}{2}F_{\mu\nu}\,dx^\mu dx^\nu\right)}$$ change (see 921)

take the coefficient of $dx^1\dots dx^n$ and multiply by $\left(\frac{i}{2\pi t}\right)^m$. So I am saying that I seem to get

$$\cancel{\lim_{h\to 0}\operatorname{tr}\left(\varepsilon\, \frac{e^{-t(L^2+hA+h^2B)}}{(2\pi h^2 t)^m}\right) dx^1\dots dx^n = \left(\frac{i}{2\pi t}\right)^m \text{coeff of } dx^1}$$

that the local index is given by capping the above differential form with $\left(\frac{i}{2\pi t}\right)^{n/2}$ the element of $\Lambda^n T$ corresponding to W with its oriented volume.

Let's check this against the formula for the curvature of the determinant line bundle.

$$i(\delta D)\,\Theta = \operatorname{Tr}\left(e^{-tD^*D}D^{-1}\delta D\right)_{0\text{th coeff.}}$$

$$\delta_1 \operatorname{Tr}\left(e^{-tD^*D}D^{-1}\delta D\right) = \operatorname{Tr}\left(-\int_0^t dt_1\, e^{-(t-t_1)(D^*D)}\,\delta_1(D^*D)\,e^{-t_1 D^*D}\,D^{-1}\delta D\right)$$

$$+ \operatorname{Tr}\left(e^{-tD^*D} D^{-1}(-\delta_1 D) D^{-1}\delta D\right)$$

$$= -\int_0^t dt_1 \operatorname{Tr}\left(e^{-(t-t_1)(D^*D)} \delta_1 D^* e^{-t_1 DD^*} \delta D\right)$$

$$-\int_0^t dt_1 \operatorname{Tr}\left(e^{-(t-t_1)D^*D} D^* \delta_1 D \, e^{-t_1 D^*D} D^{-1} \delta D\right)$$

$$- \operatorname{Tr}\left(e^{-tD^*D} D^{-1}\delta_1 D \, D^{-1}\delta D\right)$$

I guess I have to place myself in the Riemann surface situation where I know that I only have to compute $d''\Theta$. Then the curvature form is simply

$$-\left[\int_0^t dt_1 \operatorname{Tr}\left(e^{-(t-t_1)D^*D} \delta_1 D^* e^{-t_1 DD^*} \delta D\right)\right]_{\text{0th coeff.}}$$

I seem to be off by a factor of t. This for the following reason.

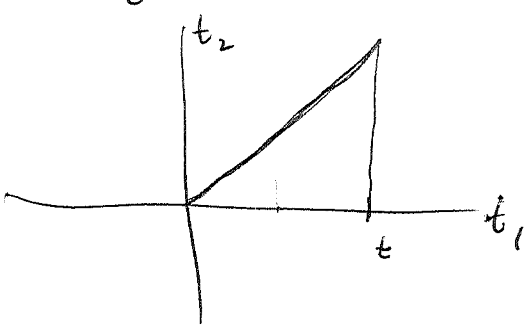
$$L = \begin{pmatrix} & D^* \\ D & \end{pmatrix} \qquad L^2 = \begin{pmatrix} D^*D & \\ & DD^* \end{pmatrix}$$

$$e^{-t(L^2 + \frac{1}{i}\delta_1 L + \frac{1}{i}\delta L)}$$

I want the term of second order which involves $\delta_1 D^*$, δD.

$$\int_0^t dt_1 \int_0^{t_1} dt_2 \operatorname{Tr}\left(e^{-(t-t_1)D^*D} \tfrac{1}{i}\delta_1 D^* e^{-(t_1-t_2)DD^*} \tfrac{1}{i}\delta D \, e^{-t_2 D^*D}\right)$$

$$= \int_0^t dt_1 \int_0^{t_1} dt_2 \operatorname{Tr}\left(e^{-(\overbrace{t-t_1}^{-u}+t_2)D^*D} \tfrac{1}{i}\delta_1 D^* e^{-(\overbrace{t_1-t_2}^{u})DD^*} \tfrac{1}{i}\delta D\right)$$



$$= \iint\limits_{\underbrace{(t-t_1)}_{t-t_2-u}+\underbrace{(t_1-t_2)}_{u}+\underbrace{(t_2)}_{t_2}=t} \qquad = \iint$$

??

Try again

$$\iint_{a+b+c=t} \operatorname{Tr}\left(e^{-aD^*D}\,\delta D_1^*\,e^{-bDD^*}\,\delta D\,e^{-cD^*D}\right)$$

$$= \iint_{a+b+c=t} \operatorname{Tr}\left(e^{-(a+c)D^*D}\,\delta D_1^*\,e^{-bDD^*}\,\delta D\right)$$

$$= \int_{u+b=t} \operatorname{Tr}\left(e^{-uD^*D}\,\delta D_1^*\,e^{-bDD^*}\,\delta D\right) \underbrace{\int_{a+c=u} 1}_{u}$$

I can check the measures as follows

$$\iint_{a+b+c=t} = \int_0^t db \int_0^{t-b} dc$$

$$\int_{u+b=t} = \int_0^t db \qquad\qquad \int_{a+c=u} = \int_0^u dc$$

So in precise terms

$$\int_0^t db \int_0^{t-b} dc \; \operatorname{Tr}\left(e^{-(t-b-c)D^*D}\,\delta D_1^*\,e^{-bDD^*}\,\delta D\,e^{-cD^*D}\right)$$

$$= \int_0^t db \int_0^{t-b} dc \; \operatorname{Tr}\left(e^{-(t-b)D^*D}\,\delta D_1^*\,e^{-bDD^*}\,\delta D\right)$$

$$= \int_0^t db \; (t-b)\operatorname{Tr}\left(e^{-(t-b)D^*D}\,\delta D_1^*\,e^{-bDD^*}\,\delta D\right)$$

But there is another contribution which comes from the other half. To be more precise

$$\operatorname{Tr}\left(\varepsilon_M\, e^{-tL^2 + \frac{1}{i}\delta_1 L\,dx^1 + \frac{1}{i}\delta_2 L\,dx^2}\right)$$

is the sum of four terms. To simplify suppose

$$\delta_1 L = \begin{pmatrix} 0 & 0 \\ \delta D & 0 \end{pmatrix} \quad \text{and} \quad \delta_2 L = \begin{pmatrix} 0 & \delta_1 D^* \\ 0 & 0 \end{pmatrix}.$$

Then we get two terms in the coeff. of $dx^1 dx^2$:

from $\mathcal{E}$ [redacted]

$$-\iint_{a+b+c=t} \mathrm{Tr}\left(e^{-aDD^*} \tfrac{1}{i}\delta D \; e^{-bD^*D} \tfrac{1}{i}\delta_1 D^* \; e^{-cDD^*}\right)$$

from dx^2dx^1

$$-\iint_{a+b+c=t} \mathrm{Tr}\left(e^{-aD^*D} \tfrac{1}{i}\delta_1 D^* \; e^{-bDD^*} \tfrac{1}{i}\delta D \; e^{-cD^*D}\right)$$

$$= + \iint \mathrm{Tr}\left(e^{-bD^*D} \delta_1 D^* \; e^{-(c+a)DD^*} \delta D\right)$$

$$+ \iint \mathrm{Tr}\left(e^{-(a+c)D^*D} \delta_1 D^* \; e^{-b\,DD^*} \delta D\right)$$

$$= \int_0^t db \left[(t-b) + b\right] \mathrm{Tr}\left(e^{-bD^*D}\delta_1 D^* \; e^{-(t-b)DD^*} \delta D\right)$$

which does indeed give the extra factor of t.

Lets now consider the odd case.

$$\tilde{D} = \sigma_1\left(\tfrac{1}{i}\gamma^\mu D_\mu\right) + \sigma_2 L$$

and the index is given by the trace with σ_3. Here $\sigma_1 \otimes \gamma^\mu$, $\sigma_2 \otimes 1$, $\sigma_3 \otimes 1$ operates on $\mathbb{C}^2 \otimes S$ where S is a module of odd dimensional spinors. Maybe the notation would be simpler if the dimension is denote $n-1$ where $n = 2m-1$, and that

$$\tilde{D} = \frac{h}{i}\gamma^\mu D_\mu + \gamma^n L \qquad 1 \le \mu < n$$

---

where the γ^μ, γ^n act on S_n. Then

$$\tilde{D}^2 = -\hbar^2 D_\mu^2 + \frac{\hbar}{i}\gamma^\mu\gamma^n[D_\mu, L] + L^2 - \hbar^2 \tfrac{1}{2}\gamma^\mu\gamma^\nu F_{\mu\nu}$$

$$= \underbrace{\left(-\hbar^2 D_\mu^2 + L^2\right)}_{H} + \hbar\,\underbrace{\frac{1}{i}\gamma^\mu\gamma^n[D_\mu, L]}_{A} + \hbar^2\underbrace{\left(-\tfrac{1}{2}\gamma^\mu\gamma^\nu F_{\mu\nu}\right)}_{B}$$

which is essentially the same notation as in the even case. This time the denominator will be

$$(4\pi\hbar^2 t)^{\frac{n-1}{2}} = \text{const.}\,\hbar^{n-1}\cdot t^{\frac{n-1}{2}}$$

~~We need to know about tr(ε) on S_n. Since γ^n occurs only in A we have to use an odd number p of the $\gamma^\mu\gamma^n$ in A. The corresponding γ^μ have to be distinct and the missing γ's comes from the B factors, say there are q of these. Total number of γ~~

We need to see what products of factors of the form $\hbar\gamma^\mu\gamma^n$, $\hbar^2\gamma^\mu\gamma^\nu$ will give a non-trivial trace(ε ⋯). If we take the product and write it in minimal form, then γ^n must appear, hence we have an odd number p of factors $\hbar\gamma^\mu\gamma^n$ contributing and $\hbar^p$. Then q or more factors of the second type contribute $\hbar^{2q}$, so the total power of $\hbar$ is $\hbar^{p+2q}$. We must also have all γ^μ and γ^n present in the minimal form, so that $p + 2q \geq n-1$. Thus since the denominator has $\hbar^{n-1}$ we get a limit as $\hbar \to 0$, and the limit occurs when $p+2q = n-1$, and the corresponding γ^μ fill out all the possibilities. The trace is again $\pm i^m 2^m$. $m = \frac{n}{2}$

The recipe seems to be to take the $(n-1)$ dimensional component of

$$e^{-t\left(L^2 + \frac{1}{i}[D_\mu, L]\,dx^\mu - \frac{1}{2}F_{\mu\nu}\,dx^\mu dx^\nu\right)} \qquad ?$$

and multiply by
$$\frac{i^m 2^m}{(4\pi t)^{m-\frac{1}{2}}} = \left(\frac{i}{2\pi}\right)^m \frac{\sqrt{4\pi}}{t^{m-1/2}}.$$

March 31, 1983

Review $\zeta(-k) \qquad k = 0, 1, 2, \dots$

$$\frac{x}{e^x - 1} = \frac{1}{1 + \frac{x}{2} + \frac{x^2}{6} + \frac{x^3}{24} + \cdots}$$

$$= 1 + \left(-\frac{x}{2} - \frac{x^2}{6} - \frac{x^3}{24} - \cdots\right) + \left(\frac{x}{2} + \frac{x^2}{6} \cdots\right)^2 - \left(\frac{x}{2}\right)^3$$

$$= 1 - \frac{x}{2} + x^2\left(-\frac{1}{6} + \frac{1}{4}\right) + x^3\left(-\frac{1}{24} + 2\frac{1}{2}\frac{1}{6} - \frac{1}{8}\right) + \cdots$$

$$= 1 - \frac{x}{2} + \frac{x^2}{12} + 0x^3 + \cdots$$

In general the odd powers of x are absent except for the $-\frac{x}{2}$.

$$\frac{1}{e^x - 1} + \frac{1}{2} = \frac{1}{2}\frac{e^x + 1}{e^x - 1} = \frac{1}{2}\underbrace{\frac{e^{\frac{x}{2}} + e^{-\frac{x}{2}}}{e^{\frac{x}{2}} - e^{-x/2}}}_{\text{odd fn. of } x.}$$

Thus

$$\boxed{\frac{1}{e^t - 1} = \frac{1}{t} - \frac{1}{2} + \frac{t}{12} + 0t^2 + a_3 t^3 + 0t^4 + \cdots}$$

$$\Gamma(s)\zeta(s) = \Gamma(s)\sum_1^\infty \frac{1}{n^s} = \int_0^\infty \underbrace{\sum_1^\infty e^{-nt}}_{\frac{e^{-t}}{1-e^{-t}}} t^s \frac{dt}{t}$$

$$= \int_0^\infty \frac{1}{e^t - 1} t^s \frac{dt}{t} \sim \sum_{k=-1}^\infty \int_0^1 a_k t^k t^s \frac{dt}{t} = \sum \frac{a_k}{s+k}$$

$$\Gamma(s)\, s(s+1)\cdots(s+k) = \Gamma(s+k+1) \xrightarrow[\text{as } s \to -k]{} \Gamma(1) = 1$$

$$\Gamma(s)(s+k) \xrightarrow[\text{as } s \to -k]{} \frac{(-1)^k}{k!}$$

Thus $\quad \zeta(s) \sim \dfrac{1}{s-1} \quad$ as $\quad s \to 1$

$k \geq 0 \qquad \zeta(-k) = (-1)^k k!\, a_k$

$$\zeta(0) = -\tfrac{1}{2} \qquad\qquad \zeta(-2) = \zeta(-4) = \zeta(-6) = \cdots 0$$

$$\zeta(-1) = -\tfrac{1}{12}$$

Connection with rank of $K_*(\mathbb{Z})$:

n	0	1	2	3	4	5	6	7	8	9	
$\operatorname{rank}(K_n)$	1	0	0	0	0	1	0	0	0	1	$\cdots$

In general one hopes for a connection as follows:

$$\zeta(0) \longleftrightarrow K_0, K_1$$
$$\zeta(-1) \quad '' \quad K_2, K_3$$
$$\zeta(-2) \quad '' \quad K_4, K_5$$

This is too simple-minded. Somehow I want to look at the weights of the Adams operations on the K-groups, and then ζ at $s = -k$ is related to weight $k+1$, or better:

$$\text{weight } p \longleftrightarrow \zeta(-p+1).$$

$\therefore \quad K_1 \quad 1 \longleftrightarrow \zeta(0)$

$\qquad K_3 \quad 2 \longleftrightarrow \zeta(-1)$

$\qquad K_5 \quad 3 \longleftrightarrow \zeta(-2).$

also $\quad K_0 \quad 0 \longleftrightarrow \zeta(1)$

One sees that there is a kind of duality shift:

$$\text{weight } p \longleftrightarrow \zeta(1-p)$$

which might be explained by the fact that ζ is related to cohomology with compact supports, whereas K is related to cohomology.

The lesson for the future seems to be that there is to be found an arithmetic cohomology with compact supports which will have a weight decomposition directly related to ζ.

So we should look for something contravariant in the ring A, something like a Kasparov theory, or K-homology. Whereas K cohomology is related to Fredholm operators between infinite dimensional A-modules, K homology should deal with representations of A modulo compact operators. It is difficult to imagine what this might be for $\mathbb{Z}$, but I think the compactness has to be put in somewhat like a cocompact lattice.

Put in the points at ∞

$$\pi^{-s}\Gamma(s) = \int_0^\infty e^{-\pi t} t^s \frac{dt}{t}$$

$$\pi^{-s/2}\Gamma(s/2) = 2\int_0^\infty e^{-\pi t^2} t^s \frac{dt}{t}$$

$$\pi^{-s/2}\Gamma(s/2)\,\zeta(s) = \int_0^\infty \left[\theta(t^2)-1\right] t^s \frac{dt}{t}$$

$$\theta(t) = \sum e^{-\pi n^2 t} \qquad \text{satisfies} \qquad \theta(t) = \frac{1}{\sqrt{t}}\,\theta\left(\frac{1}{t}\right)$$

This leads immediately to the functional equation asserting that $\pi^{-s/2}\Gamma(s/2)\zeta(s)$ is invariant under $s \leftrightarrow 1-s$. Also we have the asymptotics

$$\theta(t^2) - 1 \sim \frac{1}{t} - 1 \qquad \text{to infinite order as } t \downarrow 0.$$

hence $\hat{\zeta}(s) = \pi^{-s/2}\Gamma(s/2)\zeta(s)$ has simple poles at $s = 1, 0$ with the behavior $\frac{1}{s-1} - \frac{1}{s}$, otherwise is regular.

Since we know $\hat{\zeta}(s)$ doesn't vanish for $Re(s) > 1$, we conclude that it also doesn't vanish for $Re(s) < 0$ by the functional equation, so again $\zeta(s)$ has simple zeroes at $-2, -4, -6, \cdots$.

Finite field $\mathbb{F}_q$ has $\zeta(s) = \dfrac{1}{1-q^{-s}}$ and K-groups

	$\mathbb{Z}/(q^2-1)$	0	$\mathbb{Z}/(q-1)$	$\mathbb{Z}$
weight	2		1	0

$$\text{weight } p \longleftrightarrow \zeta(-p) = \frac{1}{1-q^p}$$

This checks the feeling that

$$\text{weight } p \longleftrightarrow \zeta(d-p)$$

for something regular of dimension d.

Also I want to think of $\hat{\zeta}(s)$ as being the ζ function of $\mathbb{Z}$ completed at ∞. Then

$$\pi^{-s/2}\Gamma(s/2)$$

is the ζ function of the residue field at ∞. So what I would like to find is a cohomology to be associated to $\mathbb{R}$ of which $\pi^{-s/2}\Gamma(s/2)$ is some kind of characteristic polynomial (Fredholm determinant). Of course $\pi^{-s}\Gamma(s)$ is to be associated with $\mathbb{C}$.

Knowing the ζ-function I should be able to reconstruct the K-groups. An possible point is that we probably want to use Pontrygagin duals of the K-groups to get what should be the cohomology with compact support and it's the size of this which should be given by the ζ fn.

Let's proceed by analogy with the finite field.

The fact that $\pi^{-s}\zeta(s)$ has simple poles at $s = 0, -1, -2, \cdots\cdots$ suggests that we have something infinite of weights $0, 1, 2, \cdots\cdots$ and that weight p should occur in dimension $2p$ in order to produce a pole. So therefore we expect the K-groups belonging to $\mathbb{C}$ to be

$$\begin{array}{ccccccccc} \cdots & 0 & * & 0 & * & 0 & * & 0 & * \\ & & 6 & & 4 & & 2 & & 0 \end{array}$$

where $*$ is something infinite of first order because we have simple poles. I expect $* = \mathbb{R}$ or $\mathbb{C}^*$. For $\mathbb{R}$ we expect that only the $0, 4, 8, \cdots$ groups survive.

So next write down the localization sequence in analogy with a curve over a finite field

$$\to K_i(\mathbb{Q}_\infty) \to K_i(\hat{\mathbb{Z}}) \to K_i(\mathbb{Z}) \xrightarrow{\delta} K_{i-1}(\mathbb{Q}_\infty) \to$$

I think it is reasonable to expect that δ cancels the infinities for $i = 5, 9, 13, \cdots$. At the bottom we have

$$\begin{array}{ccccccccccc} K_1(\mathbb{Q}_\infty) & \to & K_1(\hat{\mathbb{Z}}) & \to & K_1(\mathbb{Z}) & \to & K_0(\mathbb{Q}_\infty) & \to & K_0(\hat{\mathbb{Z}}) & \to & K_0(\mathbb{Z}) \\ \|? & & & & \| & & \|? & & & & \| \\ 0 & & & & \{\pm 1\} & & \mathbb{R} & & & & \mathbb{Z} \end{array}$$

From the analogy with curves over finite fields, we expect that

$$K_0(\hat{\mathbb{Z}}) = \underbrace{\mathbb{Z}}_{\substack{\text{weight}\\0}} + \underbrace{\mathbb{R}}_{\substack{\text{weight}\\1}}$$

which produces the pole in $\hat{\zeta}$ at $s = 1$ and the pole in $\hat{\zeta}$ at $s = 0$. Note that

$$K_i(\mathbb{Z}) \xrightarrow{\delta} K_{i-1}(\mathbb{Q}_\infty)$$

decreases weight by 1, which is consistent with its cohomological interpretation as involving a Gysin homomorphism.

But I still feel that the K-groups are not the good gadget, that they are only reasonable because of some duality between cohomology and cohomology with compact supports. So the above guesses are perhaps wrong. $\hat{\mathbb{Z}}$ being non-singular and compact should have pure weights.

The problem is to construct $H_c^*(\mathbb{Z}, T(i))$ in some form consistent with K-theory.

In order to get some insight let's consider a non-singular curve over a finite field and let's calculate cohomology, cohomology with compact supports and K-groups.

Start with the complete curve X smooth over k. Then we have the spectral sequence

$$E_2 = H(\mathrm{Gal}(\bar{k}/k), H(\bar{X})(i)) \Longrightarrow H(X, T(i))$$

where
$$H(\bar{X}) = \mathbb{Z}_\ell,\ \mathbb{Z}_\ell^{2g},\ \mathbb{Z}_\ell(+1)$$
Now $\mathrm{Gal}(\bar{k}/k) = \hat{\mathbb{Z}}$ generated by F and most of the time one has

$$H^1(\mathrm{Gal}, H(\bar{X})(i)) = H(\bar{X})(i)/(1-F)$$

whereas the $H^{\neq 1}$ is zero, so the spectral sequence is deg.

$$\zeta_X(s) = \frac{\prod_1^{2g}(1-\omega_i q^{-s})}{(1-q^{-s})(1-q^{1-s})} \qquad |\omega_i| = \sqrt{q}$$

Here the numerator of $\zeta_X(s)$ is $\det(1-q^{-s}F$ on $H^1(\bar{X}))$.
So now what can I do is to write down the Atiyah-Hirzebruch spectral sequence.

But already things are complicated. I know that $K_i(X)$ contains $K_i(k) \oplus K_i(k)$ and these are of different weights.

The AH spectral sequence has the form

$$
\begin{array}{c|cccc}
\hline
 & H^0(T(0)) & & & \\
 & 0 & 0 & 0 & 0 \\
0 & \cancel{H^0(T(1))} & H^1(T(1)) & H^2(T(1)) & H^3(T(1)) \\
 & 0 & 0 & 0 & 0 \\
0 & \cancel{H^0(T(2))} & H^1(T(2)) & H^2(T(2)) & H^3(T(2))
\end{array}
$$

So one gets

$$K_1 = \underset{\substack{\| \\ H^1(gal,\, H^0(\bar{X})(1)) \\ \text{order } [redacted]\ q-1}}{H^1(T(1))} \oplus \underset{\substack{\| \\ H^1(gal,\, H^2(\bar{X})(2)) \\ \text{order } [redacted]\ q-1}}{H^3(T(2))}$$

(Notice: Geometric Frobenius acts as q on $H^2(\bar{X})$, hence Galois Frobenius acts as q^{-1}.)

Also

$$K_2 = H^2(T(2)) = H^1(gal, H^1(\bar{X})(2))$$

Now

$$K_0 \sim \underset{\substack{\| \\ \mathbb{Z}_\ell}}{H^0(T(0))} \oplus \underset{\substack{\| \\ H^0(gal,\, H^2(\bar{X})(1)) \\ \| \\ \mathbb{Z}_\ell}}{H^2(T(1))} \oplus \underbrace{H^1(gal, H^1(\bar{X})(1))}_{\text{order } h}$$

$\therefore$ [redacted]

__Summary:__ Even for a complete non-singular curve the K-groups are not pure for the weight filtration.

The real important thing for me to do now is to get away from the K-theory. You should try to guess the cohomology of $\mathbb{Z} \cup \infty$.

The long exact K-sequence

$$\to K_5(\hat{\mathbb{Z}}) \to K_5(\mathbb{Z}) \to K_4(\mathbb{Q}_\infty) \to K_4(\hat{\mathbb{Z}}) \to K_4(\mathbb{Z}) \to K_3(\mathbb{Q}_\infty)$$

$$K_3(\hat{\mathbb{Z}}) \to K_3(\mathbb{Z}) \to K_2(\mathbb{Q}_\infty) \to K_2(\hat{\mathbb{Z}}) \to \underset{\{\pm 1\}}{\overset{\|}{K_2(\mathbb{Z})}} \to K_1(\mathbb{Q}_\infty)$$

$$K_1(\hat{\mathbb{Z}}) \to \underset{\{\pm 1\}}{\overset{\|}{K_1(\mathbb{Z})}} \to \underset{\mathbb{R}}{\overset{\|}{K_0(\mathbb{Q}_\infty)}} \to K_0(\hat{\mathbb{Z}}) \to \underset{\mathbb{Z}}{\overset{\|}{K_0(\mathbb{Z})}} \to 0$$

It is reasonable for $K_0(\hat{\mathbb{Z}}) = \mathbb{Z} \oplus \mathbb{R}$ especially if we consider a bundle over $\hat{\mathbb{Z}}$ to be a free f.t. $\mathbb{Z}$-module with quadratic form (positive-def.). Then the analogue of the degree is the covolume of the lattice with respect to the measure defined by the quadratic form. This should give

$$H^2(\hat{\mathbb{Z}}, T(1)) = \mathbb{R}$$

Actually we should try to guess the order of $K_{4p+2}(\mathbb{Q}_\infty)$ from the "ζ-fn." $\pi^{-s/2}\Gamma(s/2)$ at $s=-2p-1$.

$$\pi^{1/2}\Gamma(-1/2) = \pi^{1/2}\frac{\Gamma(\frac{1}{2})}{(-\frac{1}{2})} = -2\pi$$

$$\pi^{3/2}\Gamma(-3/2) = \pi^{3/2}\frac{\Gamma(\frac{1}{2})}{(-\frac{3}{2})(-\frac{1}{2})} = \frac{(2\pi)^2}{1\cdot 3}$$

In general we get

$$\pi^{2p+1/2}\,\Gamma(-(2p+1)/2) = \frac{(-2\pi)^{p+1}}{1\cdot 3\cdots(2p+1)}.$$

Somehow you have got things all wrong. Recall that until one proved that $K_3\mathbb{Z} = \mathbb{Z}/48\mathbb{Z}$ one believed that

$$|\zeta(-1)| = \frac{\# K_2}{\# K_3} \left(= \frac{2}{48} = \frac{1}{24} \neq \frac{1}{12}\right)$$

You want something similar for the $\zeta(-1)$. But then you have to explain the irrational character of $\pi^{-s/2}\Gamma(s/2)$ at $s=-1$.

The explanation is as follows. For a curve over [redacted] a finite field, the functional equation is

$$\zeta\left(\frac{1}{qz}\right) = \frac{\prod\left(1-\omega_i\frac{1}{qz}\right)}{\left(1-\frac{1}{qz}\right)\left(1-q\frac{1}{qz}\right)} = \frac{\frac{1}{(\text{[redacted]}z)^{2g}}\prod\left(z-\frac{\omega_i}{q}\right)}{\frac{1}{qz^2}\,(qz-1)(z-1)}$$

$$= (z^2)^{1-g}\, q\, \frac{\prod(z-\omega_i^{-1})}{(1-z)(1-qz)} = (z^2)^{1-g}\, q\, \frac{1}{\underset{q^g}{\prod \omega_i}}\, \frac{\prod(\omega_i z-1)}{(1-z)(1-qz)}$$

$$= (qz^2)^{1-g}\,\zeta(z). \Longrightarrow \boxed{\left(\frac{1}{qz}\right)^{1-g}\zeta\left(\frac{1}{qz}\right) = z^{1-g}\zeta(z)}$$

Thus the function which satisfies strict symmetry under $s \longleftrightarrow 1-s$ or $z \longleftrightarrow \frac{1}{qz}$ is the function

$$z^{1-g}\zeta(z) = q^{-(1-g)s}\,\zeta(s)$$

This means that instead of $\pi^{-s/2}\Gamma(s/2)\zeta(s)$ having its values related to order of K-groups, one might change this by an exponential factor. The simplest thing to do is to eliminate the factors of π occurring at $s=-1,-3,\dots$ is to use the function

$$\pi^{-1/2}\Gamma(s/2)\,\zeta(s) \qquad \text{[redacted]}$$

which will "predict"

$$\#\, K_{\text{[redacted]}+2}(\mathbb{R}_\infty) = \left|\pi^{-1/2}\Gamma(-1/2)\right| = \left|\pi^{-1/2}\,\frac{\Gamma(\frac{1}{2})}{-\frac{1}{2}}\right| = 2$$

Assuming $K_{odd}(\mathbb{Q}_\infty) = 0$ we have the exact sequence

$$0 \longrightarrow K_3(\hat{\mathbb{Z}}) \longrightarrow K_3(\mathbb{Z}) \longrightarrow K_2(\mathbb{Q}_\infty) \longrightarrow K_2(\hat{\mathbb{Z}}) \rightarrow K_2(\mathbb{Z}) \rightarrow 0$$

hence
$$\frac{\# K_2(\hat{\mathbb{Z}})}{\# K_3(\hat{\mathbb{Z}})} = \frac{\# K_2(\mathbb{Q}_\infty) \cdot \# K_2(\mathbb{Z})}{\# K_3(\mathbb{Z})} = \frac{2 \cdot 2}{48} = \frac{1}{12}$$

which is OK. In general the prediction is

$$\# K_{4p-2}(\mathbb{Q}_\infty) = \frac{2^p}{1 \cdot 3 \cdots (2p-1)}$$

which makes the whole business look like non-sense.

April 1, 1983

Arakelov – Faltings intersection theory on an arithmetic surface. The basic idea behind this is that a curve over a number field should be viewed as the generic fibre of a 2-diml scheme over the ring of integers. Hence one has a surface over a curve situation. Now one wants to construct an analogue of the intersection theory of curves on this surface.

So the first thing to discuss is the theory of curves on an ordinary surfaces. The topics are: the intersection number $D_1 \cdot D_2$ which is a quadratic function on $\mathrm{Pic}(X)$, the RR thm, the Hodge signature theorem.

RR thm says

$$\chi(X, E) = \int_X ch(E)\, Todd(X)$$

$$Todd(x) = \frac{x}{1-e^{-x}} = 1 + \tfrac{1}{2}x + \tfrac{1}{12}x^2 + 0x^3 + \cdots$$

$$Todd(E) = \prod Todd(x_i) = \left(1 + \tfrac{1}{2}x_1 + \tfrac{1}{12}x_1^2 + \cdot\right)\left(1 + \tfrac{1}{2}x_2 + \tfrac{1}{12}x_2^2 + \cdots\right)$$

$$= 1 + \tfrac{1}{2}(x_1 + x_2) + \left(\tfrac{1}{12}x_1^2 + \tfrac{1}{4}x_1x_2 + \tfrac{1}{12}x_2^2\right) + \cdots$$

$$\boxed{Todd = 1 + \tfrac{1}{2}c_1 + \tfrac{1}{12}(c_1^2 + c_2) + \cdots\cdots}$$

Hence

$$\chi(E) = \int\left\{\tfrac{1}{12}\left(c_1(X)^2 + c_2(X)\right)\cdot \mathrm{rank}(E) + \tfrac{1}{2}c_1(X)\cdot c_1(E) + \tfrac{1}{2}ch_2(E)\right\}$$

so for a line bundle

$$\chi(L) = \chi(\mathcal{O}_X) + \tfrac{1}{2}c_1(L)\cdot(c_1(L) + c_1(X))$$

$$\boxed{\chi(D) = \chi(\mathcal{O}_X) + \tfrac{1}{2}D\cdot(D - K)}$$

$\chi(D) = \chi(\mathcal{O}(D))$
$c_1\mathcal{O}(D) = D$

In Mumford's book one starts with $\chi(L)$ and its properties relative to exact sequences. Then one can obtain the intersection product by polarization:

$$D_1 \cdot D_2 \text{ [redacted] } = \chi(D_1+D_2) - \chi(D_1) - \chi(D_2) + \chi(0)$$

If $D>0$, then realize D by a curve and compare $D \cdot c_1(L)$ with $\deg(L|D)$.

$$0 \longrightarrow \mathcal{O}(D^{-1}) \longrightarrow \mathcal{O} \longrightarrow \mathcal{O}_D \longrightarrow 0$$

$$\chi(L \otimes \mathcal{O}(D^{-1})) - \chi(L) + \chi(L|D) = 0$$

$$\chi(\mathcal{O}(D^{-1})) - \chi(\mathcal{O}) + \chi(\mathcal{O}_D) = 0$$

$$c_1(L)\cdot(-D) = \chi(\mathcal{O}_D) - \chi(L|D) = -\deg(L|D).$$

$$\Longrightarrow \quad c_1(L)\cdot D = \deg(L|D).$$

Hodge signature theorem: Assume $D^2>0$. Then

$$\chi(nD) \longrightarrow +\infty \qquad \text{as } n \longrightarrow \infty$$

$$\| $$

$$h^0(nD) - h^1(nD) + h^2(nD) \leq h^0(nD) + h^0(K-nD)$$

so for arbitrarily large n $h^0(nD)>0$, or for arb. large n, $h^0(K-nD)>0$. In the former case

nD contains a pos. divisor $\Longrightarrow$ $nD\cdot H>0$

$K-nD$ ———————— $\Longrightarrow$ $(K-nD)\cdot H>0$

where H is the hyperplane. Thus

$$D^2>0 \Longrightarrow D\cdot H \neq 0,$$

[redacted] which is the signature theorem.

Let's now look at the arithmetic situation. In this case we have a curve over a number field F which we thicken out to be a 2-diml scheme X over $\mathrm{Spec}(A)$. In addition we "complete" $Sp(A)$ to a arithmetic curve Y by "adding" the points at ∞.

I know what vector bundles on Y look like. They are f.t. projective A-modules together with positive definite forms at the infinite completion. There is a RR thm. for Y given by the transformation formula for Θ functions.

To be specific take $A = \mathbb{Z}$, so that a vector bundle over Y is a free f.t. abelian gp. M together with $q: M \longrightarrow \mathbb{R}$ a positive definite quadratic fn. Then we can form

$$\Theta(M,q) = \sum_{m\in M} e^{-\pi q(m)}$$

which is the analogue of $q^{h^0(E)} = \text{card } H^0(E)$. To be specific take $M = \mathbb{Z}$ with $q(n) = tn^2$. Then

$$\Theta(t) = \sum_{n\in\mathbb{Z}} e^{-\pi t n^2}$$

satisfies the functional equation

$$\Theta(t) = \frac{1}{t}\Theta\left(\frac{1}{t}\right)$$

which one should interpret as

$$q^{h^0(E) - h^0(E^*\otimes K)} = q^{\deg E + (\mathrm{rg}\, E)(1-g)}$$

Thus

$$q^{\deg E} = \frac{1}{t} = \text{covolume of } M \text{ in } M\otimes\mathbb{R} \text{ with respect to } q.$$

Now I know that with respect to the obvious notation of exact sequence one has

$$K_0(\mathbb{Z}\cup\infty) = \mathbb{Z}\oplus\mathbb{R}.$$

Question: Is it possible to use Connes ideas in roughly the following way? Invent a C^*-algebra $\mathcal{A}$ or something similar so that the kind of vector bundles (M, q) just described actually are described by idempotents in $\mathcal{A}$. ~~[illegible]~~

In addition I would like the first Chern class of (M,q) which is this covolume to be given by a curvature in the Connes homology of $\mathcal{A}$.

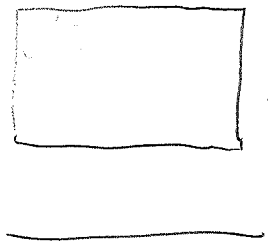
Since $\Theta(t) > 1$ it would be possible that $\log \Theta(t)$ could be interpreted as the dimension of the space of holomorphic sections in some sense.

Ideas: 1) To understand the K-theory of $\mathbb{Z} \cup \infty$ well, it will be necessary to find some sort of gadget like Fredholm operators.

2) One might get some idea of where to look by considering what sort of index f, E you get from a bundle over X = an arithmetic surface over Y.

3) Are there p-adic versions of Baker-Akhieser? Probably not in general, but one might carry out the parametrization of an elliptic curve with j not integral à la Tate. Also Barry described in the Arbeitstagung a version of p-adic τ-function related to yours. This suggests one might be able to make p-adic Green's functions and their regularizations.

This raises the problem of the relation of heights and the arithmetic intersection theory which you never sorted out.



$$\begin{array}{c} X \\ \downarrow f \\ Y \end{array}$$

Suppose this map is smooth proper curve over Y and that Y is non-singular projective.

$$0 \to T_{X/Y} \to T_X \to f^* T_Y \to 0$$

RR thm. for f and a line bundle L over X is

$$ch(f_! L) = f_* \{ ch(L) \, Todd(T_{X/Y}) \}$$

$$c_0(f_! L) = f_* \{ c_1(L) + \tfrac{1}{2} c_1(T_{X/Y}) \} \qquad \text{is an integer}$$

$$c_1(f_! L) = f_* \{ \tfrac{1}{2} c_1(L)^2 + \tfrac{1}{2} c_1(L) c_1(T_{X/Y}) + \tfrac{1}{12} c_1(T_{X/Y})^2 \}$$

RR thm. for Y gives

$$\chi(X, L) = \int_Y c_0(f_! L) \tfrac{1}{2} c_1(T_Y) + c_1(f_! L)$$

$$= \int_X \tfrac{1}{2} c_1(L)^2 + \tfrac{1}{2} c_1(L) c_1(T_{X/Y}) + \tfrac{1}{12} c_1(T_{X/Y})^2$$

$$+ \left(c_1(L) + \tfrac{1}{2} c_1(T_{X/Y}) \right) \tfrac{1}{2} f^* c_1(T_Y)$$

which is just RR for the surface.

Let M be a compact Riemann surface with metric. Given a holomorphic v.b. E over M it has a determinant line canonically associated, and ~~to~~ a metric on E is associated an analytic-torsion metric on the determinant line.

Problem 1: Given an exact sequence of holom. vector bundles on M

$$0 \longrightarrow E' \longrightarrow E \longrightarrow E'' \longrightarrow 0$$

one has a canonical isom.

$$\mathcal{L}_E = \mathcal{L}_{E'} \otimes \mathcal{L}_{E''}$$

If E has a metric and one takes the induced metrics on E' and E'', then one can ask whether this can. isom. respects the metrics.

Here is a proof that this is so. NO One considers two C^∞-bundles with metric E', E'' and then considers the family of all $\bar\partial$-operators on $E = E' \oplus E''$ which induce a $\bar\partial$-operator on E' as a subbundle (hence E'' as a quotient bundle. Maybe simpler would be to fix $\bar\partial$ operators in E' and E'' and look at all $\bar\partial$-ops. on E compatible with the maps $E' \rightarrowtail E \twoheadrightarrow E''$. This is essentially a complex vector space isom. to $\mathrm{Hom}(E'', E' \otimes T^{0,1})$. Now I use the curvature thm. for the curvature of $\mathcal{L}_E$. Over this family I have two holomorphic line bundles represented by $\mathcal{L}_E$ and $\mathcal{L}_{E'} \otimes \mathcal{L}_{E''}$, and a canonical holom. isomorphism between them. ??

This isn't going to work, because clearly one can have a holomorphic isomorphism between trivial bundles. Consequently it **is not** enough to work with the curvature - one must work with the connections.

Let us suppose that E', E'' have trivial cohomology whence from the coh. exact sequence the same is true for E. Then we know the canonical isom.

$$\mathcal{L}_E \cong \mathcal{L}_{E'} \otimes \mathcal{L}_{E''}$$

preserves the canonical sections, so we have to check that

$$|s_E|^2 = |s_{E'}^{\text{[redacted]}}|^2 |s_{E''}^{\text{[redacted]}}|^2$$

which is a statement about ζ-function determinants.

Summary: An exact sequence of holomorphic v.b. $0 \to E_I^{\text{[redacted]}} \to E \to E_{II}^{\text{[redacted]}} \to 0$ furnishes us with a canonical isom

(1) $$\mathcal{L}_E \cong \mathcal{L}_{E_I}^{\text{[redacted]}} \otimes \mathcal{L}_{E_{II}}$$

under which one has

$$s_E \longleftrightarrow s_{E_I} \otimes s_{E_{II}}$$

when [redacted] canonical sections are defined. When E has a metric and we equip E_I, E_{II} with the induced metric then both sides of (1) have metrics, so the problem arises as to whether (1) is compatible with metrics.

Last year (p. 745) you considered the family of all $\bar\partial$-operators on E compatible with the maps

$$E_I \rightarrowtail E \twoheadrightarrow E_{II}.$$

The parameter space of this family $\mathcal{V}$ can be identified with the complex vector space of ~~vector~~ bundle maps $E \to E \otimes T^{0,1}$ carrying $E_I \to E_I \otimes T^{0,1}$. (Choose a splitting of the sequence, say the one given by the metric, and choose also $\bar\partial$ operators in E_I and E_{II}.)

Over $\mathcal{V}$ I have three holomorphic line bundles with fibres

$$\mathcal{L}_E \ , \quad \mathcal{L}_{E_I} , \text{[redacted]} \ \mathcal{L}_{E_{II}}$$

and metrics given by analytic torsion. I know how to compute the curvature forms of these line bundles say using the GRR formula. [redacted] If I want the [redacted] Chern form of the bundle with fibre $\mathcal{L}_E$, then I [redacted] have

$$c_1(\{\mathcal{L}_E\}) = \int_M ch_2(\tilde{E})$$

where $\tilde{E}$ denotes the tautological bundle over $\mathcal{V} \times M$.

One[redacted] sees that to prove $\{\mathcal{L}_E\}$, $\{\mathcal{L}_{E_I} \otimes \mathcal{L}_{E_{II}}\}$ have the same curvature, [redacted] it would suffice to have

$$ch_2(\tilde{E}[redacted]) = ch_2(\tilde{E}_I) + ch_2(\tilde{E}_{II}).$$

But from what I know about Bott-Chern, this is apt to be false. It is therefore necessary to go thru things very carefully.

Let's consider the following situation. Consider two bundles E_I[redacted], E_{II}[redacted] with holomorphic structures D_I, D_{II} and hermitian metrics. [redacted] On $E = E_I \oplus E_{II}$ we consider the family $\mathcal{W}$ of all holom. ~~structures~~ of the form

$$D_W = \begin{pmatrix} D_I & W \\ 0 & D_{II} \end{pmatrix} \qquad W \in \Gamma\{Hom(E_{II}, E_I \otimes T^{0,1})\}$$

and we equip E with the direct sum inner product. [redacted] Over the space $\mathcal{W}$ we have the determinant line bundle of the family $\{D_W\}$ which is holomorphic and has the analytic torsion metric. [redacted] I also know that the curvature of this line bundle is the Kahler 2-form on $\mathcal{W}$ associated to the inner product

$$\|W\|^2 = \frac{i}{2\pi}\int tr(W^* W).$$

In particular this curvature is not zero, so we can't have a canonical isomorphism preserving metrics:

$$\mathcal{L}_E = \mathcal{L}_{E_I} \otimes \mathcal{L}_{E_{II}}$$

as hoped for.

Put another way I can consider an exact sequence of C^∞ bundles with metric

$$0 \longrightarrow E_I \longrightarrow E \longrightarrow E_{II} \longrightarrow 0$$

and consider all holom. structures on it. Suppose the index is zero and that we choose a basepoint D_0 in the space of these holom. structures corresponding to a structure compatible with the isomorphism $E = E_I \oplus E_{II}$ given by the metric. Then the analytic determinant should satisfy

$$\det(D; D_0) = \det(D_I; D_{0I}) \cdot \det(D_{II}; D_{0II}).$$

On the other hand it is not true that

$$\|D - D_0\|^2 = \|D_I - D_{0I}\|^2 + \|D_{II} - D_{0II}\|^2$$

because we are missing the term $\|W\|^2$.

<u>Question:</u> What kind of RR theorem can we possibly have for <u>arithmetic surfaces?</u>

Consider now $X = Y \times M$ where Y, M are two Riemann surfaces with metric, and suppose we have an exact sequence of holomorphic bundles

$$0 \longrightarrow E' \longrightarrow E \longrightarrow E'' \longrightarrow 0$$

and X. Put a metric on E and consider the induced metrics on E' and E''. The difference between the given holomorphic structure on E and the one obtained from the ones on E' and E'' via the isom. $E = E' \oplus E''$ obtained from the metric is a map of vector bundles

$$W : E'' \longrightarrow E' \otimes T_X^{0,1}.$$

I can take $\mathrm{tr}(W^*W)$ which is a section of $T_X^{1,1}$ and integrate over the fibres to get a function on Y. This function seems to give the **discrepancy** of the canon. isom.

$$\mathcal{L}_E = \mathcal{L}_{E'} \otimes \mathcal{L}_{E''}$$

from being compatible with the analytical torsion metrics. Also

$$c_1(\mathcal{L}_E) = f_*(ch_2 E)$$

and similarly for E', E'' and I think that

$$c_1(\mathcal{L}_E) - c_1(\mathcal{L}_{E'} \otimes \mathcal{L}_{E''}) = d''d' \left(f_* \, \mathrm{tr}(W^*W)\right)$$

up to some constants.

So what is happening is that $E \mapsto c_1(\mathcal{L}_E) = f_*(ch_2 E)$ is not additive for exact sequences because $ch_2 E$ isn't.

Let's interpret things as follows. We are working with metrics on bundles, or better, I should say bundles with metrics, because we don't have the correct way to think about virtual bundles. A bundle with metric gives ~~us~~ a virtual bundle, but we have just seen that the natural notion of exact sequence of bundles with metric is not going to satisfy an addition relation in the K-theory, because there is going to be the above W-correction.

Problem 2: ~~Now~~ I should look at the problem of the compatibility of the metrics with the canonical isom

$$\mathcal{L}(E \otimes \mathcal{O}(-P)) \otimes \lambda(E \otimes k(P)) = \mathcal{L}(E)$$

associated to the exact sequence

$$0 \longrightarrow E \otimes \mathcal{O}(-P) \longrightarrow E \longrightarrow E \otimes k(P) \longrightarrow 0$$

where P is a point on the Riemann surface. We suppose $\mathcal{O}(-P)$ equipped with a metric

April 4, 1983

Technical question: Let D be a $\bar\partial$-operator on the Riemann surface M and suppose metrics given so that we have the Laplacean D^*D and heat operator e^{-tD^*D}. Consider a first order differential operator $B: \Gamma(E) \to \Gamma(E\otimes T^{0,1})$ and assume D invertible. Then I want to compute the constant term in the asymptotic expansion for

$$\operatorname{Tr}\left(e^{-tD^*D}D^{-1}B\right) \qquad \text{as } t \to 0.$$

I would like to calculate some simple examples in order to understand what to expect.

Let's start with $D = \partial_{\bar z}$ on $\mathbb{C}$ and ignore the "infrared" problems arising from the non-compactness.

$$D = \frac{\partial}{\partial \bar z} = \tfrac12(\partial_x + i\partial_y) \qquad D^* = -\partial_z$$

$$D^*D = -\tfrac14(\partial_x^2 + \partial_y^2). \qquad \text{not the right normalization}$$

so I will take $D^*D = -\frac12(\partial_x^2+\partial_y^2)$. Then

$$\langle z|\,e^{-tD^*D}\,|z'\rangle = \frac{1}{2\pi t}\,e^{-\frac{|z-z'|^2}{2t}}$$

$$\langle z|\,D^{-1}\,|z'\rangle = \frac{1}{\pi(z-z')}$$

$$B = a\partial_z + b\partial_{\bar z} + c \qquad a,b,c \text{ fns. of } z$$

It would appear that I get a well-defined problem if a,b,c are of compact support.

Digression: Suppose we want to compute $\operatorname{Tr}(e^{-tD^*D}D^{-1}B)$ in the case of $M = \mathbb{C}/\Gamma$ where D is a constant coeff. $\bar\partial$-operator. I know that the kernel for e^{-tD^*D} over M is obtained by summing translates of the kernel over $\mathbb{C}$. Thus if $\bar x = x+\Gamma$ for $x\in\mathbb{C}$ we have

$$K(\bar x,\bar y) = \sum_{\gamma\in\Gamma} K_0(x, y+\gamma)$$

hence
$$\operatorname{Tr}(KB) = \int_M d^2\bar{x} \underbrace{\int_M d^2\bar{y}\, K(\bar{x},\bar{y})\, B(\bar{y},\bar{x})}$$

$$= \int_{\substack{x\in \text{fdl}\\ \text{domain}}} d^2x \int_{\substack{y\in \text{fdl}\\ \text{domain}}} d^2y \sum_\gamma K_0(x, y+\gamma)\, B(\bar{y},\bar{x})$$

$$= \int_{\substack{x\in \text{fdl}\\ \text{domain}}} d^2x \int_{y\in\mathbb{C}} d^2y\, K_0(x,y)\, B(y,x)$$

where we **define** $B(y, x) = B(\bar{y},\bar{x})$.

Let's see if I can make this more transparent. ~~Start~~ with $K_0(x,y)$ on $\mathbb{C}\times\mathbb{C}$ invariant under the Γ-action:

$$K_0(x+\gamma, y+\gamma) = K_0(x,y),$$

e.g. K_0 a function of $x-y$. Then given f on $\mathbb{C}$ invariant under Γ, the same is true for

$$(K_0 * f)(x) = \int K_0(x,y)\, f(y)\, d^2y.$$

(The only problem is whether $K_0(x,?)$ decays enough so that the integral converges)

The kernel corresponding ~~to~~ this operator is found as follows

$$\int K_0(x,y)\, f(y)\, d^2y = \sum_\gamma \int_\Delta K_0(x,y+\gamma)\, f(y)\, d^2y$$

or
$$(K_0 * f)(\bar{x}) = \int_\Delta \sum_\gamma K_0(x,y+\gamma)\, f(y)\, d^2y$$

$$= \int_{\mathbb{C}/\Gamma} K(\bar{x},\bar{y})\, f(\bar{y})\, d^2\bar{y}$$

where
$$K(\bar{x},\bar{y}) = \sum_\gamma K_0(x, y+\gamma).$$

Finally the trace of this operator on functions on $\mathbb{C}/\Gamma$ is

$$\int_{\mathbb{C}/\Gamma} K(\bar{x},\bar{x})\, d^2\bar{x} = \int_{\Delta} d^2x \sum_{\gamma} K_0(x,x+\gamma)$$

which is not too useful. However if I also have $B(\bar{x},\bar{y})$, then

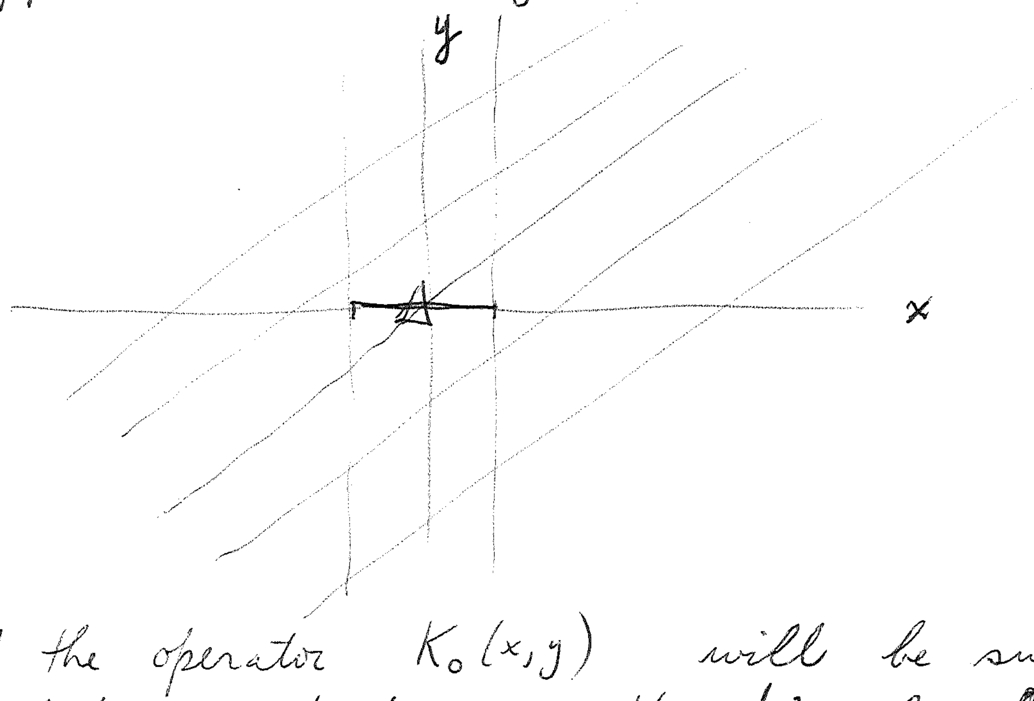
$$\mathrm{Tr}(KB) = \int d^2\bar{x} \underbrace{\int d^2\bar{y}\;\, K(\bar{x},\bar{y})\, B(\bar{y},\bar{x})}_{\displaystyle \int d^2y\;\, K_0(x,y)\, B(y,x)}$$

In other words, if I want the trace $\mathrm{Tr}(KB)$ where B is an operator on $\mathbb{C}/\Gamma$, and K is the operator obtained from K_0 on $\mathbb{C}$, then I compute

$$\int d^2y\;\, K_0(x,y)\, B(y,x)$$

and this will be periodic and I can integrate it over $\mathbb{C}/\Gamma$ to get the trace.

When B is a differential operator $B(y,x)$ will be supported on lines $y = x+\gamma \quad x \in \mathbb{R}$.



And the operator $K_0(x,y)$ will be supported all over, but concentrated near the diagonal. The trace is kernel $K_0(x,y)\,B(y,x)$ integrated over $x \in \Delta$, $y \in \mathbb{C}$ and the important part is the piece where

$x = y \in \Delta$.

So now I want to take up the important case where $B = \partial_z$.

$$\mathrm{Tr}\left(e^{-tD^*D} D^{-1} B\right) = \mathrm{Tr}\big(B \underbrace{e^{-tD^*D}}_{K} D^{-1}\big)$$

$$= \int_\Delta d^2 z \int_{\mathbb{C}} d^2 z' \, \partial_z \left(\frac{e^{-\frac{|z-z'|^2}{2t}}}{2\pi t} \right) \frac{1}{\pi(z'-z)} \quad + \quad \text{terms exp. small in } t.$$

So the key thing to be evaluated is

$$\int_{\mathbb{C}} d^2 w \quad \frac{e^{-\frac{|z-w|^2}{2t}}}{2\pi t} \left(-\frac{\overline{z-w}}{2t} \right) \frac{1}{\pi(w-z)}$$

$$= \int_{\mathbb{C}} d^2 w \quad \frac{e^{-\frac{|w|^2}{2t}}}{4\pi t^2} \quad \frac{\bar w}{w} \quad = \quad 0$$

by reasons of symmetry.

Therefore we have made a mistake, the mistake being to assume that the kernel of $e^{-tD^*D} D^{-1}$ is ~~[illegible]~~ concentrated near the diagonal.

Consider the problem of regularizing $D^{-1}B$ where D is an invertible $\bar\partial$-operator and B is a first order operator. I wish to regularize using the behavior of

$$\mathrm{Tr}\left(e^{-tD^*D} D^{-1} B\right)$$

as $t \to 0$.

Let's first assume that the metric on the Riemann surface is flat on the support of B. Actually I can break B down using a partition of unity, so in some sense the calculation is to be done locally. More precisely

$$\mathrm{Tr}\left(e^{-tD^*D} D^{-1} B\right) = \iint_{z\; z'} \mathrm{tr} \, \langle z | e^{-tD^*D} | z' \rangle \langle z' | D^{-1} B | z \rangle$$

and so it is enough to understand

$$\mathrm{tr}\,\langle z|\,e^{-tD^*D}D^{-1}B\,|z\rangle = \int_{z'} \mathrm{tr}\left(\langle z\,|e^{-tD^*D}|z'\rangle\langle z'|D^{-1}B|z\rangle\right)$$

for z fixed. Put $z=0$ and change z' to z. Use the asymptotics

$$\langle 0\,|\,e^{-tD^*D}|z\rangle \doteq \frac{e^{-\frac{h(z)^2}{2t}}}{2\pi t}\left\{F_0(0,z) + F_1(0,z)\,t + \dots\right\}$$

where $F_0(z,z')$ denotes the parallel translation radially between z, z' with respect to the unitary connection extending D.

I need next some understanding of $\langle z|D^{-1}B|0\rangle$. If I suppose that $D = \partial_{\bar z}$, then I know that

$$\langle z|\,D^{-1}|z'\rangle = \frac{1}{\pi(z-z')} + \text{smooth fn. of } (z,z').$$

A typical B is

$$a\,\partial_z + b\,\partial_{\bar z} + c$$

where a, b, c are functions of z which are smooth.

Now

$$(D^{-1}Bf)(z) = \int \left[\frac{1}{\pi(z-z')} + h(z,z')\right]\left[a(z')\partial_{z'} + b(z')\partial_{\bar z'} + c(z')\right] f(z')\,dz'$$

and we can do integration by parts to move the derivatives off f onto the kernel $\langle z|D^{-1}|z'\rangle$. So we get terms

$$-\partial_{z'}\frac{a(z')}{\pi(z-z')} \qquad -\partial_{\bar z'}\frac{b(z')}{\pi(z-z')}$$

which gives for $\langle z|D^{-1}B|z'\rangle$ the following singular terms

$$-\frac{a(z')}{\pi(z-z')^2} - \delta(z-z')\,b(z') + \frac{\tilde c(z')}{z-z'}$$

and the rest is a completely smooth.

So we must next understand

$$\int \langle 0|\,e^{-tD^*D}|z\rangle\left[\frac{-a(0)}{\pi z^2} + \frac{\tilde c(0)}{z} - \delta(z)\,b(0) + h(z,0)\right]$$

as $t \to 0$.