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Letter to Iz 253

November 21, 1983

Dear Iz,

I have a few comments concerning your draft, especially with the computation of the transgression class in $H^*(\mathcal{G})$ obtained from $\int_M ch(\mathcal{F})$ in $H^*(\mathcal{A}/\mathcal{G})$.

First of all I had some difficulty with the process $\int_P$ (Chern-Simons form)$\wedge vol_B(G)$ which goes as follows. Suppose to fix the ideas that P is a principal G-bundle over M and that $A, B \in \Omega^1(P)\otimes\mathfrak{g}$ are two connection forms. If $p \in S(\mathfrak{g}^*)^G$ is an invariant polynomial one has the formula

$$p(F_A) - p(F_B) = du(A,B)$$

1)
$$u(A,B) = \int_0^1 dt \;\; p'\left((1-t)F_B + tF_A + (t^2-t)(A-B)^2 ; A-B\right)$$

expressing the fact that the characteristic class $p(P)$ in $H^*(M)$ is independent of the choice of connection. This formula is the one I think has to be used in computing the (inverse) transgression map $H^*(\mathcal{A}/\mathcal{G}) \to H^*(\mathcal{G})$.

Now the operation of multiplying by vol_B followed by integration over the fibre is the same as the composition

2)
$$\Omega(P) \underset{\substack{\text{connection}\\ B}}{\xleftarrow{\;\sim\;}} \Omega(P)_{horiz}\otimes\Lambda\mathfrak{g}^* \xrightarrow{1\otimes\varepsilon} \Omega(P)_{horiz} \xrightarrow{\natural} \Omega(P)_{basic}$$

where ε is the augmentation of $\Lambda\mathfrak{g}^*$ and $\natural$ is averaging over the compact group G. The Chern-Simons form for the connection A is

$$\int_0^1 dt \quad p'(tF_A + (t^2-t)A^2 ; A).$$

Applying the map $(1\otimes\varepsilon)\frac{\sim}{B}$ from $\Omega(P)$ to $\Omega(P)_{\text{horiz}}$, which is a ring homomorphism such that

$$F_A \longmapsto F_A \quad , \quad A \longmapsto A-B \quad ,$$

to the Chern-Simons form, we get

$$3) \qquad \int_0^1 dt \quad p'(tF_A + (t^2-t)(A-B)^2 ; A-B).$$

This is already G-invariant, so is preserved by $\not{q}$. Thus your process 2) applied to the Chern-Simons class yields the good form $u(A,B)$ of 1) if $F_B = 0$, but in general I expect F_B to occur in the formulas for t_{2j-1} on p.6 of your draft.

Next I should explain why 1) is used in computing the transgression. We have a square of principal G-bundles

$$\begin{tikzcd} \mathcal{G}\times P \arrow[r, phantom, "\subset"] \arrow[d, "m"] & \mathcal{a}\times P \arrow[d] \\ P \arrow[r, phantom, "\subset"] & (\mathcal{a}\times P)/\mathcal{G} = \mathcal{Q} \end{tikzcd} \quad \text{over} \quad \begin{tikzcd} \mathcal{G}\times M \arrow[r, phantom, "\subset"] \arrow[d, "pr_2"] & \mathcal{a}\times M \arrow[d] \\ M \arrow[r, phantom, "\subset"] & (\mathcal{a}/\mathcal{G})\times M \end{tikzcd}$$

Let $p \in S^k(g^*)^G$; we are interested in the transgressing class $\int_M p(\mathcal{Q}) \in H^{2k-d}(\mathcal{a}/\mathcal{G})$, where $d = \dim M$, so $2k > d$. Then $p(\mathcal{Q})$ lifted to $H^{2k}(\mathcal{a}\times M)$ is zero; also $p(\mathcal{Q})$ restricted to M is zero, so a secondary class t in $H^{2k-1}(\mathcal{G}\times M)$ is defined and $\int_M t$ is the transgression of $\int_M p(\mathcal{Q})$.

In order to compute on the form level, let $\hat{A} \in \Omega^{0,1}(\mathcal{A}\times P, \mathfrak{g})$ be the canonical connection form. It is $\mathcal{G}$-invariant but must be modified to $\bar{A} = \hat{A} + \Theta$, where Θ is essentially a connection in $\mathcal{A} \to \mathcal{A}/\mathcal{G}$, before it descends to a connection form on $\mathcal{Q}$. The vanishing of $p(\mathcal{Q})$ lifted to $\mathcal{A}\times M$ is expressed by

$$4) \qquad p(F_{\bar{A}}) - \underbrace{p(F_B)}_{=0} = du(\bar{A}, B)$$

where B is a connection on $\mathcal{A}\times P$ obtained by pulling back a fixed connection on P. When this equation is restricted to $\mathcal{G}\times M$, $p(F_{\bar{A}})$ goes to zero, because $\bar{A}$ becomes $m^*(A_0)$, where $m: \mathcal{G}\times P \to P$ is the multiplication and $A_0 \in \mathcal{A}$ is the point defining the map $\mathcal{G} \to \mathcal{A}$. So $t \in H^{2k-1}(\mathcal{G}\times M)$ is represented by the form $u(m^* A_0, B)$.

Finally one computes this form and finds that over $\phi \times P$, it is (when P is trivial and $B =$ zero connection)

$$\int_0^1 dt \; p'\left(tF_A + (t^2-t)(A+\Theta)^2 ; A+\Theta\right)$$

where $A = \phi \cdot A_0$ and Θ is the Maurer-Cartan form of $\mathcal{G}$. When $p(F) = tr(F^{n+1})$ and we take the component of degree 1 in $\mathcal{G}$, this gives the linear function of $v \in Lie(\mathcal{G})$ of Zumino Wu + Zee's preprint

$$-\omega^1_{2n}(v, A) = (n+1)\int_0^1 dt \left\{ str(v, F_t^n) - (t^2-t)\, n\, st(A, [v, A], F_t^{n-1}) \right\}.$$

I mention this, because the formula on p. 6 of your draft apparently, doesn't contain a $[v, A]$-term.

My final comment is that when the degree k of the polynomial p is $>$ dim M, then instead of choosing B to compute the transgression we can use

$$p(F_{\bar{A}}) - \underbrace{p(F_{\hat{A}})}_{=0} = du(\bar{A}, \hat{A})$$

since $F_{\hat{A}}$ has only components of degrees $(1,1)$ and $(0,2)$. This shows that the transgression class $\int_M t \in H^{2k-d-1}(\mathcal{G})$ is represented by a right-invariant form, because both $\hat{A}$ and $\bar{A}$ are $\mathcal{G}$-invariant. Similarly one shows that if $\gamma \in H_i(M)$ and $k > i$, then the class $\int_\gamma t \in H^{2k-i-1}(\mathcal{G})$ is represented by a right-invariant form on $\mathcal{G}$.

Best regards,

Dan

November 21, 1983 (continued)

Let's see if I can get right and left straight in the gauge transformation situation. First of all if P comes from a vector bundle, or better if we think of connections in P in terms of the associated vector bundles, then we have

$$\begin{array}{ccc} \Omega(M,E) & \xrightarrow{\sim} & (\Omega(P)\otimes V)_{basic} \\ D & & d+A \end{array} \qquad E = P\times^G V.$$

If g is a gauge transformation then we can think of it either as an autom. of E or of P, the relation being

$$\begin{tikzcd} P\times V \arrow[r, "g\times id"] \arrow[d] & P\times V \arrow[d] \\ E \arrow[r, "g"] & E \end{tikzcd}$$

It follows that the action $s \mapsto gs$ on sections of E corresponds to the map $f \mapsto f\circ g^{-1}$ for $f \in (\Omega^0(P)\otimes V)^{\mathcal{U}}$. (Details: If $m\in M$, let $p\in P_m$. Then $s(m) = (p,f(p))G$ and $g(s(m)) = (gp, f(p))G$. But $gp = pu$ for some! $u\in\mathcal{U}$ and then

$$g(s(m)) \equiv (gp, f(p)) \equiv (pu, f(p)) \equiv (p, uf(p))$$
$$\equiv (p, f(pu^{-1})) = (p, f(g^{-1}p))$$

so we see gs corresponds to $f\circ g^{-1}$).

Thus g on $\Omega(M,E)$ corresponds to $(g^*)^{-1}$ on $\Omega(P)\otimes V$.

so $$g^{-1}\cdot D\cdot g \longleftrightarrow g^*\cdot(d+A)\cdot(g^*)^{-1} = d + g^*A.$$

Thus we get a right action of $\mathcal{G}$ on $\mathcal{A}$ in keeping b

with its intended role as a principal bundle. The infinitesimal version of the action is

$$D \longmapsto [D, v] \quad \longleftrightarrow \quad dv + [A, v]$$

Notice that this action of $\tilde{\mathfrak{g}}$ on the space of connections is an anti- (or right) action, since we have v acting on D is $[D, v] = -[v, D]$.

Next I consider $\mathcal{P} \times P$ where $\mathcal{P}$ is a principal $\mathcal{G}$-bundle. On this U-bundle $\mathcal{P} \times P$ over $\mathcal{P} \times M$ I have a tautological connection $\hat{A}$ associated to a $\mathcal{G}$-map $\mathcal{P} \to \mathcal{A}$ which I suppose given. In effect

$$\Omega^{0,1}(\mathcal{P} \times P, \mathfrak{u}) = \Omega^0(\mathcal{P}, \Omega^1(P, \mathfrak{u}))$$

and $\mathcal{A}$ is a subset of $\Omega^1(P, \mathfrak{u})$. Now $\mathcal{G}$ acts on $\mathcal{P} \times P$ by $(\xi, p)g = (\xi g, g^{-1}p)$ so associated to $v \in \tilde{\mathfrak{g}}$ we get a vector field v on $\mathcal{P} \times P$. This has a horizontal and vertical component, the $\mathcal{P}$-part being the basic vector field on a principal $\mathcal{G}$-bundle, the P part being minus the vector field v on P.

I should have recorded the following formulas for the vector field v on P:

$$\iota_v A = v$$

No, this is the formula that involves an identification equivalent to the Higgs field. On the left is an element of $\Omega^0(P, \mathfrak{u})^U$ and on the right is an element of $\tilde{\mathfrak{g}}$. The isomorphism

$$\tilde{\mathfrak{g}} \xrightarrow{\sim} \Omega^0(P, \mathfrak{u})^U$$

should be given a name call it $v \mapsto \psi_v$ for the moment.

Now when we come to the action of $\mathcal{G}$ on $P \times P$ we have the formula

$$\iota_v \hat{A} = -\psi_v$$

The left side is in $\Omega^0(P \times P, \mathfrak{u})^{U}$ and the right side is in $\Omega^0(P, \mathfrak{u})^{U}$ embedding by pr_2^*. So now it is clear that

$$\tilde{\mathfrak{g}} \xrightarrow{\underset{\sim}{\psi}} \Omega^0(P, \mathfrak{u})^{U} \xrightarrow{pr_2^*} \Omega^0(P \times P, \mathfrak{u})^{U}$$

is the Higgs field belonging to the connection $\hat{A}$.

Next we take $\Theta \in \Omega^1(P, \tilde{\mathfrak{g}})$ a connection form and consider the modified connection

$$\bar{A} = \hat{A} - \Theta \iota \hat{A} = \hat{A} + \psi\Theta \quad \in \Omega^1(P \times P, \mathfrak{u})$$

which descends to $\tilde{P} = P \times^{\mathcal{G}} P$ over $Y \times M$.

But now I know that when I restrict to a fibre of P over Y, the connection form becomes the Maurer-Cartan form of $\mathcal{G}$. It is trivially left invariant and has a slightly subtler right invariance. On the other hand $\hat{A}$, $\bar{A}$ which come from $P \times P$ are only right invariant forms on $\mathcal{G} \times P$, i.e. for the action $(g, p)g_1 = (gg_1, g_1^{-1}p)$. So therefore we are not interested in the left invariance of Θ on $\mathcal{G}$ but rather the right-invariance of

$$\Theta \text{ in } \Omega^1(\mathcal{G}, \tilde{\mathfrak{g}}) \stackrel{\psi}{\subset} \Omega^{1,0}(\mathcal{G} \times P, \mathfrak{u})$$

November 26, 1983 (after Oxford trip)

I reviewed formulas for the local index of a family of operators L.

Even case: Here L is of degree 1 on $\mathcal{H}^+ \oplus \mathcal{H}^-$ which are Hilbert bundles; L is self-adjoint. The local index is the form on the base obtained as follows. You consider the differential form on the base

$$e^{-tL^2 + \sqrt{t}[D,L] + F}$$

whose values are endos. of the Hilbert bundle $\mathcal{H}$. Here D and F are the connection and curvature resp. in the bundles $\mathcal{H}$. Next apply $\operatorname{tr}_{\mathcal{H}} \varepsilon$ to get an even differential form on the base. Then take the component of degree $2m$ and multiply it by $\left(\frac{i}{2\pi}\right)^m$.

Odd case: Here L is [redacted] a self-adjoint operator on an ungraded Hilbert space $\mathcal{H}$. One takes the same differential form and applies $\operatorname{tr}_{\mathcal{H}}$ to get a differential form on the base. This time however it is not clear that one gets an odd form on the base. However we ultimately take the coefficient of t^0 in the asymptotic expansion, so that when L is a family of Dirac operators on an odd diml manifold one does obtain an odd form in this way. Finally the component of degree $2m+1$ is multiplied by $-\frac{1}{\sqrt{\pi}}\left(\frac{i}{2\pi}\right)^m$

Let's review the heuristic calculations of these formulas. In the even case, think of L as [redacted] a family of Dirac operators on an even dimensional man.

Then the total Dirac operator is

$$\not{D} = \gamma^\mu \frac{1}{i} D_\mu + \varepsilon L. \qquad \text{(even case)}$$

The index of the total Dirac operator doesn't change if we rescale:

$$\not{D} = \gamma^\mu \frac{h}{i} D_\mu + \varepsilon \sqrt{t}\, L$$

Now I take the index of the total Dirac op

$$\mathrm{Ind}(\not{D}) = \mathrm{Tr}\left(\varepsilon_{tot}\, e^{-\not{D}^2}\right)$$

and let $h \to 0$, and I calculate that the trace can be computed as

$$\int_{\text{base}} \left(\frac{i}{2\pi}\right)^m \mathrm{Tr}_{\mathcal{H}}\left(\varepsilon_{\mathcal{H}}\, e^{-tL^2 + \sqrt{t}[D,L] + F}\right)$$

where $2m$ is the dimension of the base. So this number should be independent of t, which suggests

<u>Conjecture</u>: The forms

$$\mathrm{Tr}_{\mathcal{H}}\left(\varepsilon\, e^{-tL^2 + \sqrt{t}[D,L] + F}\right) \qquad \text{even case}$$

$$\mathrm{Tr}_{\mathcal{H}}\left(e^{-tL^2 + \sqrt{t}[D,L] + F}\right)_{(ev)} \qquad \text{odd case}$$

are closed and their DR classes are independent of t.

In the above derivations the following formulas were used. If $n = 2$ recall that one has the Pauli choices

$$\gamma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \gamma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \qquad \varepsilon = \gamma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

satisfying $\gamma^1\gamma^2\gamma^3 = i$. Thus in two dimensions $\varepsilon\gamma^1\gamma^2 = i$ and so taking tensor powers we get in n dimensions, n even

$$\boxed{\mathrm{tr}\left(\varepsilon\gamma^{\mu_1}\cdots\gamma^{\mu_n}\right) = (i)^{\frac{n}{2}}\,\varepsilon^{\mu_1\cdots\mu_n} \qquad n \text{ even}}$$

Next consider the odd case, say the base is of dimension $n-1$, where n is even. Then the [redacted] C_n module S_n splits into two C_{n-1} modules [redacted] which are the eigenspaces of

$$\varepsilon\gamma^n = (-i)^{n/2}\gamma^1\cdots\gamma^{n-1}$$

(Note: [redacted] $\varepsilon\gamma^1\gamma^2 = i$ in 2 dim $\Longrightarrow$ $\varepsilon\gamma^1\cdots\gamma^n = i^{n/2}$ $\Longrightarrow$ $\gamma^1\cdots\gamma^{n-1} = i^{n/2}\varepsilon\gamma^n$.)

The Dirac operator with coefficients in L is

$$D\!\!\!/ = \gamma^\mu\left(\frac{1}{i}D_\mu\right) + \gamma^n L \qquad \text{on } S_n\otimes\mathcal{H}.$$

where $\mu = 1,\dots,n-1$. Notice for $n=2$, [redacted] we get

$$D\!\!\!/ = \begin{pmatrix} 0 & \frac{1}{i}\partial_x - iL \\ \frac{1}{i}\partial_x + iL & 0 \end{pmatrix}$$

so if $L = x$ on $\mathcal{H} = \mathbb{C}$ we get an operator of index -1 (since $\frac{1}{i}\partial_x + ix = \frac{1}{i}(\partial_x - x)$.) According to the recipe above we take

$$e^{-tL^2 + \sqrt{t}\,dL} = e^{-tx^2 + \sqrt{t}\,dx} = e^{-tx^2}\left(1 + \sqrt{t}\,dx\right)$$

and multiply the component of degree 1 by $-\frac{1}{\sqrt{\pi}}$ to get the 1-form

$$-\frac{\sqrt{t}}{\sqrt{\pi}}e^{-tx^2}\,dx$$

whose integral is -1.

Notice that the even degree component of this form is e^{-tx^2} which is only exact at $t=+\infty$. What seems to be most reasonable is that in the odd case the analogue of trace ε is to apply the trace to the components of odd degree and kill the even degree components.

Question: Is it actually possible that one doesn't have to take the asymptotic expansion in t but rather one can take the limit as $t \to 0$? Does the Witten alv. Getzler result give any kind of cancellation in the odd-dimensional case.

New idea: Up to now you have been thinking of the local index for families in terms of a "smooth" (i.e. submersion) map $X \to Y$. Perhaps it is possible to formulate a theorem about an embedding.

Let's suppose then that we have an embedding $i: Y \longrightarrow X$ and try to formulate a theorem like GRR. Atiyah and Singer do exactly something like this to prove the index theorem in K-theory. But I am going to try for a local theorem which expresses an equality of differential forms.

Let's begin with the right hand side of the topological formula

$$ch\{i_!(E)\} = i_*\{T(\nu)\, ch(E)\}$$

The first problem concerns defining the Gysin homom. i_*.

For simplicity assume i is the embedding of the zero section in a vector bundle and let $\pi: X \to Y$ be the projection. Then $i_*(y) = i_* 1 \cdot \pi^*(y)$ where $i_* 1 \in H_{pr/Y}(X)$ is the Thom class.

Bott explained many years back in a course how to construct the Thom class of an oriented vector bundle in DR cohomology. One uses transgression. Let e be the closed form on Y computed using the connection which represents the Euler class of the vector bundle X/Y. Then $\pi^* e = du$ where u is the transgression form on $X - Y$.

This means that we want to work in the mapping cone of $\pi^*: \Omega(Y) \to \Omega(X-Y)$ which consists of pairs (α, β), $\alpha \in \Omega(Y)$, $\beta \in \Omega(X-Y)$ with $\deg(\alpha) = 1 + \deg(\beta)$. The diff'l is

$$d(\alpha,\beta) = (d\alpha, \pi^*\alpha - d\beta)$$

(Check $d^2(\alpha,\beta) = d(d\alpha, \pi^*\alpha - d\beta) = (d^2\alpha, \pi^* d\alpha - d(\pi^*\alpha - d\beta)) = 0$.)

Then the pair (e, u) is a cycle ~~in~~ this mapping cone. Define the Gysin map from $\Omega(Y)$ into this cone by

$$\alpha \cdot (e, u) = (\alpha e, (-1)^{\deg \alpha} \pi^*(\alpha) \cdot u).$$

(The sign is due to the fact that u is really σu where σ has degree $+1$.) Then this is a map of complexes

$$d\{\alpha \cdot (e,u)\} = \left(d(\alpha e), \pi^*(\alpha e) - d[(-1)^{\deg \alpha} \pi^*(\alpha) u] \right)$$

$$= (d\alpha \cdot e, (-1)^{\deg \alpha + 1} \pi^*(d\alpha)\, u)$$

$$= d\alpha \cdot (e, u)$$

Now suppose h is a function on X such that $h = 1$ near Y and $h = 0$ outside a tubular nbd of Y in X. Then consider the following map from the mapping cone to the forms on X with support in the tubular nbd.

$$(\alpha, \beta) \longmapsto h\,\pi^*(\alpha) + dh \cdot \beta$$

One has

$$d\big(h\,\pi^*(\alpha) + dh \cdot \beta\big) = dh \cdot \pi^*(\alpha) + h\,\pi^*(d\alpha) - dh \cdot d\beta$$

$$d(\alpha, \beta) = (d\alpha, \pi^*\alpha - d\beta) \longmapsto h\,\pi^*(d\alpha) + dh\,(\pi^*\alpha - d\beta)$$

so this is a map of complexes. I forgot to mention that $dh \cdot \beta$ will extend by zero from $X - Y$ to Y since $dh = 0$ near Y.

Finally we should note that the Thom class goes into

$$(e, u) \longmapsto h\,\pi^*(e) + dh \cdot u = d(hu)$$

and

$$\alpha \cdot (e, u) = \big(\alpha e, (-1)^{\deg \alpha}\pi^*(\alpha) \cdot u\big) \mapsto h\,\pi^*(\alpha e) + dh\,(-1)^{\deg\alpha}\pi^*(\alpha) \cdot u$$
$$= \pi^*(\alpha) \cdot d(hu).$$

as expected.

Conclusion: Provided we work in the mapping cone, we have a [redacted] fairly canonical Thom class, so we can expect a kind of local index thm.

But now that we have made sense out of how to interpret $i_*(T(\nu)\,\mathrm{ch}\,E)$ as a form, we still have to make sense out of [redacted] $\mathrm{ch}(i_! E)$. At best, this is supported on Y. Also I have to take some kind of classical limit in order to get the theorem. This suggests some kind of Morse theory and leads to the

Next idea: Let's try to do Witten's analytical version of Morse theory, but using the Dirac operator instead of DR. Thus take a Dirac operator over X and consider the gauge transformations $e^{tf(x)}$, where f is a Morse function with a non-degenerate critical submanifold Y.

I forgot another ideathat emerged on the type-writer. How do we know that $d(hu)$ is the Thom class? It's closed so the thing we need to see is that when integrated over the fibre it gives ± 1. But by Stoke's thm.

$$\int_{\text{fibre}} d(hu) = \int_{\substack{\text{small}\\ \text{sphere}}} u$$

and this is ± 1 by topology, i.e. the definition of the Euler class in terms of transgression. This suggested a connection with taking the trace in the case of pseudo-diffl operators. The form u is homogeneous so that the integral around any sphere gives the same answer.

Problems with the Morse theory idea:

<u>November 27, 1983</u>

I want to review the construction of characteristic classes for the Lie algebras of gauge transformations that I gave in the Loday letter.

A left-invariant form on $\mathcal{G}$ is to be identified with a natural transformation from flat connections on the trivial $\mathcal{G}$-bundle to forms on the base. So consider the trivial $\mathcal{G}$-bundle $P = Y \times \mathcal{G}$ over Y. It corresponds to $\tilde{P} = pr_2^*(P) = Y \times P$ over $Y \times M$. In order to do calculations, let's work with associated vector bundles $\tilde{E} = Y \times E$, $E = P \times^{\mathcal{U}} V$.

We are given a flat partial Y-connection in $\tilde{E}$. This can be compared to the obvious Y-connection d' coming from the trivialization. So the given Y-connection D' can be written

$$D' = d' + \theta$$

where $\theta \in \Omega^{1,0}(Y \times M, \operatorname{End} \tilde{E}) = \Omega^1(Y, \Omega^0(M, \operatorname{End} E))$. Flatness means

$$d'\theta + \theta^2 = 0.$$

In order to define characteristic classes, we extend D' to a full connection. In practice we do this by choosing a connection D_0 on E and taking its pull-back to $pr_2^*(E)$ which we can write $d' + D''$. Then we consider the 1-parameter family joining $d' + D''$ to $d' + \theta + D''$.

Let's lift things up to the principal bundle $Y \times P$. Then $d' + D'' \longmapsto d + A$ $\quad A = pr_2^*(A_0)$ and so we are dealing with the 1-parameter family of connection forms

$$t\theta + A .$$

with the curvature

$$\Omega_t = t\,d\theta + dA + t^2\theta^2 + t[\theta, A] + A^2$$
$$= \underset{2,0}{(t^2-t)\theta^2} + \underset{1,1}{t\left(d''\theta + [\theta, A]\right)} + \underset{0,2}{F_A}$$

Then the transgression formula is

$$tr(\Omega_1^j) - tr(\Omega_0^j) = j\,d\int_0^1 dt\; tr\left(\theta\,\Omega_t^{j-1}\right).$$

Since Ω_1, Ω_0 are of filtration 1, we see that

$$u = j\int_0^1 tr\left(\theta\,\Omega_t^{j-1}\right) \in \Omega^{2j-1}(Y\times M)\Big/F_j\Omega^{2j-1}(Y\times M)$$

is a cocycle. Here $F_j\Omega = \bigoplus_{q\geq j}\Omega^{p,q}$. This means that if γ is a closed current of degree $d < j$, then $\int_\gamma u \in \Omega^{2j-1-d}(Y)$ will be a closed form. Thus we get a natural transf. from flat connections on the trivial G-bundle to forms on Y, hence a left invariant diff'l form on G.

This is obviously very close to what we did in the Singer letter.

Let us first get straight the relations between Lie algebra cohomology and the cohomology of the group straight.

Take a Lie group G. We have decided to think of left-invariant differential forms on G as natural transf. from flat connections on the trivial bundle to forms on the base:

$$\Lambda\mathfrak{g}^* \xrightarrow[\text{conn.}]{} \Omega(P) \xrightarrow{s^*} \Omega(Y)$$

The connection is written $d+\theta$, $\theta \in \Omega(Y)\otimes\mathfrak{g}$; actually one should say the connection in an associated vector bundle. Flatness means

$$d\theta + \theta^2 = 0 .$$

Thus given $\theta \in \Omega(Y)\otimes\mathfrak{g}$ satisfying this equation we get a map of DG algebras

$$\Lambda\mathfrak{g}^* \longrightarrow \Omega(Y)$$

and hence characteristic classes in $H^*(Y)$ associated to elements of $H^*(\mathfrak{g})$.

Next we consider the homomorphism $H^*(\mathfrak{g}) \to H^*(G)$. We want a geometric interpretation, hence given a map $Y \xrightarrow{g} G$ we want a form $\theta \in \Omega(Y)\otimes\mathfrak{g}$ as above. There are two choices, namely, the connections

$$g^{-1}\cdot d\cdot g = d + g^{-1}dg$$

$$\text{or} \quad g\cdot d\cdot g^{-1} = d - dg\cdot g^{-1}.$$

Each will give a map of differential algebras from $\Lambda\mathfrak{g}^*$ to $\Omega(Y)$. If we take the universal situation $Y=G$, then we get two maps $\Lambda\mathfrak{g}^* \to \Omega(Y)$

the former with image the left invariant forms, and the latter with image the right-invariant forms.

Conclusion: Given $g : Y \to G$ we associate to it the form $\theta \in \Omega(Y) \otimes \mathfrak{g}$ which is the inverse image of the MC form $g^{-1}dg$ on G.

Next consider the (inverse) transgression. Topologically we go from $H^*(BG)$ to $H^*(G)$. Geometrically if a characteristic class is $p(\Omega)$ where $p \in S(\mathfrak{g})^*$, then $p(\Omega)$ lifted to the principal bundle becomes d of the Chern-Simons form. Up in P we use the linear path $d + t\theta$, which has curvature

$$\Omega_t = t\Omega + (t^2 - t)\theta^2$$

and leads to

$$p(\Omega) - p(0) = d\int_0^1 dt \; p'(\Omega_t; \theta).$$

Then on restricting to a fibre we see that

$$\int_0^1 dt \; p'((t^2-t)\theta^2; \theta)$$

is closed. Notice that this transgression process produces a biinvariant form on G, i.e. lies in $(\Lambda\mathfrak{g}^*)^G$.

We can see directly that transgressing a class of the form $p(\Omega)$ leads to an element of $H^*(\mathfrak{g})$ as follows. (?) A flat connection on the trivial G-bundle/Y has the class represented by $p(\Omega)$ equal to zero for two reasons, hence there is an odd class in $H^*(Y)$. This doesn't quite give a class in $H^*(\mathfrak{g})$

unfortunately.

Question: Recall the arrows:

$$G \longrightarrow \mathfrak{g} \longrightarrow B_c G \longrightarrow BG$$

which are successive fibrations. Is transgression the Toda bracket?

If so then classes P on BG of high enough degree to vanish in $B_c G$ will then transgress to classes in $H^*(\mathfrak{g})$ showing that the transgression in $H^*(G)$ of P comes from $H^*(\mathfrak{g})$.

Let's now turn to the case of the gauge transformation group $\mathcal{G}$.

It just occurred to me how we might get at the van Est fibrations, or really the van Est + Bott spectral sequences. Suppose $\mathcal{a}$ in the following denotes a contractible principal $\mathcal{G}$-bundles. Then we should be able to establish that

$$\Omega(\mathcal{a}/\mathcal{G}) = \Omega(\mathcal{a})_{basic}$$

is homotopy equivalent to

$$[W(\tilde{\mathfrak{g}}) \otimes \Omega(\mathcal{a})]_{basic} = [S(\tilde{\mathfrak{g}}^*) \otimes \Omega(\mathcal{a})]^{\mathcal{G}}.$$

In fact a choice of connection form in $\mathcal{a} \to \mathcal{a}/\mathcal{G}$ gives a map $W(\tilde{\mathfrak{g}}) \otimes \Omega(\mathcal{a}) \longrightarrow \Omega(\mathcal{a})$ which should become a quis upon passing to basics. On the other hand $[S(\tilde{\mathfrak{g}}^*)\otimes\Omega(\mathcal{a})]^{\mathcal{G}}$ has for its cohomology $H^*_{cont}(\mathcal{G}, S(\tilde{\mathfrak{g}}^*))$ if we just looked at the differential on $\Omega(\mathcal{a})$. Thus one should get Bott's spectral sequence.

As for the van Est spectral sequence, consider

$$\Omega(\mathcal{G}\times\mathcal{a}) = \Omega(\mathcal{G})\otimes\Omega(\mathcal{a})$$

Because $\mathcal{G}$ acts freely on itself, we should have

$\Lambda(\tilde{\mathfrak{g}}^*) = \Omega(\mathcal{G})^{\mathcal{G}}$ [redacted] $\longrightarrow$ [redacted] $[\Omega(\mathcal{G})\otimes\Omega(\mathcal{a})]^{\mathcal{G}}$

is a quis, i.e. $\Omega(\mathcal{G})$ is already acyclic for the continuous cohomology. Then the standard Postnikov filtration of $\Omega(\mathcal{G})$ gives the desired spectral sequence.

So let us now consider the Lie alg. cohomology of $\mathcal{G}$. This means that I consider a flat connection form $\Theta\in\Omega^1(Y$ [redacted] $,\tilde{\mathfrak{g}})$ on the trivial $\mathcal{G}$-bundle. Then we use the isomorphism

$\Omega^1(Y,\tilde{\mathfrak{g}}) = \Omega^1(Y, \Omega^0(M,$ [redacted] $P\times^U \mathfrak{u}))$

$$\subset \Omega^{1,0}(Y\times P, \mathfrak{u})$$

to interpret Θ as a horizontal connection on $Y\times P/Y\times M$. In order to construct characteristic forms associated to Θ we need to extend the partial connection to a full connection natural in Y. The easiest way to do that is to choose $A_0\in\mathcal{a}$ and then pull it back to $Y \times P$ to get a connection form. Then I use the linear path A_0 to $A_0 + t\theta$ [redacted]

[redacted] $$\mathrm{tr}(F_1^j) - \mathrm{tr}(F_0^j) = d\underbrace{\int_0^1 dt\; j\,\mathrm{tr}(\theta F_t^{j-1})}_{t}$$

and the fact that F_0, F_1 are of filtration 1 so that the left side belongs to $F_j\Omega(Y\times M)$. Hence for γ a closed current of degree $< j$, $\int_\gamma t$ will be a closed form on Y.

This [redacted] construction might be understood from the viewpoint that we are transgressing in the fibration

$$\tilde{\mathfrak{g}} \longrightarrow B_c\mathcal{G} \longrightarrow B\mathcal{G}.$$

The relevant cochain complexes are

$$\Lambda(\tilde{\mathfrak{g}})^* \qquad \Omega(\mathcal{A})^{\mathcal{G}} \qquad [S(\tilde{\mathfrak{g}}^*)\otimes\Omega(\mathcal{A})]^{\mathcal{G}}.$$

Actually what seems to be happening is that we have the diagram

$$\begin{array}{ccccc}
\mathcal{G} & \longrightarrow & P\mathcal{G} & \longrightarrow & [redacted]\ B\mathcal{G} \\
\downarrow & & \downarrow & & \| \\
\mathfrak{g} & \longrightarrow & B_c\mathcal{G} & \longrightarrow & B\mathcal{G}
\end{array}$$

realized by the complexes

$$\begin{array}{ccccc}
\Omega(\mathcal{G}) & \longleftarrow & \Omega(\mathcal{A}) & \longleftarrow & \Omega(\mathcal{A}/\mathcal{G}) \\
\cup & & \cup & & \| \\
\Omega(\mathcal{G})^{\mathcal{G}} & \longleftarrow & \Omega(\mathcal{A})^{\mathcal{G}} & \longleftarrow & \Omega(\mathcal{A}/\mathcal{G})
\end{array}$$

One thing that happens is that $\mathcal{G}$ acts on the right of $\mathcal{A}$, hence if $\mathcal{G}$ is embedded in $\mathcal{A}$ as an orbit, $\mathcal{G}$ acts on itself by right translation. Thus $\Omega(\mathcal{G})^{\mathcal{G}}$ is the space of right-invariant vector fields.

Now things more or less fall into place. We start with the **form** $\int_\gamma p(F_{\bar{A}})$ on $\mathcal{A}/\mathcal{G}$. Here $\bar{A} = \hat{A} + \theta\varphi$ is the [redacted] connection on $\mathcal{A}$ that descends. Now we lift it up to $\mathcal{A}$ and write it as d of something. The point is that we can always do this in $\Omega(\mathcal{A})$ using a connection B in the principal bundle $\mathcal{A}\times P$ over $\mathcal{A}\times M$ which comes from P/M: [redacted]

$$\int_\gamma p(F_{\bar{A}}) - \underbrace{\int_\gamma p(F_B)}_{=0} = d\int_\gamma [redacted]\; u(\bar{A}, B)$$

$=0$ as F_B comes from M and

$2k = \deg p(F_B) > \dim \gamma$ so $p(F_B)|\mathcal{A}\times\gamma = 0$.

Then the transgression of $\int_\gamma p(\bar{A})$ will be given by $\int_\gamma u(\bar{A}, B)$ restricted to the $\mathcal{G}$-orbit.

However when [redacted] $k > \dim\gamma$, then we can actually write $\int_\gamma p(F_{\bar{A}})$ lifted to $\mathcal{A}$ as a coboundary within $\Omega(\mathcal{A})^{\mathcal{G}}$, namely

$$\int_\gamma p(F_{\bar{A}}) - \underbrace{\int_\gamma p(F_{\hat{A}})}_{=0} = d\int_\gamma u(\bar{A}, \hat{A})$$

Here $F_{\hat{A}}$ is of filtration 1, hence $p(F_{\hat{A}}) \in F_k\Omega(\mathcal{A}\times M)$, so is zero when ~~restricted~~ to $\mathcal{A}\times\gamma$ as $\dim\gamma < k$. Then the transgression of $\int_\gamma p(F_{\hat{A}})$ will be given by the right-invariant form $\int_\gamma u(\bar{A},\hat{A})$ restricted to the $\mathcal{G}$-orbit.

November 28, 1983

Let's return to the problem of constructing left invariant forms on $\mathcal{G}$. We consider the a flat connection θ on the trivial principal $\mathcal{G}$-bundle $\mathcal{P}$. As usual we replace $\mathcal{P}$ by the corresponding $\mathcal{U}$-bundle $\tilde{P} = pr_2^*(P) = Y \times P$ over $Y \times M$ and θ by the corresponding Y-connection form

$$\theta \in \Omega^1(Y, \tilde{\mathfrak{g}}) = \Omega^1(Y, \Omega^0(M, P \times^{\mathcal{U}} \mathfrak{u})) \subset \Omega^{1,0}(Y \times M, \mathfrak{u}).$$

To construct forms on Y in a natural way we fix a connection A_0 on P and let $A = pr_2^*(A_0) \in \Omega^{0,1}(Y \times P, \mathfrak{u})$. Then we use the path $A + t\theta$ which joins two Y-flat connections. The curvature at time t is

$$d(A+t\theta) + (A+t\theta)^2 = \underset{0,2}{F_A} + t\underset{1,1}{\{d''\theta + [A,\theta]\}} + \underset{2,0}{(t^2-t)\theta^2}$$

Notice now that when P is trivial and A is the zero connection we get the familiar form

$$t\, d''\theta + (t^2-t)\theta^2$$

which has exactly the same appearance as the Chern-Simons term

$$F_t = t\,dA + t^2A^2 = tF_A + (t^2-t)A^2.$$

This shouldn't be a surprise because we are joining the connection $d+\theta$ linearly to d, i.e. the curvature of $d+\theta$ is $d\theta + \theta^2 = d''\theta$.

Now all I have to do is to correlate the terms in $\int_0^1 dt\, tr\{e^{t d''\theta + (t^2-t)\theta^2}\theta\}$ with Connes's S operator.

November 29, 1983

I start with Atiyah's idea that your formula

$$e^{-tL^2 + \sqrt{t}\,[D,L] + F}$$

(*)

$$= e^{-tL^2 + \sqrt{t}\, dx^\mu [D_\mu, L] + \frac{1}{2} dx^\mu dx^\nu F_{\mu\nu}}$$

and the Getzler proof should be part of the same framework. This framework should involve a vector bundle over M and an elliptic operator on M together with some sort of Clifford module structure belonging to the vector bundle. In my situation with the local index formula the vector is the normal bundle to the fibre M in X over y and it is trivial because it is isomorphic to the pull-back of the tangent space to Y at y.

The analytical situation is as follows in my case. We trivialized X, in fact I worked mainly in the case $X = Y \times M$. The operator I started with on X is the Dirac operator on $S_X \otimes \tilde{E}$ for some connection on $\tilde{E}$. Then $S_X = S_Y \otimes S_M$ and we assume $\tilde{E} = pr_2^*(E)$ so that we are operating on spinors on Y with coefficients in $\mathcal{H} = L^2(M, S_M \otimes E)$.

So the set up is as follows. We have the vector bundle $S_M \otimes E$ and hence the Hilbert module $\mathcal{H} = L^2(M, S_M \otimes E)$ over $C^\infty(M)$. And on the Hilbert module we have the Dirac operator which I have been calling L. In the normal direction I have been taking the spinors S_y and tensoring with $\mathcal{H}$. This is what I need to write down the expression

$$e^{-L^2 + dy^\mu [D_\mu, L] + \frac{1}{2} dy^\mu dy^\nu F_{\mu\nu}}.$$

(This isn't quite correct maybe).

The point will now be to try to [redacted] generalize the formulas to the case where the normal bundle to M in X is not trivial. Here instead of S_y one should have the spinor bundle S_ν over M associated to the normal bundle ν. I want to then take the operator L over M, [redacted] think of it as being our Hamiltonian, then combine it with fermion operators belonging to ν to get a kind of "super" situation.

So we have to start by treating the fermions in the normal direction. Start with the manifold M and a vector bundle V over it. V is our normal bundle, so we may as well think of M as the zero section in V. V is a real vector bundle with inner product, so we can form the Clifford algebra bundle $C(V)$. Let S be a $C(V)$-module, i.e. a vector bundle over M on which the algebra bundle $C(V)$ acts. Locally on M we can choose a trivialization of V, i.e. an orthonormal frame e^μ [redacted] and then we get corresponding operators γ^μ on S. [redacted]

Geometrically this situation might become interesting when we put a connection on V. This will then induce a connection in S. Let's recall why. We form the principal $SO(n)$ bundle P belonging to V. If S is the spinor bundle, then we have supposed given a "reduction" of P to a $Spin(n)$ bundle $\tilde{P}$, and then

$$S = \tilde{P} \times^{Spin(n)} \Delta$$

where Δ is the spin representation. Now a connection in V is the same as a connection in P or $\tilde{P}$, since $SO(n)$ and $Spin(n)$ have the same Lie algebra.

So far I have described the normal fermion structure. The following picture emerges. We ultimately will take the bundle S tensor it with another bundle and take the Hilbert space $\mathcal{H}$ of sections. Then $\mathcal{H}$ has the operators $\Gamma(M, C(V))$ which includes $C^\infty(M)$ as well as certain operators [redacted] γ_v, $v \in \Gamma(V)$.

A standard example is the following. Take V to be the [redacted] tangent bundle $T \simeq T^*$ and S to be the bundle of forms ΛT^* with the usual Clifford [redacted] action $v \mapsto i(v) + e(v)$.

The next step will be to bring in diffl operators on the base M. [redacted]

Next we want the differential operators on M. There seem to be two possible approaches.

i) continue to think of V as a normal bundle. What this means is that we have an embedding $M \subset X$ with normal bundle V, we take a Dirac operator on X and rescale in the vertical direction to make it classical. Then we see what kind of limit we get on M.

Note that in order to rescale we don't need to have a foliation. We need only the splitting of the tangent bundle into horizontal and vertical. When X is Riemannian there is a natural fibring of a tubular nbd of M over M whose tangent spaces give a notion of vertical. The perpendicular directions are horizontal.

ii) follow the model given by $V = T$ and the $C(V)$-module ΛT^*. In this case we have a connection in V hence in S hence we can write differential operators D_μ on the sections of S.

Suppose M^n is a submanifold of $\mathbb{R}^N$. Then we have the Gauss map

$$M \longrightarrow \mathrm{Grass}_n(\mathbb{R}^N)$$

which assigns to each point of M its tangent plane translated to the origin. In other words the Gauss map is the classifying map for the tangent bundle using the fact that the tangent bundle of $\mathbb{R}^N$ is trivial. I think the Levi-Civita connection on T_M is induced by the natural Grassmannian connection on the subbundle over $\mathrm{Grass}_n(\mathbb{R}^N)$. Hence the Riemannian curvature of M is the pull-back of the connection e de de on the subbundle.

In the case of a surface embedded in $\mathbb{R}^3$ the Gauss map assigns to a point on the 2-surface its unit normal vector (the surface being supposed oriented). The Gaussian curvature is then the pull-back of the volume form on the 2-sphere (up to a constant).

If at a point on the surface we draw the tangent plane and consider the height of the surface above the tangent plane to second order, then we obtain the second fundamental form

$$E\,du^2 + 2F\,du\,dv + G\,dv^2$$

on the surface. This should be enough to compute the differential of the Gauss map. This form should be contrasted with the first fundamental form

$$ds^2 = A\,du^2 + 2B\,du\,dv + C\,dv^2$$

which gives the metric.

November 30, 1983

Idea: Recall the formula for the determinant of a $\bar{\partial}$-operator over $\mathbb{C}$ that Jackiw showed me. It is the sum of a σ-model [redacted] and Chern-Simons term just like the Wess-Zumino Lagrangian as described by Witten. Thus it might be possible to understand the low energy version of a gauge theory as an effective Lagrangian theory taking place on one of the $\mathcal{G}$-orbits. The effective Lagrangian comes from just the fermion integral in the presence of the gauge-field.

~~In any case we can see now why the two ways the names Wess-Zumino occur are related. On one hand the Wess-Zumino Lagrangian~~

In any case we see a connection between the Wess-Zumino Lagrangian and the Zumino work on anomalies. [redacted] Also why Zumino (maybe Wess-Zumino)'s name occur in connection with super-symmetry. This might result from differential form calculations in the physicists notation.

Next Tuesday I am supposed to present the differential form calculations where one computes the transgression. I have to find clear formulas and notation.

$M, P, \mathcal{U}, \mathcal{a}, \mathcal{G}$

One likes to compute in an associated v.b. $E = P \times^{\mathcal{U}} V$.

$$\Omega(M, E) \xrightarrow{\sim} \{\Omega(P) \otimes V\}_{basic} \subset \Omega(P) \otimes V$$

$$D_A \longleftrightarrow d + A$$

Gauge transf $g \in \mathcal{G}$ act naturally on P to the left (as $P \subseteq \mathrm{Isom}(V,E)$) hence to the right on $\Omega(P)$ and on connections. One has

$$g_* \text{ on } \Omega(M,E) \longleftrightarrow (g^{-1})^* \text{ on } \Omega(P)\otimes V;$$

[redacted] the action of g on $\mathcal{A}$ is

$$g^{-1} D_A \cdot g \longleftrightarrow g^* A$$

[redacted]

(The notation differs from physics usage where $P = M \times U$ comes provided with a section and the connection form is written $\theta + A$ (more accurately $pr_2^*\theta + pr_1^* A$) with $A \in \Omega^1(M, \mathfrak{u})$. Then a gauge transf. is a map $g : M \longrightarrow U$ and the action on connections is

$$g \cdot A = g^{-1}dg + g^{-1}Ag.)$$

I have the definitions

$\mathcal{G}$ = group of autos of P

$\tilde{\mathfrak{g}}$ = Lie alg. of $\mathcal{G}$

For each $v \in \tilde{\mathfrak{g}}$ we get a vector field on P which is vertical an U-invariant. This is an anti-isomorphism of $\tilde{\mathfrak{g}}$ with the U-invariant vertical vector field, because $\mathcal{G}$ acts to left of P.

December 1, 1983

I need formulas for the vector fields on $\mathcal{A}\times P$ belonging to elements σ of $\tilde{\mathcal{G}}$. I also have a problem describing $\tilde{\mathcal{G}}$. On one hand it is the Lie algebra of vertical vector fields on P which commute with the right U-action. On the other hand it should be $\Omega^0(M, P\times^U \mathfrak{u})$. If $M = pt$ so that $P = G$, then the former is the Lie algebra of right-invariant vector fields and the latter is the Lie algebra of left-invariant vector fields. Hence there is a sign somewhere.

So let's see if we can get precise formulas. Let's adopt the physics situation and work with a trivialization of P. Then $\mathcal{G}$ will be maps from M to U.

Actually this suggests that I correlate sections of $P\times^U U$ with autos. of P. Such a section is the same as a map $f: P\longrightarrow U$ s.t. $f(pu) = u^{-1}f(p)u$. To such a section we can associate the automorphism

$$\tilde{f} : P \longrightarrow P \qquad : \qquad px \longmapsto px\,f(px) = p\cancel{x\,x^{-1}}f(p)x$$
$$p \longmapsto p\,f(p)$$

Then $$\widetilde{f_1f_2}(p) = p\,f_1(p)f_2(p)$$
$$\tilde{f}_1\big(\tilde{f}_2(p)\big) = \tilde{f}_1\big(p\,f_2(p)\big) = p\,f_2(p)\cdot f_1\big(p\,f_2(p)\big) = p\,f_1(p)\,f_2(p)$$

hence pointwise product of f corresponds to composition of $\tilde{f}$. A further check: Suppose $P = U$. Then $f(x) = x^{-1}f(e)x$ so $\tilde{f}(x) = x\,f(x) = f(e)x$. Thus
$$\widetilde{f_1f_2}(x) = (f_1f_2)(e)\,x = f_1(e)\big[f_2(e)x\big] = \tilde{f}_1\big(\tilde{f}_2(x)\big).$$

Hence if we define $\mathcal{G} = \Gamma(P \times^U U_c)$, then $\mathcal{G}$ acts naturally on the left of P. When P is trivialized $P = M \times U$ we can identify $\mathcal{G}$ with the maps $M \xrightarrow{g} U$, and then the corresponding auto ~~of~~ of P is $(m, x) \longmapsto (m, g(m)x)$.

The Lie algebra of $\mathcal{G}$ then is naturally

$$\tilde{\mathfrak{g}} = \Gamma(P \times^U \mathfrak{u})$$

and when P is trivialized can be identified with maps $v : M \longrightarrow \mathfrak{u}$. The only problem will be that because $\mathcal{G}$ is acting to the left [redacted] on P, the natural map from $\mathrm{Lie}(\mathcal{G}) = \tilde{\mathfrak{g}}$ to vector fields on P will be an anti-Lie homomorphism.

It seems I want to have $\mathcal{G}$ acting on $\mathcal{A}$ to the right so that it acts to the left on functionals on $\mathcal{A}$.

December 2, 1983

Let $\mathcal{G} = \Gamma(M, P\times^U U)$ be the gauge transf group of P. We are interested in the cohomology of $B\mathcal{G}$ and $\mathcal{G}$. From topology we know that generators for $H^*(B\mathcal{G})$ can be obtained by taking a characteristic class $\varphi(\tilde{P}) \in H(B\mathcal{G}\times M)$, where $\tilde{P} = P\mathcal{G}\times^{\mathcal{G}} P$ over $B\mathcal{G}\times M$, and integrating over homology classes $\gamma \in H_*(M)$:

$$1) \qquad \int_\gamma \varphi(\tilde{P}) \in H^{2k-d}(B\mathcal{G}) \qquad \begin{array}{l} d = \dim\gamma \\ 2k = \deg \varphi \in H^*(BU) \end{array}$$

We want to compute 1) as a differential form, or better, in DR cohomology. Let's suppose we choose a basepoint on M and consider only gauge transf. which are the identity at the basepoint. Then $\mathcal{G}$ acts freely on $\mathcal{A}$, so I can take $P\mathcal{G} = \mathcal{A}$, $B\mathcal{G} = \mathcal{A}/\mathcal{G}$. I need a connection on $\tilde{P}$ in order to realize $\varphi(\tilde{P})$ in $\Omega(\mathcal{A}/\mathcal{G}\times M)$.

$$\begin{tikzcd} \mathcal{A}\times P \arrow[r] \arrow[d] & \tilde{P} \arrow[d] \\ \mathcal{A}\times M \arrow[r] & \mathcal{A}/\mathcal{G}\times M \end{tikzcd}$$

The bundle $\mathcal{A}\times P = pr_2^*(P)$ over $\mathcal{A}\times M$ has a canonical connection $\tilde{A}$. (General assertion: Given a family of connections A_y on P parametrized by Y, there is unique connection $\tilde{A}$ on $Y\times P = pr_2^*(P)$ over $Y\times M$ such that (i) In the Y-direction it coincides with the "zero" connection (ii) $\tilde{A}$ rest. to $y\times M$ coincides with A_y on P.)

The connection $\tilde{A}$ over $\mathcal{A}\times M$ is $\mathcal{G}$-invariant as it is canonical, but not horizontal. Let $\Theta \in \Omega^1(\mathcal{A})\otimes\tilde{\mathfrak{g}}$

be a connection in the principal $\mathcal{G}$-bundle $\mathcal{A} \to \mathcal{A}/\mathcal{G}$.

Let $\rho : \tilde{\mathfrak{g}} \longrightarrow \Omega^0(M, P \times^{U} \mathfrak{u}) \subset \Omega^0(P) \otimes \mathfrak{u} \subset \Omega^0(\mathcal{A} \times P) \otimes \mathfrak{u}$ be the Higgs field of $\tilde{A}$. Then $\bar{A} = \tilde{A} + \theta\rho$ is a connection on $\mathcal{A} \times P$ which descends to $\tilde{P}$.

(Formulas: Let G acts on a principal U-bundle P, [redacted] and let $A \in \Omega^1(P) \otimes \mathfrak{u}$ be a G-invariant connection. Then the defining formula for the Higgs field

$$\mathcal{L}_X = [\iota_X, D] + \rho X \qquad \text{on } \Omega(M, P \times^{U} \mathfrak{u})$$

when considered upstairs is

$$\mathcal{L}_X = [\iota_X, d + A] + \rho X \qquad \text{or} \qquad \iota_X A + \varphi_X = 0.$$

Thus $\rho X = -\iota_X A$.)

At this point I have ~~[redacted]~~ produced the connection on [redacted] $\tilde{P}$ over $\mathcal{A}/\mathcal{G} \times M$. So I can then plug in to the Chern-Weil machine and get $\varphi(\tilde{P}) \in H^{2k}(\mathcal{A}/\mathcal{G} \times M)$ realized by differential forms.

My problem now is to get a more concrete realization of these things that I can explain to physicists. Above all I need formulas for the vector fields on $\mathcal{A}$ and $\mathcal{A} \times P$ associated to $v \in \tilde{\mathfrak{g}}$.

Now from a physicist's viewpoint, in order to describe tangent vectors ~~or~~ forms on $\mathcal{A} \times P$ I must give coordinates. This more or less means that I might as well assume P is trivial.

So we assume $P = M \times U$ is trivial. Then $\mathcal{G} =$ Maps(M, U), $\tilde{\mathfrak{g}} =$ Maps$(M, \mathfrak{u})$ [redacted] $= \Omega^0(M) \otimes \mathfrak{u}$, $\mathcal{A} = \Omega^1(M) \otimes \mathfrak{u}$, and we have

$$g * A = g^{-1}dg + g^{-1}Ag$$
$$v * A = dv + [A, v] = [d + A, v]$$

The last formula means that if $v : M \longrightarrow \mathfrak{u}$ and A is a connection, then vector field on $\mathcal{A}$ assoc. to v assigns to A the vector

$$\boxed{\delta_v A = [d + A, v]}$$

If $\Phi(A)$ is a functional of the gauge field A, the corresponding change in Φ is

$$(\delta_v \Phi)(A) = \Phi(A + \delta_v A) - \Phi(A)$$

$$= \int dx \, \frac{\delta \Phi(A)}{\delta A^a_\mu(x)} \, \delta_v A^a_\mu(x) \qquad = \int dx \, \frac{\delta \Phi(A)}{\delta A_\mu(x)} \, [\partial_\mu + A_\mu, v](x)$$

Physicists have the nice notation

$$A_\mu \times v = [A_\mu, v]$$

So

$$(\delta_v \Phi)(A) = \int dx \, \frac{\delta \Phi}{\delta A_\mu(x)}(A) \, [\partial_\mu + A_\mu, v](x)$$

$$= -\int dx \, \left[\partial_\mu + A_\mu, \frac{\delta \Phi}{\delta A_\mu(x)}(A)\right](x) \, v(x)$$

$$= -\int dx \, (\partial_\mu + A_\mu \times) \frac{\delta \Phi}{\delta A_\mu(x)} \cdot v(x)$$

Thus δ_v acting on functionals Φ on $\mathcal{A}$ is the vector field

$$\boxed{\begin{gathered} X_v = \int dx \, v^a(x) \, X_a(x) \\ \text{where} \quad -X_a(x) = \partial_\mu \frac{\delta}{\delta A^a_\mu(x)} + \left[A_\mu \times \frac{\delta}{\delta A_\mu(x)}\right]_a \end{gathered}}$$

In other words $\tilde{\mathfrak{g}}$ has the basis $\lambda_a \delta_x$ and $X_a(x)$ is the corresponding vector field on $\mathcal{A}$. Now maybe I should use the notation X_v for the vector field δ_v on $\mathcal{A}$. Since $\mathcal{G}$ is acting to the right on $\mathcal{A}$ we know it acts on $\Omega(\mathcal{A})$ to the left, hence

$$[X_u, X_v] = X_{[u,v]} \qquad u, v \in \tilde{\mathfrak{g}}.$$

At this point we have the $\tilde{\mathfrak{g}}$-action on $\mathcal{A}$ and hence $\mathcal{A} \times M$. But now we have over $\mathcal{A} \times M$ [redacted] a $\mathcal{G}$-equivariant [redacted] U-bundle $\mathcal{A} \times P = \mathcal{A} \times M \times U$, with the invariant connection $\tilde{A}$. Let's think in terms of the associated vector bundle $\mathcal{A} \times M \times V$ over $\mathcal{A} \times M$. A section is a functional $\Phi(A, x)$; i.e. $\Phi \in \Omega^0(\mathcal{A} \times M) \otimes V$. The $\mathcal{G}$-action is

$$(g * \Phi)(A, x) = [redacted]\ g(x)\Phi(g * A, x)$$
$$= g(x)\, \Phi(g^{-1}dg + g^{-1}Ag,\ x)$$

Infinitesimally we have

$$\boxed{\begin{aligned} (v * \Phi)(A, x) &= v(x)\Phi(A, x) + \delta_v \Phi(A, x) \\ \text{or} \quad v * \Phi &= (v + X_v)\, \Phi \\ &\text{where } \Phi(A,x) \text{ is a section of } \mathcal{A} \times M \times V \text{ over } \mathcal{A} \times M. \end{aligned}}$$

The next thing we must do is to describe the connection [redacted] $d + \tilde{A}$ on $\mathcal{A} \times M \times V$:

$$\left((d+\tilde{A})\Phi\right)(A,x) = (d\Phi)(A,x) + (\tilde{A}\Phi)(A,x)$$

$$= \int dy\, \frac{\delta\Phi(A,x)}{\delta A(y)}\, \delta\,[redacted]\,A(y) + \partial_\mu \Phi(A,x)\,dx^\mu + A(x)\Phi(A,x)$$

$$= (d_a \Phi)(A,x) + [redacted]\,(\partial_\mu + A_\mu)\Phi(A,x)\,dx^\mu$$

Thus $\boxed{(d+\tilde{A})\Phi(A,x) = d_a\Phi(A,x) + D_A\Phi(A,x)}$

Since the vector field X_v points in the a-direction it follows that the lift of X_v via the connection is just X_v acting on the A-variable:

$$[\iota_{X_v}, d+\tilde{A}] = [\iota_{X_v}, d]$$

or

$$i_{X_v}(d+\tilde{A})\Phi = X_v\Phi \text{ on functions.}$$

Thus the Higgs field is

$$\boxed{(\rho_{v^-}\Phi)(A,x) = v(x)\,\Phi(A,x)}$$

Now I must describe the connection $d+\bar{A}$. This means I need the form Θ

<u>Remark</u>: On p. 287 when you consider an equivariant for G principal bundle P/M you should assume G acts to the right, so that it acts on the left of forms in the usual way

December 3, 1983

Here is probably what Iz had in mind with his vol_B construction

$$\begin{tikzcd}
 & \tilde{\alpha} & \alpha \arrow[l, mapsto] & d\alpha \\
\mathcal{G}\times P \arrow[r]\arrow[d] & \mathcal{A}\times P \arrow[r]\arrow[d] & \mathcal{A}\times^{\mathcal{G}} P \arrow[d] & \\
\mathcal{G}\times M \arrow[r]\arrow[d] & \mathcal{A}\times M \arrow[r]\arrow[d] & \mathcal{A}/\mathcal{G}\times M \arrow[d] & ch_k(\tilde{E}) \arrow[uu, mapsto] \arrow[d, mapsto, "\int_M"] \\
\mathcal{G} \arrow[r] & \mathcal{A} \arrow[r] & \mathcal{A}/\mathcal{G} & \int_M ch_k(\tilde{E})
\end{tikzcd}$$

Here α is the Chern-Simons form for the connection on $\mathcal{A}\times^{\mathcal{G}}P$ descended from $\bar{A}$. α lifts back to $\tilde{\alpha}$ = Chern-Simons of $\bar{A}$. Now if you form

$$\int_P \tilde{\alpha}\cdot vol_B$$

then you obtain a form on $\mathcal{A}$ such that

$$d\int_P \tilde{\alpha}\, vol_B = \pm\int_P d\tilde{\alpha}\cdot vol_B \quad \tilde{\alpha}\, d vol_B$$

Now suppose $F_B = 0$. Then $\cdot vol_B$ is closed

~~so~~ so we get

$$\pm\int_P ch_k(\mathcal{A}\times E;\bar{A})\, vol_B = \pm\int_M ch_k(\mathcal{A}\times E;\bar{A})$$

We can get the signs straight as follows: Use the operator $\int_{P/M} vol_B \wedge \cdots = \pi_*(vol_B\wedge\cdots)$ where

$\pi : P \longrightarrow M$. Then we have

$$\pi_*\left(\mathrm{vol}_B \wedge \pi^*\omega\right) = \omega \qquad \text{for } \omega \in \Omega(M) \text{ or } \Omega(\mathcal{a}\times M).$$

Then if vol_B is closed, this map will be compatible with d. Hence

$$d\int_{P/M} \mathrm{vol}_B \cdot \tilde{\alpha} = \int_{P/M} \mathrm{vol}_B \, d\tilde{\alpha} = \pi_*\left(\mathrm{vol}_B\, \pi^* c_k\right) = c_k$$

where $c_k = ch_k(pr_2^* E, \bar{A})$. Hence in $\Omega(\mathcal{a})$ we have

$$d\int_P \mathrm{vol}_B \cdot \tilde{\alpha} = (-1)^{\dim M} \int_M d \int_{P/M} \mathrm{vol}_B \cdot \tilde{\alpha}$$

$$= (-1)^{\dim M} \int_M c_k$$

and this is $(-1)^{\dim M}$ times the inverse image of $\int_M ch_k(\tilde{E}, \bar{A})$ in $\Omega(\mathcal{a}/\mathcal{G})$.

So I conclude that Iz's procedure works when P/M has a flat connection. This is what gives the reason for $ch_k(\tilde{E})$ to vanish when lifted to $\mathcal{a}\times M$.

December 4, 1983

Summary of the formulas

$M, P, \mathcal{U}, \mathcal{a}, \mathcal{G}, \tilde{\mathfrak{g}}, E = P \times^{\mathcal{U}} V$

The group $\mathcal{G}$ acts on $P, \mathcal{a}$ and hence on forms over these spaces. On $pr_2^*(P) = \mathcal{a} \times P$ over $\mathcal{a} \times M$ is a tautological connection $\tilde{A}$ which is $\mathcal{G}$-invariant. Its equivariant curvature consists of its curvature and the Higgs field ρ. Assume $\mathcal{G}$ acts freely on $\mathcal{a}$ and Θ is a connection in the principal $\mathcal{G}$-bundle $\mathcal{a} \longrightarrow \mathcal{a}/\mathcal{G}$. Then $\bar{A} = \tilde{A} + \rho\Theta$ is a connection in the principal $\mathcal{U}$-bundle $\mathcal{a} \times P / \mathcal{a} \times M$ which is invariant and has zero Higgs field. Hence it descends to a connection $\bar{\bar{A}}$ on $\mathcal{a} \times^{\mathcal{G}} P \,/\, (\mathcal{a}/\mathcal{G}) \times M$.

Here are the formulas with respect to a given trivialization $P = M \times \mathcal{U}$ of P, and in terms of $E = M \times V$.

$$\mathcal{G} = \text{Maps}(M, \mathcal{U})$$

$$\tilde{\mathfrak{g}} = \text{Maps}(M, \mathfrak{u}) = \Omega^0(M) \otimes \mathfrak{u}$$

$$\mathcal{a} = \Omega^1(M) \otimes \mathfrak{u}$$

To each $A \in \mathcal{a}$ we associate the cov. derivative

$$D_A = d + A \quad \text{on} \quad \Omega(M, E) = \Omega(M) \otimes V.$$

The group $\mathcal{G}$ acts to the right on $\mathcal{a}$:

$$g * D_A = g^{-1} \cdot D_A \cdot g \quad \text{or} \quad g * A = g^{-1} dg + g^{-1} A g$$

$$v * D_A = [D_A, v] \quad \text{or} \quad v * A = dv + [A, v].$$

The vector field X_v on $\mathcal{a}$ associated to $v \in \tilde{\mathfrak{g}}$ is

$$X_v \Phi(A) = \Phi(A + v * A) - \Phi(A)$$

$$= \int dx \, [D_A, v] \frac{\delta \Phi}{\delta A(x)}(A) = -\int dx \, v(x) \left[D_A, \frac{\delta \Phi}{\delta A(x)}(A)\right](x)$$

or
$$X_v = \int dx \; v^a(x) X_a(x)$$
where
$$-X_a(x) = \partial_\mu \frac{\delta}{\delta A^a_\mu(x)} + \left[A_\mu \times \frac{\delta}{\delta A_\mu(x)} \right]_a$$

We describe the taut. connection $\tilde{A}$ on $pr_2^*(E)$ by the covariant derivative operator on $\Psi(A,x) \in \Omega^0(\mathcal{A}\times M)\otimes V$
$$\left(d_{\mathcal{A}\times M} + \tilde{A}\right)\Psi(A,x)$$
where
$$\tilde{A}\,\Psi(A,x) = A(x)\Psi(A,x) = A_\mu(x)\,dx^\mu\,\Psi(A,x).$$
The $\mathcal{G}$-action on sections of $\mathcal{A}\times E$ is
$$g\cdot\Psi(A,x) = g(x)\,\Psi(g*A,x)$$
$$v\cdot\Psi(A,x) = X_v\Psi(A,x) + v(x)\,\Psi(A,x)$$
The curvature of $\tilde{A}$ is
$$F_{\tilde{A}} = d_{\mathcal{A}}\tilde{A} + \underbrace{d_M\tilde{A} + \tilde{A}^2}_{\tilde{F}}$$
where $\quad \tilde{F}$ at A,x is $F_A(x) \qquad \left(\tilde{F}\Psi(A,x) = F_A(x)\Psi(A,x)\right)$

$d_{\mathcal{A}}\tilde{A}$ at A,x is $\delta A(x) = \delta A_\mu(x)\,dx^\mu$.

The Higgs field is $\quad \rho v = \tilde{v} \in \Omega^0(\mathcal{A}\times M)\otimes u$

where $\quad \tilde{v}$ at A,x is $v(x) \qquad \left(\tilde{v}\Psi(A,x) = v(x)\Psi(A,x)\right)$

Next I want a formula for $\bar{A} = \tilde{A} + \rho\theta$, and this requires a choice of θ. $\theta \in \Omega^1(\mathcal{A},\tilde{g})$, and the formula is
$$\iota_{\delta A}\theta = (D_A^* D_A)^{-1} D_A^*\,\delta A \qquad D_A = [D_A,\cdot\,] : \underset{\Omega^0(M,u)}{\tilde{g}} \to \Omega^1(M,u)$$

Now we want to restrict to a $\mathcal{G}$-orbit. There are two approaches possible at this point.

i) Use the map $\mathcal{G} \longrightarrow \mathcal{A}$, $g \longmapsto g^* A^0$ and pull-back $\tilde{A}$, $\bar{A}$ to forms on $\mathcal{A} \times M$.

ii) Fix a point A^0 of $\mathcal{A}$ and calculate the forms $\tilde{A}$, $\bar{A}$ etc. to the tangent space to the $\mathcal{G}$-orbit thru A^0.

Let's start with (i). Let $\hat{g} : \mathcal{G} \times M \longrightarrow U$ be the evalation map $\hat{g}(g,x) = g(x)$. The formulas for the two connections are

$$\boxed{\begin{aligned} d_{\mathcal{G}\times M} + \tilde{A} &= d_{\mathcal{G}} + \hat{g}^{-1}(d_M + A^0)\cdot\hat{g} \\ d_{\mathcal{G}\times M} + \bar{A} &= \hat{g}^{-1}\cdot(d_{\mathcal{G}\times M} + A^0)\cdot\hat{g} \end{aligned}}$$

where $A^0 \Psi(g,x) = A^0(x)\Psi(g,x)$. Proof:

$$\begin{aligned} (d_M + \tilde{A})\Psi(g,x) &= d_M \Psi(g,x) + (g^* A^0)(x)\Psi(g,x) \\ &= (d_M + g^{-1} d_M g + g^{-1} A^0 g)\Psi(g,x) \\ &= g^{-1}\cdot(d_M + A^0)\cdot g\, \Psi(g,x) \end{aligned}$$

which shows $\boxed{d_M + \tilde{A} = \hat{g}^{-1}\cdot(d_M + A^0)\hat{g}}$ and proves the first formula. As for the second $\bar{A} = \tilde{A} + \rho\,\theta$ and θ restricted to $\mathcal{G}$ becomes the Maurer-Cartan form $\theta_{\mathcal{G}}$. One has by checking definitions and $\rho_0 = \check{v}$ that

$$\boxed{\rho\,\theta_{\mathcal{G}} = \hat{g}^{-1} d_{\mathcal{G}} \hat{g} \in \Omega^1(\mathcal{G}\times M)\otimes \mathfrak{u}.}$$

The second formula follows.

Next we consider the Chern-Simons form for $\bar{A}$.

since
$$F_{\bar A} = \hat g^{-1} F_{A_0} \hat g$$
$$\bar A = \hat g^{-1} d_{\mathcal G} \hat g + \hat g^{-1} d_M \hat g + \hat g^{-1} A^0 \hat g$$
we have at the point g, x of $\mathcal G \times M$
$$F_{\bar A}(g,x) = (g^{-1} F_{A_0} g)(x) = F_A(x) \quad \text{where } A = g^* A_0$$

$\cancel{\bar A(g,x) = (\omega(g))^* A(x) \quad \text{where } \omega = \hat g^{-1} d_{\mathcal G} \hat g \ \text{at} \ g}$

$\bar A(g,x) : \tilde{\mathfrak g} \oplus T_M(x) \longrightarrow$ [redacted] $\mathfrak u$ is the sum of $v \longmapsto v(x)$ and $A(x)$. Thus

$$\bar A \text{ at } (g,x) \quad \text{is} \quad \omega(x) + A(x)$$

where $\omega = \hat g^{-1} d_{\mathcal G} \hat g \in \Omega^0(\mathcal G \times M) \otimes \mathfrak u$.

Thus the Chern-Simons [redacted] form is

$$(*) \qquad \int_0^1 dt \ \operatorname{tr}\left\{ (A+\omega)\, e^{tF_A + (t^2-t)(A+\omega)^2} \right\}$$

along $g \times M$, where [redacted] $A = g^* A_0$.

(The above might be clearer without the $\hat g$:
$$\bar A = g^{-1} d_{\mathcal G}\, g + \tilde A = \tilde A + \omega$$
$$F_{\bar A} = g^{-1} F_{A^0} g = \tilde F$$
so Chern-Simons is
$$\int_0^1 dt \ \operatorname{tr}\left\{ (\tilde A + \omega)\, e^{t\tilde F + (t^2-t)(\tilde A + \omega)^2} \right\}$$
which restricts to the form $(*)$ along $g \times M$.)

Paradox: Over $\mathcal{G}\times M$ we have the two invariant connections $\tilde{A}$ and $\bar{A} = \tilde{A} + \Theta$. The transgression class we are after in $H^*(\mathcal{G}\times M)$ is obtained by joining $\bar{A}$ to 0. I used the linear path but could have used the path from 0 to Θ and Θ to $\tilde{A}+\Theta = \bar{A}$. The second path leads to the form

$$1)\qquad \int_0^1 dt \; tr\{\tilde{A}\, e^{(1-t)F_\Theta + tF_{\bar{A}} + (t^2-t)\tilde{A}^2}\}$$

and the first path to

$$2)\qquad \int_0^1 dt \; tr\{\Theta\, e^{tF_\Theta + (t^2-t)\Theta^2}\}.$$

Now restrict both of these to $\mathcal{G}\times M$. Then we have

$$F_{\bar{A}} = \tilde{F} \qquad\qquad \tilde{F} \text{ at } g^*A_0 = A \text{ is } F_A$$

$$\Theta = g^{-1}d_{\mathcal{G}}g$$

so
$$F_\Theta = (d_{\mathcal{G}} + d_M)\Theta + \Theta^2 = d_M\Theta.$$

So 1), 2) becomes

$$1)'\qquad \int_0^1 dt \; tr\{\tilde{A}\, e^{t\tilde{F} + (t^2-t)\tilde{A}^2 + (1-t)\, d_M\Theta}\}$$

$$2)'\qquad \int_0^1 dt \; tr\{\Theta\, e^{t d_M\Theta + (t^2-t)\Theta^2}\}$$

The paradox arises because the form 1)' should be right $\mathcal{G}$-invariant as a form on $\mathcal{G}\times M$ since $\tilde{F}$ and $\tilde{A}$ are $\mathcal{G}$-invariant. This maybe is the mistake, because $\tilde{A}$ is not invariant, rather the connection

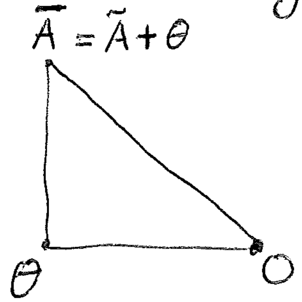
$d_{\mathcal{A}\times M} + \tilde{A}$ is invariant.

Thus it is important to keep in mind that $\tilde{A}, \bar{A}$ when viewed in $\Omega^1(\mathcal{A}\times M, \mathfrak{u})$ are connections forms and hence not $\mathcal{G}$-invariant for the natural action of $\mathcal{G}$ on forms on $\mathcal{A}\times M$ with values in $pr_2^*(E)$. However $F_{\tilde{A}}$ and $F_{\bar{A}}$ as well as $\bar{A}-\tilde{A} = \theta$ are invariant in this way. That is why the path from $\tilde{A}$ to $\bar{A}$ leads to a right-invariant form:

$$u(\bar{A},\tilde{A}) = \int_0^1 dt \; tr\left\{\theta \, e^{(1-t)F_{\tilde{A}} + tF_{\bar{A}} + (t^2-t)\theta^2}\right\}$$

But the path from θ to $\theta + \tilde{A} = \bar{A}$ will not give a right-invariant form, because $\tilde{A}$ is not right-invariant.

Next we should compare the above with Zumino. He actually works out the triangle



and so finds a formula for the difference of

$$\int_0^1 dt \; tr\left\{(\tilde{A}+\theta)\, e^{t\tilde{F} + (t^2-t)(\tilde{A}+\theta)^2}\right\}$$

and $1)' + 2)'$.