DANIEL G. QUILLEN (RESEACH SUMMARY 1982-83)

1. Determinants of $\bar{\partial}$-operators as a Riemann surface

Let M be a closed Riemann surface, let E be a vector bundle with hermitian metric over it, and consider the space $\mathcal{A}$ of unitary connections on E. Over $\mathcal{A}$ is a holomorphic line bundle $\mathcal{L}$, called the determinant line bundle, which is the highest exterior power of the index of the family of $\bar{\partial}$-operators D_A on E associated to the different connections. There is a natural metric on $\mathcal{L}$ defined by using analytic torsion, or zeta-function determinants. I have proved that the curvature of $\mathcal{L}$ for the connection associated to this metric and the holomorphic structure is equal to the natural Kähler form on $\mathcal{A}$.

Suppose that the degree of E is such that the $\bar{\partial}$-operators D_A have index zero, and let A_o be a basepoint in $\mathcal{A}$. Using the curvature theorem I construct an analytic function $\det_{A_o}(D_A)$ on $\mathcal{A}$, unique up to a scalar factor, such that

$$|\det_{A_o}(D_A)|^2 e^{-\|A-A_o\|^2} = \text{zeta-function determinant of } D_A^* D_A .$$

The determinant function $\det_{A_o}(D_A)$ is a generalization to vector bundles of the Θ-functions used in the study of line bundles on a Riemann surface. It depends on the choice of basepoints, and is not gauge-invariant, but rather transforms with a simple exponential factor. Because the group $\mathcal{G}$ of gauge transformations acts on $\mathcal{L}$ preserving the connection, the lack of gauge-invariance is closely related to the moment map for the $\mathcal{G}$-action on the symplectic manifold $\mathcal{A}$. The fact that the moment map is given by assigning to a connection its curvature is equivalent to a formula giving the 'gauge anomaly', i.e. the change in the determinant under infinitesimal gauge transformations.

It would be interesting to extend the above to the situation where the metric, and hence holomorphic structure, on M also is allowed to vary. The corresponding determinant function should generalize modular functions for elliptic curves.

2. Cyclic cohomology and cohomology of the Lie algebra of infinitesimal gauge transformations.

In his recent work on the index theorem for foliated manifolds, A. Connes defined a new cohomology theory for associative algebras called cyclic homology. This year Loday and I proved that the cyclic cohomology of A is the same as the primitive part of the cohomology of the Lie algebra $\mathfrak{gl}_n(A)$ in the stable range, where n is large relative to the cohomology degree. By virtue of Connes' results on cyclic cohomology, the Lie algebra cohomology of $\mathfrak{gl}_n(A)$ in the stable range can be computed for many interesting algebras (see [1,2]).

Let $\tilde{\mathfrak{g}}$ be the Lie algebra of infinitesimal gauge transformations of a principal G-bundle over a compact manifold M, where G is a compact Lie group. By curvature methods one can associate to a homogenous invariant polynomial ϕ of degree i on the Lie algebra of G a class ϕ' in $H^{2i-1}(\tilde{\mathfrak{g}}, \Omega_M^{<i})$, where $\Omega_M^{<i}$ is the truncation of the De Rham complex of M which is zero in degrees $\geq i$. Integrating ϕ' over closed currents in M gives Lie algebra cohomology classes for $\tilde{\mathfrak{g}}$, and one might hope to obtain generators for the Lie algebra cohomology in this way. In the case of the

trivial U_n-bundle where $\tilde{\mathfrak{g}} = \mathfrak{gl}_n(C^\infty(M))$, this can be proved in the stable range using the above results on cyclic cohomology. One obtains the formual $(n > p + 1)$:

$$\mathrm{Prim}\{H^{p+1}(\tilde{\mathfrak{g}})\} = \left\{\begin{matrix}\text{closed currents}\\ \text{of dimension } p\end{matrix}\right\} \oplus H_{p-1}(M,\mathbb{R}) \oplus H_{p-4}(M,\mathbb{R}) \oplus \cdots$$

References

1. J.-L. Loday and D. Quillen, Homologie cyclique et homologie de l'algèbres de Lie des matrices, Comptes Rendus Acad. Scie., 296 (1983), Sér. I, 295-297.

2. D. Quillen, On the cyclic homology of algebras (to appear).

Lectures given in England

Warwick (Colloquium)	October, 1982.
British Mathematical Colloquium (Aberdeen)	April 7, 1983
Liverpool (Colloquium)	April 22.
Cambridge (Adam's seminar)	April 28.
Edinburgh (Colloquium)	May 3.

In Oxford

Atiyah's seminar	January 31.
Atiyah's seminar	May 9.
Topology seminar	June 6 (?)

Seven lectures during Hilary term on "Determinants of $\bar{\partial}$-operators on a Riemann surface".

Conferences outside England

Marseilles (Luminy) - C* algebras	March 21-25.
Marseilles (Luminy) - Algebra K-theory	May 23-27.
Bonn- Arbeitstagung	June 16-23.