May 18, 1986

Recall that by using a smooth approximation to the Heaviside function I constructed a connection on the bundle E over $U_n \times S^1$ such that its character forms when integrated over the circle gave the odd character forms on U_n. Recall also that the connection extended the tautological partial connection in the S^1-direction, and that hence the construction of this connection is equivalent to a connection in the principal $\mathcal{G}_*$-bundle $\mathcal{A} \to U_n$. Recall further that Atiyah-Singer showed how to transgress the odd forms on the parameter space U_n to even forms on the $\mathcal{G}_*$-orbits, and that I in my letter to Singer showed how to do this $\mathcal{G}$-invariantly for the odd forms of degree ≥ 3.

Conclude: There is a way to lift the odd character forms γ_{2k+1} on U_n to $\mathcal{A}$ and write it as $\gamma_{2k+1} = d\mu_{2k}$ where μ_{2k} is $\mathcal{G}$-invariant.

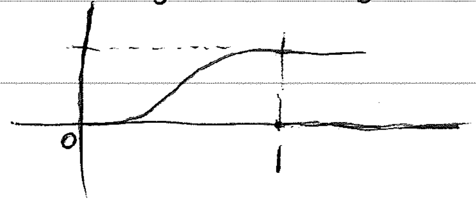
Recall that over $\mathcal{A} \times S^1$ one has a tautological connection $\tilde{A}$ which one modifies to $\tilde{A} + \varphi\theta$ so it descends to $U_n \times S^1$. The character forms associated to $\tilde{A}$ vanish by Bott's theorem, so a deformation from $\tilde{A}$ to $\tilde{A} + \varphi\theta$ leads to a way of writing γ_{2k+1} as $d\mu_{2k}$ for $k \geq 1$. Unfortunately the formulas are messy:

The connection constructed on E over $U_n \times S^1$ when lifted to $U_n \times \mathbb{R}$, or say, $U_n \times [0,1]$ where E becomes trivial is

$$\delta + dt\,\partial_t + \hat{A}(g,t) \qquad \delta = d_{U_n}$$

where $\hat{A}(g,t) = \rho(t)\, g^{-1}\delta g \qquad 0 \le t \le 1$

ρ:



This lifts back to the connection over $\mathcal{A} \times [0,1]$ given by the same formula with $g = h(1)$. We think of $\mathcal{A}$ as paths $h: \mathbb{R} \to U_n \ni h(t+1) = h(1)h(t)$ and $h(0) = 1$. This gives the connection $\tilde{A} + \varphi\theta$ which descends.

On the other hand $\tilde{A}$ is the tautological connection

$$\delta + dt\,\partial_t + dt\, h(t)^{-1}h'(t).$$

Thus we have the two connection forms

$$\rho(t)\, g^{-1}\delta g \in \Omega^{1,0}(U_n \times S^1, \mathfrak{g}) \subset \Omega^{1,0}(\mathcal{A} \times S^1, \mathfrak{g})$$

$$dt\, h(t)^{-1}h'(t) \in \Omega^{0,1}(\mathcal{A} \times S^1, \mathfrak{g})$$

Something seems strange because

$$\tilde{A} \in \Omega^{0,1}(\mathcal{A} \times M, \mathfrak{g})$$

$$\theta \in \Omega^1(\mathcal{A}, \tilde{\mathfrak{g}}) \subset \Omega^{1,0}(\mathcal{A} \times M, \mathfrak{g})$$

$\varphi \in$ [redacted] $\mathrm{Hom}(\tilde{\mathfrak{g}}, \Omega^0(M, \mathfrak{g}))$ is ess. the identity

so that $\tilde{A} + \varphi\theta$ has components of both types $(1,0)$ and $(0,1)$.

?

May 19, 1986

I would like to understand the relation between loops in the Grassmannian and the loop group $\mathcal{G}^\sigma$.

Let's start with the continuous level. An element of $\mathcal{G}^\sigma_*$ is a periodic $g(t): \mathbb{R} \longrightarrow U(N)$ such that $g(0) = 1$ and $\varepsilon g(-t)\varepsilon = g(t)$. A continuous $g(t)$ with these properties is the same as a map $g: [0,\frac{1}{2}] \longrightarrow U(N) = G$ such that $g(0)=1$ and $g(\frac{1}{2}) \in G^\varepsilon$. In effect given such a g on $[0,\frac{1}{2}]$ one extends it to $[-\frac{1}{2},\frac{1}{2}]$ so that $g(-t) = \varepsilon g(t)\varepsilon$; the two values at $t=0$ coincide; similarly the values $g(-\frac{1}{2}) = \varepsilon g(\frac{1}{2})\varepsilon = g(\frac{1}{2})$ coincide, so one can extend to all of $\mathbb{R}$ so as to be periodic.

Next one can ask about smoothness, and for this one should pass to the connection $A = g^{-1}g'$. We have

$$\begin{aligned} \varepsilon A(-t)\varepsilon &= \varepsilon\, g(-t)^{-1} g'(-t)\, \varepsilon \\ &= g(+t)^{-1}\left(-\frac{d}{dt}\right) \varepsilon g(-t)\varepsilon \\ &= -A(t) \end{aligned}$$

so that we want $A^+(t)$ to be odd and $A^-(t)$ to be even. Thus given a smooth $g(t): [0,\frac{1}{2}] \longrightarrow U(N)$ with $g(0)=1$, we want to extend it to $[-\frac{1}{2},\frac{1}{2}]$ so that $\varepsilon g(-t)\varepsilon = g(t)$. Thus we want to extend $A(t)$ smoothly to $[-\frac{1}{2},\frac{1}{2}]$ so that $\varepsilon A(-t)\varepsilon = -A(t)$. There are obviously an infinite number of derivative conditions on A at $t=0$ to be satisfied. The same should be true at $t=\frac{1}{2}$.

Given $g \in \mathcal{G}^\sigma_*$ we can send it to the path

$$F_t = g(t)\,\varepsilon\, g(t)^{-1} = g(t)\, g(-t)^{-1}\,\varepsilon$$

in the Grassmannian. This satisfies

$$\star \quad \begin{cases} F_{t+1} = F_t & F_0 = F_{1/2} = \varepsilon \\ \varepsilon F_t \varepsilon = F_{-t} \end{cases}$$

In general, $\mathcal{G}^\sigma$ acts on these paths in $G_n(\mathbb{C}^N)$ by

$$(g \cdot F)_t = g(t) F_t\, g(t)^{-1}$$

and a natural question is whether it is onto.

So let us start with a smooth F_t satisfying $\star$ and construct a lifting:

$$h \varepsilon h^{-1} = F$$

$$h' \varepsilon h^{-1} - h \varepsilon h^{-1} h' h^{-1} = F'$$

$$[h^{-1}h', \varepsilon] = h^{-1} F' h \quad \text{anti-comm. with } h^{-1}Fh = \varepsilon$$

so we take $\quad h^{-1}h' = \frac{1}{2} h^{-1} F' h \varepsilon$, whence

$$h' = \tfrac{1}{2} F' h \varepsilon$$

$$h' h^{-1} = \tfrac{1}{2} F' F$$

Actually the formulas might have been simpler with h^{-1}:

$$h^{-1} (h^{-1})' = \tfrac{1}{2} F F'.$$

Note that $\quad \varepsilon F(-t) F'(-t) \varepsilon = F(t) - \frac{d}{dt} \varepsilon F(-t) \varepsilon$

$$= - F(t)\, F'(t)$$

belongs to $\mathcal{A}^\sigma$, so that $\varepsilon h^{-1}(-t) \varepsilon = h^{-1}(t)$ whence also $\varepsilon h(-t) \varepsilon = h(t)$. Also $h(0) = 0$.

Finally $h^{-1}(t+1) = h^{-1}(1)\,h^{-1}(t)$ and

$$\varepsilon = h(\tfrac{1}{2})\,\varepsilon\, h(+\tfrac{1}{2})^{-1} = h(\tfrac{1}{2})\,h(-\tfrac{1}{2})^{-1}\varepsilon \quad \Rightarrow h(-\tfrac{1}{2}) = h(\tfrac{1}{2})$$

And $h(1) = h^{-1}(t)\,h(t+1)$, $h(t)h(1) = h(t+1)$,
$h(-\tfrac{1}{2})\,h(1) = h(\tfrac{1}{2}) \Longrightarrow h(1) = 1$. Thus $h \in \mathcal{G}^{\sigma}_{*}$.

<u>Conclusion:</u> Any smooth path F_t in the Grassmannian such that

$$F_0 = F_{\frac{1}{2}} = \varepsilon, \qquad F_{t+1} = F_t, \qquad \varepsilon F_t \varepsilon = F_{-t}$$

comes from an element of $\mathcal{G}^{\sigma}$ which is unique up to right multiplication by a $g(t) : \mathbb{R} \to U(N)^{\varepsilon}$ which is periodic and $\ni$ $g(t) = g(-t)$.

May 22, 1986

I returned to the study of BG where $G = U_{res}$. This space classifies Hilbert bundles with splitting mod $\mathcal{K}$. To such a thing belongs an element of K^1 of the base Y which may be represented by a map $Y \longrightarrow \mathcal{I}(\mathcal{Q})$ by using Kuiper's thm. to trivialize the Hilbert bundle.

A goal of mine is to replace Kuiper's thm. by the weaker stable triviality of Hilbert bundles. One can always embed a Hilbert bundle into a trivial one easily; then infinite repetition gives stable triviality. Specifically if the Hilbert bundle is $\mathrm{Im}(e)$, e projector on the trivial bundle with fibre H, then one has

$$E \oplus H' \simeq 1 \otimes eH \oplus (z \otimes e + 1 \otimes (1-e))H' = H'$$

$$H' = H^2(S^1) \otimes H.$$

From past experience with $\mathcal{G}'$, we learn we can do the following. Given E over Y with K-splitting represented by $A \in \mathcal{F}_1(E)$, we can find an embedding of E into a trivial graded Hilbert bundle (H, F) such that the grading F compresses to A. Then inf. repetition gives

$$E \oplus H' \simeq H' \qquad H' = H^2(S^1) \otimes H.$$

Now let $F' = 1 \otimes F$ on H' and use the above isom. to transport $A \oplus F'$ to a $\tilde{A}$ on H' having the same zeroes as A. Thus we get a map $Y \to \mathcal{F}_1(H')$ and a fortiori, a map $Y \longrightarrow \mathcal{I}(\mathcal{Q}(H'))$.

This construction differs from one we gave before in that on the complementary bundle we previously used $F = +1$ or -1. The significance of the refined construction isn't clear.

Another description of $\tilde{A}$ is probably the following. On $L^2(S^1) \otimes H$ one has the automorphism given by the loop $ze + 1 - e$, and one conjugates $1 \otimes F$ by this automorphism to get a new involution. Then $\tilde{A}$ is the compression of this to H'.

<u>Summary:</u> We have used the infinite repetition to construct a map $B\mathcal{G} \longrightarrow \mathcal{J}(2)$ which is more symmetrical. However the significance isn't clear.

It seems that we keep coming back to the following problem. Consider $H = L^2(S^1, \mathbb{C}^N)$ with the Hilbert involution F, and suppose given a family of loops in $Gr_n(\mathbb{C}^N)$, i.e. a rank n ~~n-dim~~ vector subbundle of the trivial rank N bundle over $Y \times S^1$. Then we have the whole index thm. for families problem. We have a subbundle $\{\operatorname{Im} e_y\}$ of the trivial Hilbert bundle with fibre H; this subbundle has a Grassm. connection curvature. The problem is how are we supposed to get forms on Y out of F? I can consider a connection on the trivial bundle of rank N over S^1 such that the Hilbert involution is the phase of the corresponding Dirac operator. The Dirac operator on S^1 can be compressed to the Dirac on the bundle described by e_y.

May 23, 1986 (Alice is 24)

Before I can make any further progress, it is necessary to understand what happens over S^1. To fix the notation, we consider a vector bundle V over S^1 with a connection ∇. This gives us a Hilbert space H and a Dirac operator D. The latter determines a mod $\mathcal{K}$ splitting of H which we can lift to an involution F if we like. Now we consider a subbundle of the given vector bundle V. This gives a projector e on H such that $[F,e]\in\mathcal{K}$. The connection ∇ on V induces a connection $e\nabla e$ on the subbundle eV, and the Dirac op. on eV ~~for~~ this connection is eDe.

The problem is to determine the relation between the ~~self-adjoint~~ contractions one looks at for D and for eDe. We have nice behavior for the unbounded operators: D compresses to the Dirac on the subbundle. But what should one do in the case of the involution F?

Next let's become more specific. The pair (V,∇) is ~~classified by~~ determined up to isom. by the holonomy $\tau\in U_N$, so we can suppose sections of V are smooth $f: V\longrightarrow \mathbb{C}^N \ni f(t+1)=\tau f(t)$, and $\nabla f = f'(t)dt$. The projector e is then just a idempotent $N\times N$ matrix (hermitian) $e(t)$, $t\in\mathbb{R}$, satisfying

$$e(t+1) = \tau e(t)\tau^{-1}$$

I need the parallel transport in eV for the induced connection $e\nabla e$. I might as well find the parallel transport $h(t)$ in V for the connection

$$e\nabla e + (1-e)\nabla(1-e) = dt\{\partial_t + ee' + (1-e)(-e')\}$$

$$e\nabla e + (1-e)\nabla(1-e) = dt\{\partial_t + (2e-1)e'\}$$

Put $F(t) = 2e(t) - 1$, whence

$$e\nabla e + (1-e)\nabla(1-e) = dt\{\partial_t + \tfrac{1}{2}FF'\}$$

and the parallel transport $h(t)$ is to satisfy

$$h^{-1}h' = [redacted] \ \tfrac{1}{2}FF' \qquad h(0)=1$$

It then follows that $F = h^{-1}\varepsilon h$. [redacted]
where $\varepsilon = F(0)$. __Proof:__ ~~It's enough to check that~~

$$\cancel{F' \overset{?}{=} (h^{-1}\varepsilon h)' = -h^{-1}h'h^{-1}\varepsilon h + h^{-1}\varepsilon h'}$$

$$\cancel{= h^{-1}[\varepsilon, h'h^{-1}]h}$$

$$\cancel{= h^{-1}[\varepsilon,}$$

$$(h^{-1}\varepsilon h)' = h^{-1}\varepsilon h' - h^{-1}h'h^{-1}\varepsilon h$$

$$= [h^{-1}\varepsilon h, h^{-1}h'] = [h^{-1}\varepsilon h, \tfrac{1}{2}FF']$$

$$[F, \tfrac{1}{2}FF'] = \tfrac{1}{2}(F^2F' - FF'F) = F'$$

so $h^{-1}\varepsilon h$ and F satisfy the same initial value problem.

[redacted]

Notice that because $e(t+1) = \tau e(t)\tau^{-1}$, we have

$$F(t+1) = \tau F(t)\tau^{-1}$$

and hence $h^{-1}h' = \frac{1}{2}FF'$ is not periodic unless τ is a scalar. This is a result of the particular description of the original bundle V.

At this point we have found the [redacted] parallel transport in the bundle eV so in principle we know everything about the Dirac op. eDe. In practice we might have difficulty identifying the near zero modes. It is not clear what to do next.

May 26, 1986

Let $H = H^+ \oplus H^-$, $\varepsilon = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ be given and let us consider the model for $B\mathcal{G}$, $\mathcal{G} = U_{res}$, consisting of subspaces $eH \subset H$ such that $[\varepsilon, e] \in \mathcal{K}$ and ε induces non-trivial involutions ~~mod~~ $\mathcal{K}$ on both eH and $(1-e)H$.

Then over $B\mathcal{G}$ we have the Hilbert bundle $e \longmapsto eH$ with splitting mod $\mathcal{K}$. To define the index of this gadget, we embed the Hilbert bundle in a trivial one and extend the splitting mod $\mathcal{K}$ on eH by a trivial one on the complement.

Infinite repetition gives the embedding

$$eH \subset eH \oplus \underbrace{(ze+1-e)\,H^2(S^1)\otimes H} = H^2(S^1)\otimes H$$

and we can use the involution on given by

$$(ze+1-e)(1\otimes\varepsilon)(ze+1-e)^{-1}.$$

Combining this involution on the complement of eH in ~~The above process suggests~~ $H' = H^2(S^1)\otimes H$ with the self-adjoint contraction $e\varepsilon e$ on eH, we obtain a self-adjoint contraction on H', which is in $\mathcal{F}_1(H')$ since by assumption $e\varepsilon e$ is an involution mod $\mathcal{K}$. Thus we obtain a map

$$B\mathcal{G} \longrightarrow \mathcal{F}_1(H')$$

which is our index map.

Another way to go from e to a self-adjoint contraction which is suggested by the above formulas is to take the involution $F = (ze+1-e)(1\otimes\varepsilon)(ze+1-e)^{-1}$ on $L^2(S^1)\otimes H'$ and compress it to H'. I claim we get exactly $e\varepsilon e$ on eH together with this involution on the complement.

To see this note that on compressing an involution to a subspace one can separate the intersection of the subspace with the eigenspaces of the involution. Clear F is an involution on $(ze+1-e)H'$ and so πF preserves the complement $eH \subset H'$. Here π is the projection of $L^2(S^1)\otimes H$ onto H'.

We now now compute πF on eH.

$$1\otimes eH \xrightarrow{(ze+1-e)^{-1}} z^{-1}\otimes eH \subset z^{-1}\otimes H \xrightarrow{1\otimes\varepsilon} z^{-1}\otimes H$$

$$z^{-1}\otimes H \xrightarrow{(ze+1-e)} 1\otimes eH + z^{-1}\otimes(1-e)H \xrightarrow{\pi} 1\otimes eH$$

So we see $1\otimes eh$ goes to $z^{-1}\otimes eh$ goes to $z^{-1}\otimes \varepsilon eH$ goes to $1\otimes e\varepsilon eh$.

Thus $\pi F = e\varepsilon e$ on eH proving the claim.

May 29, 1986

It seems wise to write down some ideas found this past week when Jeanie was away before I go on my trip.

The persistant problem is how, given a Fredholm module (A, H, F), to construct differential forms on the space of projectors over A. These forms should be transgressions of the left-invariant forms on the group of unitaries over A which one has via cyclic theory.

Let's consider the case where $A = C^\infty(S^1)$, $H = L^2(S^1)$ and $F =$ Hilbert's involution. A projector over A is the same thing as a subbundle of a trivial bundle over S^1. Suppose $e \in M_N(A)$ has rank n, and change notation so that $H^{\oplus N}$, $F^{\oplus N}$ become H, F. Let $G = \mathcal{L}U_n$; we have already seen how to define left-invariant forms on G as follows. Given an embedding of the trivial rank n bundle into the trivial rank N bundle we get a homomorphism from G to the gp. $\mathcal{L}U_N$, which has the forms corresponding to the cyclic cocycles defined by F.

Now one of the things I don't know is exactly what these cyclic cocycles are. To be more clear, I have defined certain left-invariant forms on $\mathcal{L}U_N$ using connections, etc. Presumably the forms associated to the Hilbert involution are of this type. If this were so however, then I would have an understanding in diffl geometric terms of the cyclic cocycles on $G = \mathcal{L}U_n$ associated to points in the principal bundle P of embeddings of $\tilde{\mathbb{C}}^n$ in $\tilde{\mathbb{C}}^\infty$.

The idea here is that up to now I have been operating without an understanding of the forms on LU_n associated to the simplest involution.

The hope would be that these forms are simply related by means of the S-operator, in which case once we knew the 2-form the rest would be known.

June 1, 1986

Ideas: I have recently obtained a lot of experience with the Hilbert space point of view towards cyclic cocycles and periodicity. As yet I have no understanding of cyclic cocycles attached to naturally occurring operators such as the Hilbert involution on $L^2(S^1)$. Also no real understanding of the S-operator and the role of Hochschild cohomology.

However I have played around with involutions & projectors, dilations, Milnor's model. The ideas involved are different from the standard differential geometric approach. In particular I have seen how the familiar way of using a partition of unitary to construct a connection has a natural Hilbert space interpretation, e.g. the connection form over the Milnor model is

$$\sum t_j \, g_j^* dg_j$$

and this is related to embedding $H \hookrightarrow \oplus H$ by $v \longmapsto \sum \sqrt{t_j}\, in_j(v)$

Thus the t parameter used in deformations of connections $(1-t)\nabla_0 + t\nabla_1$ now has a new interpretation.

The idea now is to go back over the construction of cyclic cocycles using differential geometric methods and to try to work in these new ideas.

First let's review how we previously constructed left invariant forms on $\mathcal{G} = \mathrm{Aut}(E)$, E a vector bundle with inner product over the manifold M. We used the fact that a left-invariant form on $\mathcal{G}$, or Lie algebra cochain on $\tilde{\mathfrak{g}}_c = (\mathrm{Lie}\,\mathcal{G})_c$, is the same thing a natural transformation from flat connections on the trivial $\mathcal{G}$-bundle over Y to forms on Y. And this is the same as a natural way to go from a flat partial connection in the Y-direction on $pr_2^*(E)$ over $Y \times M$

$$d_Y + \theta \qquad\qquad \theta \in \Omega^1(Y, \tilde{\mathfrak{g}}) = \Omega^{1,0}(Y \times M, \mathrm{End}(pr_2^*E))$$

to forms on Y. To do this we can make choices over M, i.e. choose a connection ∇ or operator.

The construction done previously starts with choosing a connection ∇ [redacted] on E over M. Then one has a 1-parameter family of connections on $pr_2^*(E)$

$$(*) \qquad \tilde{\nabla}_t = \delta + \nabla + t\theta$$

where $\delta = d_Y$ and $\delta + \nabla$ is the pull-back of ∇. So

$$\tilde{\nabla}_t^2 = \nabla^2 + t[\nabla, \theta] + (t^2 - t)\theta^2$$

$$\mathrm{tr}\left(e^{\nabla^2 + [\nabla,\theta]}\right) - \mathrm{tr}\left(e^{\nabla^2}\right) = (\delta + d)\int_0^1 dt\ \mathrm{tr}\left(e^{\nabla^2 + t[\nabla,\theta] + (t^2-t)\theta^2}\,\theta\right)$$

and as usual this leads to closed left-invariant forms on $\mathcal{G}$ [redacted] by applying Bott's theorem.

But [redacted] the 1-parameter family $(*)$, as I have mentioned, has a new interpretation which frees us from the parameter t.

Let's recall the Bott map

$$U_n \longrightarrow \Omega(Gr_n(\mathbb{C}^{2n}), \varepsilon, -\varepsilon)$$

$$(+) \qquad I \times U_n \longrightarrow Gr_n(\mathbb{C}^{2n})$$

$$(\theta, g) \longmapsto (\cos\theta)\varepsilon + (\sin\theta)\begin{pmatrix} 0 & g^* \\ g & 0 \end{pmatrix} \qquad 0 \leq \theta \leq \pi$$

$$= \begin{pmatrix} \cos\theta & (\sin\theta)g^* \\ (\sin\theta)g & -\cos\theta \end{pmatrix}$$

The $+1$ eigenspace of the involution is the image of the isom. embedding

$$j_{\theta,g} = \begin{pmatrix} \cos\frac{\theta}{2} \\ (\sin\frac{\theta}{2})g \end{pmatrix}$$

so the subbundle of the Grassmannian pulls back to the trivial bundle over $I \times U_n$ with the connection form

$$j^* dj = \cos\tfrac{\theta}{2}\, d(\cos\tfrac{\theta}{2}) + g^*(\sin\tfrac{\theta}{2})\, d((\sin\tfrac{\theta}{2})g)$$

$$= (\sin^2\tfrac{\theta}{2})\, g^* dg .$$

So we see $t = \sin^2(\frac{\theta}{2})$, which is nice, as it puts t and $1-t$ on an equal footing.

Suppose we now generalize the above Bott map $(+)$ from $\mathbb{C}^n, U_n$ to $E, \mathcal{G}$. This means we look at the map from $I \times \mathcal{G}$ to subbundles of $E \oplus E$ which sends θ, g to the image of $j_{\theta,g} = \begin{pmatrix} \cos\frac{\theta}{2} \\ (\sin\frac{\theta}{2})g \end{pmatrix}$.

What I am doing is to go through the first stage of the Milnor model for BG. Over the join $G * G$ one has a family of embeddings of E into $E \oplus E$, and so we have a canonical vector bundle $\tilde{E}$ over $\mathrm{Susp}(G) \times M$ which is a subbundle of $pr_2^*(E \oplus E)$. If ∇ is a connection on E over M, then there is an induced connection on $pr_2^*(E \oplus E)$ and hence also one on $\tilde{E}$.

It should be easy to determine this connection on $\tilde{E}$ by pulling back to $I \times G \times M$ where $\tilde{E}$ becomes $pr_{12}^*(E)$. The connection should be

$$j^*(d\theta\,\partial_\theta + d_G + \nabla)\,j$$

$$= \begin{pmatrix} \cos\frac{\theta}{2} & (\sin\frac{\theta}{2})g^* \end{pmatrix} (d\theta\,\partial_\theta + d_G + \nabla) \begin{pmatrix} \cos\frac{\theta}{2} \\ (\sin\frac{\theta}{2})g \end{pmatrix}$$

$$= \cos\tfrac{\theta}{2}\left(d\cos\tfrac{\theta}{2}\right) + \sin\tfrac{\theta}{2}\, d\left(\sin\tfrac{\theta}{2}\right) + d\theta\cdot\partial_\theta$$

$$\left(\sin\tfrac{\theta}{2}\right)^2 g^*(dg + [\nabla, g]) \qquad + d_G + \nabla.$$

$$= pr_{12}^*(\nabla) + \left(\sin\tfrac{\theta}{2}\right)^2 g^*\left(dg + [\nabla, g]\right)$$

Thus we have the straight line path between ∇ and $g^*\cdot\nabla\cdot g$. This we know leads to the odd forms on $G \times M$, hence to forms on G generating the cohomology at least rationally by integrating over cycles in M. <u>But we don't get</u>

the connection family $\tilde{\nabla}_t = \delta + \nabla + t\Theta$ which gives rise to the left invariant forms.

Summary: Recall that we want to use the model for $B\mathcal{G}$, $\mathcal{G} = U_{res}$ consisting of suitable embeddings into H, where H comes with a fixed involution F. Today's idea was to use instead of (H, F) the bundle $\oplus E$ together with the fixed connection $\oplus\nabla$ on $\oplus E$. This means we replace the operator by a connection. We see this leads to forms on $B\mathcal{G} \times M$ like those constructed by Atiyah + Singer, [redacted] hence to non-left-[redacted] invariant forms on $\mathcal{G}$.

Let's return to another idea: Consider an isom embedding of vector bundle $j : E \longrightarrow E'$ and the induced homomorphism of grps of gauge transfs.

$$j_* : \mathcal{G} \longrightarrow \mathcal{G}'.$$

We have seen how associated to a connection ∇' in $\mathcal{G}'$ there are closed left-invariant forms on $\mathcal{G}'$. These can be pulled back to $\mathcal{G}$ and a natural question is whether they can be determined from the induced connection $j^* \cdot \nabla' \cdot j$ on E.

To fix the ideas suppose $E' = \mathbb{C}^N$ with $\nabla' = d$, and let j be the inclusion of a subbundle E of E' and put $e = jj^*$ so that $E = \operatorname{Im}(e)$. The basic forms associated to E', d are

$$(**) \qquad \int_0^1 dt \; \operatorname{tr}\left(e^{t\,d\theta + (t^2 - t)\theta^2}\, \theta\right)$$

and we propose to restrict them to E. This means that θ takes values in

$$e\,\Omega^0(M, \mathrm{End}\,E')\,e = e\big(\Omega^0(M) \otimes M_N\big)e = \Omega^0(M, \mathrm{End}\,E).$$

Note that we have

$$\Omega^0(M, \mathrm{End}\,E) = e\big(\Omega^0(M) \otimes M_N\big)e$$

but that this subalgebra is not closed under d. We have for $a \in \Omega^0(M, \mathrm{End}\,E)$ $\qquad [\nabla, a]$, $\nabla = e\,d\,e$

$$da = d(eae) = de.ae + e.da.e + ea.de$$

$$= (de.e)\,a + e.da.e + a(e\,de)$$

In terms of the splitting $\tilde{\mathbb{C}}^N = E \oplus E^\perp$ we have

$$da = \begin{pmatrix} [\nabla, a] & a(e.de) \\ (de.e)a & 0 \end{pmatrix}$$

The basic form $(**)$ on $\mathcal{G}'$ pulls back to

$$\int_0^1 dt \; \mathrm{tr}\left(e^{t[de.e\,\theta + \theta\,e\,de] + t[\nabla, \theta] + (t^2-t)\theta^2}\,\theta\right)$$

If one expands the exponential out as a series, then for the usual reason $(e\,de\,e = 0)$ the only monomials which are non-zero must involve the terms $\theta e de$ and $de\,e\,\theta$ consecutively, so the result will monomials in $[\nabla, \theta]$, θ^2 and

$$\theta\,e\,de\,de\,e\,\theta = \theta K \theta$$

where $K =$ curvature of ∇.

Let's attempt to understand the above exponential in terms of the resolvent

$$\frac{1}{\lambda - \Gamma}, \qquad \Gamma = t[de.e\,\theta + \theta\,e\,de] + t[\nabla, \theta] + (t^2-t)\theta^2$$

Then $\Gamma = \begin{pmatrix} H & Y \\ X & 0 \end{pmatrix} = \underbrace{\begin{pmatrix} H & 0 \\ 0 & 0 \end{pmatrix}}_{\tilde{H}} + \underbrace{\begin{pmatrix} 0 & Y \\ X & 0 \end{pmatrix}}_{V}$

and $$e\frac{1}{\lambda-\Gamma}e = e\left(\frac{1}{\lambda-\tilde{H}} + \frac{1}{\lambda-\tilde{H}}V\frac{1}{\lambda-\tilde{H}} + \cdots\right)e$$

$$= \frac{1}{\lambda-H} + \frac{1}{\lambda-H}\left(Y\frac{1}{\lambda}X\frac{1}{\lambda-H}\right) + \frac{1}{\lambda-H}\left(Y\frac{1}{\lambda}X\frac{1}{\lambda-H}\right)\left(Y\frac{1}{\lambda}X\frac{1}{\lambda-H}\right) +$$

$$= \frac{1}{\lambda-H}\cdot\frac{1}{1-Y\frac{1}{\lambda}X\frac{1}{\lambda-H}} = \frac{1}{\lambda-H-\frac{1}{\lambda}YX}$$

This is simple but not simple enough to use. In any case it seems we have proved the following (at least for E', ∇' trivial)

Proposition: Let $j: E \hookrightarrow E'$ be an isometric embedding and $j_*: \mathcal{G} \to \mathcal{G}'$ the induced homomorphism of gauge tr. groups. Then the left-invariant forms on $\mathcal{G}'$ determined by a connection ∇' on E' when pulled back to $\mathcal{G}$ via j_* are determined explicitly from the [redacted] connection on E [redacted] induced from ∇' via j.

The conjecture is that there might be a nicer way to define left-invariant forms on $\mathcal{G}$ associated to a connection ∇, which would work also for an operator D instead of ∇, and which would behave in a transparent manner relative to isometric embeddings.

Connes has done this for connections by noting [redacted] once one has a cycle in his sense, i.e. diffl graded algebra $\Omega^\cdot$, $\mathcal{A} \to \Omega^\cdot$; graded closed trace, then one can restrict to $e\mathcal{A}e$. The subalgebra closed under d in $\Omega^\cdot$ ~~containing~~ generated by $e\mathcal{A}e$ can be reconstructed from the "connection" $e\,d\,e$ and its curvature.

June 3, 1986 (married 25 years)

Basic invariant forms: On the Grassmannian we have

$$\operatorname{tr} F(dF)^{2n}$$

This is closed because applying d gives $\operatorname{tr}(dF)^{2n+1}$ which is zero as dF is of odd degree relative to the F-grading. Specifically if $F\alpha + \alpha F = 0$, then

$$\operatorname{tr}(\alpha) = \operatorname{tr}(F^2\alpha) = -\operatorname{tr}(F\alpha F) \overset{\text{trace symm.}}{=} -\operatorname{tr}(F^2\alpha) = -\operatorname{tr}(\alpha).$$

In the case of involutions anti-commuting with a fixed involution ε we have the form

$$\operatorname{tr} \varepsilon F(dF)^{2n-1}.$$

This is closed as applying d gives $\operatorname{tr} \varepsilon F(dF)^{2n}$, which is zero as ε and dF are odd relative to the F grading.

Let us compose the two Bott maps

$$\mathrm{Grass}(\mathbb{C}^n) \longrightarrow \Omega U_n \qquad e \mapsto (e^{i\theta})e + 1 - e \qquad 0 \le \theta \le 2\pi$$

$$U_n \longrightarrow \Omega\left(Gr_n(\mathbb{C}^{2n}), \varepsilon, -\varepsilon\right) \qquad g \mapsto \begin{pmatrix} \cos\theta & (\sin\varphi)g^* \\ (\sin\varphi)g & -\cos\theta \end{pmatrix} \qquad 0 \le \varphi \le \pi.$$

We then get the map

$\cancel{S^2 \times \mathrm{Grass}(\mathbb{C}^n) \longleftrightarrow Gr_n(\mathbb{C}^{2n})}$

~~which we can describe as follows. Given $e \in \mathrm{Grass}(\mathbb{C}^n)$ the map~~

$$(\theta, \varphi, e) \longmapsto \underbrace{\begin{pmatrix} \cos\varphi & (\sin\varphi)e^{i\theta} \\ (\sin\varphi)e^{i\theta} & -\cos\varphi \end{pmatrix}}_{\substack{\text{involution whose} \\ +1 \text{ eigenspace is} \\ \text{spanned by}}} \otimes e + \begin{pmatrix} \cos\varphi & \sin\varphi \\ \sin\varphi & -\cos\varphi \end{pmatrix} \otimes (1-e)$$

$$\begin{pmatrix} \cos\frac{\varphi}{2} \\ (\sin\frac{\varphi}{2})e^{i\theta} \end{pmatrix}$$

Note that

$$(\theta, \varphi) \longrightarrow z = \tan\left(\tfrac{\varphi}{2}\right) e^{i\theta} \qquad \begin{array}{l} 0 \le \theta \le 2\pi \\ 0 \le \varphi \le \pi \end{array}$$

is the standard parameterization of the 2-sphere $S^2 = \mathbb{CP}^1$.

Thus we see that we have the map

$$\mu : S^2 \times \mathrm{Grass}(\mathbb{C}^n) \longrightarrow Gr_n(\mathbb{C}^2 \otimes \mathbb{C}^n)$$

which classifies the bundle $L \otimes S \oplus L' \otimes S^\perp$, where $L =$ subbundle $\mathcal{O}(-1)$ over $\mathbb{CP}^1$, $S =$ subbundle over $\mathrm{Grass}(\mathbb{C}^n)$, and L' is the trivial line bundle obtained by mapping S^2 onto the great semi-circle $\theta = 0$. L^1 can be trivialized using the connection over this semi-circle.

Thus it appears that the subbundle S' of $Gr_n(\mathbb{C}^2 \otimes \mathbb{C}^n)$ pulls back under μ to a vector bundle isomorphic to

$$\mathcal{O}(-1) \otimes S \oplus \mathcal{O} \otimes S^\perp \subset \overbrace{\mathbb{C}^2 \otimes \mathbb{C}^n}$$

and moreover this isomorphism is compatible with the connections, $\mathcal{O}(-1)$ and S being equipped with the Grassmannian connections. Thus on the form level

$$\mu^* ch_k(S') = 1 \otimes ch_k(S) + \alpha \otimes ch_{k-1}(S) + 1 \otimes ch_k(S^\perp)$$

$$= \alpha \otimes ch_{k-1}(S)$$

where α is the curvature form of $\mathcal{O}(-1)$

There are several problems: Firstly it would be nice to have this Bott map on the level of the restricted Grassmannian. But even if we did, it is not clear that such a Bott map would be helpful with understanding the S-operator.

The map goes the wrong way, because suppose have cyclic cocycles obtained from a map $G \longrightarrow \mathcal{I}_{res}$ by pulling back the invariant forms. Then a Bott map would give

$$S^2 \times G \longrightarrow S^2 \times \mathcal{I}_{res} \longrightarrow \mathcal{I}_{res} \quad ?$$

June 7, 1986

Problem: Link the S operator to periodicity.

The first approach involves trying to use a Bott map $S^2 \times \text{Grass} \longrightarrow \text{Grass}$. Let $\mathcal{I}^p$ be the p-th Schatten class Grassmannian, i.e. the space of F congruent to a fixed F_0 mod $\mathcal{L}^p$. On $\mathcal{I}^p$ one has the invariant character forms

$$\omega_{2k} = \text{Const} \ \operatorname{tr}\left(F(dF)^{2k}\right) \qquad 2k > p.$$

It would appear that the best sort of Bott map one could hope for is a map

$$\beta : S^2 \times \mathcal{I}^p \longrightarrow \mathcal{I}^{p+2} \qquad \text{such that}$$

$$(*) \qquad \int_{S^2} \beta^*(\omega_{2k+2}) = \omega_{2k} \qquad \text{for } 2k > p$$

Otherwise we would be able to define ω_{2k} on $\mathcal{I}^p$ for $2k \leq p$ which we know is impossible.

If a Bott map $(*)$ were to exist, then we see one can determine ω_{2k-2} from ω_{2k}, which suggests the higher cyclic cocycles associated to a Fredholm module determine the lower ones. As this is the wrong direction, <u>we conclude that a Bott</u> map is not likely to provide the link we seek.

June 8, 1986

The problem is to link the S-operator with periodicity. Or to explain the S-operator in terms of left-invariant forms on the groups $U_n(\mathcal{Q})$.

The S operator raises degree by 2. Thus it appears that a Bott map $S^2 \times \mathcal{G} \longrightarrow \mathcal{G}$, $\mathcal{G} = U(\mathcal{Q})$ is not likely to provide the mechanism desired.

The only way I can see how to raise the degree of a class is by transgression. By transgression a primitive class of degree k on $\mathcal{G}$ becomes a class of degree $k+1$ on $B\mathcal{G}$. Repeating one gets a class of degree $k+2$ on $B^2\mathcal{G}$ which is $\sim \mathcal{G}$ by periodicity.

Alternatively there is a map $\Omega^2 \mathcal{G} \longrightarrow \mathcal{G}$ given by the Dirac on S^2. So a primitive form on $\mathcal{G}$ of degree k gives also one on $\Omega^2\mathcal{G}$ which corresponds to a $(k+2)$-degree primitive form on $\mathcal{G}$.

It is clear that the S-operator is closely related to the Dirac on S^2. For a cyclic cocycle on $\mathcal{G}$ = group of gauge transf over M coming from a Dirac op. on M, the Dirac corresponding operator on $S^2 \times M$ gives the S-transform.

However one must recall that S is a well-defined operation on cyclic cocycles in general and is not just defined on cocycles coming from operators. So it appears I have to start with a left-invariant primitive form on $\mathcal{G}$ and carry out the transgression of it to $B^2\mathcal{G}$.

~~Now this raises the problem of what~~ primitive means. It makes sense for a cohomology class on $\mathcal{G}$ to be primitive and these classes are the transgressive classes when $\mathcal{G}$ is connected. (One way perhaps to prove this is to use the spectral sequence associated to the Milnor model and the Hopf structure theory. One might even be able to take a primitive class on $\mathcal{G}$ and then successively extend it to classes of degree one more on the various skeleta of $B\mathcal{G}$.)

However it doesn't make sense for a Lie algebra class to be primitive. This has a meaning only for matrices in the stable range, where one has the Whitney sum operation

It would be nice if we could find a differential form model for the fibration $\mathcal{G} \longrightarrow P\mathcal{G} \longrightarrow B\mathcal{G}$ where $\mathcal{G}$ is the inf. unitary group over $\mathcal{A}$. A first step would be to consider

$$U(n) \longrightarrow U(N)/U(N-n) \longrightarrow Gr_n(\mathbb{C}^N)$$

Note that $Gr_n(\mathbb{C}^{2n}) \sim BU$ as $n \longrightarrow \infty$. What are the invariant forms on the Stieffel manifold?

<u>Problem</u>: What is the relation between the van Est picture and Connes exact sequence?

June 11, 1986

Question: Does there exist a [redacted] nice fibration

$$\mathbb{Z} \times BU \longrightarrow * \longrightarrow U \,?$$

Here $\mathbb{Z} \times BU$ and U are to be realized as the restricted Grassmannian and the group of unitaries congruent to 1 mod $\mathcal{K}$.

Idea: Go back to algebraic K-theory and use the link between U and the Q-category. This leads to the following.

For the contractible space $*$ we take the self adjoint contractions A on H such that A lies over the given involution η ~~mod~~ $\mathcal{K}$ of H. This is the convex set $\mathcal{F}_{1,\eta}$. We then have the map $A \longmapsto -\exp(i\pi A)$ from $\mathcal{F}_{1,\eta}$ to $U(\mathcal{K})$.

Consider a $g \in U(\mathcal{K})$ having finitely many eigenvalues $\neq 1$, and let's compute ~~the~~ fibre over such a g. If $-\exp(i\pi A) = g$, then the eigenspaces of A [redacted] belonging to eigenvalues λ in $(-1,1)$ are completely determined. The $+1$ eigenspace of g has to be divided into the $+1$ and -1 eigenspaces of A. Thus [redacted] the fibre over g is the restricted Grassmannian of the $g = 1$ eigenspace.

Thus [redacted] it would appear that we have a quasi-fibration

$$\mathcal{I}_{res} \longrightarrow \mathcal{F}_{1,\eta} \xrightarrow{-\exp(i\pi ?)} U(\mathcal{K})$$

which is quite natural. The following

diagram relates this quasi-fibration to another one studied before

$$\begin{tikzcd}
\mathcal{J}_{res} \arrow[r] \arrow[d] & \mathcal{F}_{1,\eta} \arrow[r] \arrow[d] & \mathcal{U}(\mathcal{K}) \arrow[d, equal] \\
\mathcal{J}(H) \arrow[r] \arrow[d] & \mathcal{F}_1 \arrow[r, "\sim"'] \arrow[d, "\sim"] & \mathcal{U}(\mathcal{K}) \\
\mathcal{J}(Q) \arrow[r, Rightarrow] & \mathcal{J}(Q).
\end{tikzcd}$$

In the future it might be more useful to have a fibration rather than quasi-fibration. Such a fibration might arise from considering the loop group fibration

$$\Omega U_n \longrightarrow \mathcal{A}_n \longrightarrow U_n$$

and taking the large n limit. We already know ~~how~~ to embed ΩU_n in $\mathcal{J}_{res}$ and how to recognize $Gr(\mathbb{C}^n)$ inside the loop group.

Recall in general that given a Hilbert space V we can form $L^2(S^1)\otimes V$. Then to a subspace W we associate the loop

$$\lambda_W = ze + 1 - e \qquad e = \text{proj on } W.$$

which transforms $H^2(S^1)\otimes V$ into

$$\lambda_W\left(H^2(S^1)\otimes V\right) = 1\otimes W^\perp + zH^2(S^1)\otimes V.$$

In this way the restricted Grass. of V relative to W_0 maps into the restricted Grass of $L^2(S^1)\otimes V$ relative to $1\otimes W_0^\perp + zH^2(S^1)\otimes V$.

Idea: A Dirac operator on $\tilde{\mathbb{C}}^n$ over S^1 is a self-adjoint operator D on $L^2(S^1)\otimes\mathbb{C}^n$ satisfying

$$z^{-1}Dz = D+1$$

(assuming $D = \frac{1}{2\pi i}(\partial_t + A)$, $z = e^{2\pi i t}$). The converse is essentially true, because suppose given A such that $z^{-1}Az = A+1$. Then we consider the eigenspace decomposition of A:

$$L^2(S^1)\otimes\mathbb{C}^n = \bigoplus_{\lambda\in Sp(A)} W_\lambda$$

Then $zW_\lambda = W_{\lambda+1}$ which means that for any $a\in\mathbb{R}$ we get an outgoing subspace

$$L_a = \bigoplus_{\lambda\geq a} W_a$$

Assuming L_a is in the (suitably) restricted Grassmannian, its scattering operator is a gauge transformation on $\tilde{\mathbb{C}}^n$, so we can suppose $L_0 = H^2(S^1)\otimes\mathbb{C}^n$. Then the eigenspace W_λ for $0\leq\lambda<1$ will, or should, enable me to construct a constant coefficient connection whose Dirac operator will be A.

<u>Prop</u>. A self-adjoint operator A on $L^2(S^1)\otimes\mathbb{C}^n$ satisfying $z^{-1}Az = A+1$ and such that the gen. eigenspace where $A\geq 0$ has a smooth scattering operator is a Dirac operator.

June 12, 1986

Yesterday we considered a quasi-filtration

$$I_{res} \longrightarrow \mathcal{F}_{1,\eta} \longrightarrow U(\mathcal{K})$$

and posed the question of relating it to a loop fibration. In finite dimensions one has a principal fibration

$$\Omega U_n \longrightarrow \mathcal{A}_n \longrightarrow U_n,$$

and one seeks an infinite dimensional version.

Thus we want to look at an infinite dimensional vector bundle over S^1 and connections and gauge transformations. Suppose the bundle is $S^1 \times V$ over S^1 where V is Hilbert space. Gauge transfs. are smooth maps $S^1 \to U(V)$, but we want loops with values in $U(\mathcal{K})$. Clearly if $g: S^1 \to U(\mathcal{K})$ then $g^{-1}g' : S^1 \longrightarrow \mathcal{K}$. Thus the orbits of $\mathcal{G} = L_{an}U(\mathcal{K})$ the space of connections are at most cosets relative to $L(\mathcal{K})$. It also is pretty clear that if we have two connections $\partial_t + \alpha$, $\partial_t + \alpha'$ such that $\alpha \equiv \alpha'$ mod $\mathcal{K}$, then the parallel transports h_t and h'_t differ by $U(\mathcal{K})$. Thus its clear that we can expect a principal fibring

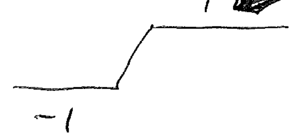
$$\Omega U(\mathcal{K}) \longrightarrow \mathcal{A}\left(\equiv \partial_t + \alpha_0(\mathcal{K})\right) \longrightarrow U(\mathcal{K})$$

for any fixed connection $\partial_t + \alpha_0$.

In practice the connection α_0 is constant and is a constant times an involution on V, the involution lying over the involution mod $\mathcal{K}$ η. This enables the

Bott map to be defined.

Interesting point: Let's compare the two quasi-fibrings over $U(\mathcal{K})$. One consists of connections which give rise to operators $D \ni z^{-1}Dz = D+1$. The other consists of $-1 \leq A \leq 1$ which lie over η. There is a rough correspondence between the two, because [redacted] I can associate to A a constant Dirac operator, and because I, [redacted] can compress D via a function



June 13, 1986

I recall I have been looking for a fibration (concrete) of the form

$$\mathbb{Z} \times BU \longrightarrow * \longrightarrow U$$

so that I might understand the behavior of differential forms relative to periodicity. So far the best I can do is to obtain a quasi-fibration. This may suffice, but there is another candidate based on loop groups:

$$\Omega U \longrightarrow \mathcal{A}_{res} \longrightarrow U$$

which is a limit of the principal fibrations

$$(*) \qquad \Omega U_n \longrightarrow \mathcal{A}_n \longrightarrow U_n$$

Now I recall that I never properly understood the differential forms on $(*)$, in particular, why the even characters forms on ΩU_n transgress to the odd character forms on U_n. It should be possible to treat this problem by the differential form methods of Atiyah + Singer's note. There is a problem because the principal bundle $(*)$ doesn't seem to have a canonical connection.

There's another reason to investigate transgression in the principal fibration $(*)$. I eventually want to understand transgression in the universal bundle associated to a group of gauge transformations, and the above is essentially the case of gauge transformations over the circle.

To begin let $\mathcal{G} = LU_n$, $\mathcal{G}' = \Omega U_n$. We are going to be interested in differential forms on $B\mathcal{G}'$. What does this mean? $B\mathcal{G}'$ classifies principal $\mathcal{G}'$-bundles which are the same thing over a space Y as vector bundles over $Y \times S^1$ of rank n trivialized along $Y \times \{0\}$. A particular model for $B\mathcal{G}'$ results by rigidifying such bundles by choosing a structure of some sort which is unique up to homotopy. Let's discuss two examples.

First consider the model U_n for $B\mathcal{G}'$. Given E over $Y \times S^1$ trivialized over $Y \times \{0\}$, we can choose a partial connection on E in the S^1 direction. This determines a holonomy map from Y to U_n. Over $U_n \times S^1$ is a canonical vector bundle E_{can} equipped with partial connection in the S^1-direction and the bundle E is isomorphic canonically to the pull-back of E_{can} over $U_n \times S^1$. In this example the rigidifying structure is a connection and the classifying space is U_n.

Secondly we can consider the model $\Omega(BU_n)$ for $B\mathcal{G}'$ where BU_n denotes $Gr_n(H)$. Given E over $Y \times S^1$ trivialized over $Y \times \{0\}$ we embed E inside the trivial bundle with fibre H such that over $Y \times \{0\}$ we get a constant embedding $\mathbb{C}^n \hookrightarrow H$. This gives a map $Y \times S^1 \longrightarrow BU_n$ such that $Y \times \{0\}$ goes to the basepoint of BU_n. Hence we obtain a map $Y \longrightarrow \Omega(BU_n)$. In this example the rigidifying structure is an embedding of the vector bundle over

S^1 into the trivial bundle $\tilde{H}$ doing the right thing at the basepoint.

This example is close to the Milnor model where the rigidification consists of a partition of unity and local trivializations. Incidently the Milnor model of $B\mathcal{G}$, $\mathcal{G} = LU_n$, maps to the free loop space of the Milnor model for U_n. This holds more generally for $\mathcal{G} = C^\infty(M, G)$.

Question: To what extent is

$$B\{Map(M,G)\} \longrightarrow Map(M, BG)$$

an equivalence?

This is sort of an open-ended question as I haven't specified equivalence. I know that the above map is a homotopy equivalence of the component of $Map(M,BG)$ corresponding to the trivial G-bundle. I also have in mind that if one uses the Milnor model, or the simplicial model for BG, in a naive fashion it looks like the above map is an equality (isomorphism). So the question really comes down to one of understanding why the naive picture breaks down and whether some smoothness conditions are involved.

Let's now return to our two examples for $B\mathcal{G}'$ namely U_n and $\Omega(BU_n)$.

First we should review the even forms on $\mathcal{G}'$. It's easier to construct them on $\mathcal{G} = LU_n$ and then restrict to $\mathcal{G}'$.

The first method is to use the evaluation map
$$\mathcal{G} \times S^1 \longrightarrow U_n$$
to pull-back the odd char. forms on U_n and then integrate over S^1 to give even forms on $\mathcal{G}$. Recall the description of the odd forms on U_n. Over U_n one has a canonical automorphism g of $\tilde{\mathbb{C}}^n$ and hence one has two flat conns. on $\tilde{\mathbb{C}}^n$, namely d and $g^{-1}.d.g = d + \theta$. Then one uses the linear path $d + t\theta$ between these connections to get a difference elt:
$$0 = d \int_0^1 dt \, \mathrm{tr}\{e^{(t^2-t)\theta^2}\theta\}$$

When we pull this ~~back~~ over $\mathcal{G}\times S^1$ we then get the path of connections
$$\delta + d + t(g^{-1}\delta g + g^{-1}dg)$$
Because of the term $g^{-1}dg$ the resulting forms on $\mathcal{G}\times S^1$ and on $\mathcal{G}$ are not likely to be left $\mathcal{G}$-invariant.

The second method is to use the path of connections on $\tilde{\mathbb{C}}^n$ over $\mathcal{G}\times S^1$ given by
$$\nabla_t = \delta + d + t\,\underbrace{g^{-1}\delta g}_{\theta} \qquad \nabla_t^2 = t\,d\theta + (t^2-t)\theta^2$$
which leads to the formula
$$\mathrm{tr}(e^{d\theta}) - \mathrm{tr}(e^0) = (\delta+d)\int_0^1 dt\, \mathrm{tr}\{e^{t\,d\theta + (t^2-t)\theta^2}\theta\}$$
or
$$\boxed{\mathrm{tr}(d\theta) = (\delta + d)\int_0^1 dt\left[\mathrm{tr}\left(e^{(t^2-t)\theta^2}\theta\right) + \mathrm{tr}\left(t\,d\theta\, e^{(t^2-t)\theta^2}\theta\right)\right]}$$

Thus we get on $\mathcal{G}$ the closed left-invariant forms

$$\text{const} \int_{S^1} tr\left(\theta^{odd} d\theta\right)$$

Next, now that we have reviewed the even forms on $\mathcal{G}$, we [redacted] turn to the odd forms on [redacted] U_n and $\Omega(BU_n)$; maybe it would be better to think about $B\mathcal{G}'$. This means that we have to construct characteristic forms on Y associated to a vector bundle E over $Y \times S^1$ of rank n trivialized along $Y \times \{0\}$. The method is to choose a connection on E take the even character forms assoc. to this connection and then integrate over the circle.

In the case of $\Omega(BU_n)$ one has the evaluation map

$$\Omega(BU_n) \times S^1 \longrightarrow BU_n$$

which induces the universal bundle. This bundle has a canonical connection which is a Grassmannian connection. So for this model for $B\mathcal{G}'$ the universal bundle E_{univ} over $B\mathcal{G}' \times S^1$ carries a canonical connection, and thus we obtain definite odd forms on $\Omega(BU_n)$.

In the case of U_n the canonical bundle E_{can} over $U_n \times S^1$ carries only a canonical partial connection in the S^1-direction. One can extend to a full connection. Question: Do the forms obtained on U_n depend on this extension?

At the moment we have two ways to produce odd forms on $L(BU_n)$. The first is via the evaluation map

$$L(BU_n) \times S^1 \longrightarrow BU_n$$

Concretely a map $Y \longrightarrow L(BU_n)$ gives rise to an E over $Y \times S^1$ embedded in $\tilde{H}$. This gives a Grass. connection on E and we integrate the even character forms assoc. to this connection over S^1 to get odd forms on Y.

The second is to restrict E + its connection to $Y \times \{0\}$ to obtain a v.b. + connection over Y, and then uses the connection in the vertical direction to define the monodromy. In this way we obtain a bundle with connection and automorphism and we have a difference construction which gives odd forms on Y.

In both case we ~~start with~~ have a bundle E over $Y \times S^1$ with connection. It's this much we use from the map $Y \longrightarrow L(BU_n)$. And we have these two ways of getting **odd forms on** Y: The first is to take the character forms of the connection and integrate over S^1, the second is to use the monodromy along the circles ~~to~~ get a bundle + auto, + then use the connection over $Y \times \{0\}$.

Something else we could do would be to use the superconnection forms for the family of Diracs on S^1 parametrized by Y. These forms are equivalent in a known way to the ones obtained by integrating the total curvature of E over S^1.

[redacted] In order to obtain some insight we fix things over [redacted] $\in S^1$ and work with $\mathcal{G}' = \Omega U_n$ instead of $\mathcal{G}$. We have ~~this~~ [redacted] bundle E over $Y \times S^1$ trivialized over $Y \times \{0\}$ and it has a connection. Such a connection consists of a horizontal and vertical partial connection. Let us introduce the principal $\mathcal{G}'$ bundle P defined by E, so that

$$P \times^{\mathcal{G}'} (\tilde{\mathbb{C}}^n_{S^1}) = E \qquad \text{over } Y \times S^1.$$

Because of the vertical connection in E we have a map of principal $\mathcal{G}'$ bundles

$$\begin{tikzcd} P \arrow[r] \arrow[d] & \mathcal{A}_n \arrow[d] \\ Y \arrow[r] & U_n \end{tikzcd}$$

The horizontal connection in E is equivalent to a connection in P/Y.

I think it true that we can find a connection in $\mathcal{A}_n/U_n$, i.e. a connection in E can over $U_n \times S^1$ extending the canonical vertical connection, such that the character forms integrated over S^1 are the [redacted] odd character forms on U_n. If we [redacted] use the induced connection in [redacted] P/Y (or $E/(Y \times S^1)$, then the two sets of odd forms on Y agree.

First idea: We know that the classifying space of $\mathcal{G} = C^\infty(M, U_n)$ is a component of $C^\infty(M, BU_n)$. The only way I know how to produce cohomology on $B\mathcal{G}$ is via the characteristic classes of the canonical vector bundle over $B\mathcal{G} \times M$, i.e. via the evaluation map

$$B\mathcal{G} \times M \longrightarrow BU_n$$

The corresponding classes on $\mathcal{G}$ are obtained via the evaluation map $\mathcal{G} \times M \longrightarrow U_n$. The real question then becomes what is the significance of the classes on $\mathcal{G}$ which represented by left-invariant forms when we go to $B\mathcal{G}$. The only thing I can think of is that we are looking at cohomology classes on $B\mathcal{G}$ which are trivial in the continuous cohomology.

So far what we have done for $B\mathcal{G}$ sheds no light on the significance of being able to represent certain classes on $\mathcal{G}$ by left-invariant forms

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Let's now go back to the case $\mathcal{G} = \mathcal{L}(U_n)$. We have two ways to define odd forms on the model $\mathcal{L}(BU_n)$, $BU_n = Gr_n(H)$, for $B\mathcal{G}$. Both use the connection on the universal bundle over $B\mathcal{G} \times S^1$, so one might as well consider more generally the case of a v.b. + connection E over $Y \times S^1$.

The second idea I had was to try to bring the Bismut forms, which also involve the holonomy. Let's review his construction. One supposes given a vector bundle E with connection ∇ over a manifold M on which S^1 acts, but where the action is not defined on E. Then one considers

$$d + u\iota_X \quad \text{on} \quad \Omega^\cdot(M)$$

$$\nabla + u\iota_X \quad \text{on} \quad \Omega^\cdot(M,E)$$

where u is a parameter, and X is the generator of the circle action. We have

$$(\nabla + u\iota_X)^2 = u\nabla_X + \nabla^2 \qquad (d+u\iota_X)^2 = u\mathcal{L}_X .$$

Suppose $e^X = 1$. Then $e^{\nabla_X + u^{-1}\nabla^2} \in \Omega(M, \text{End}\, E)$ and we can form

$$\tau = \operatorname{tr}\left(e^{\nabla_X + u^{-1}\nabla^2}\right)$$

This satisfies

$$(d+u\iota_X)\tau = \operatorname{tr}\left(\left[\nabla + u i_X, e^{\nabla_X + u^{-1}\nabla^2}\right]\right) = 0$$

and so τ is an equivariantly closed form with the reservation that it involves u^{-1}.

Now the hope is to do something similar in the case of a bundle with connection over $Y \times S^1$.

Maybe I should review the previous work on equivariant cohomology of $\mathcal{L}(M)$. The idea is that an S^1-equivariant map $Z \to \mathcal{L}(M)$ is the same as an ordinary map $Z \to M$. A map

$$Y \longrightarrow PS^1 \times^{S^1} \mathcal{L}(M)$$

is the same as a principal S^1 bundle Z/Y with an equivariant map $Z \to \mathcal{L}(M)$, i.e. an ordinary map $Z \to M$. Thus an equivariant class on $\mathcal{L}(M)$ is the same as a characteristic class for pairs consisting of a principal S^1-bundle and a map of the total space to M. Clearly by pulling back a coh. class on M and integrating over the fibre we see that any coh. class on M gives rise to an equiv. class of degree 1 less:

$$H^q(M) \longrightarrow H^{q-1}_{S^1}(\mathcal{L}(M))$$

Let us now see what we can do concretely. Suppose given E over $Y \times S^1$ with connection.

June 17, 1986

It seems that there might be a simpler proof of periodicity based on quasi-fibration ideas, and not using Kuiper's theorem. The idea is to exhibit a quasi-fibration with contractible total space, fibre $\mathbb{Z} \times BU$, and base U. This I essentially did when I showed that $K_0 A \times BGL(A)^+$ is the loop space of the Q-category.

Let's now try to describe a version of these ideas, keeping it as finite as possible. Fix the Hilbert space $H = L^2(S^1)$ and the Hardy space $H^+ = H^2(S^1) =$ span $1, z, z^2, \ldots$. Our version of U will be the group of unitaries on H which are supported in $z^{-N}H^+ \ominus z^N H^+$ for some N. (In fact I might as well replace H by the space of Laurent polynomials.)

Another way to say this is to introduce the ideal $\mathcal{K}$ of operators of finite support

$$\mathcal{K} = \lim_N \operatorname{End}\left(z^{-N}H^+ \ominus z^N H^+\right)$$

and then $U = U(\mathcal{K}) =$ group of unitaries $\equiv 1 \mod \mathcal{K}$.

The total space $\mathcal{A}$ of our fibring will consist of self-adjoint contractions A on H such that

$$A = \begin{cases} -1 & \text{on } z^{-N}H^- \\ +1 & \text{on } z^N H^+ \end{cases}$$

for some N. Alternatively $\mathcal{A}$ consists of self-adjoint contractions A such that $A \equiv \varepsilon \pmod{\mathcal{K}}$ where ε is the involution corresponding to $H = H^+ \oplus H^-$. It clear $\mathcal{A}$ is convex hence contractible.

The map from $\mathcal{A}$ to U sends A to

$-\exp(i\pi A)$. It is probably easier to think of the map

$$\mathcal{A} \longrightarrow -\mathcal{U}$$

$$A \longmapsto \exp(i\pi A)$$

What are the fibres? Given $g \in -\mathcal{U}$ the only arbitrariness in lifting g to an $A \in \mathcal{A}$ occurs over the -1 eigenspace of g, where one must choose the splitting of this eigenspace into the $+1$ and -1 eigenspaces of A. These eigenspaces must be chosen to contain $z^N H^+$ and $z^{-N} H^-$ resp. for some N. So it is clear that the fibre over g is the restricted Grassmannian of the -1 eigenspace.

What are the specialization maps? When g approaches g' the -1 eigenspace of g' is the limit of the sum of the eigenspaces of g belonging to eigenvalues approaching -1. Some eigenvalues approach -1 from the UHP and some from the LHP. Thus the -1 eigenspace of g' splits into the -1 eigenspace of g direct sum the eigenspaces comes from the UHP and direct sum the LHP. We can cover this specialization by the map from the restricted Grass of $g = -1$ to the restricted Grass of $g' = -1$ which takes an involution on $g = -1$ and adds the involution corresponding to whether the approach is from the Upper or lower half plane.

Since this specialization map is a homotopy equivalence, it should follow that the map is a quasi-fibn.

June 18, 1986

Next we construct a quasi-fibration [redacted] of the form

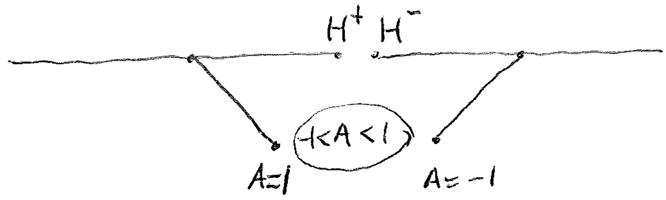
$$U \longrightarrow * \longrightarrow BU,$$

by putting a grading on the previous quasi-fibn. It's convenient to change notation a little. We will need a Hilbert space H with two anti-commuting involutions ε, [redacted] γ together with an exhaustive filtration $F_N H$ by finite-diml subspaces stable under ε, γ. To be specific put

$$H = z^{\frac{1}{2}} L^2(S^1) = \underbrace{H^+}_{\substack{\text{span} \\ \text{of } z^{n+\frac{1}{2}} \\ n \geq 0}} \oplus \underbrace{H^-}_{\substack{\text{span} \\ \text{of } z^{-n-\frac{1}{2}} \\ n \geq 0}} \qquad \varepsilon \text{ [redacted] grading}$$

$$\gamma\left(z^{n+\frac{1}{2}}\right) = z^{-n-\frac{1}{2}}$$

Now [redacted] let $\mathcal{a}^\gamma = \{ A \in \mathcal{a} \mid \gamma A + A\gamma = 0 \}$. Recall that $\mathcal{a}$ consists of self adjoint contractions congruent to ε module $\mathcal{K} = \bigcup_N \operatorname{End}(F_N H)$. Thus the $A = \pm 1$ space is commensurable with $H^\pm$. If $A \in \mathcal{a}^\gamma$, then γ interchanges these eigenspaces and the complement is even-diml



The unitary $g = \exp(i\pi A)$ is in $-U(\mathcal{K})$ and is reversed by γ, so we get a map from $\mathcal{a}^\gamma$ to the restricted Grassmannian consisting of involutions congruent mod $\mathcal{K}$ to $-\gamma$ by

$$A \longmapsto \exp(i\pi A)\gamma$$

I want now to see the image and fibres of this map.

The first thing to note is that for $A \in \mathcal{A}^\gamma$ the null space, which is stable under γ, has equal numbers of ± 1 eigenvalues for γ. This is because on the complement

$$H^+ \ominus \{h \in H^+ \mid Ah = h\} \overset{\gamma}{\rightleftarrows} H^- \ominus \{h \in H^- \mid Ah = -h\}$$

there are equally many ± 1's, and the non-zero part of A also has equally many ± 1's for γ. Thus for $g = \exp(i\pi A)$ the $+1$ eigenspace has equally many ± 1's for γ. This is the index zero condition that a $g \in -U(\mathcal{K})$ reversed by γ must satisfy in order that it came from an $A \in \mathcal{A}^\gamma$.

Conversely suppose $g \in -U(\mathcal{K})$ is reversed by γ and has index zero. We know what A must be except on the $g = 1$ eigenspace where we have to choose A to be an involution anti commuting with γ which is also $\equiv \varepsilon \mod \mathcal{K}$. Intersect the $g = +1$ eigenspace with H^+ and with H^-, take the direct sum of these intersections, and take the complement in the $g = 1$ eigenspace. This complement is stable under γ and the index zero condition implies there are equally many ± 1s for γ. Hence we can find an involution in this complement anti-commuting with γ. This means we can find an $A \in \mathcal{A}^\gamma$ over g.

The fibre over g is clearly the space of involutions on the $g = 1$ eigenspace anti-commuting with γ and congruent to a fixed one mod $\mathcal{K}$. This is isomorphic to a group $U(\mathcal{K})$.

The rest of the picture is fairly clear. Specialization maps are isomorphic to embeddings $U(\mathcal{K}') \subset U(\mathcal{K})$ assoc. to an embeddings of finite codim.

I want now to describe the above quasi fibration in more detail. Notice that it is a quasi-fibration of the form

$$(*) \qquad U \longrightarrow * \longrightarrow BU$$

and it is not the same as the limit of the principal fibrations

$$U_n \longrightarrow U_{2n}/1\times U_n \longrightarrow U_{2n}/U_n \times U_n$$

At least this is the way it appears.

On the other hand I think I can construct a principal fibration of the form $(*)$ as follows. Let V, H be sep. Hilbert spaces, let $\eta \in \mathcal{J}(Q_H)$ and let's choose an isom α modulo K of V with the $\eta = +1$ part of H. Then we consider all (isom) embeddings $V \longrightarrow H$ lying over α; call this space P. The group U of unitaries $\equiv 1$ mod compacts on V acts freely on the right of P and the quotient is a subspace of the restricted Grass $\mathcal{J}_{res}(H,\eta)$. What subspace? Given $W \subset H$ in the restricted Grassmannian we then get an isom. of V with W modulo K. It can be lifted to a unitary isom iff the index is zero. Thus P/U = a component of $\mathcal{J}_{res}(H,\eta)$.

So let's now try to describe the quasi-fibration

$$U \longrightarrow A^{\eta} \longrightarrow (\text{res Grass})_{(0)}$$

in a bit more detail so that we can work out the relations. We first consider the Atiyah-Singer map

$$\mathcal{F}_0 \longrightarrow \mathcal{J}_{res}(-\eta)$$

$$A \longrightarrow \exp(i\pi A)\eta$$

which they prove is a h.eq. Instead of this exponential map I want to use a Cayley transform which looks as follows. Given

$$A = \begin{pmatrix} 0 & T^* \\ T & 0 \end{pmatrix} \quad \text{a self-adjoint contraction}$$

we map it to the subspace

$$\operatorname{Im} \begin{pmatrix} \sqrt{1-T^*T} \\ T \end{pmatrix}$$

Maybe it's better to say that we assign to A the isometric embedding

$$\begin{pmatrix} \sqrt{1-T^*T} \\ T \end{pmatrix} : \; H^+ \longrightarrow \begin{matrix} H^+ \\ \oplus \\ H^- \end{matrix} \qquad (H^{\pm} \text{ rel. to } \eta)$$

Notice that when T is unitary this is just the subspace of H with involution $-\eta$.

We are also given $\varepsilon = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ relative to some identification of H^+ and H^-. Then $\mathcal{F}_{0,\varepsilon}$ is the space of contractions T on $H^+ = H^-$ congruent to 1 mod $\mathcal{K}$.

Next what is P? It consists of isom. embedding $H^+ \hookrightarrow H$ which mod $\mathcal{K}$ reduce to $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$. Thus it consists of $\begin{pmatrix} S \\ T \end{pmatrix}$ with $S^*S + T^*T = 1$ and $T \equiv 1 \ (\mathcal{K})$

It is clear now that $\mathcal{A}^\eta$ is going to sit inside of P as the subspace where $S \geq 0$.

Now effectively S is of finite rank for many purposes. The passage from P to $\mathcal{A}^\eta$ cuts the fibre down from the group $\mathcal{U}$ on H^+ to the group $\mathcal{U}$ on the orthogonal complement of the image of S.

So we see that our quasi-fibration sits inside a fibration and the inclusion of the corresponding fibres is a homotopy equivalence.

June 19, 1986

Let's recall one of the problems which we haven't got anywhere on really. Suppose we have an ungraded Fredholm module $(\mathcal{A}, H, F)$. Then $U_n(\mathcal{A})$ acts on H^n and there is a map

$$U_n(\mathcal{A}) \longrightarrow \text{rest. Grass}(H^n, F^n)$$

which we use to construct even degree forms on $U_n(\mathcal{A})$ which turn out to be cyclic cocycles. Instead of the group $U_n(\mathcal{A})$ I want to consider the Grass. of $\mathcal{A}$-linear projectors, or better idempotent matrices over $\mathcal{A}$. The problem is to construct differential forms on this Grassmannian which are of odd degree.

I still think it ought to be possible to go from forms on the unitary group to forms on the Grassmannian in a fairly canonical way, assuming some sort of primitivity. But I haven't got anywhere yet.

It seems desirable to look at this problem in the case of the circle. A stumbling block has been to somehow relate a projector e on $L^2(S^1)^N$ to the Hardy involution F. I don't have enough feeling for this involution.

I do have some feeling for connections and Dirac operators. I recall that the space of connections $\mathcal{A}$ is the analogue of the Bruhat-Tits building. I recall from the time when I studied buildings that a splitting of the vector bundle over the circle is related to a nice "flat" part of the building. To be specific if we have a splitting $\tilde{\mathbb{C}}^N = E \oplus E^{\perp}$, then we get the affine subspace of $\mathcal{A}$ consisting of connections compatible with this splitting. Call this subspace $\mathcal{A}'$. We also have a ~~projection~~ retraction $\mathcal{A} \longrightarrow \mathcal{A}'$ which assigns to a connection the induced connections on E and E'.

June 22, 1986 (46 years old)

Let E be a vector bundle with connection ∇ over $Y\times S^1$, or more generally a principal S^1-bundle P with base Y. Let $\pi: P \longrightarrow Y$ be the projection. Then recall that we have odd forms on Y given by the components

$$\pi_* \operatorname{tr}(\nabla^2)^k/k! \qquad \text{of} \qquad \pi_* \operatorname{tr}(e^{\nabla^2})$$

In the case $P = Y\times S^1$ we get another set of odd forms by using the holonomy map on the bundle restricted to $Y\times\{0\}$; this gives an autom of a v.b. over Y and the bundle has a connection, so we obtain odd forms.

Today I found a third way to construct odd forms on Y which is closely related to Bismut's construction. Let X generate the circle action on P, and assume that $\exp(X) = e^X$ is the identity. I recall one considers the operator

$$\nabla + u\,\iota_X \qquad \text{on} \qquad \Omega(P,E)$$

where u is a real parameter. Then

$$(\nabla + u\,\iota_X)^2 = u\,\nabla_X + \nabla^2.$$

The operator

$$e^{\nabla_X + u^{-1}\nabla^2}$$

belongs to $\Omega(P, \operatorname{End} E)$, because it commutes with multiplication by forms as $e^{L_X} = 1$. Thus we have defined

$$\operatorname{tr}\left(e^{\nabla_X + u^{-1}\nabla^2}\right) \in \Omega^{ev}(P)$$

and it is equivariantly closed as

$$(d + u\,\iota_X)\operatorname{tr}\left(e^{\nabla_X + u^{-1}\nabla^2}\right) = \operatorname{tr}\left[\nabla + u\,\iota_X,\, e^{u^{-1}(\nabla + u\,\iota_X)^2}\right] = 0.$$

Now integrate over the fibre and use that

$$\pi_* \iota_X = \iota_X \pi_* = 0$$

as $X = 0$ on Y. (alternatively use the construction of π_* as ι_X followed by averaging.) This gives

$$d\left\{\pi_* \operatorname{tr}\left(e^{\nabla_X + u^{-1}\nabla^2}\right)\right\} = \pi_*\left\{(d + u\iota_X)\operatorname{tr}\left(e^{\nabla_X + u^{-1}\nabla^2}\right)\right\} = 0$$

So we therefore obtain a closed odd form

$$\pi_* \operatorname{tr}\left(e^{\nabla_X + u^{-1}\nabla^2}\right) \in \Omega^{odd}(Y)$$

which is obviously very close to

$$\pi_* \operatorname{tr}\left(e^{u^{-1}\nabla^2}\right).$$

The difference is that in the latter one takes the trace first after exponentiating the curvature and integrates over the circle; in the former one integrates over the circle first in a time-ordered way, using the holonomy to relate different points on a fibre.

Obvious question: What's the link between these two forms? Also with the superconnection forms where π_* is somehow replaced by a Hilbert space trace. Note ∇_X is the Dirac operator along the fibres.

June 23, 1986

Apparently the forms $\pi_*\left(\mathrm{tr}\, e^{\nabla_X + \nabla^2}\right)$ are exact. The reason is that the Bismut form $\alpha = \mathrm{tr}\left(e^{\nabla_X + \nabla^2}\right)$ is exact as an equivariant form since S^1 acts freely on P. So if we write

$$\alpha = (d + \iota_X)\beta$$

then $\pi_* \alpha = \pi_* d\beta = d(\pi_* \beta)$.

We can check this for line bundles in which case I believe one have

$$\alpha = \mathrm{tr}\left(e^{\nabla_X + \nabla^2}\right) = \quad \tau e^{\overline{\nabla^2}}$$

where $\overline{\nabla^2}$ is the average of the curvature of the line bundle for the S^1 action. One has

$$d \log \tau + \iota_X \overline{\nabla^2} = 0$$

so $\quad \pi_*\left(\tau \overline{\nabla}^{2k}\right) = \tau\, \pi_*\left(\overline{\nabla}^{2k}\right)$.

Now in computing π_* one applies ι_X to get something horizontal and then one averages ~~average~~ over the circle action. Thus

$$\tau\, \pi_*\left(\overline{\nabla}^2\right) = \tau\left(-\frac{d\tau}{\tau}\right) = -d\tau$$

which is exact.

What I should do now is to relate the odd forms on Y constructed in two ways. Recall we have a vector bundle with connection $(\tilde{E}, \tilde{\nabla})$ over $Y \times S^1$. One set of odd forms comes from integrating the character forms over the fibre:

$$\pi_* \, tr\left(e^{\tilde{\nabla}^2}\right)$$

The other is obtained by using the monodromy, or holonomy, to obtain a v.b. over $Y \times \{0\}$ equipped with connection and automorphism

It's necessary to review how to attach odd forms to a vector bundle E over Y equipped with connection D and auto τ. One considers the one parameter family of connections

$$D_t = (1-t)D + t\,\tau^{-1}D\tau$$

and on the bundle $pr_1^*(E)$ over $Y \times [0,1]$ one forms the connection

$$\tilde{D} = dt\,\partial_t + D_t$$

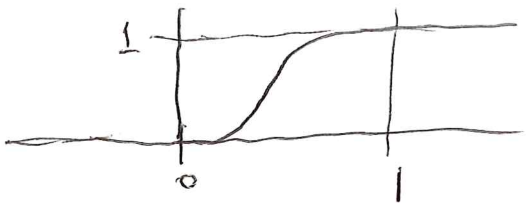
Then the character forms $tr\left(e^{\tilde{D}^2}\right)$ are integrated over the fibre $[0,1]$ to obtain odd forms on Y. These are closed because the connections $D_1 = \tau^{-1}D\tau$ and $D_0 = D$ are gauge equivalent, so the boundary contribution cancels.

The next goal will be to at least outline a proof that the odd forms constructed via the holonomy are special cases of the odd forms constructed by integrating the character forms of a connection on $Y \times S^1$ over the fibre. All we have to show how $\tilde{D}$ on the bundle over $Y \times [0,1]$ can be effectively replaced by a bundle with connection over $Y \times S^1$.

I want to glue the ends of the cylinder $Y \times [0,1]$ together using the autom. τ. One can do this by the family D_t won't be smooth at $t=0$. Thus we shall want to know that if we use

$$D_t = (1-\rho(t))D_0 + \rho(t) D_1 \qquad D_1 = \tau^{-1} D_0 \tau$$

where $\rho(t)$:



then the forms on Y obtained from this family are the same as the ones obtained from $(1-t)D_0 + tD_1$. But the forms are obtained by applying π_*, $\pi = pr_1 : Y \times [0,1] \to Y$, to

$$\int_0^1 dt \, tr\left(e^{D_t^2} \rho'(t)(D_1 - D_0)\right) = \int_0^1 d\rho \, tr\left(e^{[(1-\rho)D_0 + \rho D_1]^2} D_1 - D_0\right)$$

so it's clear.

Now proceed as follows. We start with (E, ∇) over $Y \times S^1$ and let $E_0 = \iota_0^* E$, where $\iota_0 : Y \to Y \times S^1$ is $\iota_0(y) = (y, 0)$. The holonomy of ∇ in the circle direction sets up an isomorphism of E with the quotient of $pr_1^*(E_0)$ over $Y \times [0,1]$, where $(\xi, 0)$ is identified with $(\tau^{-1}\xi, 1)$. This isomorphism is compatible with the vertical partial connection $dt\, \partial_t$ on $pr_1^*(E_0)$ and the vertical partial connection determined by ∇. Next let D_0 = conn. on $E_0 = \iota_0^* E$ induced by τ and $D_1 = \tau^{-1} D_0 \tau$

We have this canonical quotient setup

$$\begin{array}{ccc} pr_1^*(E_0) & \longrightarrow & E \\ \downarrow & & \downarrow \\ Y \times [0,1] & \longrightarrow & Y \times S^1 \end{array}$$

and the horizontal partial conn. determined by ∇ gives a path of connections in E_0 joining D_0 to D_1. On the other hand we could use $D_t = D_0 + \rho(t)(D_1 - D_0)$ as before. So we see that we have a different choice of horizontal partial connection in E.

<u>Conclusion</u>: The odd forms on $B\mathcal{G}$, $\mathcal{G} = \mathcal{L}(U_n)$, defined by the holonomy autom. are special cases of the ones defined by integrating over S^1 the character forms of a connection over $B\mathcal{G} \times S^1$. (In practice this means that the diffl geom. leads to only one candidate as equivariant forms for $\mathcal{G}$ acting on $\mathcal{A}$?)

Let's try to sort out the last statement. The first remark to make is that if $H \triangleleft G$ and X is a G on which H acts freely, then homotopy orbit spaces X_G and $(X/H)_{G/H}$ are homotopy equivalent. Another way to say this is that there is an equivalence between G-bundles P/Y equipped with an equivariant map $P \longrightarrow X$ (equivalently a section of $P \times^G X / Y$) and between G/H-bundles Q/Y equipped with an equivariant map $Q \longrightarrow X/H$. The equivalence sends P to P/H, and Q to $Q \times_{(H/X)} X$. Another way to see the equivalence is

$$\begin{tikzcd} P(G/H) \times^{G/H} X/H \arrow[d, equal] & & PG \times^G X \\ P(G/H) \times^G X & & (P(G/H) \times PG) \times^G X \arrow[ll, "\sim"'] \arrow[u, "\sim"'] \end{tikzcd}$$

The reason the indicated maps are h.eq.'s is that they are quotients by free G actions of maps with contractible fibres (i.e. $P(G/H) \times X \longleftarrow P(G/H) \times PG \times X$)

The above principal tells us that the classifying space for $\mathcal{G}$ acting on $\mathcal{A}$ (circle case) is the same as for U_n acting on itself by conjugation. In concrete terms a bundle $E / Y \times S^1$ equipped with vertical connection is equivalent to the vector bundle $E_0 = E | Y \times \{0\}$ equipped with the holonomy automorphism.

However this equivalence does not extend to the connections in the principal $\mathcal{G}$-bundle versus the principal U_n bundle.

In any case we could very well start with a bundle + autom, to be specific, take $Y = U_n$ and the tautological autom. of the trivial bundle $\widetilde{\mathbb{C}^n}$.