The problem. Consider a loop group $\mathcal{K}$ = alg. maps $S^1 \to K$. Then it has a central extension $\tilde{\mathcal{K}}$ which presumably [redacted] acts on the Kac highest weight modules. So, what I should be able to do is to [redacted] find over the homog. space

$\tilde{\mathcal{K}}/\tilde{T}$ = $S^1 \ltimes \mathcal{K}/$ [redacted] $S^1 \ltimes \tilde{T} \cong \mathcal{K}/T$

a line bundle on which $\tilde{\mathcal{K}}$ but not $\mathcal{K}$ acts. Then the highest weight module should be something like holomorphic sections of this line bundle.

The idea goes as follows. Given a point of $\mathcal{K}/T$ one gets an outgoing subspace L and a flag in L/zL, hence we get a subgroup $\cong \mathbb{Z}^n \times (S^1)^n$ in $\mathcal{K}$. The group $\mathbb{Z}^n \times (S^1)^n$ has a canonical Heisenberg central extension with a unique irreducible modules. So the line bundle should have for fibre the maps of the standard Heisenberg module over $\mathbb{Z}^n \times (S^1)^n$ with the one for this subgroup.

Analogy: Consider $SL_2(\mathbb{R})$ acting on the symmetric space. A point of the symmetric space determines a circle group in $SL_2(\mathbb{R})$ and also a [redacted] line in the holomorphic representation. Thus you get a line bundle over the symmetric space. [redacted] If the stabilizer of a point of the symmetric space is the circle gp. K, then the double covering of K acts on the line in the holom. repn.

The tricky point is the following: There is nothing to stop us from forming over the symmetric space, a line bundle associated to the n-fold covering of the circle group. But it's not clear we get any holomorphic sections. ? ?

October 6, 1981

Irreducible repns of $SL_2(\mathbb{R})$. Adopt Lie algebra viewpoint, and maybe it's good to work with $SU(1,1)$ picture. This group consists of $\begin{pmatrix} \alpha & \beta \\ \bar\beta & \bar\alpha \end{pmatrix}$ with $|\alpha|^2 - |\beta|^2 = 1$. Thus one finds the Lie algebra ~~is generated by~~ consists of $\begin{pmatrix} a & b \\ \bar b & \bar a \end{pmatrix}$ with $a + \bar a = 0. \Rightarrow a \in i\mathbb{R}$. Introduce the generators

$$J = iH = \begin{pmatrix} i & \\ & -i \end{pmatrix}, \quad X+Y = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad i(X-Y) = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}.$$

for the Lie algebra.

The group $K = \begin{pmatrix} \alpha & \\ & \alpha^{-1} \end{pmatrix}$ is compact so given any reasonable topological representation of G is can be decomposed into characters under K. In fact one knows the K-finite vectors form a module for the complexified Lie algebra. So one classifies such modules which are irreducible. Thus $V = \bigoplus_\lambda V_\lambda$ where $H = \lambda$ on V_λ.

$$X V_\lambda \subset V_{\lambda+2} \qquad Y V_\lambda \subset V_{\lambda-2}$$

so that the λ for which $V_\lambda \neq 0$ is contained in a coset $\lambda_0 + 2\mathbb{Z}$. Also I know by irreducibility that the Casimir operator

$$\tfrac{1}{2}H^2 + XY + YX = \tfrac{1}{2}H^2 + H + 2YX$$

is a scalar and therefore YX is a scalar on V_λ. This shows that we have a submodule generated by any $v \in V_\lambda$, hence irred $\Rightarrow$ $\dim(V_\lambda) = 1$

Case 1: support of V is a full coset $\lambda_0 + 2\mathbb{Z}$. This means that XY, YX never zero. Call ε the value of Casimir so that

$$\varepsilon = \tfrac{1}{2}\lambda^2 + \lambda + 2 \text{ value of } YX \text{ on } V_\lambda$$

$$\varepsilon = \tfrac{1}{2}\lambda^2 - \lambda \quad + 2\cdot\text{value of } XY \text{ on } V_\lambda$$

So
$$\varepsilon \neq \tfrac{1}{2}(\lambda_0 + 2n)^2 \pm (\lambda_0 + 2n)$$
$$= \tfrac{1}{2}(\lambda_0 + 2n)(\lambda_0 + 2n \pm 2).$$

Can concentrate on the $+$.

$$\varepsilon \neq \tfrac{1}{2}\left[(\lambda_0 + 2n + 1)^2 - 1\right]$$

[redacted]

Because the compact K is a circle one must have $\lambda_{[redacted]} \in \mathbb{Z}$, and then one picks λ_0 to be the minimum possible for $(\lambda+1)^2$. Thus $\lambda_0 = 0$ or -1. If I use the covering group of degree d, then $\lambda \in \mathbb{Z}\frac{1}{d}$.

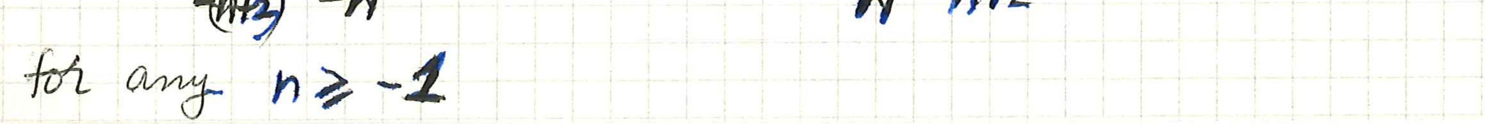
Thus we have that
$$\varepsilon \notin \tfrac{1}{2}\left[(\lambda_0 + 1 + 2\mathbb{Z})^2 - 1\right] \implies$$ we get an irreducible representation with [redacted] weights $\lambda_0 + 2\mathbb{Z}$.

Case 2: $\varepsilon = \frac{1}{2}\left[(\lambda+1)^2 - 1\right] = \frac{1}{2}\lambda^2 + \lambda$. Then we have that $YX = 0$ on $V_\lambda \implies$ one of $V_\lambda \underset{Y}{\overset{X}{\rightleftarrows}} V_{\lambda+2}$ is zero $\implies YX = 0$ on $V_{\lambda+2}$. By irreducibility either $V_\lambda \neq 0$ or $V_{\lambda+2} \neq 0$ but not both. If λ is a root of $\varepsilon = \frac{1}{2}\left[(\lambda+1)^2 - 1\right]$ so is $-\lambda - 2$. Both roots will be in the coset $\lambda + 2\mathbb{Z}$ if

$$-\lambda - 2 = \lambda + 2n$$

or [redacted] $2\lambda \in 2\mathbb{Z} \implies \lambda \in \mathbb{Z}$.

Therefore in the case of Sl_2 where $\lambda \in \mathbb{Z}$ necessarily we have that the string of weights can be of three types



for any $n \geq -1$

Unitary representations: This puts the restriction that the matrices in the Lie alg of $SU(1,1)$, such as, iH, $X+Y$, $i(X-Y)$ have to be represented by skew-hermitian operators. Thus H must be hermitian $\Rightarrow$ all λ appearing are real. Also

$$X+Y = iA \qquad A = A^*$$
$$i(X-Y) = iB \qquad B = B^*$$
$$X = \frac{B+iA}{2} \qquad Y = \frac{iA-B}{2} \Rightarrow \boxed{Y = -X^*}$$

Thus

$$\varepsilon = \tfrac{1}{2}H^2 + H + \underbrace{2YX}_{-2X^*X} \leq \tfrac{1}{2}H^2 + H$$

so that $\varepsilon \leq \frac{1}{2}\lambda^2 + \lambda$ for all weights λ.

For SL_2 $\lambda \in \lambda_0 + 2\mathbb{Z}$ where $\lambda_0 = 0$ or -1.

$$\lambda_0 = 0 \qquad \varepsilon \leq \tfrac{1}{2}\left[(2n+1)^2 - 1\right] \Rightarrow \varepsilon \leq 0$$
$$\lambda_0 = -1 \qquad \varepsilon \leq \tfrac{1}{2}\left[(2n)^2 - 1\right] \Rightarrow \varepsilon \leq -\tfrac{1}{2}$$

in the case where the full coset $\lambda_0 + 2\mathbb{Z}$ occurs. This eliminates the cases $\varepsilon = 0$ $\lambda_0 = 0$ and $\varepsilon = -\frac{1}{2}$ $\lambda_0 = -1$.

Thus for $SL_2(\mathbb{R})$ one has only the following possibilities for unitary representations.

weights	$2\mathbb{Z}$	$\varepsilon < 0$
weights	$1 + 2\mathbb{Z}$	$\varepsilon < -\frac{1}{2}$
weights	$n + 2\mathbb{Z}_{\geq 0}$ $n \geq 1$	$\varepsilon = \frac{1}{2}n^2 - n$
	$-(n + 2\mathbb{Z}_{\geq 0})$ "	$\varepsilon = \frac{1}{2}n^2 - n$

Next try to understand the metaplectic representations. We have the following commutation relations.

$$\left[\frac{a^2}{2}, \frac{a^{*2}}{2}\right] = a^*a + \frac{1}{2}$$

$$\left[a^*a + \frac{1}{2}, \frac{a^2}{2}\right] = -2\,\frac{a^2}{2} \qquad \left[a^*a + \frac{1}{2}, \frac{a^{*2}}{2}\right] = 2\,\frac{a^{*2}}{2}$$

which gives a representation of the SL_2 Lie algebra as follows. Compute matrix of $\left[\frac{a^2}{2}, \ \right]$ etc. in the space with basis $(a \ a^*)$.

$$\left[\frac{a^2}{2}, (a \ a^*)\right] = (0 \ \ a) = (a \ a^*)\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

$$\left[\frac{a^{*2}}{2}, (a \ a^*)\right] = (-a^* \ \ 0) = (a \ a^*)\begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix}$$

Thus $\quad \frac{a^2}{2} \longleftrightarrow X \qquad \frac{a^{*2}}{2} \longleftrightarrow -Y \qquad$ and so

$$a^*a + \frac{1}{2} = \left[\frac{a^2}{2}, \frac{a^{*2}}{2}\right] \longleftrightarrow [X, -Y] = -H$$

Now these operators $a^*a + \frac{1}{2}$, etc. work on the space of even holomorphic fns. which has the highest weight vector 1, which is killed by X. Then

$$H z^{2n} = -\left(a^*a + \tfrac{1}{2}\right) z^{2n} = -\left(2n + \tfrac{1}{2}\right) z^{2n}$$

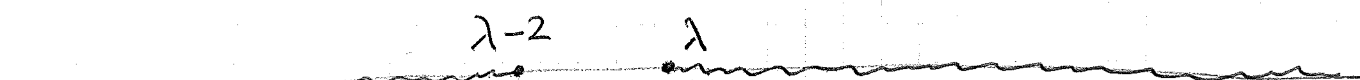
So the weights of this representation are

$$-\frac{1}{2}, \ -\left(\frac{1}{2} + 2\right), \ \ldots$$

Also we have the odd holomorphic functions with weight vectors $\quad z^{2n+1}, \quad n \geq 0 \quad$ giving us the weights

$$-\frac{3}{2}, \ -\left(\frac{3}{2} + 2\right), \ -\left(\frac{3}{2} + 4\right), \ \ldots$$

For a covering group of SL_2 and weights $\lambda \notin \mathbb{Z}$ we then have two irreducible strings:



$$\Sigma = \frac{1}{2}\lambda^2 - \lambda = \frac{1}{2}\left[(\lambda - 1)^2 - 1\right]$$

For the right string to be unitary we must have that

$$\frac{1}{2}\left[(\lambda-1)^2-1\right] < \frac{1}{2}\left[(\lambda+2n-1)^2-1\right]$$

$$|\lambda-1| < |\lambda+2n-1| \qquad n \geq 1.$$

which forces $\lambda > 0$, and conversely. For the left string to be unitary we must have

$$\frac{1}{2}\left[(\lambda-2n)^2-1\right] \geq \frac{1}{2}\left[(\lambda-1)^2-1\right] \qquad n \geq 1$$

or

$$|\lambda-2n| \geq |\lambda-1| \qquad n \geq 1$$

which forces $\lambda < 2$ or $\lambda - 2 < 0$.

So therefore if one takes $\lambda \in \mathbb{R}$ not zero, say >0 you should get an irreducible unitary repn. of a covering group with weights $\lambda, \lambda+2, \lambda+4, \ldots$, and dually for $\lambda < 0$ a unitary repn. with weights $\lambda, \lambda-2, \lambda-4, \ldots\ldots$

Let's go back to the holom. functions representation of $SL_2(\mathbb{R})$. Thus $X = \frac{a^2}{2}$ $Y = -\frac{a^{*2}}{2}$, $-H = a^*a + \frac{1}{2}$.
Let's now take the tensor product of this representation with itself. Then you get polys $z_1^{n_1} z_2^{n_2}$ with

$$X \longleftrightarrow \frac{1}{2}\left(\frac{\partial^2}{\partial z_1^2} + \frac{\partial^2}{\partial z_2^2}\right)$$

$$Y \longleftrightarrow \frac{1}{2}\left(z_1^2 + z_2^2\right)$$

$$-H \longleftrightarrow z_1\frac{\partial}{\partial z_1} + z_2\frac{\partial}{\partial z_2} + 1$$

So the subrepresentation generated by 1 has weights $-1, -3, -5, \ldots$. It consists of the powers

$$1, r^2, r^4, \cdots \qquad r^2 = z_1^2 + z_2^2$$

on which we have

$$X \longleftrightarrow \frac{1}{2}\frac{1}{r}\frac{\partial}{\partial r} r \frac{\partial}{\partial r}, \qquad Y \longleftrightarrow \frac{r^2}{2}, \quad -H \longleftrightarrow r\frac{\partial}{\partial r} + 1$$

In general one gets a Lie algebra representation with

$$X \longleftrightarrow \underbrace{\frac{1}{2}\,\frac{1}{r^{n-1}}\frac{\partial}{\partial r}r^{n-1}\frac{\partial}{\partial r}}_{\text{radial part of Laplacean}} \qquad Y \longleftrightarrow \frac{r^2}{2} \qquad -H \longleftrightarrow r\frac{\partial}{\partial r}+\frac{n}{2}$$

and possible one gets a uniform way of constructing representations this way.

Ideas about the representations of the loop groups. I would like to try to proceed by analogy with $SL_2(\mathbb{R})$. For SL_r the center of the Kac-Moody covering is spanned by $h_1 + \cdots + h_r$. The basic representations are when $\lambda(h_1) + \cdots + \lambda(h_r) = \varepsilon = 1$, so that exactly one $\lambda(h_i) = 1$ and the others are zero. Then the general Kac module will probably occur as a component in a tensor product. The key problem is therefore to construct the basic representations.

The next idea which comes from the SL_2 analogy is that the group in question $\mathrm{Maps}(S^1, SU_r)$ should be the automorphism group of a linear-like space which has a canonical Heisenberg style representation. Thus I would like to see something looking like the translation operators on the holomorphic representation

---

Question: Take a local field F like $\mathbb{Q}_p$. Is there an analogue of a metaplectic representation? I know that in some sense the Gaussian functions on $\mathbb{R}$ can be replaced by characteristic functions of lattices in a local field.

Try to take $L_2(F)$ on which one has a action of F as translations and F^* as scale transf.

It is known that F is self-dual as a locally compact abelian groups; hence we have the Fourier transform on $L_2(F)$. So do we get a representation of a covering of SL_2?

So [redacted] why is F self-dual? Look first at $\mathbb{Q}_p$. We have

$$\mathbb{Z}_p = \varprojlim_n \mathbb{Z}/p^n\mathbb{Z}$$

hence
$$(\mathbb{Z}_p)^\wedge = \varinjlim_n \mu_{p^n} = \mu_{p^\infty} \subset S^1$$
discrete top. because $\mathbb{Z}_p$ is compact.

[redacted] Then
$$\mathbb{Q}_p = \varinjlim (\mathbb{Z}_p \xrightarrow{p} \mathbb{Z}_p \xrightarrow{p} \cdots)$$

so
$$\hat{\mathbb{Q}}_p = \varprojlim (\mu_{p^\infty} \xleftarrow{p} \mu_{p^\infty} \xleftarrow{p} \cdots)$$

Now up to a non-canonical isomorphism one has
$$\mu_{p^\infty} \cong \mathbb{Q}_p/\mathbb{Z}_p,$$
but over the complex numbers it [redacted] might be reasonably canonical since one has the map
$$\mathbb{Q}/\mathbb{Z} \hookrightarrow S^1$$
$$q \longmapsto \exp(2\pi i q)$$
More precisely the group μ_{p^n} has a relatively distinguished generator namely $\exp\left(\frac{2\pi i}{p^n}\right)$ and these behave well under n. Thus one sees that

$$\begin{aligned}\hat{\mathbb{Q}}_p &= \varprojlim (\mu_{p^\infty} \xleftarrow{p} \mu_{p^\infty} \longleftarrow \cdots \\ &= \varprojlim (\mathbb{Q}_p/\mathbb{Z}_p \xleftarrow{p} \mathbb{Q}_p/\mathbb{Z}_p \longleftarrow \cdots) \\ &= \mathbb{Q}_p\end{aligned}$$

So therefore there is a reasonably canonical pairing [redacted] $\mathbb{Q}_p \otimes_{\mathbb{Z}} \mathbb{Q}_p \longrightarrow S^1$ setting up this duality. So what it seems to be is the following. You start with the map

$$(*) \qquad \mathbb{Q}_p \longrightarrow \mathbb{Q}_p/\mathbb{Z}_p \xrightarrow{\sim} \mu_{p^\infty} \subset S^1$$

and then combine this with the product on $\mathbb{Q}_p$. Thus the $\mathbb{Z}_p$ is its own annihilator.

In the case of a finite extension F of $\mathbb{Q}_p$ one knows because F is separable over $\mathbb{Q}_p$ that the [redacted] pairing $F \times F \to \mathbb{Q}$, $(x, y) \mapsto tr_{F/\mathbb{Q}_p}(xy)$ is non-degenerate.

This construction makes sense for adeles. So if you start with $\prod_p' \mathbb{Q}_p \times \mathbb{R}$ you have an obvious map to [redacted] S^1 which can be used for the basic duality.

Now we need to be able to compute some Fourier transforms. [redacted] Let us denote by $e(x)$, the basic map $(*)$ from $\mathbb{Q}_p$ to S^1. Then the Fourier transform is

$$\hat{f}(x) = \int e(xy) f(y)\, dy$$

and it makes good sense when f has compact support. Take f to be the characteristic function of $\mathbb{Z}_p$. If $x \in \mathbb{Z}_p$ then we get

$$\int_{\mathbb{Z}_p} 1 = 1$$

by the standard normalization. If $x \notin \mathbb{Z}_p$, then

$$y \longmapsto e(xy)$$

is a non-trivial character of $\mathbb{Z}_p$, so it integrates to give zero. Thus the characteristic function of $\mathbb{Z}_p$ is self-dual

In general the characteristic function of a lattice will have as Fourier transform the characteristic function of the dual lattice with a scalar. More precisely if one takes [redacted] the lattice $p^n\mathbb{Z}$, then the Fourier transform has value $\frac{1}{p^n}$ on $p^{-n}\mathbb{Z}$. This is a special case of

$$\int e(xy)\, f(\beta y)\, dy = \int e\left(\tfrac{x}{\beta} y\right) f(y) \frac{dy}{|\beta|} = \frac{1}{|\beta|} \hat{f}\left(\tfrac{x}{\beta}\right)$$

<u>October 7, 1981</u>

Let's carefully understand what we can say about the group $\mathrm{Maps}(S^1, S^1)$ and more generally $\mathrm{Maps}(S^1, T)$ where T is a torus. The first idea is that one has a canonical map (group homom)

$$\begin{aligned} \mathrm{Maps}(S^1,S^1) &\longrightarrow \mathbb{Z}\times S^1 \\ f &\longmapsto (\deg f,\ f(1)) \end{aligned}$$

which is a homotopy equivalence because it's onto and the kernel is loops in S^1 of degree $0 \cong$ loops in $\mathbb{R}$. The kernel of the degree map

$$0 \to \mathbb{Z} \longrightarrow \underset{\substack{\| \\ \text{Lie algebra}}}{\mathrm{Maps}(S^1,\mathbb{R})} \longrightarrow \mathrm{Maps}(S^1,S^1) \xrightarrow{\ \deg\ } \mathbb{Z} \longrightarrow 0$$

is essentially the Lie algebra.

Another idea was that for the torus T of diagonal matrices in SU_2 one gets a central extension of $\mathrm{Map}(S^1,T)$ from a Heisenberg extension. How does this go?

Let's begin by trying to understand the canonical central extension of $SL_2(\mathbb{C}[z,z^{-1}])$ restricted to the diagonal subgroup $\cong \mathbb{Z}\times\mathbb{C}^*$. Better to work with $SL_2(F)$, $F =$ Laurent series? Now from symbol theory I know this central extension has the commutator pairing

$$\begin{pmatrix} f & \\ & f^{-1}\end{pmatrix}\begin{pmatrix} g & \\ & g^{-1}\end{pmatrix} \longmapsto \left(\frac{f^{\deg g}}{g^{\deg f}}\right)^2$$

The next question is what to do with the Lie algebra.

The problem is that we can start with a central extension of F^* given by the commutator pairing

$$(f,g) \longmapsto \frac{f^{\deg g}}{g^{\deg f}}$$

This corresponds to pulling back via the canon. map

$$\circledast \qquad F^{\circ} \longrightarrow \mathbb{Z}\times\mathbb{C}^{\circ}$$

the Heisenberg central extension. [redacted]

The question is what to do with the Lie algebra. Things may be subtle, because it seems that F has a central Lie algebra extension.

We know that central extensions of an abelian Lie algebra F are given by [redacted] alternating maps $F\times F \to \mathbb{C}$. This is because $H^*(\mathfrak{g}) = \Lambda^*(\mathfrak{g}^\vee)$. On F we have the canonical alternating [redacted] form

$$(f,g) \longmapsto \int f\,dg$$

So it seems natural to conjecture that the central Lie extension of F should be compatible in a certain sense with the pull-back of the Heisenberg extension via $\circledast$.

For example we have over $SL_2($[redacted]$F)$ the central extension by $\mathbb{C}^{\circ}$ defined by the tame symbol. There is also the Kac-Moody central extension of the Lie algebra $sl_2(F)$ by $\mathbb{C}$. Somehow these two gadgets should be compatible. If so get an extension of $GL_2(F)$.

Possible answer based on the Poisson summation formula. [redacted] The idea should be some way of relating the Heisenberg extension of $\mathbb{Z}\times\mathbb{C}^{\circ}$ with that of $\mathbb{C}\times\mathbb{C}$. Better to use $\mathbb{Z}\times S^1$ related to $\mathbb{R}\times\mathbb{R}$ relative to the embedding $\mathbb{Z}\subset\mathbb{R}$.

$$\log\left(\frac{f^{\deg g}}{g^{\deg f}}\right) = \log f \ \mathrm{res}(d\log g) - \log g \ \mathrm{res}(d\log f)$$

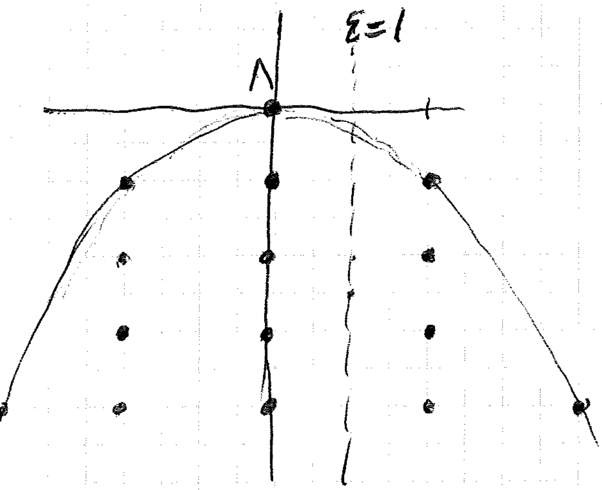
Spencer's dilogarithm ?

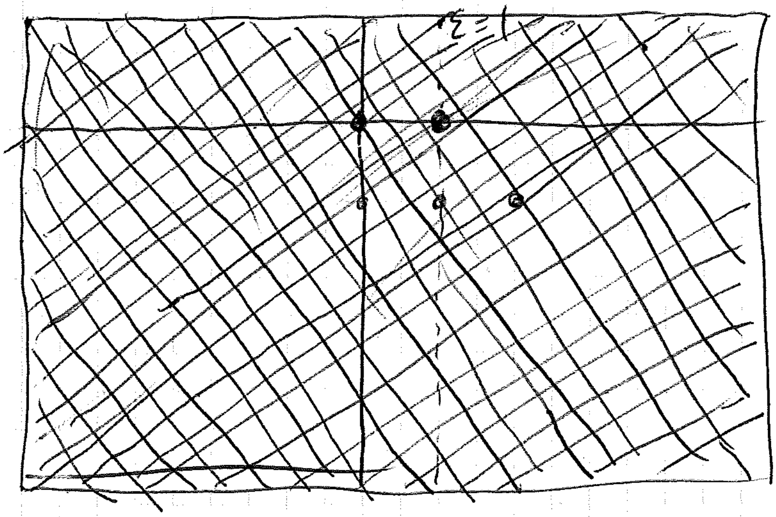
Project: Inside of the loop algebra is the diagonal subalgebra, which is the central extension of the abelian Lie algebra $\sum (z^n H)$, $n \in \mathbb{Z}$ defined by the cocycle

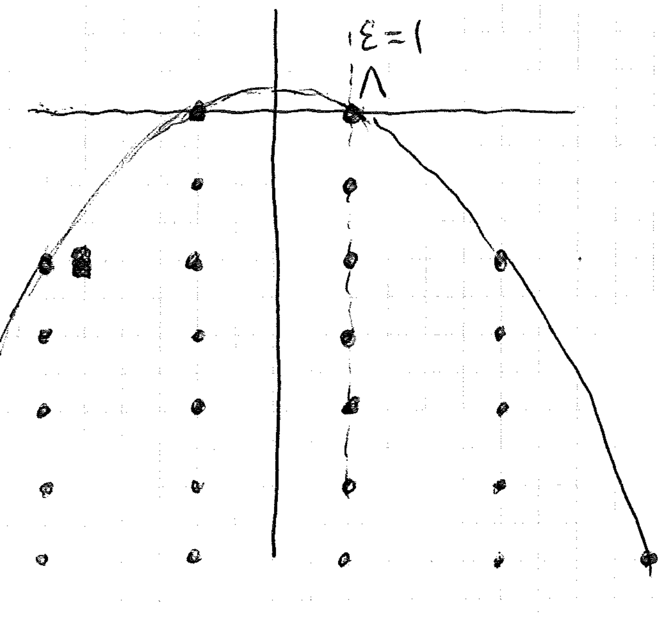
$$f(z)H,\ g(z)H \longmapsto 2\int f\,dg \cdot \frac{1}{2\pi i} = 2\,\mathrm{res}(f dg)$$

The 2 comes from $(H,H) = 2$. Now we perhaps can see how this diagonal subalgebra acts on the Kac modules, in particular what can be generated from the highest weight vector.

Recall that we are above all interested in the standard representations which belong to the cases where $\varepsilon = \Lambda(h_1 + h_2) = 1$. Thus one has $\lambda(h_1, h_2) = (1,0)$ or $(0,1)$, and the weights look as follows.







We should compute the characters of these representations so that we know the multiplicities. This means working out the Weyl-Kac character formula.

$$\mathrm{ch}\,V(\Lambda) = \frac{\sum_{w}' \det(w)\, e^{w(\Lambda+\rho)}}{\sum_{w} \det(w)\, e^{w\rho}}$$

and the denominator formula

$$\prod_{\alpha>0} (1 - e^{-\alpha}) = \sum_{w} \det(w)\, e^{w\rho - \rho}$$

October 8, 1981:

[redacted] There is a problem: Let F be the local field of Laurent series. Then we get a central extension of $SL_2(F)$ defined by the tame symbol, and we know from the theory of symbols, that the restriction of this central extension to the diagonal subgroup H of $\begin{pmatrix} f & 0 \\ 0 & f^{-1} \end{pmatrix}$ $f \in F^*$ is the extension with the commutator pairing

$$f, g \longmapsto \left(\frac{f^{\deg g}}{g^{\deg f}}(0) \right)^2$$

Thus we have this central extension

$$(1) \qquad 1 \longrightarrow \mathbb{C}^* \longrightarrow E \longrightarrow F^* \longrightarrow 1.$$

Now there is a central extension of $sl_2(F) = Lie(SL_2(F))$ defined by the cocycle

$$x, y \longmapsto \operatorname{res} tr(x\,dy)$$

and if we restrict this to $Lie(H) = Lie(F^*) = F$ we get the [redacted] central Lie alg extension of F by $\mathbb{C}$ with commutator pairing

$$\begin{pmatrix} 1 & \\ & -1 \end{pmatrix}$$

$$f, g \longmapsto \operatorname{res}(f\,dg)\ tr(H^2)$$

$$= 2 \operatorname{res}(f\,dg).$$

I was hoping that the central group extension of F^* would be compatible with the central Lie alg extension of F. [redacted] However if you apply Lie to (1) you end up with [redacted] cutting F^* down to its identity [redacted] component, i.e. to $\deg(f) = 0$, and then one gets an abelian extension of Lie algebras.

However we do know that

$$K_2(\mathbb{C}[z, z^{-1}]) = K_2(\mathbb{C}) \oplus \underbrace{K_1(\mathbb{C})}_{\mathbb{C}^*}$$

and therefore the only stable central extensions of $SL_n(\mathbb{C}[z,z^{-1}])$ come from a universal such extension [redacted] with kernel $\mathbb{C}^\circ$.

So morally I am convinced there has to be a good way to correlate these two extensions. And the point somehow ought to involve the fact that only for $\deg(f)$ [redacted] ≥ 0 can we make sense of e^f. Thus the local isomorphism of F and F° takes place over the ring of [redacted] power series $\mathbb{C}[[z]]$.

$$F \supset \mathbb{C}[[z]] \xrightarrow{\exp} \mathbb{C}[[z]]^\circ \subset F^\circ$$

And over $\mathbb{C}[[z]]^\circ$ the extension of F° is abelian, and over $\mathbb{C}[[z]]$ the Lie extension of F is abelian.

So what would be really nice is a situation where we could see directly the compatibility of these extensions. Possibility: find a common representation of both group and Lie algebra.

How to do it for the group. I will take the simplest case, namely, where F is replaced by $\mathbb{C}[z,z^{-1}]$ and then

$$F^\circ \longmapsto \mathbb{C}[z,z^{-1}]^\circ \cong \mathbb{Z} \times \mathbb{C}^\circ$$

This group has a natural representation of a central extension on $\mathbb{C}[z,z^{-1}]$ with $\mathbb{Z} = \{z^n\}$ acting as multiplication and $\mathbb{C}^\circ = \{\zeta\}$ acting as translation on the circle.

Next look at the Lie algebra. On $F = \mathbb{C}[z,z^{-1}]$ we have this skew-symmetric form $\operatorname{res}(f\,dg)$, and

$$\operatorname{res}(z^m dz^n) = n \operatorname{res}\left(z^{m+n}\frac{dz}{z}\right)$$
$$= \begin{cases} 0 & m \neq -n \\ n & m = -n \end{cases}$$

so it is non-degenerate off $\mathbb{C}\cdot 1$. The standard trick is to take a maximal isotropic subspace $\mathbb{C}[z] = W$

and then construct the representation on $S(W)$. On $S(W)$ one has W acting as multiplication a^*_w and W^* acting as derivations a_λ with the usual commutation relations:

$$[a_\lambda, a^*_w] = \langle \lambda, w \rangle .$$

Actually we have to be more careful because the pairing is degenerate. So therefore what you want to do is put $W = z\mathbb{C}[z]$. Then [redacted] W^* can be identified with $z^{-1}\mathbb{C}[z^{-1}]$ using the residue pairing. So we get W, W^* acting on $S(W)$ and finally you can specify how $\mathbb{C}1 \in F$ should act on $S(W)$.

We can use this idea as follows. Inside the Kac Moody algebra is the inverse image of $\mathbb{C}[z,z^{-1}]H$, i.e. the centralizer of $\mathfrak{h}$. Now we can restrict the Kac module $V(\Lambda)$ to this subalgebra. Call it L. It has the basis $z^n H$, $n \neq 0$ and h_1, h_2 and the bracket is completely defined by the pairing

[redacted] $$fH, gH \longmapsto -2\,\mathrm{res}(f\,dg)(h_1+h_2).$$

(The point is that the bracket $L \times L \to L$ is equivalent to a skew-symmetric map $F \times F \to (h_1+h_2)$.) Check of -2:

$$[e_1, e_2] \longmapsto [X, zY] = zH$$
$$[f_2, f_1] \longmapsto [z^{-1}X, Y] = z^{-1}H.$$

Now $\mathrm{res}(z^{-1}dz) = 1$, and

$$[[f_2,f_1],[e_1,e_2]] = [f_2,\underbrace{[f_1,[e_1,e_2]]}] - [f_1,[f_2,[e_1,e_2]]]$$

[redacted]

$$= [f_2,\underbrace{[\underbrace{[f_1,e_1]}_{-h_1},e_2]}_{2e_2}] - [f_1,\underbrace{[e_1,\underbrace{[f_2,e_2]}_{-h_2}]}_{[h_2,e_1]=-2e_1}]$$

$$= 2[f_2, e_2] + 2[f_1, e_1] = -2(h_2 + h_1)$$

Now L is a Heisenberg algebra essentially and so we know what its irreducible modules are. The central elements h_1, h_2 act as scalars and as long as $h_1 + h_2 \neq 0$ there is only one module.

So what I want to do next is to calculate the character of the L submodule of $V(\Lambda)$ generated by a weight vector v_λ, assuming $\lambda(h_1) + \lambda(h_2) \neq 0$.

Put $L = W_- \oplus (h_1, h_2) \oplus W_+$ where $W_+ \xrightarrow{\sim} z\mathbb{C}[z]H$ and $W_- \xrightarrow{\sim} z^{-1}\mathbb{C}[z^{-1}]H$ are D-invariant.

So what do we find? The representation should be $S(W_-) \otimes v_\lambda$ and so its character should be

$$\frac{e^\lambda}{\prod(1 - e^{-\alpha})}$$

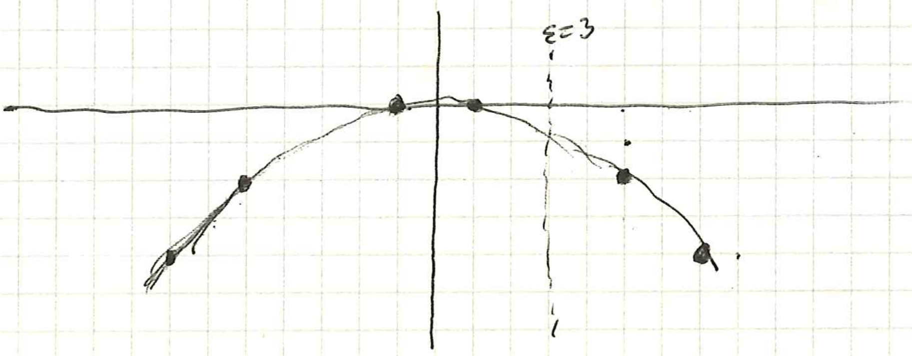
where $e^{-\alpha}$ runs of the characters of W_-. So we get, using the convention that $q =$ character of $z^{-1}H$, that

$$\mathrm{char}\,(U(L)\cdot e_\lambda) = \frac{e^\lambda}{\prod_{n=1}^{\infty}(1 - q^n)}$$

Let's go back to computing the character of the basic Kac modules. First take $\Lambda = \begin{pmatrix} 0 & 1 & 0 \\ h_1 & h_2 & D \end{pmatrix}$

and recall $\rho = (1 \;\; 1 \;\; 0)$, so $\Lambda + \rho = (1 \;\; 2 \;\; 0) \Rightarrow \varepsilon = (\Lambda + \rho)(h_1 + h_2) = 3$

So the orbit looks like



$$y + \frac{1}{12}x^2 = \frac{1}{12}$$

$$-y = \frac{x^2 - 1}{12} \in \mathbb{Z} \text{ for when } x \in \mathbb{Z}$$

$$x = 6n \pm 1$$

Recall our convention that to the point (x,y) we associate $u^{-x}q^{-y}$. $-y = \frac{(6n\pm1)^2-1}{12} = 3n^2 \pm n$. Thus we get the numerator series

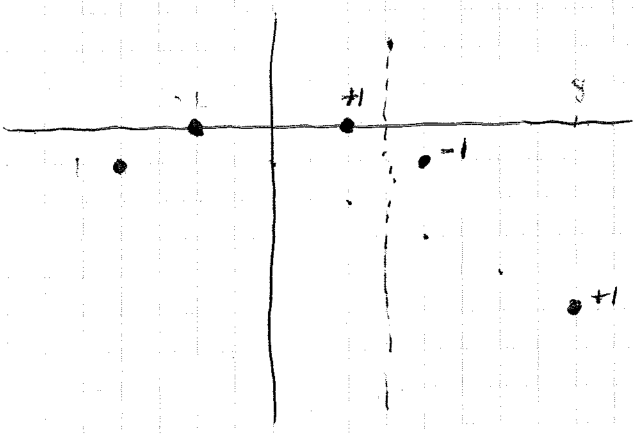
$$\sum_n u^{-(6n+1)+1} q^{3n^2+n} - \sum_n u^{-(6n-1)+1} q^{3n^2-n}$$

$$= \sum_n \left(u^{6n} - u^{-6n+2}\right) q^{3n^2-n}$$

And so

$$\mathrm{char}(V(0,1,0)) = \frac{\sum_n \left(u^{6n} - u^{-6n+2}\right) q^{3n^2-n}}{\underbrace{\sum_n (-1)^n u^{2n} q^{\frac{n^2-n}{2}}}_{\prod_{n\geq 0}(1-u^2q^n)\prod_{n\geq 1}(1-q^n)(1-u^{-2}q^n)}}$$

The other case is $\Lambda = (1,0,0)$ $\quad \Lambda+\rho = (2,1,0)$ $\quad \varepsilon = 3$



$$y + \frac{x^2}{12} = \text{const} \Rightarrow -y = \frac{1}{3}\left(\left(\frac{x}{2}\right)^2 - 1\right)$$

$\in \mathbb{Z}$ for $x \in \mathbb{Z}$

when $x = 2(3n\pm1) = 6n\pm2$

$$-y = \frac{1}{3}\left[(3n\pm1)^2 - 1\right] = 3n^2 \pm 2n$$

$\therefore$ numerator series is

$$\sum_n u^{-(6n+2)+1} q^{3n^2+2n} - \sum u^{-(6n-2)+1} q^{3n^2-2n}$$

$$= \sum_n \left(u^{6n-1} - u^{-6n+3}\right) q^{3n^2-2n}$$

Put

$$F(u) = \sum_n u^{6n} q^{3n^2-n}$$

$$= \sum_n u^{6n+6} q^{3n^2+6n+3-n-1}$$

$$= u^6 q^2 \sum_n u^{6n} q^{6n} q^{3n^2-n}$$

$$= u^6 q^2 F(qu)$$

Put

$$F_1(u) = \sum_n u^{-6n+2} q^{3n^2-n}$$

$$= \sum u^{-6n+6+2} q^{3n^2-6n+3-n+1}$$

$$= u^6 q^2 \sum' u^{-6n+2} q^{3n^2-n} q^{-6n+2}$$

$$= u^6 q^2 F_1(qu)$$

Thus $\quad u^2 F(u) = u^6 (qu)^2 F(qu)$ and similarly for F_1

Try $\quad F_2(u) = \sum' u^{6n-1} q^{3n^2-2n}$

$$= \sum' u^{6n+6-1} q^{3n^2+6n+3-2n-2}$$

$$= u^6 \sum u^{6n-1} q^{3n^2-2n} q^{6n-1+2}$$

$$= u^6 q^2 F_2(qu)$$

$$F_3 = \sum' u^{-6n+3} q^{3n^2-2n}$$

$$= \sum' u^{-6n+6+3} q^{3n^2-6n+3-2n+2}$$

$$= u^6 \sum' u^{-6n+3} q^{3n^2-2n} q^{-6n+3+2}$$

$$= u^6 q^2 F_3(qu)$$

So $u^2 F_i(u) = G(u)$ satisfies the identity $\quad G($[redacted]$u) = u^6 G(qu)$.
How many Laurent series satisfy this $\quad G(u) = \sum' u^n a_n$

$$a_n u = u^6 a_{n-6} q^{n-6} u^{n-6} \Rightarrow a_n = a_{n-6} q^{n-6}$$

A basis for the[redacted] space of solutions is

$$\sum' u^{6n} q^{3n^2-3n}$$

$$\sum u^{6n+1} q^{3n^2-2n}$$

$$\sum' u^{6n+2} q^{3n^2-n}$$

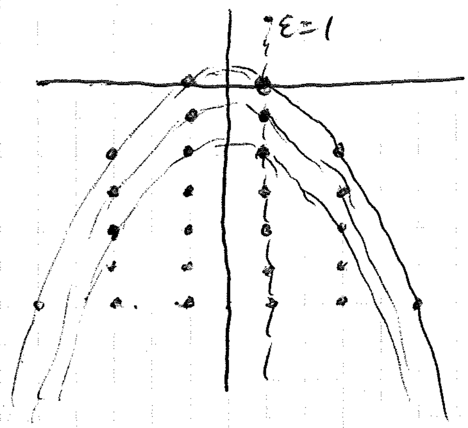
$$\sum u^{6n+3} q^{3n^2}$$

$$\sum u^{6n+4} q^{3n^2+n}$$

$$\sum u^{6n+5} q^{3n^2+2n}$$

October 9, 1981

Look directly at the character of the representations in question, and use the fact that the multiplicities are invariant under the Weyl group. For example, take $\Lambda = (0\ 1\ 0)$ so that the weights are odd. The Weyl gp orbits are the different parabolas



$$y + \frac{x^2}{4} = \frac{1}{4}, -\frac{3}{4}, -\frac{7}{4}, \ldots\ldots$$

over the odd integers. Hence

$$\text{char } V(\Lambda) = \varphi(q) \sum_{x \in 1+2\mathbb{Z}} u^{-x} q^{\frac{x^2-1}{4}}$$

where $\varphi(q)$ = multiplicities in any vertical direction.

$$\sum_{x\in 1+2\mathbb{Z}} u^{-x} q^{\frac{x^2-1}{4}} = \sum u^{-2n-1} q^{n^2+n} = \sum u^{2n-1} q^{n^2-n}$$

Put $\quad F(u) = \sum u^{2n} q^{n^2-n} = \sum u^{2n+2} q^{n^2-n+(2n+1-1)}$

$$= u^2 \sum u^{2n} q^{2n} q^{n^2-n} = u^2 F(qu)$$

Now we know that the Laurent series solutions of

$$F(u) = u^2 F(qu)$$

form a 2-diml space whose solutions can be described either in terms of the basis

$$\sum u^{2n} q^{n^2-n}, \quad \sum u^{2n+1} q^{n^2}$$

or as products $\quad \theta\left(\frac{u}{a}\right)\theta(au)$

where θ is the standard solution of $\theta(u) = -u\,\theta(qu)$

$$\theta(u) = \sum (-1)^n u^n q^{\frac{n^2-n}{2}} = \prod_{n\geq 0}(1-q^n u) \prod_{n\geq 1}(1-q^n)(1-q^n u^{-1})$$

Now $\quad F(u) = \sum u^{2n} q^{n^2-n} = \frac{1}{2}\sum (u^{2n} + u^{2-2n}) q^{n(1-n)}$

has zeroes at $u^2 = -1$, i.e. $u = \pm i$. So therefore if $\quad F(u) = c\,\Theta(au)\,\Theta\left(\frac{u}{a}\right) \qquad \Theta(au)=0 \quad au \in q^{\mathbb{Z}}$

one can take $a = i$. Thus it is clear that

$$F(u) = \text{const.}\ \Theta(iu)\Theta(-iu)$$

$$\Theta(iu)\,\Theta(-iu) = \prod_{n\geq 0}(1-q^n iu)(1-q^n(-iu)) \prod_{n\geq 1}(1-q^n)^2(1-q^n(iu)^{-1})(1-q^n(-iu)^{-1})$$

$$= \prod_{n\geq 0}(1+q^{2n}u^2)\prod_{n\geq 1}(1-q^n)^2 \prod_{n\geq 1}(1+q^{2n}u^{-2})$$

Now $\quad F(u) = \sum u^{2n} q^{n^2-n} = \sum (u^2)^n (q^2)^{\frac{n^2-n}{2}} = \Theta(u^2; q^2)$

$$= \prod_{n\geq 0}(1+q^{2n}u^2)\prod_{n\geq 1}(1-q^{2n})\prod_{n\geq 1}(1+q^{2n}u^{-2})$$

which checks.

Now go back to the Weyl-Kac formula where we know that

$$\operatorname{char} V(\Lambda) = \frac{\sum_n (u^{6n-1} - u^{-6n+3})\,q^{3n^2-2n}}{\Theta_q(u^2)}$$

Now the numerator vanishes for $u^{6n-1} = u^{-6n+3} \Leftrightarrow u^{12n} = u^4$ $\Leftrightarrow u^4 = 1$. So we get immediately the roots $u = \pm 1, \pm i$

If I multiply the numerator $N(u)$ by u^2 I get an odd function $G(u)$ of u satisfying $G(u) = u^6 G(qu)$, and these form a three-dimensional space.

Look at

$$uN(u) = \frac{G(u)}{u} = \sum (u^{6n} - u^{-6n+4})\,q^{3n^2-2n}$$

and call this $K(u^2)$. Then

$$K(u^2) = \frac{G(u)}{u} = q\frac{u^6 G(qu)}{qu} = q(u^2)^3 K(q^2u^2)$$

has the solution

$$\Theta_{q^2}\left(\frac{u^2}{a}\right)\Theta_{q^2}\left(\frac{u^2}{b}\right)\Theta_{q^2}\left(\frac{u^2}{c}\right)$$

where $\left(-\frac{1}{a}\right)\left(-\frac{1}{b}\right)\left(-\frac{1}{c}\right) = q$. So we have the roots $a=1, b=-1$

and so $c = 1/q$, hence $q^2 c = q$ must be a root also of $K(u^2)$, so that $\pm\sqrt{q}$ should be a root of $uN(u)$. ~~Actually this is clear because since $K(u^2)$ has the root~~ Check this

$$uN(u) = \sum \left(u^{6n} - u^{-6n+4}\right) q^{3n^2-2n}$$

$$K(q) = \sqrt{q}\, N(\sqrt{q}) = \sum q^{3n+3n^2-2n} - \sum q^{-3n+2+3n^2-2n}$$

$$= \sum q^{3n^2+n} - \underbrace{\sum q^{3n^2-5n+2}}_{\sum q^{\underbrace{3n^2+6n+\cancel{3}-5n-\cancel{5}+\cancel{2}}_{3n^2+n}}} = 0$$

OKAY so now I have

$$\text{char } V(\Lambda) = \frac{u^{-1}(\text{const})\, \Theta_{q^2}(u^2)\, \Theta_{q^2}(-u^2)\, \Theta_{q^2}(qu^2)}{\Theta_q(u^2)}$$

$$= \text{const } u^{-1}\, \Theta_{q^2}(u^2)$$

which agrees with the previous calculation. But the problem is still to determine the constant which is a function of q.

This means one has to take the function

$$uN(u) = K(u^2) = \sum \left(u^{6n} - u^{-6n+4}\right) q^{3n^2-2n}$$

and effectively write it out ~~[illegible]~~ as

$$c(q)\, \Theta_{q^2}(u^2)\, \Theta_{q^2}(-u^2)\, \Theta_{q^2}(qu^2)$$

~~This means that I [illegible]~~ The only way I can see how to do this is to specialize the variable u in some way. Kac has a method based on modular functions.

So we are going to have to learn about modular functions. Let us begin the example of a Θ-fn. defined by a lattice in $\mathbb{R}$. Call the lattice M

and suppose given a quadratic function of M with a negative definite real part. $\operatorname{Im}\tau > 0$

$$Q(x) = i\tau \frac{x^2}{2} + ax + \text{const.}$$

Then one gets a Θ-function

$$\sum_{x \in M} e^{Q(x)}$$

Review: Poisson summation formula

$f(x) \in C_0^\infty(\mathbb{R})$ $\qquad g(x) = \sum_{m \in \mathbb{Z}} f(x+m)$ $\qquad$ periodic

$$= \sum a_n e^{2\pi i n x}$$

where $\quad a_n = \int_0^1 g(x) e^{-2\pi i n x} dx = \int_{-\infty}^{\infty} f(x) e^{-2\pi i n x} dx = \hat{f}(2\pi n)$

So

$$\boxed{\sum_{m \in \mathbb{Z}} f(x+m) = \sum_m \hat{f}(2\pi n) e^{2\pi i n x}}$$

In particular $\quad f(x) = e^{-a\frac{x^2}{2}} \implies \hat{f}(\xi) = \frac{\sqrt{2\pi}}{\sqrt{a}} e^{-\frac{1}{a}\frac{\xi^2}{2}}$

$$\sum_{m \in \mathbb{Z}} e^{-a\frac{m^2}{2}} = \sqrt{\frac{2\pi}{a}} \sum_n e^{-\frac{1}{a}\frac{(2\pi n)^2}{2}}$$

Put $a = 2\pi t$

$$\boxed{\sum_{\mathbb{Z}} e^{-\pi t m^2} = \frac{1}{\sqrt{t}} \sum_{\mathbb{Z}} e^{-\frac{\pi}{t} n^2}}$$

which is the familiar one used with ζ

$$\sum_{m \in \mathbb{Z}} e^{-\pi t (m+x)^2} = \frac{1}{\sqrt{t}} \sum_{n \in \mathbb{Z}} e^{-\frac{\pi}{t} n^2 + 2\pi i n x}$$

So now what can be said about our friend

$$\theta_q(-u) = \sum_{n \in \mathbb{Z}} u^n q^{\frac{n^2 - n}{2}}$$

Put $q = e^{2\pi i \tau}$ with $\tau \in UHP$.

New approach. Let's go back to the problem of the compatibility of the extensions of F° and F. Here I am thinking of F as $\mathbb{C}[z,z^{-1}]$ with the skew-form Res(fdg) which gives the central extension. And I am thinking of F° as having the central extension with commutator pairing $f,g \longmapsto f^{\deg g}/g^{\deg f}$. Somehow these two extensions should be compatible in a very good way. It would also be nice if I can make sense of $F =$ the local field $\mathbb{C}[[z]][z^{-1}]$ or $F^\circ = \text{maps}\{S^1 \to \mathbb{C}^\circ\}$, $F = \text{maps}\{S^1 \to \mathbb{C}\}$. In this latter case however one really has a good exponential map, so one can take the Lie algebra of the extension of F°, and one finds it is abelian.

Yesterday I worked out the representation of the Lie algebra. The central extension has a double center because the element $1 \in \mathbb{C}[z,z^{-1}]$ is isotropic for res(fdg).

Still very confusing: You could take $F =$ analytic functions on S^1 and then $F \to F^\circ$ would be well-defined. The real problem is going to be how to exponentiate the Lie algebra action. Note that if we take analytic functions over S^1, then the duality defined by res(fdg) is good. So we want a Hilbert space representation. Thus I need to have a Hilbert space W with which to form $S(W)$.

So let $S = \mathfrak{h}[z, z^{-1}]$ be the diagonals in $\mathfrak{g}_0[z, z^{-1}]$ and let $\tilde{S}$ be the central extension. [redacted] In the representation I am interested in $\varepsilon = \lambda(h_1 + h_2) = 1$ for all weights, [redacted] so the $\tilde{S}$ submodule generated by a weight vector has to be the Heisenberg representation. More precisely take a weight vector v_λ on the highest parabolic = Weyl orbit of Λ. Then v_λ is killed by $z^n H$ for $n \geq 1$ and the center $h_1 + h_2$ acts as 1. Therefore the elements $z^n H$ for $n \geq 1$ are going to act like annihilation operators and the ~~operators~~ $z^n H$ for $n \leq -1$ will act as creators. Since the pairing is

$$fH, gH \longmapsto 2 \operatorname{res}(g\,df) \cdot h_1 + h_2$$

we have

$$(z^n H, z^{-n} H) \longmapsto 2 \underbrace{\operatorname{res}(z^{-n} dz^n)}_{n} \cdot (h_1 + h_2)$$

and therefore we have to put factors $\sqrt{\frac{2}{n}}$ in order to get the actual creation and annihilation operators. So how does this all work?

October 10, 1981

One thing we learn from Kac-Frenkel construction is that the central extensions of the Lie algebra F and of the group F° are separate. Specifically we find that the basic representation module will be a tensor product $S(F^-) \otimes \mathbb{C}[\mathbb{Z}]$ of the repns of $\tilde{F}$ and $\tilde{F^\circ}$.

More precisely $S(F^-)$ is a representation of $\widetilde{F^- + F^+}$ and $\mathbb{C}[\mathbb{Z}]$ is a representation of $\widetilde{\mathbb{Z} \times \mathbb{C}^\circ}$, so the tensor product is a representation of the product amalgamated over the center.

However one should really think of F as something like analytic functions on S^1, and then exp maps F onto analytic maps $f : S^1 \to \mathbb{C}^\circ$ of degree 0, whereas $F^+ + F^-$ is mapped by exp isomorphically onto those f of degree 0 such that $\int_{S^1} \log f \in 2\pi i \mathbb{Z}$.

More precisely, let $\mathcal{K} =$ analytic maps $S^1 \to \mathbb{C}^\circ$. Then
$F =$ " " $S^1 \to \mathbb{C}$

$$0 \longrightarrow \mathcal{K}_0 \longrightarrow \mathcal{K} \xrightarrow{\deg} \mathbb{Z} \longrightarrow 0$$

$$\begin{array}{ccccccccc}
0 & \longrightarrow & 2\pi i \mathbb{Z} & \longrightarrow & F & \xrightarrow{\exp} & \mathcal{K}_0 & \longrightarrow & 0 \\
 & & & & \| & & & & \\
 & & & & F^- \oplus \mathbb{C} \oplus F^+ & & & & \\
 & & & & \cup & & & & \\
0 & \longrightarrow & 2\pi i \mathbb{Z} & \longrightarrow & \mathbb{C} & \longrightarrow & \mathbb{C}^\circ & \longrightarrow & 0
\end{array}$$

so that
$$F^- \oplus F^+ \xrightarrow{\sim} \mathcal{K}_0 / \mathbb{C}^\circ .$$

Hence we see that the residue pairing on $F^- \oplus F^+$ defines a central group extension of $\mathcal{K}_0/\mathbb{C}^\circ$. Specifically given $f, g \in \mathcal{K}_0$ we can define $\log f$ and $\log g$ and then our commutator is $(f,g) = \exp\{ \operatorname{Res}(\log g \, d \log f)\}$.

Thus we get the formula for the commutator pairing

$$\boxed{(f,g) = \exp\left\{\frac{1}{2\pi i}\int_{S^1} \log(g)\,\frac{df}{f}\right\}}$$

which is the <u>dilogarithm</u>. The point is that it makes sense for $f,g \in \mathcal{K}$. ? $\log(g)$ is defined up to $2\pi i n$ for $n \in \mathbb{Z}$, then $\frac{1}{2\pi i}\int_{S^1} 2\pi i n \frac{df}{f} = (2\pi i) n \,(2\pi i \deg f) \in 2\pi i \mathbb{Z}$.

It also makes sense for Laurent series. For $f, g \in \mathbb{C}[z,z^{-1}]^\circ = \mathbb{Z} \times \mathbb{C}^\circ$. ? Actually, it's not clear what (f,g) means if g is not of degree 0.

Let's try to understand what one needs to get a pairing defined on all of $\mathcal{K}$. We have canonical maps

$$\mathbb{C}^\circ \xhookrightarrow[\text{constant}]{} \mathcal{K} \xrightarrow{\deg} \mathbb{Z}$$

and $\mathbb{C}^\circ$ is the kernel of the form on $\mathcal{K}_0$ defined by the dilogarithm function above. So we have the following situation, or rather a situation analogous to the following:

Given a space V a hyperplane W and a pairing $V \times W \longrightarrow k$, skew-symmetric on $W \times W$, having a line $L \subset W$ for kernel such that the pairing on $V/W \times L \to k$ is non-degenerate. What do I need to do to extend the pairing to all of V? Answer: choose a complement L' to W in V.

Better version. Start with pairing $V \times W \longrightarrow k$ with W a hyperplane in V, and the pairing skew-symmetric on $W \times W$. Let $L =$ **annihator of** W in W and suppose that $V/L \times W \longrightarrow k$, $V/W \times L \longrightarrow k$ are non-degenerate. Then if I choose an L' complementary to W, I get an obvious extension of the pairing to $V \otimes V = V \otimes W \oplus V \otimes L'$

$= V\otimes W \oplus W\otimes L' \oplus L'\otimes L'$, because its given already on $V\otimes W + W\otimes V$ by skew-symmetry and on $L'\otimes L'$ it has to be zero. Actually we don't have to pick L', because given v_1, v_2 we know what (v_1, v_2) is if $v_2 \in W$ or if $v_1 \in W$ and if both are outside W, then $v_2 - a v_1 \in W$ for a unique $a \in k$ and so

$$(v_1, v_2) = (v_1, v_2 - a v_1)$$

is defined. This proves uniqueness of the extension.

Alternatively one can argue that

$$\begin{tikzcd} W\otimes W \arrow[r, hook] \arrow[d] \arrow[dr, phantom, "\text{cocart}"] & V\otimes W \arrow[d] \\ \Lambda^2 W \arrow[r] & V\cdot W \subset \Lambda^2 V \end{tikzcd}$$

equal because the cokernel is $\Lambda^2(V/W) = 0$.

So now put $\mathcal{K}$ = continuous maps $S^1 \to \mathbb{C}^*$ and look at

$$0 \to \mathcal{K}_0 \to \mathcal{K} \xrightarrow{\deg} \mathbb{Z} \to 0$$

and the pairing

$$(f,g) = \exp\left\{\frac{1}{2\pi i}\int_{S^1} \log(g)\frac{df}{f}\right\}$$

defined for $f \in \mathcal{K}$, $g \in \mathcal{K}_0$ bilinear and alternating on $\mathcal{K}_0$. Then we have

$$\begin{tikzcd} \mathcal{K}_0 \otimes_{\mathbb{Z}} \mathcal{K}_0 \arrow[r] \arrow[d] \arrow[dr, phantom, "\text{cocart}"] & \mathcal{K}_0 \otimes_{\mathbb{Z}} \mathcal{K} \arrow[d] \\ \Lambda^2 \mathcal{K}_0 \arrow[r] & \Lambda^2 \mathcal{K} \end{tikzcd}$$

note $\mathcal{K}_0 \oplus \mathbb{Z} = \mathcal{K}$ as $\mathbb{Z}$ is projective

and so the dilogarithm is defined for all $f, g \in \mathcal{K}$.

So the way it works is to define (f,g) by

$$(f,g) = (f, g z^{-\deg g}) \cdot \underbrace{(f, z^{+\deg g})}_{(f z^{-\deg f}, z^{\deg g})\underbrace{(z^{\deg f}, z^{\deg g})}_{1}}$$

So for example

$$(z^m \xi, z^n \mu) = (z^m \xi, \mu)(z^m \xi, z^n) = \mu^m / \xi^n \qquad \xi, \mu \in \mathbb{C}^*$$

October 11, 1981

Poisson summation formula somehow expresses the compatibility of the Heisenberg extensions of $\mathbb{Z}\times S^1$ and $\mathbb{R}\times\mathbb{R}$ with respect to the map $\mathbb{Z}\subset\mathbb{R}$. Let's use the following convention:

$$\begin{aligned} \mathbb{R} &\xrightarrow{\sim} \hat{\mathbb{R}} \\ y &\longmapsto (x\mapsto e^{ixy}) \end{aligned}$$

Then we get

$$\begin{array}{ccccc} \mathbb{R} & \xrightarrow{\sim} & \hat{\mathbb{R}} & \longrightarrow & \hat{\mathbb{Z}} = S^1 \\ y & \longmapsto & e^{iy?} & \longmapsto & e^{iy\epsilon} \end{array}$$

so that the dual lattice to $\mathbb{Z}$ in $\mathbb{R}$ is $2\pi\mathbb{Z}$. So then we have

$$\begin{tikzcd} f(m) \arrow[r, mapsto] & \sum' f(m)e^{imx} = \sum' \hat{f}(x+2\pi m) \\ \mathcal{S}(\mathbb{Z}) \arrow[r, "F"] & \mathcal{S}(S^1) \\ \mathcal{S}(\mathbb{R}) \arrow[u, "\text{rest}"] \arrow[r, "F"'] & \mathcal{S}(\mathbb{R}) \arrow[u] \\ f(x) \arrow[uuu, mapsto] \arrow[r, mapsto] & \hat{f}(x) \arrow[uuu, mapsto] \end{tikzcd} \qquad (*)$$

Start with $f(x)\in\mathcal{S}(\mathbb{R})$, put $\hat{f}(\xi)=\int e^{ix\xi}f(x)\,dx$. Then

$$g(x)=\sum_{n\in\mathbb{Z}}{}' \hat{f}(x+2\pi n) = \sum{}' a_m e^{imx} \qquad a_m = \frac{1}{2\pi}\int_0^{2\pi} g(x)e^{-imx}\,dx = \frac{1}{2\pi}\int_{-\infty}^{\infty}\hat{f}(x)e^{-imx}\,dx = f(m)$$

so

$$\boxed{\sum_{n\in\mathbb{Z}}{}' \hat{f}(x+2\pi n) = \sum_{m\in\mathbb{Z}}{}' f(m)e^{imx}}$$

Note the F is unitary when S^1 is given Haar measure $\frac{d\theta}{2\pi}$, and the same is true if we use the Haar measure in the second $\mathcal{S}(\mathbb{R})$ for which $2\pi\mathbb{Z}$ has covolume 1.

Interesting question (maybe): What does the above Poisson formula square look like when $\mathcal{S}(\mathbb{R})$ is described in the holomorphic representation? Both vertical maps are natural quotient maps for an action of $\mathbb{Z}$; on the left $n\in\mathbb{Z}$ multiplies by $e^{2\pi i n x}$, and on the right it translates by $2\pi n$.

On the holomorphic representation we have for any $\gamma \in \mathbb{C}$ a translation operator

$$(T_\gamma f)(z) = e^{-\frac{1}{2}|\gamma|^2 + \bar{\gamma} z} f(z-\gamma)$$

Thus

$$T_\gamma = e^{-\frac{1}{2}|\gamma|^2} e^{\bar{\gamma} a^*} e^{-\gamma a}$$

and one has

$$T_\beta T_\gamma = e^{-\frac{1}{2}|\beta|^2 - \frac{1}{2}|\gamma|^2} \cdot e^{\bar{\beta} a^*} \cdot \underbrace{e^{-\beta a} e^{\bar{\gamma} a^*}}_{e^{\bar{\gamma} a^*} e^{-\beta a} e^{[-\beta a, \bar{\gamma} a^*]}} e^{-\gamma a}$$

$$= e^{-\frac{1}{2}|\beta|^2 - \frac{1}{2}|\gamma|^2 - \beta\bar{\gamma} + \frac{1}{2}|\beta+\gamma|^2} T_{\beta+\gamma}$$

$$= e^{\frac{1}{2}(\bar{\beta}\gamma - \beta\bar{\gamma})} T_{\beta+\gamma}$$

Now $\frac{1}{2}(\bar{\beta}\gamma - \beta\gamma) = i\,\mathrm{Im}(\bar{\beta}\gamma) = 0$ if $\beta \in \mathbb{R}\gamma$. Therefore if we fix a $\gamma \neq 0$ and take the cyclic group $\mathbb{Z}\gamma$, the corresponding $T_{n\gamma} = (T_\gamma)^n$ give an action of $\mathbb{Z}$ on the Hilbert space.

Question: What might an outgoing space look like?

First look for invariant: Take $\gamma = 1$.

$$(*) \qquad (T_\gamma f)(z) = e^{-\frac{1}{2}} e^{z} f(z-1) = f(z)$$

Obvious soln. is $e^{z^2/2}$, which we know is not in the Hilbert space (the functions $e^{\alpha \frac{z^2}{2}}$ with $|\alpha| < 1$ are)

$$e^{-\frac{1}{2} + z + \frac{1}{2}(z^2 - 2z + 1)} = e^{\frac{z^2}{2}}$$

Then if I put $g(z) = e^{-z^2/2} f(z)$, I have

$$g(z-1) = g(z)$$

so that

$$g(z) = \sum_{n \in \mathbb{Z}} a_n e^{2\pi i n z}$$

There we have found all analytic solutions of $(*)$ and they are of the form

$$e^{\frac{z^2}{2}} \sum_{n\in\mathbb{Z}} a_n e^{2\pi i n z} \qquad a_n \in \mathbb{Z}$$

Next given an $f(z)$ we want to form $\sum_n (T_\gamma)^n f(z)$, because this is the analogue of the map $\mathcal{S}(\mathbb{R}) \longrightarrow \mathcal{S}(S^1)$. But

$$\Big(\sum_n T_\gamma^n f\Big)(z) = \sum_n e^{-\frac{n^2}{2}} e^{nz} f(z-n)$$
$$= e^{z^2/2} \sum_n e^{-\frac{1}{2}(z-n)^2} f(z-n).$$

Thus we are taking analytic functions $+ L^2$ wrt $e^{-|z|^2}\frac{dx\,dy}{\pi}$ and [redacted] sending f to $e^{-z^2/2} f$ which is now analytic in the plane but L^2 wrt the measure $e^{-y^2}\frac{dx\,dy}{\pi}$, and then we are averaging with respect to [redacted] the translation subgroup [redacted] $\mathbb{Z}$.

Back to the **dilogarithms**. Think of F now as the analytic $\mathbb{C}$-valued fns. on S^1 and F° as the invertible ones. We have

$$0 \longrightarrow (F^\circ)_0 \longrightarrow F^\circ \xrightarrow{\deg} \mathbb{Z} \longrightarrow 0$$

and

$$(F^\circ)_0 \xleftarrow{\sim} \underset{\text{const}}{\mathbb{C}^\circ} \times \underset{\text{by exponential}}{(F^+ \oplus F^-)}$$

so that if we use the obvious way to lift $\mathbb{Z}$ into F°, namely by $1 \mapsto z$ we get an isomorphism

$$F^\circ \xleftarrow{\sim} (F^+ \oplus F^-) \times (\mathbb{C}^\circ \times \mathbb{Z})$$

Now we ~~[illegible]~~ treat F° as an analogue of the symplectic vector space generated by p's and q's and construct the corresponding Heisenberg repn. Here are the requirements. On F and F° we have an obvious conjugation, [redacted] $z^n \mapsto z^{-n}$, and the unitary elements in F° are those f with $f^* = f^{-1}$. These should go to unitary operators so that in any repn. $*$ should correspond to adjoint. Thus [redacted] if to the elements z^n, z^{-n} of F belong operators b, b' we want $b' = b^*$. also we want

$$[b, b^*] = (z^n, z^{-n}) = \operatorname{res}(z^{-n} dz^n) = n$$

which will force b to be an annihilator operator.

[redacted]

The idea will be to associate to any $f \in F^+$ an annihilator operator a_f as follows. To $z^n/\sqrt{n}$ belongs a_n, to $z^{-n}/\sqrt{n}$ belongs a_n^*. Thus to

$$f = \sum_{n \geq 1} c_n z^n \quad \text{belongs} \quad a_f = \sum_{n \geq 1} c_n \sqrt{n}\, a_n .$$

[redacted] Now I want to exponentiate and associate to any $f \in F^+$ a unitary translation operator

$$T_f = e^{\frac{1}{2}\|f\|^2} e^{(a_f)^*} e^{-a_f}$$

Then [redacted] it's clear we must define

$$\|f\|^2 = \sum_{n=1}^{\infty} |c_n|^2 n \qquad \text{if} \quad f = \sum_1^\infty c_n z^n$$

But in some sense we should think of T_f as being the translation associated to $e^{f+\bar{f}}$ in F°.

One idea is to form something like the Heisenberg algebra generated by the operators p, q with the relation $[p,q] = 1/i$. Now when you try to do this in the Weyl form it seems you have to give the actual cocycle i.e.

$$T_\beta T_\gamma = e^{i \operatorname{Im}(\bar{\beta}\gamma)} T_{\beta+\gamma}$$

Atiyah's L^2 index thm. Let Γ be a discrete group acting freely on a manifold X with compact quotient X/Γ, and let $D: E \to F$ be a Γ-invariant differential operator on X. Put metrics on $\bar{E}, \bar{F}$ over X/Γ and a volume on X/Γ, then we get Hilbert spaces and operators

$$0 \longrightarrow \operatorname{Ker}(D) \longrightarrow L^2(X,E) \longrightarrow L^2(X,F) \longrightarrow \operatorname{Cok}(D) \to 0$$

Now because Γ acts freely on X, one knows that $L^2(X,E)$ is in some sense a free Γ-module. Precisely we can choose an isomorphism

$$L^2(X,E) \cong L^2(X/\Gamma, \bar{E}) \hat{\otimes} L^2(\Gamma)$$

algebraic analysis: Take $\Gamma = \mathbb{Z}$. I think the effect of all the analysis is to prove that the map $L^2(X,E) \to L^2(X,F)$ is an analogue of a perfect complex of $\mathbb{C}[\mathbb{Z}]$-modules. If so, it follows that there is defined an element in

$$K_0(\mathbb{C}[\mathbb{Z}]) = \mathbb{Z}$$

October 14, 1981

I would like to find evidence that this business of central extensions is an SL and not a GL theory, or the contrary. We have produced over the group F° of maps $S^1 \longrightarrow \mathbb{C}^*$ an alternating pairing given by the dilogarithm. This should give us a central extension of F° having this pairing as commutator pairing, once you produce a cocycle. One would like the group of diffeomorphisms of S^1 to act on this central extension, so it would be nice to have the cocycle invariant under diffeos as the dilogarithm is.

There is no obvious way to take a square root of the dilogarithm. If we try

$$f,g \longmapsto \exp\left\{\frac{1}{2}\,\frac{1}{2\pi i}\int \log(g)\,\frac{df}{f}\right\}$$

then this extends to an alternating pairing on the group of $f \in F^\circ$ of even degree. e.g. on $\mathbb{C}^* \times (2\mathbb{Z})$ it gives

$$(f,g) \longmapsto \frac{g^{(\deg f)/2}}{f^{(\deg g)/2}}$$

but it's not clear what to do next.

Interesting point. In the algebraic examples, $F =$ formal Laurent series $\mathbb{C}[[z]][z^{-1}]$, the dilogarithm doesn't see much. Thus if $f = z^m f_0$, $g = z^n g_0$ one has

$$(z^m, g_0) = \exp\left\{\frac{1}{2\pi i}\int \log(g_0)\,\frac{dz}{z}\,m\right\} = g_0(0)^m$$

because $\log g_0 \in \mathbb{C}[[z]]$.

$$(f_0, g_0) = 1.$$

Thus
$$(f,g) = \left(\frac{g^{\deg f}}{f^{\deg g}}\right)(0)$$

and so the dilog comes from $\mathbb{C}^* \times \mathbb{Z}$.

Idea: In the basic representation we expect unitary operators associated to $f \in F^{\circ}$ which are actually maps $S^1 \to S^1$. The simplest of these are Blaschke factors, namely

$$f(t) = \frac{t-\alpha}{-\bar{\alpha}t+1} \qquad \text{for } |\alpha| < 1$$

which have degree 1, and which obviously generate all rational maps from S^1 to S^1 along with constants. Now

$$\log\left(\frac{t-\alpha}{-\bar{\alpha}t+1}\right) = \log(t) + \log\left(\frac{1-\alpha t^{-1}}{1-\bar{\alpha}t}\right)$$

$$= \log(t) - \sum_{n\geq 1}' \frac{\alpha^n}{n} t^{-n} + \sum_{n\geq 1}' \frac{\bar{\alpha}^n}{n} t^n$$

Here I follow Kac-Frankel and call the variable t.

To such an element will belong a translation operator analogous to $e^{\frac{1}{2}|\gamma|^2} e^{\bar{\gamma}a^*} e^{-\bar{\gamma}a}$ considered before. Recall that $\frac{t^n}{\sqrt{n}} \longmapsto a_n \quad \frac{t^{-n}}{\sqrt{n}} \mapsto a_n^* \quad n \geq 1$ so that $-\bar{\gamma}\cdot a =$ Image of $\sum \frac{\bar{\alpha}^n}{n} t^n \longmapsto \sum \frac{\bar{\alpha}^n}{n}\sqrt{n}\, a_n$. Thus $\gamma = \left(-\frac{\alpha^n}{\sqrt{n}},\ n \geq 1\right)$ and so

$$\|\gamma\|^2 = \sum' \frac{|\alpha|^{2n}}{n} = \log\left(\frac{1}{1-|\alpha|^2}\right)$$

$$e^{\frac{1}{2}\|\gamma\|^2} = \frac{1}{\sqrt{1-|\alpha|^2}}$$

Now what remains is to understand what happens to the middle, i.e. to the subgroup of $\zeta t^n \quad \zeta \in S^1, n \in \mathbb{Z}$. ∎

October 15, 1981

Trying to understand vertex operators. The situation is as follows. One has a alg. torus H ($\simeq \mathbb{G}_m^d$) with Lie algebra $\mathfrak{h}$. Then the group $H(S^1)$ of maps $S^1 \to H$ has Lie algebra $\mathfrak{h}(S^1) = s_+ \oplus \mathfrak{h} \oplus s_-$, and we have

$$H(S^1) \cong \exp(s_+ \oplus s_-) \times (H \times \pi_1 H)$$

I am using here that $\pi_1(H) = \mathrm{Hom}(\mathbb{G}_m, H)$ for a torus, and hence $\pi_1(H)$ embeds in $H(S^1)$ in a natural way. Now I want to construct the Heisenberg repn. of $H(S^1)$, and for this I need ~~a~~ quadratic form $(\, , \,)$ on $\mathfrak{h}$, which will give me a duality of s_+ and s_- via

$$f, g \longmapsto \mathrm{Res}\,(g, df).$$

It should also give a map of $\pi_1 H$ to the dual group $\mathrm{Hom}(H, \mathbb{G}_m)$ of H. Maybe better is to have a map $H \to \mathrm{Hom}(\pi_1 H, \mathbb{G}_m)$, so that I get a projective representation of $H \times \pi_1 H$ on $\mathbb{C}[\pi_1 H]$.

Let's set this up carefully. Put $Q = \pi_1 H$ and identify with the subset of $\gamma \in \mathfrak{h}$ such that $e^{2\pi i \gamma} = 1$. Then $\mathbb{C}[\pi_1 H]$ has the basis δ_γ for $\gamma \in Q$ and Q acts by translations $\gamma' \cdot \delta_\gamma = \delta_{\gamma' + \gamma}$. Then make H act on $\mathbb{C}[\pi_1 H]$ by

$$\exp(h)\, \delta_\gamma = e^{2\pi i (h, \gamma)}.$$

For this to make sense I need $(\, , \,)$ to be $\mathbb{Z}$-valued on the lattice Q.

Different notation. Instead of δ_γ write e^γ. Then $\mathbb{C}[Q]$ has the basis e^γ with $\gamma \in Q$, and elements of β of Q give rise to translation operators

$$e^\beta e^\gamma = e^{\beta + \gamma}$$

Next define differentiation operators

$$\partial_\beta (e^\gamma) = \langle \beta, \gamma \rangle e^\gamma$$

or when exponentiated $\quad e^{\omega \partial_\beta}(e^\gamma) = (e^\omega)^{\langle \beta, \gamma \rangle} e^\gamma$

or putting $z = e^{w}$ we get

$$e^{(\ln z)\partial_\beta}\,(e^\gamma) = e^{(\ln z)\langle\beta,\gamma\rangle} = z^{\langle\beta,\gamma\rangle}.$$

What is happening? Consider the case where $H = \mathbb{C}^*$, $Q = \mathbb{Z}$. Somehow what's been done is to preserve the basic Lie notation as much as possible, so therefore one writes the operators in the form e^A where A might not exist. ~~Thus on $\mathbb{C}(\mathbb{Z})$ we have the translation operator by any $\gamma \in \mathbb{Z}$. Actually it seems that instead of $\mathbb{C}[\mathbb{Z}]$ we should write algebra functions on $G_m = \mathrm{Hom}(\mathbb{Z}, G_m)$.~~ Thus on $\mathbb{C}(\mathbb{Z})$ we have the basis e^n with translation operators e^n, even though the infinitesimal translation n doesn't exist. We also have

$$\partial_m(e^n) = mn\,e^n$$

and hence

$$e^{(\ln z)\partial_m}(e^n) = z^{mn}\,e^n.$$

Better notation in this case is to denote the basis of $\mathbb{C}(\mathbb{Z})$ by t^n and then ∂_m the operator $m t\frac{\partial}{\partial t}$ and e^n the operator t^n.

On to the vertex operator

$$X(\gamma, z) = \exp\Big(\sum_1^\infty \frac{z^k}{k}\gamma(-k)\Big)\exp\big(\gamma + (\ln z)\partial_\gamma\big)\exp\Big(-\sum_1^\infty \frac{z^{-k}}{k}\gamma(k)\Big)$$

e^γ here is a translation, $e^{(\ln z)\partial_\gamma} = z^{\partial_\gamma}$ is a multiplication. Formally we have been thinking of the following correspondence

$$\exp\Big(-\sum_1^\infty \frac{z^{-k}}{k}\gamma(k)\Big) \longleftrightarrow e^{-\sum_1^\infty \frac{z^{-k}}{k}t^{-k}} = (1 - z^{-1}t^{-1})$$

$$\exp\Big(\sum_1^\infty \frac{z^{k}}{k}\gamma(-k)\Big) \longleftrightarrow e^{\sum_1^\infty \frac{z^{k}}{k}t^{k}} = \frac{1}{1-zt}$$

so the product is

$$\frac{1 - z^{-1}t^{-1}}{1 - zt} = -z^{-1}t^{-1}$$

What corresponds to $\exp(\gamma + (\ln z)\partial_\gamma)$ on $\mathbb{C}[\mathbb{Z}]$. It corresponds up to a scalar to $e^\gamma z^{\partial_\gamma} \sim tz$ for $\gamma = 1$.

The operator $\exp(\gamma + (\ln z)\partial_\gamma)$ on $\mathbb{C}[\mathbb{Z}]$: uses the formula

$$e^{A+B} = e^A e^B e^{-\frac{1}{2}[A,B]}$$

$[A,B]$ scalar

to give a sense to this. Thus

$$e^{(\gamma+(\ln z)\partial_\gamma)} = e^\gamma z^{\partial_\gamma} z^{-\frac{1}{2}[\gamma, \partial_\gamma]}$$

$$= z^{\frac{1}{2}\|\gamma\|^2} e^\gamma z^{\partial_\gamma}$$

Operating on a function $f(n)$ $n \in \mathbb{Z}$ gives

$$\left(e^{\gamma+(\ln z)\partial_\gamma} f\right)(n) = z^{\frac{1}{2}\|\gamma\|^2} \underbrace{\left(e^\gamma z^{\partial_\gamma} f\right)(n)}_{\left(z^{\partial_\gamma} f\right)(n-\gamma)}$$

$$= z^{\frac{1}{2}\|\gamma\|^2} z^{(\gamma, n-\gamma)} f(n-\gamma)$$

$$\boxed{\left(e^{\gamma+(\ln z)\partial_\gamma} f\right)(n) = z^{-\frac{1}{2}\|\gamma\|^2} z^{\gamma n} f(n-\gamma).}$$

Let's now look at the vertex operator

$$X(\gamma, z) = \exp\left(\sum_1^\infty \frac{z^k}{k}\gamma(-k)\right) \exp\left(\gamma + (\ln z)\partial_\gamma\right) \exp\left(-\sum_1^\infty \frac{z^{-k}}{k}\gamma(k)\right).$$

We want it to act on the basic representation $S(\mathfrak{s}_-) \otimes \mathbb{C}[\mathbb{Z}]$. The latter factor is a translation operator. Kac + Frenkel think of elements of the repn. as being functions $F(h)$ where $h(t) = \sum_{n \geq 0} h_n t^n$ is holomorphic such that $h_0 \in \mathbb{Z}$.

For some reason $\gamma(k)$ is interpreted so that

$$\left[\exp\left(-\sum_1^\infty \frac{z^{-k}}{k}\gamma(k)\right) F\right](h) = F\left(h - \sum_1^\infty z^{-k} t^k \gamma\right)$$

and then applying $\exp(\gamma + \ln(z)\partial_\gamma)$ to this gives

$$e^{-\frac{1}{2}\|\gamma\|^2} z^{\gamma h(0)} F\Big(h - \underbrace{\gamma - \sum_1^\infty z^{-k} t^k \gamma}_{\frac{\gamma}{1-z^{-1}t}}\Big)$$

September 18, 1981

Kac-Frenkel formulas for vertex operator $X(\gamma, z)$.

Let's begin with a review. Consider the case of $F^\cdot$, where $F =$ maps $S^1 \to \mathbb{C}$. We consider the ~~subspace~~ F^+ as giving annihilation operators, and so $\exp(F^+)$ gives translation operators on $S(F^+)$. Then we can combine this with $\mathbb{Z}$ translating on $\mathbb{C}[\mathbb{Z}]$, to get $\exp(F^+) \times \mathbb{Z}$ acting as translations on $S(F^+) \otimes \mathbb{C}[\mathbb{Z}]$. Then $\exp(F^-) \times \mathbb{C}^\cdot$ has to be interpreted as multiplication operators ending up in a completion.

For some reason I don't understand, one defines the vertex operator $X(\gamma, z)$ by

$$\exp\Big(\big(\sum_1^\infty \frac{z^k}{\sqrt{k}} a_k^*\big)\gamma\Big) \underbrace{\exp\big(\gamma + (\ln z)\partial_\gamma\big)}_{z^{\frac{1}{2}(\gamma)^2} e^\gamma z^{\partial_\gamma} \,=\, z^{-\frac{1}{2}\gamma^2} z^{\partial_\gamma} e^\gamma} \exp\Big(\big(-\sum_1^\infty \frac{z^{-k}}{\sqrt{k}} a_k\big)\gamma\Big)$$

Here $\gamma \in \mathbb{Z}$ and $\mathbb{C}[\mathbb{Z}]$ we think of as having basis e^β, $\beta \in \mathbb{Z}$ with $e^\gamma e^\beta = e^{\gamma+\beta}$ $\quad z^{\partial_\gamma} e^\beta = z^{\gamma\beta} e^\beta$.

The far right factor is translation associated to

$$\exp\Big(\big(-\sum_1^\infty \frac{z^{-k}}{k} t^k\big)\gamma\Big) = \frac{}{(1-z^{-1}t)^\gamma} \in \exp(F^+)$$

and the far left factor is the multiplication belonging to

$$\exp\Big(\big(\sum_1^\infty \frac{z^k}{k} t^{-k}\big)\gamma\Big) = \frac{1}{(1-zt^{-1})^\gamma} \in \exp(F^-)$$

e^γ belongs to $t^\gamma \in F^\cdot$ and z^{∂_γ} to the constant $z^\gamma \in \mathbb{C}^\cdot$. Thus the operator belongs to

$$\left(\frac{1-z^{-1}t}{1-zt^{-1}}\, z t^{-1}\right)^\gamma = (-1)^\gamma$$

Possible idea. Recall Euler formula

$$\sum{}' z^n t^{-n} = 2\pi i\, \delta_t(z) \qquad \text{for } |z| = |t| = 1$$

Integration (inverting $z\frac{d}{dz}$) gives

$$(*) \qquad \sum_{n\neq 0} \frac{z^n t^{-n}}{n} + \log(z/t) = 2\pi i(\text{Heaviside})$$

Thus the idea perhaps is that the series (*) for a natural "eigenfunction basis" for the group $F^{\cdot}$.

When we apply these operators $X(\gamma, z)$ to the vacuum state v_0 we ~~get~~ coherent states. Thus any product $X(\gamma_1, z_1)\cdots X(\gamma_N, z_N)$ can be normal ordered so to get a pure multiplication times a scalar times v_0. Thus

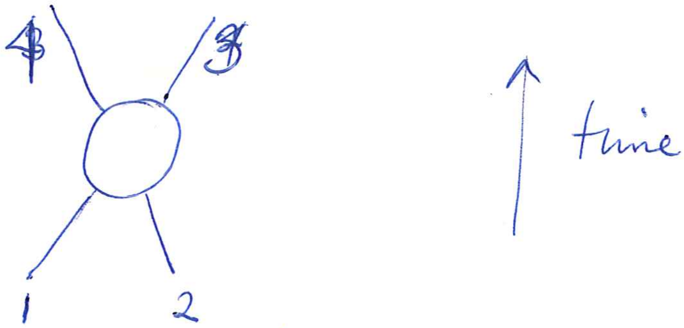
$$e^{\bar{c}a^*} e^{-ca} e^{\bar{b}a^*} \cancel{e^{-\bar{b}a}}\, v_0 = e^{\bar{c}a^*} e^{\bar{b}a^*} \cancel{e^{-ca}}\, e^{\overset{\text{scalar}}{[-ca,\,\bar{b}a^*]}}\, v_0$$

Thus there is no mystery belonging to the coherent states. In fact one should think of a coherent state as belonging to a product of factors $1 - zt^{-1}$ with $|z| < 1$, so that the function is invertible holomorphic outside $|t| = 1$.

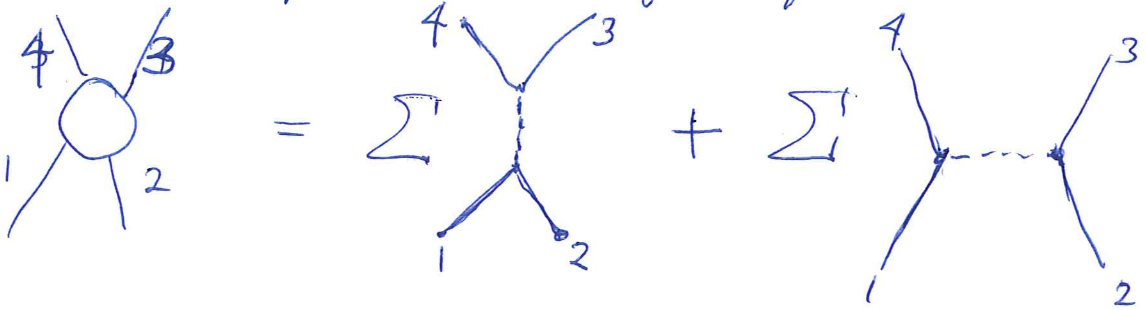
I need a more general viewpoint

dual resonance models:

The amplitude A has to do with the process

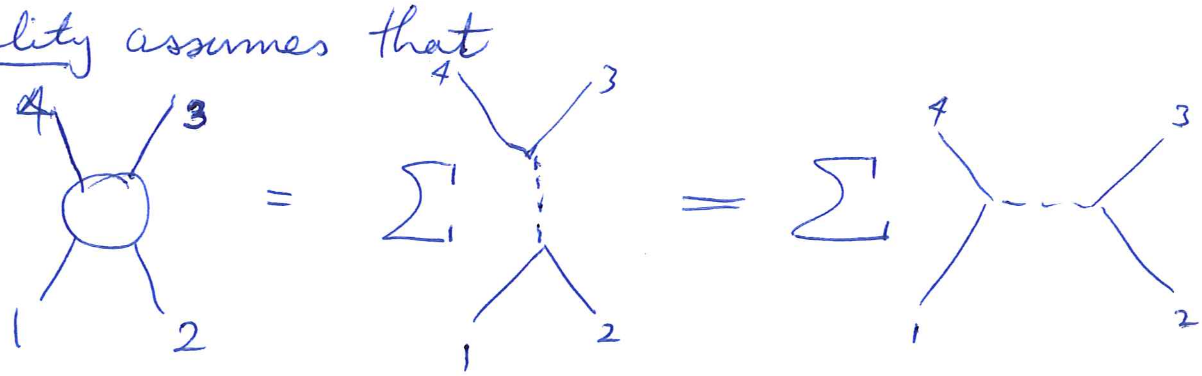


which we read as the pair 1,2 coming in and 3,4 going out. In the case of elementary particles the amplitude is of the form



where the dotted line is a sum over resonances. One picture is a sum over s channel resonances, the other a sum over t-channel resonances.

Duality assumes that



s channel	consists	of	particles	1, 2
t "	"	"	"	1, 4
u "	"	"	"	1, 3

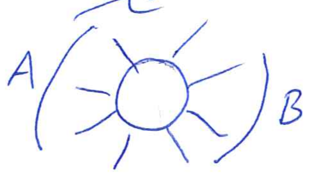
The actual amplitude is a sum of terms associated to a particular ordering of the four external particles. A single term has resonances only in channels consisting of adjacent particles, e.g. no u channels resonances above because 1,3 are not adjacent

The N-particle formula will consist of

$$\frac{N!}{2N} = \frac{1}{2}(N-1)! \quad \text{terms}$$

corresponding to the different orderings of the N ext. particles (modulo dihedral group

The N-point function can have $\leq N-3$ mutually non-overlapping channels.



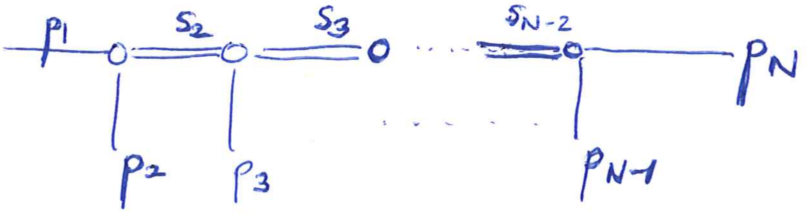
A, C overlap
A, B don't

non-overlapping means disjoint or contained in the other. (Evidently a channel is a subset of the N ext. particles having ≥ 2 and $\leq N-2$ particles, so that the max configuration is like $\{1\,2\} \subset \{1\,2\,3\} \subset \cdots \subset \{1\,2\,3 \cdots N-2\}$.)

There is a concept of factorization whereby the amplitude for

can be built up out of processes

(X)



where the p_i are the momenta of external particles and the s_i are "resonances". To such a diagram we associate an amplitude as follows

$$\underline{\underline{\ s_i\ }} \longmapsto D(s_i) = \frac{1}{R - s_i - 1}$$

$s_i = \left(\sum_{1}^{i} p_j\right)^2$ = energy of the channel $\{1, \ldots, i\}$

$$R = 1 = -\sum_{1 \leq n < \infty}' a_n^{\mu *} a_{n\mu} - 1$$

where $[a_n^\mu, a_m^\nu] = [(a_n^\mu)^*, (a_m^\nu)^*] = 0$

$[a_n^\mu, a_m^{\nu *}] = -n\delta_{nm}\, g^{\mu\nu}$

$g^{\mu\mu} = \begin{cases} 1 & \mu = 0 \\ -1 & \mu = 1, 2, \ldots \end{cases}$

The vertex for $\big|\, p_i$ is

$$\tilde V(p_i) = \exp\left\{-\sqrt{2}\, p_i \cdot \sum \frac{a_n^*}{n}\right\} \exp\left\{\sqrt{2}\, p_i \cdot \sum \frac{a_n}{n}\right\}$$

Then the amplitude associated to ⊛ is

$$A = \langle 0 |\, \tilde V(p_{N-1})\, D(s_{N-2}) \cdots \tilde V(p_2)\, | 0 \rangle$$

($|0\rangle$ means the vacuum state for the oscillators.)

Next transformation is to use

$$D(s_i) = \int_0^1 dx_i\, x_i^{R - s_i - 2}$$

and $\quad z_i = \sum_{j=i}^{N-1} x_i \quad$ and you get

$$A = \int dz_2 \cdots dz_{N-2} \prod_{\substack{i,j=1 \\ i>j}}^{N-1} (z_i - z_j)^{-2 p_i \cdot p_j}$$

where the integration is over

$$0 = z_1 < z_2 < \cdots < z_{N-1} = 1 \qquad (z_1 = 0,\ z_{N-1} = 1,\ z_N = \infty)$$

Another transformation which incorporates projective transformations goes as follows. Let p = total momentum operator (has values $p_1, p_1 + p_2, \ldots$ in intermediate states) introduce coordinate op. q, $[p^\mu, q^\nu] = i g^{\mu\nu}$. Add now the factor $\exp(-i p_i \cdot q)$ to $\tilde V(p_i, z_i)$. Gives new Vertex of

$$V(p_i, z_i) = \,:\exp\{-i\, p_i \cdot Q(z_i)\}:$$

$$Q^\mu(z_i) = q^\mu - 2 i p^\mu \ln z + i\sqrt{2} \sum_{\substack{n=-\infty \\ n \neq 0}}^{\infty} \frac{1}{n}\, a_n^\mu z_i^{-n}.$$

<u>October 21, 1981</u> More on vertex operators

$F = F^+ \oplus \mathbb{C} \oplus F^-$ and we let V be the projective representation of $F^+ \oplus F^-$ on the polynomial functions on F^+. More precisely $V = S((F^+)^\vee)$ and F^+ acts as annihilation ops, while F^- acts as creation via the given pairing of F^- and F^+. Let's use the notation $a(f^+)$, $a^*(f^-)$ for these operators, so that

$$[a(f^+), a^*(f^-)] = (f^+, f^-).$$

Hence to any $f \in F^+ \oplus F^-$ we have the translation-type operator

$$\theta(e^f) = e^{a^*(f^-)} e^{a(f^*)}$$

satisfying $\theta(e^{f+f'}) \equiv \theta(e^f)\theta(e^{f'})$ mod scalars. This operator $\theta(e^f)$ is defined on that part of F° of the form $e^{F^+ + F^-}$.

For the purpose of constructing a repn. of $\widetilde{sl_2(F)}$ on V, we want to assign to each $g \in F$ an operator $X(g)$ satisfying

$$X(g+g') = X(g) + X(g')$$

$$\theta(e^f)\, X(g)\, \theta(e^f)^{-1} = X(e^{\gamma f} g)$$

The first condition says that $X(g)$ is determined by $X(\delta_z)$ where $\delta_z = \sum_{n \in \mathbb{Z}} z^{-n} t^n$ is a δ-function. But these δ_z are eigenfunctions for multiplication:

$$e^{\gamma f} \delta_z = e^{\gamma f(z)} \delta_z$$

so that we want

$$\theta(e^f)\, X(\delta_z)\, \theta(e^f)^{-1} = e^{\gamma f(z)} X(\delta_z)$$

To solve this sort of equation, look first when there is one creation operator. You want

$$e^{f^+ a} X e^{-f^+ a} = e^{c_1 f^+} X \qquad f^{\pm}, c_1, c_2 \text{ scalars.}$$

$$e^{f^- a^*} X e^{-f^- a^*} = e^{c_2 f^-} X$$

or $\qquad [a, X] = c_1 X \qquad [a^*, X] = c_2 X.$ The solution is clearly $\quad X = \text{const}\, e^{c_1 a^*} e^{-c_2 a}.$

Thus in our case we have

$$X(\delta_z) = (\text{const})_z \; e^{a^*(\delta_z^-)} \, e^{-a(\delta_z^+)}$$

where
$$[a(f^+), a^*(\delta_z^-)] = (f^+, \delta_z^-) = \gamma f^+(z)$$
$$[a^*(f^-), -a(\delta_z^+)] = (\delta_z^+, f^-) = \gamma f^-(z)$$

If $\quad \delta_z^- = \sum_1^\infty c_n t^{-n}$, then taking $f^+ = t^n$ gives

$$(t^n, c_n t^{-n}) = n c_n = \gamma z^n \quad \Longrightarrow \quad c_n = \gamma \frac{z^n}{n}$$

If $\delta_z^+ = \sum_1^\infty c_n' t^n$, take $f^- = t^{-n}$ to get

$$(c_n' t^n, t^{-n}) = n c_n' = \gamma z^{-n} \quad \Longrightarrow \quad c_n' = \gamma \frac{z^{-n}}{n}$$

Thus
$$\delta_z^+ = \gamma \sum_1^\infty \frac{z^{-n} t^n}{n} \qquad \delta_z^- = \gamma \sum_1^\infty \frac{z^n t^{-n}}{n}$$

and we have

$$\boxed{X(\delta_z) = c(z)\, e^{a^*\left(\gamma \sum_1^\infty \frac{z^n t^{-n}}{n}\right)} \, e^{-a\left(\gamma \sum_1^\infty \frac{z^{-n} t^n}{n}\right)}}$$

Thus this explains part of the formula for the vertex operator.

Here's a good way to view this formula. Recall the situation of a harmonic oscillator perturbed by a "source". Thus

$$H_0 = \omega a^* a$$
$$H_{int} = \lambda a^* + \bar\lambda a$$

where λ is a function of time. Then the S-matrix is

$$S = T\left\{ e^{-i\int_{-\infty}^{\infty}(\lambda a^* + \bar\lambda a)\,dt} \right\}$$

which gets computed as follows: $\quad a(t) = e^{-i\omega t} a$, $\quad a^*(t) = e^{i\omega t} a$

In computing the time ordered product we must move $e^{-i\bar\lambda(t) a(t) dt}$ past $e^{-i\lambda(t') a^*(t') dt'}$ for $t > t'$, picking up

$$e^{[-i\bar\lambda(t) a(t),\, -i\lambda(t') a^*(t')]\,dt\,dt'}$$
$$= e^{-\overline{\lambda(t)}\lambda(t')\, e^{-i\omega(t-t')}\,dt\,dt'}$$

Thus

$$S = e^{\left(-i\int \lambda e^{+i\omega t} dt\right) a^*}\; e^{-\iint_{t>t'} \overline{\lambda(t)}\, e^{-i\omega(t-t')}\lambda(t')\,dt\,dt'}$$
$$\times\, e^{\left(-i\int \bar\lambda e^{-i\omega t} dt\right) a}$$

which is of form $\quad S = \zeta e^{-\frac{|\alpha|^2}{2}} e^{\alpha a^*} e^{-\bar\alpha a}$

with $|\zeta| = 1$: $\quad \alpha = -i\int \lambda e^{i\omega t} dt$.

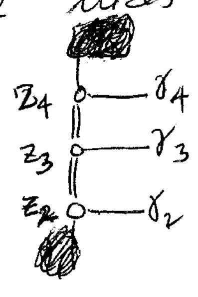
So now it is clear that we should think of z as being the time variable. Recall that

$$a_n \longleftrightarrow \frac{t^n}{\sqrt{n}} \qquad a_n^* \longleftrightarrow \frac{t^{-n}}{\sqrt{n}}$$

so that $\quad z^{-n} a_n \longleftrightarrow \dfrac{z^{-n} t^n}{\sqrt{n}} \quad$ etc.

and so we should think of z as $e^{-i\omega t}$ where t = time, the n-th oscillator has frequency $n\omega$.

Recall that one likes to think in terms of diagrams



to interpret a product

$$X(\gamma_{N-1}, z_{N-1}) \cdots X(\gamma_3, z_3)\, X(\gamma_2, z_2).$$

So there should be some way to ~~[illegible]~~ interpret the integral over different possible values of $z_2, \ldots, z_{N-1}$ maybe as a sort of Dyson expansion.

October 22, 1981

Let's calculate the vertex operators on coherent states. In general a coherent state is

$$e^{a^*(f_-)}|0\rangle$$

where

$$f_- = \sum_1^\infty c_n t^{-n} \qquad \text{analytic outside } S^1.$$

However, we really want to think of this in terms of e^{f_-} which is an invertible holom. fns. outside of S^1. with value 1 at $t=\infty$. In good cases e^{f_-} will be a rational function, hence of the form

$$e^{f_-} = \prod (1-\alpha_i t^{-1})^{\pm 1}$$

where the α_i are such that $|\alpha_i| < 1$.

Therefore we have a nice coherent state associated to any divisor $\sum n_\alpha \alpha$ in $|t|<1$. For the moment let's look just at these. The inner product

$$\langle e^{a^*(g_-)}|0\rangle \,|\, e^{a^*(f_-)}|0\rangle = \langle 0| e^{a(\bar{g}_-)} e^{a^*(f_-)}|0\rangle$$

$$= e^{(\bar{g}_-, f_-)}$$

If $\quad e^{f_-} = \dfrac{1}{1-\alpha t^{-1}} \quad e^{g_-} = \dfrac{1}{1-\beta t^{-1}}$, then

$$(\bar{g}_-, f_-) = \left(\sum_1^\infty \frac{\bar{\beta}^n t^{+n}}{n}, \sum_1^\infty \frac{\alpha^n t^{-n}}{n} \right) = \sum_1^\infty \frac{(\bar{\beta}\alpha)^n}{n}$$

$$e^{(\bar{g}_-, f_-)} = \frac{1}{1-\bar{\beta}\alpha}$$

Thus we see that the inner product of two nice coherent states belonging to the divisors $\sum_{|\beta|<1} m_\beta \beta$, $\sum_{|\alpha|<1} n_\alpha \alpha$ is

$$\langle \{m_\alpha\} \,|\, \{n_\alpha\} \rangle = \prod_{\alpha,\beta} \left(\frac{1}{1-\bar{\beta}\alpha} \right)^{m_\beta n_\alpha}$$

Thus we get a positive-definite ~~function~~ matrix on the free abelian group with basis $\{\alpha \mid |\alpha|<1\}$. But this is not deep; it amounts to essentially $e^{\bar{\lambda}\mu}$ being a positive-definite matrix over the set $\mathbb{C}$, because of the embedding $\lambda \longmapsto e^{\lambda z}$ in a Hilbert space.

Now I want to work out the vertex operator

$$X(m,z) = e^{a^*(m\delta_z^-)}\, e^{-a(m\delta_z^+)}$$

on these coherent states. $\delta_z^+ = \sum_1^\infty \frac{z^{-n}t^n}{n} \qquad \delta_z^- = \overline{\delta_z^+}$

But what I really want to do is to form the operator

$$X(m,g) = \int \frac{dz}{2\pi i z}\, g(z)\, X(m,\delta_{\bar z})$$

and then exponentiate it.

$$e^{X(m,g)} = 1 + \int \frac{dz_1}{2\pi i z_1} g(z_1) X(m,\delta_{\bar z_1})$$

$$+ \frac{1}{2!}\int \frac{dz_1}{2\pi i z_1}\frac{dz_2}{2\pi i z_2}\, g(z_1) g(z_2)\, X(m,\delta_{\bar z_1}) X(m,\delta_{\bar z_2})$$

$$+ \cdots\cdots$$

Take $m=1$ in the following.

$$X(\delta_{\bar z_1})X(\delta_{\bar z_2}) = e^{a^*\left(\overset{\delta^-_{\bar z_1}}{\sum_1^\infty \frac{\bar z_1^{\,n} t^{-n}}{n}}\right)}\, e^{-a\left(\overset{\delta^+_{\bar z_1}}{\sum_1^\infty \frac{\bar z_1^{-n} t^{n}}{n}}\right)}\, e^{a^*\left(\sum_1^\infty \frac{\bar z_2^{\,n} t^{-n}}{n}\right)}\, e^{-a\left(\sum_1^\infty \frac{\bar z_2^{-n} t^{n}}{n}\right)}$$

$$= e^{-(\delta^+_{\bar z_1},\, \delta^-_{\bar z_2})}\, e^{a^*(\delta^-_{\bar z_1} + \delta^-_{\bar z_2})}\, e^{-a(\delta^+_{\bar z_1} + \delta^+_{\bar z_2})}$$

To simplify notation change to

$$X(m,g) = \int \frac{dz}{2\pi i z}\, g(\bar z)\, X(m,\delta_z)$$

The scalar factor

$$X(\delta_{z_1})X(\delta_{z_2}) = \underbrace{e^{-(\delta^+_{z_1},\, \delta^-_{z_2})}}_{e^{-\sum \frac{z_1^{-n}}{n}\frac{z_2^{n}}{n}(t^n,t^{-n})} = (1 - z_1^{-1} z_2)}\, e^{a^*(\delta^-_{z_1} + \delta^-_{z_2})}\, e^{-a(\delta^+_{z_1} + \delta^+_{z_2})}$$

Therefore one has

$$\int \frac{dz_1}{2\pi i z_1} \frac{dz_2}{2\pi i z_2} g(\bar z_1) g(\bar z_2) X(\delta_{z_1}) X(\delta_{z_2}) \, e^{a^*(f^-)}|0\rangle$$

$$= \int \frac{dz_1}{2\pi i z_1} \frac{dz_2}{2\pi i z_2} g(\bar z_1) g(\bar z_2) (1 - z_1^{-1} z_2) \, e^{-\sum_i (\delta_{z_i}^+, f^-)} \, e^{a^*(\sum \delta_{z_i}^+ + f^-)}|0\rangle$$

and in the n-th degree term the fudge factor is

$$\prod_{i<j} (1 - z_i^{-1} z_j)$$

Except for this factor we would be able to write

$$e^{X(g)} = e^{\int \frac{dz}{2\pi i z} g(\bar z) X(\delta_z)}$$

$$e^{X(g)} e^{a^*(f^-)}|0\rangle = e^{\int \frac{dz}{2\pi i z} g(\bar z) e^{-(\delta_z^+, f^-)} e^{a^*(\delta_z^+)}} \cdot e^{a^*(f^-)}|0\rangle$$

or something like it.

Here's a problem with this thing. We want $e^{X(g)}$ to satisfy $e^{X(g_1+g_2)} = e^{X(g_1)} e^{X(g_2)}$. Hence we want the operators $X(\delta_{z_1})$ $X(\delta_{z_2})$ to commute. So far they don't, because

$$X(\delta_{z_1}) X(\delta_{z_2}) = e^{a^*(\delta_{z_1}^-)} e^{-a(\delta_{z_1}^+)} e^{a^*(\delta_{z_2}^-)} e^{-a(\delta_{z_2}^+)}$$

$$= \underbrace{e^{-(\delta_{z_1}^+, \delta_{z_2}^-)}}_{e^{-\left(\sum \frac{z_1^{-n} t^n}{n}, \sum \frac{z_2^n t^{-n}}{n}\right)} = e^{-\sum \frac{z_1^{-n} z_2^n}{n}} = 1 - z_1^{-1} z_2} e^{a^*(\delta_{z_1}^- + \delta_{z_2}^-)} e^{-a(\delta_{z_1}^+ + \delta_{z_2}^+)}$$

If we do the product in the other direction we get the factor $1 - z_2^{-1} z_1$. So now we begin to see why the $\mathbb{C}[Z]$ part has to be put in and also why one needs some kind of evenness in the lattice. Putting in the diagonal part gives the factor $z_1 - z_2$ and then

if we work with $X(n, z)$ the factor becomes $(z_1 - z_2)^{n^2}$
which is symmetric if n is even.

October 29, 1981

It is necessary to have good general viewpoints.

Problem: Is there a unified way to think about
1) Kan's G, $\overline{W}$ functors
2) bar + cobar business in rational homotopy theory?

These both arise from looking at principal bundles

$$G \longrightarrow P \longrightarrow X$$

and then rigidifying by choosing a connection. Thus one begins with a fibred category over spaces $\times$ (groups)^{op}, and one then rigidifies, and obtains then a fibred category with discrete fibres. Better: if G is fixed, then as a fibred cat over spaces, it has a final object $\overline{W}(G)$; and if X is fixed then as a cofibred category over groups it has an initial object $G(X)$.

Formally we obtain this situation whenever we have a pair of adjoint functors. In effect the fiber over X, G is the set

$$\mathrm{Hom}(G(X), G) = \mathrm{Hom}(X, \overline{W}(G)).$$

Next I ask the same question about differentiable fibre bundles with connections. This time we want to belong to a Lie gp. a classifying space BG which strictly classifies bundles with connection, and also to a pointed connected X a loop space $\Omega(X)$. We ask that $P \to X$ be pointed also.

Example: $X = S^1$. Then a principal bundle $P \to S^1$ with connection is the same as an element of G, so in this case $\Omega(X) = \mathbb{Z}$ exactly. There is no curvature over $X = S^1$.

In general the closest approximation to $\Omega(X)$ seems to be to take [redacted] piecewise smooth loops modulo some identification, but this doesn't seem to [redacted] give the right answer for $X = S^1$.

As for $B(G)$ if G is a discrete group, there is a unique connection on any principal G bundle P, and $B(G)$ clearly doesn't exist. However if G is connected and maybe semi-simple it might be the case that given $P \to X$ with connection and a point $p \to x$, then the curvature gives one [redacted] germ of structure inside G. Thus you trivialize the bundle over a neighborhood U of x

$$P = U \times G \longrightarrow U$$

and you have a connection on this. [redacted] So divide out by gauge transformations and germs of diffeos. of U around x, and maybe this set can be identified. Actually maybe one should use all germs of smooth maps.

<u>Examples</u>: $G = S^1$. Then over X we get a curvature ω which is a closed 2-form which I think determines the bundle + connection. So is there a universal space for closed 2 forms?

Back to the vertex representation. Review:

Let F = anal maps $S^1 \to \mathbb{C} \;=\; F_+ \oplus \mathbb{C} \oplus F_-$

F° = " " $S^1 \to \mathbb{C}^* \;=\; \exp(F_+ \oplus F_-) \times (\mathbb{C}^* \times \mathbb{Z})$

The $\mathbb{C}$ vector space $F_+ \oplus F_-$ has the skew form $\mathrm{Res}(g\,df)$ which sets up a duality between F_+ and F_- and which combined with conjugation gives an inner product on F_+ such that $\frac{t^n}{\sqrt{n}}$ is an orthonormal basis. We get on $S((F^+)^\vee)$ then operators $a(f_+)$ $a^*(f_-)$ with usual commutation relations, so that we get a projective repn. of $\exp(F_+ \oplus F_-)$ given by

$$\theta(e^f) = e^{a^*(f_-)}\, e^{a(f_+)} \qquad f = f^+ + f^-$$

Next we want to find operators $X(n,g)$ on the additive group $g \in F$ such that

$$X(n,g_1) + X(n,g_2) = X(n, g_1+g_2)$$

$$\theta(e^f)\,X(n,g)\,\theta(e^f)^{-1} = X(n, e^{nf}g)$$

Enough to do for $g = \delta_z$ and the solution is

$$X(n,\delta_z) = c(z)\, e^{a^*(n\delta_z^-)}\, e^{-a(n\delta_z^+)}$$

where $\delta_z^- = \sum_1^\infty \frac{z^k t^{-k}}{k}$ $\quad \delta_z^+ = \sum_1^\infty \frac{z^{-k} t^k}{k}$. We take the simplest solution $c(z) = 1$.

But now $X(n_1,\delta_{z_1})X(n_2,\delta_{z_2}) = \left(e^{-a(n_1\delta_{z_1}^+)}, e^{a^*(n_2\delta_{z_2}^-)}\right)$

$\times\; e^{a^*(n_1\delta_{z_1}^- + n_2\delta_{z_2}^-)}\, e^{-a(n_1\delta_{z_1}^+ + n_2\delta_{z_2}^+)}$. This fudge factor is

$$e^{-(n_1\delta_{z_1}^+,\, n_2\delta_{z_2}^-)} = e^{-n_1 n_2 \sum_k \frac{z_1^{-k} z_2^k}{k}} = \left(1 - \frac{z_2}{z_1}\right)^{n_1 n_2}.$$

These operators don't commute. Thus we have to add diagonal terms.

Thus look at $\mathbb{C}^* \times \mathbb{Z} = \{ct^n\}$. The commutator pairing is $\quad (f,g) = \exp\{\frac{1}{2\pi i}\log g\; d\log f\}$ provided

$\deg(g) = 0$. Thus $(t^n, c) = c^n$.

So now want to define $\Theta(t^n)$ and $\Theta(c)$ on $\mathbb{C}[\mathbb{Z}]$.
$\mathbb{C}[\mathbb{Z}]$ has basis $t^n = (e^{\log t})^n$. I might try

$$\Theta(c) = e^{(\log c)t\frac{d}{dt}} \quad \text{has eigenvalue } c^n \text{ on } t^n$$

$$\text{NO} \longrightarrow \quad \Theta(t^n) = e^{n\log t}$$

but then

$$(\Theta(t^n), \Theta(c)) = \left(e^{n\log t}, e^{(\log c)t\frac{d}{dt}}\right) = e^{-n\log c} = c^{-n}$$

so this doesn't work and instead we try

$$\boxed{\Theta(c) = e^{(\log c)t\frac{d}{dt}} \qquad \Theta(t^n) = e^{-n\log t}.}$$

Then I want

$$\Theta(c)\, X(n, \delta_z)\, \Theta(c)^{-1} = c^n X(n, \delta_z)$$

which has a solution $e^{n\log t}$. Also I want

$$\Theta(t^m)\, X(n, \delta_z)\, \Theta(t^m)^{-1} = z^{mn} X(n, \delta_z)$$

which has the solution $e^{n\log z\, t\frac{d}{dt}}$ because

$$e^{-m(\log t)}\, e^{n\log z\, t\frac{d}{dt}}\, e^{m(\log t)} = e^{n(\log z)m} = z^{mn}.$$

Thus our diagonal factor is up to a scalar.

$$X(n, \delta_z) = e^{n\log t}\, e^{n\log z\, t\frac{d}{dt}}$$

Then

$$X(n_1, \delta_{z_1}) X(n_2, \delta_{z_2}) = e^{n_1\log t + n_2\log t}\left(e^{n_1\log z_1\, t\frac{d}{dt}}, e^{n_2\log t}\right) \times e^{(n_1\log z_1 + n_2\log z_2)t\frac{d}{dt}}$$

and the fudge factor is

$$e^{n_1\log z_1 n_2} = z_1^{n_1 n_2}$$

Therefore we conclude that if put

$$\boxed{X(n, \delta_z) = e^{a^*(n\delta_z^-)}\, e^{n\log t}\, e^{n\log z\, t\frac{d}{dt}}\, e^{-a(n\delta_z^+)}}$$

then these operators for different n, z commute provided that at least one of the n's is even.

However these operators are formal, so one must be careful when one tries to compose them.

Interesting point: Depending on context, lattices of rank 1 are identifyable with elements of

$$F^\circ / \mathbb{C}^\circ \times \exp(F_+) = \mathbb{Z} \times \exp(F_-)$$

Let's check this when F = rational functions with no poles or zeroes on S^1. Then F° is made of factors

$$\begin{array}{lll} t-\alpha & |\alpha|<1 & t-\alpha = t(1-\alpha t^{-1}) \in \mathbb{Z}\times \exp(E) \\ t-\alpha & |\alpha|>1 & \\ c \neq 0 & & \end{array}$$

For $|\alpha|>1$ one has modulo $\mathbb{C}^\circ \times \exp(F^+)$, that

$$t-\alpha = (-\alpha)(1-\alpha^{-1}t) \equiv 1.$$

Thus it is very natural to put $\mathbb{C}^\circ$ with $\exp(F_+)$ and the $\mathbb{Z}$ with $\exp(F_-)$; somehow one is giving preference to the direction $z=0$. In the Laurent series case one has $\mathbb{C}^\circ \times \exp(F_+) = \mathbb{C}[[z]]^\circ$ and $F_- = 0$.

Physics of gauge fields. Let's suppose we want to understand the electromagnetic field by itself. There are the physical fields E, B which you can measure classically, but which come from the gauge field A. A is a 1-form on space-time (real-valued). It enters in quantum mechanics because a particle with charge e and normal momentum $p_0 = m\vec{v}$ has momentum $p = p_0 - eA$ as far as the Lagrangian for the coupled system is concerned. Thus the wave function

ψ has the energy operator $\frac{1}{2m}\left(\frac{\hbar}{i}\frac{\partial}{\partial x} - eA\right)^2$. The physics doesn't change if A is changed by a gauge transformation. This amounts to changing ψ to $e^{i\alpha}\psi$ and then A changes by a multiple of $d\alpha$.

What does one do normally to quantize A. A simple method is to fix a gauge, so that then one has only the relevant degrees of freedom. The kinematics requires you put these degrees of freedom into operators with standard commutation relations.

October 26, 1981

Gauge fields. Let's consider the example of the electromagnetic field in space with charges + currents. Classically it is described by fields E, B satisfying

$$\nabla \cdot B = 0 \qquad \nabla \times E = -\frac{\partial B}{\partial t}$$
$$\nabla \cdot E = \rho \qquad \nabla \times B = j + \frac{\partial E}{\partial t}$$

To solve one introduces the gauge potentials (ϕ, A) by the formulas:

$$B = \nabla \times A \qquad E = -\nabla \phi - \frac{\partial A}{\partial t}$$

and then (ϕ, A) can be altered by a gauge transformation

$$(\phi, A) \longmapsto (\phi, A) + \left(-\frac{\partial \chi}{\partial t}, \nabla \chi\right)$$

What is one doing? One has a closed 2-form on space-time given by (E, B) and one is finding a connection on the trivial S^1-bundle with this as connection form.

However, whereas the time-evolution of (E, B) is determined it is not clear how (ϕ, A) should evolve in time. To understand this, let use the Fourier transform and look at harmonic solution with space-time dependence $e^{i(kx - \omega t)}$. Then the equations are

$$ik \cdot B = 0 \qquad ik \times E = i\omega B$$

$$B = ik \times A \qquad E = -ik\phi + i\omega A$$

and the gauge transf is

$$(\phi, A) \longmapsto (\phi, A) + (\omega \chi, ik\chi)$$

Remaining Maxwell's eqns. are

$$ik \cdot E = |k|^2 \phi - \omega(k \cdot A) = \rho$$
$$ik \times B = ik(ik \cdot A) - (ik \cdot ik)A = -k(k \cdot A) + |k|^2 A$$
$$= j - i\omega(-ik\phi + i\omega A) = j - \omega k \phi + \omega^2 A$$

So get
$$\boxed{\begin{aligned} &|k|^2 \phi - \omega(k\cdot A) = \rho \\ &(|k|^2 - \omega^2)A + k(\omega\phi - k\cdot A) = j \end{aligned}}$$

These do not have unique solutions because (ϕ, A) can be gauge transformed. Hence one doesn't seem to have a well-posed problem if one works with all possible gauge fields (ϕ, A).

So one imposes a gauge (fixing) condition. Three candidates are

$\nabla\cdot A = 0$	$k\cdot A = 0$	transverse gauge (Coulomb?)
$\phi = 0$	$\phi = 0$	
$\nabla\cdot A + \frac{\partial\phi}{\partial t} = 0$	$k\cdot A - \omega\phi = 0$	Lorentz gauge

In the transverse gauge it is very easy to solve. Thus given ρ one solves $|k|^2\phi = \rho$ or $-\Delta\phi = \rho$ and there is a nice solution if spatially ρ has compact support. Then one has to solve

$$(|k|^2 - \omega^2) A = j - k\omega\phi$$

$$\left(-\Delta + \frac{\partial^2}{\partial t^2}\right) A = j - \nabla\frac{\partial\phi}{\partial t}$$

the wave equation. Thus the longitudinal + transverse degrees of freedom are separated.

Next we want to quantize. What does this mean? Ultimately I want to treat (ρ, j) as a source perturbation of a system of harmonic oscillators. So the first thing to do is to get a system of oscillators in the case $(\rho, j) = 0$. So what does this mean?

Space dim = 1: Here the equations become (E, B reduce to a single E):
$$E = -\frac{\partial \phi}{\partial x} - \frac{\partial A}{\partial t} \qquad \frac{\partial E}{\partial x} = \rho \qquad \frac{\partial E}{\partial t} = -j$$

gauge transf:
$$(\phi, A) \longmapsto (\phi, A) + \left(-\frac{\partial \chi}{\partial t}, \frac{\partial \chi}{\partial x}\right)$$

The Lorentz gauge condition is $\dfrac{\partial A}{\partial x} + \dfrac{\partial \phi}{\partial t} = 0$.

<u>Note</u>: If there are no charges: $(\rho, j) = 0$, then E is constant (hence 0 if it decays spatially). The equations
$$\begin{cases} \dfrac{\partial \phi}{\partial x} + \dfrac{\partial A}{\partial t} = 0 & (E = 0) \\[2ex] \dfrac{\partial \phi}{\partial t} + \dfrac{\partial A}{\partial x} = 0 & (\text{Lorentz}) \end{cases}$$
are the same as the transmission line equations, so one gets waves travelling both left and right. Thus 1 photon per wave vector k. However if we try to impose gauge conditions like $\dfrac{\partial A}{\partial x} = 0$ or $\phi = 0$ then the only non-trivial solutions occur with spectrum in $(k, \omega) = 0$.

Check carefully:
$$E = -ik\,\phi + i\omega A \qquad \begin{aligned} ikE &= \rho = 0 \\ i\omega E &= j = 0 \end{aligned}$$

So as I would like k or ω to be $\neq 0$, must have $E = 0$.
Hence have $k\phi = \omega A$. Now try to add $kA = 0$.
If $k \neq 0$, then $A = 0$ so $k\phi = 0$ and $\phi = 0$. If $k = 0$, then we can take $A = 0$ and ϕ, ω arbitrary.

It seems that once $(\rho, j) = 0$, then $\dfrac{\partial \phi}{\partial x} + \dfrac{\partial A}{\partial t} = 0$ so ϕ, A is gauge equivalent to 0 so a gauge invariant theory is trivial. In three space dimensions what happens?
$$\begin{cases} |k|^2 \phi - \omega (k \cdot A) = 0 \\ (|k|^2 - \omega^2) A + k(\omega \phi - k \cdot A) = 0 \end{cases}$$

If I consider $k \neq 0$, and look for $A \neq 0$ subject to the transverse gauge $k \cdot A = 0$, then $\phi = 0$ and $(|k|^2 - \omega^2) A = 0$ $\Longrightarrow \omega = \pm |k|$. Thus there are 2 photons for each $k \neq 0$.

If I use the Lorentz gauge, then $\omega\phi = k\cdot A$, so we get the equations $(|k|^2 - \omega^2)\phi = 0$ $(|k|^2 - \omega^2) A = 0$ which would give us 3 photons, except that we have to remove the [redacted] ones equivalent to 0, i.e. $(\phi, A) = (\omega, \vec{k})\chi$ so it seems we get 2 photons per k as before.

The real moral from the above is that the Lorentz gauge condition is not a good gauge condition for the free field. Also that one must work in at least 2 space dimensions to have an interesting family of photons.

October 27, 1981

$\mathcal{G}$ = gauge group
$\mathcal{X}$ = space of connections = a torsor for Lie$(\mathcal{G})$

Question: What is a joint representation of $\mathcal{G}, \mathcal{X}$?

At the least it should be a rep of $\mathcal{G}$ together with a way of assigning operators to elements of $\mathcal{X}$ such that if $D' = D + X$ with $D', D \in \mathcal{X}$, $X \in$ Lie$(\mathcal{G})$, then the operators assigned to D', D, X add in the expected way. Let take the case of the trivial bundle $S^1 \times K \to S^1$. Then

$$\text{connection:} \quad D = d + A$$
$$\text{action of } g: \quad gDg^{-1} = d + (gdg^{-1} + gAg^{-1})$$
$$\text{action of } X: \quad [X, D] = -X' + [X, A].$$

A representation will assign to each $X \in$ Lie$(\mathcal{G})$ = maps$(S^1, \mathfrak{k})$ an operator $\theta(X)$ such that $\theta([X,Y]) = [\theta(X), \theta(Y)]$; Also it will assign an operator $\theta(d)$ to the trivial connection. Then $\theta(D) = \theta(d) + \theta(A)$ if $D = d + A$. Finally we need $[\theta(X), \theta(d)] = -\theta(X')$.

However I can form the semi-direct product of $\mathbb{R}$ with Lie$(\mathcal{G})$ whose bracket is

$$[a\partial + X, b\partial + Y] = aY' - bX' + [X, Y]$$

and then if I put $\theta(a\partial + X) = a\theta(d) + \theta(X)$ I get a repn. of this semi-direct product

$$\begin{aligned}\theta([a\partial + X, b\partial + Y]) &= a\theta(Y') - b\,\theta(X') + \theta[X,Y] \\ &= a[\theta(d), \theta(Y)] + b[\theta(X), \theta(d)] + [\theta(X), \theta(Y)] \\ &= [a\theta(d) + \theta(X), b\theta(d) + \theta(Y)]\end{aligned}$$

So therefore we conclude that a joint repn of $\mathcal{G}, \mathcal{X}$ with base S^1 is the same as a repn. of $S^1 \ltimes \mathcal{G}$.

The next idea is to let $\tilde{G}$ = central extension of the gauge group, but still keep X = space of connections. Now X, A are in different places. Then our representation should assign to X and A operators $\theta(X)$, $\theta(d+A) = \theta(d) + \theta(A)$ such that

$$\theta([X,Y]) = [\theta(X), \theta(Y)]$$

$$[\theta(d), \theta(X)] =$$

Return to this later.

Idea: Look at the adjoint representation: We have the loop group acting on its Lie algebra via the adjoint action, and the Lie algebra $\mathfrak{g}_0[t,t^{-1}]$ carries the skew-form

$$(x,y) = \operatorname{Res}\,(y, dx)_{\text{in } \mathfrak{g}_0}$$

which is non-degenerate if I divide out by the constant subalgebra $\mathfrak{g}_0$. We can split naturally into complementary isotropic subspaces

$$\mathfrak{g}_0[t,t^{-1}]/\mathfrak{g}_0 = \underset{W^*}{\mathfrak{g}_0[t^{-1}]t^{-1}} \oplus \underset{W}{\mathfrak{g}_0[t]t}$$

Something is wrong because $\mathfrak{g}_0$ is certainly not an invariant subspace of $\mathfrak{g}$ under the adjoint representation, which would be the case if one had an invariant skew-form. In fact I know that (x,y) is a cocycle:

$$([x,y],z) + \underbrace{([y,z],x) + ([z,x],y)}_{(x,[z,y]) + ([z,x],y) = (z,[x,y]) \text{ instead of } 0} = 0$$

So it can't be an invariant 2-form.

However we do know that the orbits of the loop group on the building are invariant symplectic manifolds so that this idea maybe can be salvaged. Thus we take the orbit of the trivial connection d and one gets ΩK. Not clear yet how to proceed.

Simpler situation. Take a semi-simple Lie algebra $\mathfrak{g} = \mathfrak{n}_- \oplus \mathfrak{h} \oplus \mathfrak{n}_+$ and look at the adjoint repn. There is an invariant symmetric bilinear form, so that we get a map from $\mathfrak{g}$ to the orth. gp. Lie algebra of $V = \mathfrak{g}$. But $\mathfrak{o}(V)$ has a spinor representation, so you get a nice repn. of $\mathfrak{g}$.

Recall that if $V = W^* \oplus W$ with form $Q(\lambda + w) = \lambda(w)$ then the Clifford algebra $C(V)$ is isomorphic to $\mathrm{End}(\Lambda W)$ with $v = \lambda + w$ acting as $i(\lambda) + e(w)$ on ΛW. Check

$$\big(i(\lambda) + e(w)\big)^2 = i(\lambda)e(w) + e(w)i(\lambda) = (\lambda, w).$$

If V is odd dimensional, write $V = W^* \oplus W \oplus \mathbb{C}$, then $C(V)$ has center $\mathbb{C} \oplus \mathbb{C}$, so one gets two spinor modules isomorphic to ΛW with the center acting differently.

So I take $V = \mathfrak{g}$ and then I want to compute the character which means I look at only the $\mathfrak{h}$ action. In this case $V = \mathfrak{n}_- \oplus \mathfrak{n}_+ \oplus \mathfrak{h}$ is stable under $\mathfrak{h}$ so that the Clifford module is isomorphic to $\Lambda(\mathfrak{n}_+ \oplus \mathfrak{h}_+)$ where $\mathfrak{h}_+$ is half of $\mathfrak{h}$. This is isom. to $\Lambda(\mathfrak{n}_+) \otimes \Lambda(\mathfrak{h}_+)$ and $\mathfrak{h}$ has to act trivially on the second factor. So the repn. is not irreducible, and one concludes that there is a representation of $\mathfrak{g}$ on $\Lambda(\mathfrak{n}_+)$.

We need to know how the orthogonal Lie alg. $\mathfrak{o}(V)$ sits in $C(V)$. Otherwise we only have a projective repn. of $\mathfrak{o}(V)$ on ΛW. The points is to take brackets of elements of V. So for examples, V is spanned by a_i, a_i^* which gives brackets

$$a_i a_j, \quad a_i^* a_j^* \qquad i<j \qquad \text{total } 2\frac{n(n-1)}{2} = n^2 - n$$

$$a_i^* a_j \qquad i \neq j$$

$$\frac{1}{2}(a_i^* a_i - a_i a_i^*) = a_i^* a_i - \frac{1}{2} \qquad n^2 \quad \text{total } 2n^2 - n = \frac{2n(2n-1)}{2} \checkmark$$

This shows that if I look at the $\mathfrak{h}$ action on $\Lambda(\mathfrak{n}_+)$ which one gets thru the Clifford algebra, then we have

$$\Lambda(\mathfrak{n}_+) = \bigotimes_{\alpha>0} \Big(\Lambda(\mathbb{C}e_i) \text{ with } \mathfrak{h} \text{ acting as } \alpha\,(a_i^* a_i - \tfrac{1}{2})\Big)$$

Thus

$$\operatorname{char} \Lambda(\mathfrak{n}_+) = \prod_{\alpha>0} \left(e^{\alpha/2} + e^{-\alpha/2}\right)$$

and so the highest weight is $\rho = \frac{1}{2} \sum_{\alpha>0} \alpha$.

October 28, 1981

Should review a little bit about proofs of periodicity and indices. Let $T: \mathcal{H} \to \mathcal{H}$ be a Fredholm operator on a Hilbert space. Let S be projection on $\operatorname{Im}(T)$ followed by S^{-1}. Then

$$TS = \text{proj on } \operatorname{Im} T = I - \text{proj on } (\operatorname{Im} T)^\perp$$
$$ST = \text{proj on } (\operatorname{Ker} T)^\perp = I - \text{proj on } (\operatorname{Ker} T).$$

So

$$TS - ST = \text{proj on } \operatorname{Ker} T - \text{proj on } (\operatorname{Im} T)^\perp$$

$$\operatorname{tr}(TS - ST) = \operatorname{index} T$$

Now one has $\operatorname{tr}(TK - KT) = 0$ if K is compact (e.g. if $K = |a\rangle\langle b|$ is of rank 1, then

$$\operatorname{tr}(TK) = \langle b|T|a\rangle$$
$$\operatorname{tr}(KT) = \langle b|T|a\rangle \quad).$$

Thus altering S mod compacts does not affect $\operatorname{tr}[T, S]$ and so we have the formula

$$\boxed{\operatorname{tr}[T, T^*] = \operatorname{index}(T)}$$

for any T^* such that $TT^* \equiv T^*T \equiv I$ mod compacts

Tate's residue: Let A be a d.v.r. over its residue field k with quotient field F. Choose a splitting of

$$0 \longrightarrow A \underset{\xleftarrow{P}}{\longrightarrow} F \longrightarrow F/A \longrightarrow 0$$

as k-vector spaces, and then associate to any $f \in F^\times$ the operator $Pf : A \longrightarrow A$. This is automatically "Fredholm". I claim that

$$\operatorname{tr}[Pf, Pg] = -\operatorname{res}_A(g\,df)$$

Check this is well-defined, that is the left side is. $f - Pf$, $g - Pg$ are of finite rank as operators from A to F. Thus

$$fg - Pf\cdot Pg = \underbrace{(f-Pf)g}_{} + Pf\cdot(g-Pg)$$

finite rank because $(f-Pf)g$ is zero on $g^{-1}f^{-1}A$

so $\quad P(fg) - Pf\cdot Pg$ has finite rank

Now check derivation property

$$tr\,[P(f_1f_2), Pg] = tr([Pf_1\cdot Pf_2, Pg]) = tr([Pf_1, Pg]Pf_2 + Pf_1[Pf_2, Pg])$$

$$tr\,[Pf_1, Pf_2\cdot Pg] = tr([Pf_1, Pf_2]Pg + Pf_2[Pf_1, Pg])$$

$$tr\,[Pf_2, Pf_1\cdot Pg] = tr([Pf_2, Pf_1]Pg + Pf_1[Pf_2, Pg])$$

Thus $\quad tr\,[P(f_1f_2), Pg] = tr\,[Pf_1, P(gf_2)] + tr\,[Pf_2, P(gf_1)]$

Finally if f has positive degree, then $Pf = f$ and $P(f^{-1})$ maps fA onto A via f. Thus $P(f^{-1})Pf = I$ and $Pf\cdot P(f^{-1}) = I$ on fA and 0 on some complement of fA in A, so

$$tr\,[Pf, Pf^{-1}] = -\deg f. \qquad (\text{Check} = \text{index } Pf).$$

So what one has here is the familiar process of associating to an element in ΩU_n an element of a Grassmannian, either the lattice or the associated Fredholm operator. The real question is whether there is any relation between this business and the central extension of ΩSU_n. Somehow the $tr\,[Pf, Pg]$ business when exponentiated yields the central extension.

Review Atiyah-Singer proof of index thm.

$\mathcal{B}$ = bounded operators in $\mathcal{H}$

$\mathcal{K}$ = compact operators

$\mathcal{A} = \mathcal{B}/\mathcal{K}$ $\quad$ Calkin algebra

$\mathcal{F}$ = Fredholm operators = inverse image of $\mathcal{A}^\times$.

$\mathcal{B}^\times$ deforms to big unitary group $\mathcal{B}^\times_{un}$ which is cpt by Kuiper

$\mathcal{A}^\times$ deforms to $\mathcal{A}^\times_{un}$. One has

$$1 \longrightarrow \mathcal{U} \longrightarrow \mathcal{B}^\times_{un} \longrightarrow \mathcal{A}^\times_{un} \xrightarrow{\text{index}} \mathbb{Z} \longrightarrow 0$$

where $\mathcal{U} = \{\text{unitaries} \equiv 1 \bmod \mathcal{K}\}$. This sequence shows that

$$\mathcal{A}^\times_{un} \sim \mathcal{A}^\times \sim \mathbb{Z} \times B\mathcal{U}$$

On the other hand one has $\mathcal{K} \longrightarrow \mathcal{F} \longrightarrow \mathcal{A}^\times$ with $\mathcal{K}$ contractible so $\mathcal{A}^\times \sim \mathcal{F}$.

Next one uses a filtration argument

$$\left\{\begin{matrix}\text{s.a. operators}\\ \text{with essential}\\ \text{spectrum } \{-1, 1\}\end{matrix}\right\} \xrightarrow{\exp i\pi} -\mathcal{U}$$

plus Kuiper's thm. for the strata, to see the above map is a homotopy equivalence. One has a h.eq.

$$\left\{\begin{matrix}\text{s.a. operators}\\ \text{with ess. spectrum}\\ \{-1, 1\}\end{matrix}\right\} \longrightarrow \left\{ x \in \mathcal{A} \,\middle/\, \begin{matrix} x^2 = 1 & x = x^* \\ x \text{ not } 1 \text{ or } -1 & \end{matrix} \right\}$$

and the last set is $\cong \mathcal{A}^\times_{un} / \mathcal{A}^\times_{un} \times \mathcal{A}^\times_{un} \sim B\mathcal{A}^\times_{un}$.

Thus one establishes $B\mathcal{A}^\times_{un} \sim \mathcal{U}$

October 29, 1981

Quantization of the EM field. We begin with describing the field [redacted] in the transverse gauge: $\nabla\cdot A = 0$. Then the [redacted] field is a big oscillator with [redacted] two modes of frequency $|k|$ for each of the two possible transverse directions to k. So we know how to quantize it, and in particular given times t', t and field configurations $A' = \{A'(x)\}$ and $A = \{A(x)\}$ we have an amplitude that A' at t' becomes A at t. This amplitude is a Gaussian

$$e^{\int dx\,dx'\, A(x) K(tx, t'x') A(x')} \cdot \text{det factor}$$

[redacted] whose kernel should be easy to describe classically. In any case we do not want to use the x-basis, because the condition $\nabla\cdot A = 0$ is hard to describe. Rather we want to describe a field configuration $A = \{A(x)\}$ in terms of the normal modes. The modes are independent, hence the above Gaussian [redacted] should be a [redacted] product of simple factors one for each mode. There is a slight headache here because the fields A are real, hence k and $-k$ get mixed together.

Let's review how we handle a harmonic oscillator. The histories form a real vector space V with an inner product given by the energy. There is a unique complex structure on V such that the frequencies are all positive. [redacted] This means that $V = \sum_k V_k$ where $V_k \cong \mathbb{C}$ and where time acts as $e^{-it\omega_k}$, $\omega_k > 0$ in V_k. Therefore each element A of V can be written $A = \sum A_k$ with $A_k \in V_k$ and we have

$$A(t) = \sum A_k e^{-i\omega_k t}$$

Now the energy is of the form

$$E(A) = \sum_k |A_k|^2 c_k \qquad c_k > 0.$$

where I suppose V_k has been given a norm $|\ |^2$.
When we quantize the k-th mode will have energy operator $\omega_k(a_k^* a_k + \frac{1}{2})$, hence using the correspondence principle we want

$$c_k |A_k|^2 = \omega_k |a_k|^2$$

or

$$A_k = \sqrt{\frac{\omega_k}{c_k}}\, a_k$$

which means that under ~~quantization~~

$$A(t) = \sum \sqrt{\frac{\omega_k}{c_k}}\, e^{-i\omega_k t} a_k$$

So consider a free EM field described by a transverse A:

$$A(x) = \sum_k e^{ikx}(A_{k1}\varepsilon_k^1 + A_{k2}\varepsilon_k^2)$$

Here $\varepsilon_k^1, \varepsilon_k^2$ are $\perp$ unit vectors to k; say same for $-k$.
The fact $A(x)$ should be real says that $\overline{A_{ki}} = A_{-k,i}$.
It might be simpler to describe things with
$A_k = A_{k1}\varepsilon_k^1 + A_{k2}\varepsilon_k^2$ = arb. cx. vector $\perp$ k.

Time evolution is described by $\frac{\partial^2 A}{\partial t^2} = \nabla^2 A$ which is 2nd order, so I don't yet have a complete description of my solutions. A complete solution will be in the form

$$A(x,t) = \sum e^{i(kx-\omega_k t)}A_k + e^{-i(kx-\omega_k t)}\overline{A_k}$$

so in this way we see solutions can be described by a family $\{A_k\}$, where A_k is a complex vector $\perp$ k.

So to complete the quantization I need to know the energy. The energy density is proportional to $E^2 + B^2$ and $E = -\frac{\partial A}{\partial t}$, $B = \nabla \times A$ so there will be some constant such that the energy of the mode $e^{i(kx-\omega t)}A_k + c.c$ is const $\times\ \omega^2 |A_k|^2 \times$ volume.

But what should I concentrate on. The basic Hilbert space is a symmetric algebra

---

October 30, 1981:

To understand Feynman approach [redacted] for the quantization of the EM field. The possible [redacted] fields $A(t,x)$ form a vector space on which one has a quadratic function given by the action: $S(A) = \int dt\, d^3x\, (E^2 - B^2)$. Hence one can form a [redacted] functional integral which is a Gaussian and look at Green's functions, all of which should be calculable in terms of $\langle A_\mu(xt) \cdot A_\nu(x't') \rangle$. Now the gauge gp acts linearly, so the action $S(A)$ is a degenerate quadratic form which means that the Green's functions should be very singular

First one must carefully understand the simple harmonic oscillator in this Feynman language. [redacted] Instead of a field $A(t,x)$ one has a path $x(t)$ and the action is

$$S(x) = \int \left[\tfrac{1}{2}\dot{x}^2 - \tfrac{1}{2}\omega^2 x^2\right] dt .$$

What about the boundary conditions? This is handled as follows. One knows

$$\langle x | e^{-iH_0(t-t')} | x' \rangle = \int_{\substack{x(t') = x' \\ x(t) = x}} \mathcal{D}x \; e^{iS(x)} \qquad H_0 = \frac{p^2}{2} + \frac{1}{2}\omega^2 q^2$$

and $e^{-\frac{1}{2}\omega x^2}$ is the ground state so that

$$e^{-iH_0 t} \cdot e^{-\frac{1}{2}\omega x^2} = e^{-i\frac{\omega}{2}t}\, e^{-\frac{1}{2}\omega x^2}$$

When we compute Greens' functions we add a source term $J(t)q$ to the Hamiltonian H_0 with J of compact support. Then we get the path integral

$$\langle x | U_J(t,t') | x' \rangle = \int_{\substack{x(t')=x' \\ x(t)=x}} \mathcal{D}x \; e^{iS(x) - i\int Jx\,dt}$$

for the propagator between times t' and t. So let us multiply by $e^{-\frac{1}{2}\omega x'^2}$, $e^{-\frac{1}{2}\omega x^2}$ and integrate over x', x. Then we get an integral over all paths over $[t',t]$ which computes

$$\langle 0 | U_J(t,t') | 0 \rangle.$$

If we divide by $\langle 0 | U_0(t,t') | 0 \rangle = e^{-i\frac{\omega}{2}(t-t')}$, then it becomes

$$Z(J) = \langle 0 | S_J | 0 \rangle$$

which is independent of the interval $[t',t]$.

The Green's functions are

$$\begin{aligned} G(t,t') &= \langle 0 | T[x(t)\, x(t')] | 0 \rangle \\ &= \langle 0 | \frac{a+a^*}{\sqrt{2\omega}}(t) \; \frac{a+a^*}{\sqrt{2\omega}}(t') | 0 \rangle \\ &= \frac{1}{2\omega} e^{-i\omega t + i\omega t'} \langle 0 | a a^* | 0 \rangle \qquad t > t' \\ &= \frac{1}{2\omega} e^{-i\omega|t-t'|} \end{aligned}$$

$$Z(J) = e^{-\frac{1}{2}\int dt\,dt' \; J(t) \frac{1}{2\omega} e^{-i\omega|t-t'|} J(t')}.$$

Check: $H_J = H_0 + J(t) q = H_0 + J(t) \frac{a+a^*}{\sqrt{2\omega}}$

$$S_J = T\left\{ e^{-i\int dt\, J(t) \frac{e^{-i\omega t} a + e^{i\omega t} a^*}{\sqrt{2\omega}}} \right\}$$

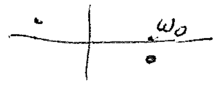
$$\langle 0 | S_J | 0 \rangle = e^{\int_{t>t'} dt\,dt' (-i)^2 J(t) J(t') \frac{e^{-i\omega t} e^{i\omega t'}}{2\omega}}$$

$$= e^{-\int_{t>t'} dt\,dt' \; J(t) e^{-i\omega(t-t')} J(t')} \qquad \text{agrees!}$$

Here's how to interpret this calculation: Look at the action

$$S(x) = \int\left(\tfrac{1}{2}\dot{x}^2 - \tfrac{1}{2}\omega_0^2 x^2\right) dt$$
$$= \tfrac{1}{2}\int x\left(-\frac{d^2}{dt^2} - \omega_0^2\right)x \;dt$$

when $x(t)$ has compact support. The device we use to make sense of the path integral amounts to choosing boundary conditions so as to invert the operator $\left(+\frac{d^2}{dt^2}+\omega_0^2\right)$. The actual inverse is the kernel



$$i\,G(t,t') = \frac{1}{-2i\omega_0} e^{-i\omega_0|t-t'|} = \int \frac{e^{-i\omega(t-t')}}{-\omega^2+\omega_0^2-i\varepsilon}\,\frac{d\omega}{2\pi}$$

Another way to obtain this kernel is to work with imaginary time $t=-i\tau$, so that the action becomes

$$iS(x) = -\tfrac{1}{2}\int x\left(-\frac{d^2}{d\tau^2}+\omega_0^2\right)x \;d\tau$$

Missing from the above is how the Green's functions $\langle T[x(t)\,x(t')]\rangle$ can be used to reconstruct the Hilbert space.

Next consider the EM field A which now consists of $(\phi, \mathbb{A})$. Take an A of compact support in space-time. It then has an action $S(A) = \int(E^2 - B^2)\,dt\,dx$. First put in imaginary time and then we get a quadratic form $iS(A)$ which is ≤ 0, but constant along gauge orbits.

First understand how the variance occurs. If I am given a quadratic function Q on a vector space V, then if non-degenerate I get an isomorphism $V \xrightarrow{\tilde{Q}} V^*$, and then I get a quadratic fn. on V^* by

$$Q^*(\lambda) = Q(\tilde{Q}^{-1}\lambda, \tilde{Q}^{-1}\lambda)$$

which matrix-wise amounts to $Q^* = Q^{-1}QQ^{-1} = Q^{-1}$.

~~[illegible]~~ If however Q is degenerate, say with kernel W, then we get

$$\tilde{Q} : V/W \xrightarrow{\sim} (V/W)^* = W^\perp$$

and so Q^* is defined on $W^\perp$. I want the moments to be calculated [redacted] for evaluation functions, but better probably are Fourier components.

So [redacted] we describe a compactly supported field on imaginary-space time by[redacted]

$$A(x) = \sum_k A_k e^{-ikx}$$

Here k, A_k are 4 vectors and $\overline{A_k} = A_{-k}$

Then our action function is

$$\|dA\|^2 = \sum_k |ik \wedge A_k|^2 = \sum_k \left(|k|^2 |A_k|^2 - |k \cdot A_k|^2 \right)$$

The obvious linear functionals are $A \longmapsto A_{k_0} \cdot v$. The space W consists of A of the form $A_k = k c_k$, c_k scalar function, so that the linear functionals of interest are ones of the form $A \longmapsto A_{k_0} \cdot v$ where $k_0 \cdot v = 0$.