Inelastic scattering situation: Let's recall the standard scattering (potential) situation. One has a free particle described by $H_0 = \frac{p^2}{2m}$, $p = \frac{\hbar}{i}\nabla$ on $\mathcal{H}_e = L^2(\mathbb{R}^3)$ and a perturbation given by a potential $V(r)$. One has the [redacted] basis

$$|k\rangle = e^{ik\cdot r}$$

for the free states, and the first order transition amplitude

$$\langle k_f | V | k_i \rangle = \int d^3r \, e^{-i(k_f - k_i)\cdot r} V(r)$$

$$= \hat{V}(k_f - k_i)$$

from the initial state k_i to the final state k_f.

Now I want to generalize to the case where the scatterer is described by a finite-dimensional Hilbert space $\mathcal{H}_s$ with Hamiltonian H_s. [redacted] Let $|\alpha\rangle$ denote the eigenstates for H_s. [redacted] The combined system is described by the Hilbert space

$$\mathcal{H}_e \otimes \mathcal{H}_s = L^2(\mathbb{R}^3; \mathcal{H}_s)$$

which has the basis $|k\rangle \otimes |\alpha\rangle = |k;\alpha\rangle$. The interaction is given by a matrix

$$\langle k_f;\beta | V | k_i;\alpha \rangle = \hat{V}_{\beta\alpha}(k_f - k_i)$$

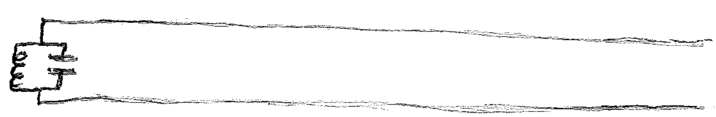
where $V_{\beta\alpha}(r)$ is a self-adjoint matrix of functions of r.

Somehow, in a way I would like to make very precise, the "Golden Rule" allows one to interpret

$$\left|\langle k_f | V | k_i \rangle\right|^2 \, \delta\left(E(k_f) - E(k_i)\right)$$

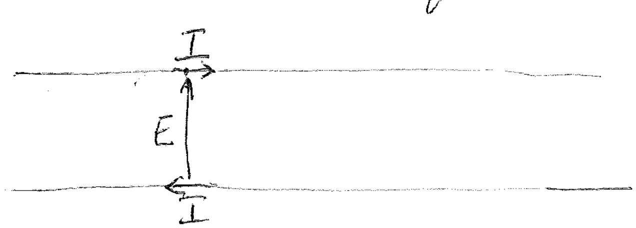
in the potential scattering case as the rate of the transition $i \to f$. I also want to generalize this to the mixed case: $\mathcal{H}_e \otimes \mathcal{H}_s$.

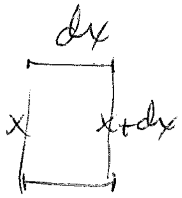
Simple example: Suppose we have light shining on an oscillator, i.e. a bound charged particles. Specifically let me imagine a transmission line terminated by a tuned circuit.



I can look at classically - waves of a given frequency are sent down the line where they do something to the oscillator and get reflected.

Recall the equations for a transmission line.





has capacitance $C\,dx$
hence stores $Q = C\,dx\,E \implies \dot{Q} = C\,dx\,\dot{E}$

$$\dot{Q} = -I(x+dx) + I(x)$$

$$\therefore \quad -\frac{\partial I}{\partial x} = C\frac{\partial E}{\partial t}$$

Also $\underset{E(x)}{\vdash}\!\!\longrightarrow\!\!\underset{E(x+dx)}{\dashv}$ has inductance $L\,dx$

so $E(x+dx) - E(x) = -L\,dx\,\dfrac{\partial I}{\partial t}$

$$\boxed{\begin{aligned} -\frac{\partial E}{\partial x} &= L\frac{\partial I}{\partial t} \\ -\frac{\partial I}{\partial x} &= C\frac{\partial E}{\partial t} \end{aligned}}$$

L = inductance / length

C = capacitance / length

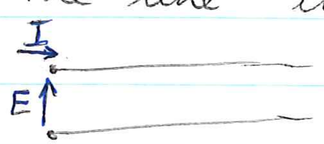
Try $\quad I = e^{i(kx-\omega t)} \qquad E = Z e^{i(kx-\omega t)}$

$$-Zik = -Li\omega \qquad [redacted] \qquad Z = L\frac{\omega}{k}$$

$$-ik = -CZi\omega \qquad\qquad Z = \frac{1}{C}\frac{k}{\omega}$$

$$\therefore\ \omega^2 = \frac{1}{LC}k^2 \qquad \text{or} \qquad \omega = \pm\frac{1}{\sqrt{LC}}k$$

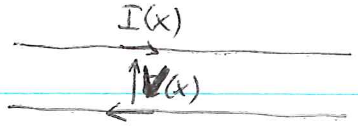
Thus the speed is $\dfrac{1}{\sqrt{LC}}$. Also $Z = \sqrt{\dfrac{L}{C}}$ is the impedance of the line in the sense that



behaves as a resistor with this resistance.

August 30, 1980

Transmission line

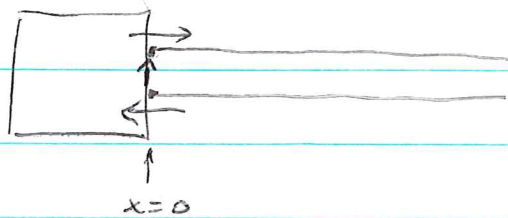


$$\frac{\partial I}{\partial x} = -\frac{\partial V}{\partial t}$$
$$\frac{\partial V}{\partial x} = -\frac{\partial I}{\partial t}$$

solutions $\quad V = \hat{V} e^{i(kx-\omega t)}$
$\qquad\qquad\quad I = \hat{I} e^{i(kx-\omega t)}$ $\qquad \hat{I} = \frac{\omega}{k}\hat{V} \qquad \hat{V} = \frac{\omega}{k}\hat{I}$

Thus $\omega = k$ and $\hat{V} = \hat{I}$; wave traveling to right
$\omega = -k$ and $\hat{V} = -\hat{I}$; " " " left.

Suppose the transmission line given for $x \geq 0$ is terminated at $x = 0$ by a circuit with impedance $Z(\omega)$:



Then $\quad I_{x=0}(t) = \hat{I}(\omega) e^{-i\omega t}$ $\quad$ and $\quad \dfrac{\hat{V}(\omega)}{\hat{I}(\omega)} = -Z(\omega)$
$\qquad\quad V_{x=0}(t) = \hat{V}(\omega) e^{-i\omega t}$

because imp. is defined with I in

Suppose the line carries incoming & outgoing waves

$$V(x,t) = A e^{-i\omega x - i\omega t} + B e^{i\omega x - i\omega t}$$
$$I(x,t) = -A e^{-i\omega x - i\omega t} + B e^{i\omega x - i\omega t}$$

$\therefore \quad \hat{V} = A + B \qquad -Z = \dfrac{A+B}{-A+B} \qquad (Z-1)A = (Z+1)B$
$\qquad \hat{I} = -A + B$

Thus the reflection coefficient is

$$\boxed{\frac{B}{A} = \frac{Z-1}{Z+1}}$$

Check: If $Z \equiv 1$ there is no reflection. If $Z = \infty$ then $B = A$ so that $I = 0$ at $x = 0$.

<u>Question</u>: Is it possible to concoct a circuit which will go into an excited state and stay? Send a pulse down the transmission line and have some of its energy trapped in one of the states of the circuit.

Somehow this doesn't happen. One can store energy in the circuit temporarily, but whatever mechanism allows the energy in also allows it to escape. I'm assuming no resistors in the box.

It should be possible to construct examples with long delay times, so-called resonances.

September 1, 1980

The problem is to understand scattering by a system that can go into excited states. Yesterday I started with an electrical analogy; a one 1-dimensional transmission line, but I had trouble seeing the scatterer.

What I was trying to do is take the Hilbert space of the line, direct sum with the Hilbert space of the scatterer. This obviously won't work because the total Hilbert space will split into bound states + scattering states. On the latter time-evolution is unitary, hence the reflection coefficient is of abs. value 1, so there is only elastic scattering.

I should take the tensor product of the Hilbert space of the line with the Hilbert space of the scatterer.

Let's try to construct a simple example which can be completely calculated. We need H_e on $\mathcal{H}_e$ to describe the particle being scattered and H_s on $\mathcal{H}_s$ to describe the scatterer.

There are two simple models for $H_e, \mathcal{H}_e$. The first is $\mathcal{H}_e = L^2(\mathbb{R})$ with basis $|k\rangle$, $k \in \mathbb{R}$ given by

$$\langle x|k\rangle = e^{-ikx}$$

and with $H_e|k\rangle = |k\rangle$. Thus

$$\langle x|e^{-itH_e}|k\rangle = \langle x|e^{-itk}|k\rangle = e^{ik(x-t)}$$

and so e^{-itH_0} represents translation on the line. This model describes a transmission line over $x \geq 0$

with an open (or short-circuit) termination at $x=0$.

(Review how this goes. Open-termination means we have the bdry condition $I(0,t)=0$, and it means that we can identify a solution of the DE'S

$$\frac{\partial E}{\partial t} = -\frac{\partial I}{\partial x} \qquad \frac{\partial I}{\partial t} = -\frac{\partial E}{\partial x}$$

over $x \geq 0$ with a solution on the whole line such that E is even and I is odd. Now $E+I$ is killed by $\frac{\partial}{\partial t}+\frac{\partial}{\partial x}$ so it is a function of $x-t$. $\therefore$

$$E+I = f(x-t)$$
$$E-I = g(x+t)$$

$$E = \frac{1}{2}\left(f(x-t)+g(x+t)\right)$$
$$I = \frac{1}{2}\left(f(x-t)-g(x+t)\right)$$

and our symmetry condition is that $g(x)=f(-x)$. Thus a solution is simply a function $f(x)$ on the line and time-evolution is just $f(x) \longmapsto f(x-t)$.)

The difficulty with the above model is that the Hamiltonian is not bounded below. Perhaps better is the radial Schroedinger equation for $l=0$ free particles. Here $\mathcal{H}_e \cong L^2(\mathbb{R}_{\geq 0})$ and H_e is

$$H_e = -\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2} \qquad \text{with } \psi=0 \text{ at } x=0$$

Thus a basis of eigenfns is $\sin kx$, $k>0$ and the energy is

$$\varepsilon_k = \frac{\hbar^2 k^2}{2m}$$

Next we need H_S on $\mathcal{H}_S$. The simplest model appears to be a 2-state system with base states $|\pm\rangle$ and $H_S|\pm\rangle = \pm\varepsilon|\pm\rangle$. Let

$$H_0 = H_e \otimes 1 + 1 \otimes H_S \qquad \text{on } \mathcal{H}_e \otimes \mathcal{H}_S$$

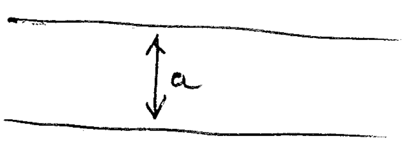
This has the basis $|k\rangle \otimes |\alpha\rangle$ $\quad \alpha = \pm$ with the eigenvalues

$$H_0\left(|k\rangle \otimes |\alpha\rangle\right) = \left(\varepsilon_k + \varepsilon_\alpha\right)|k\rangle \otimes |\alpha\rangle .$$

Let's take the radial Schr. case where $\varepsilon_k = k^2$ and $k > 0$. Then the spectrum of H_0 is the union of $\{-\varepsilon + k^2 \,/\, k \geq 0\}$ and $\{+\varepsilon + k^2 \,/\, k \geq 0\}$. So what we have is a "channel" beginning at $-\varepsilon$ and a second "channel" opening up at the energy $+\varepsilon$.

(Let's recall the old example of channels from wave-guides: I want to solve $\left(\frac{\partial^2}{\partial t^2} - \Delta\right) u = 0$ in the plane (say) with ~~period~~ a in the y-direction. Eigenfns. are

$$e^{i(k_x x + k_y y - \omega t)}$$



where $\quad \omega^2 = k_x^2 + k_y^2$

Now periodicity forces $k_y = \frac{2\pi}{a} n$ with $n \in \mathbb{Z}$ so

$$\omega^2 = k_x^2 + \left(\frac{2\pi}{a}\right)^2 n^2 \qquad n^2 = 0, 1, 4, \ldots$$

Thus with $n = 0$ one has the basic transmission line which is the only transmitting mode for $\omega^2 \leq \left(\frac{2\pi}{a}\right)^2$. At this point a new channel open up.

Actually it appears that this example occurs for the Schroed. equation $\left(i\frac{\partial}{\partial t} - \Delta\right)\psi = 0$. $\mathcal{H}_e$ describes the x dependence, while $\mathcal{H}_S$ gives the discrete modes in the y-direction.)

Finally we need a simple interaction Hamiltonian. So far we have wave functions

$$\psi = \begin{pmatrix} \psi_\uparrow \\ \psi_\downarrow \end{pmatrix} \quad \text{in } L^2(\mathbb{R}_{\geq 0})^2$$

$$H_0 = \begin{pmatrix} -\Delta + \varepsilon & 0 \\ 0 & -\Delta - \varepsilon \end{pmatrix}$$

so we need

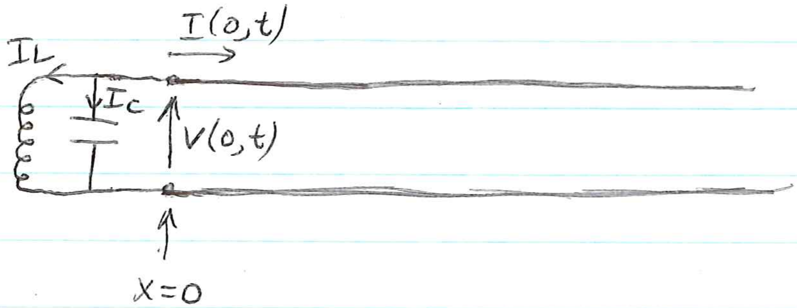
$$H_{int} = \begin{pmatrix} A & B \\ B^* & D \end{pmatrix} \qquad A = A^*, \; D = D^*$$

to be a hermitian matrix of operators on $L^2(\mathbb{R}_{\geq 0})$. Simplest to have $A = D = 0$, and one would expect, if they are small relative to Δ, that $-\Delta + A$ is unitary equivalent to $-\Delta$, so that by a unitary equivalence we can make $A = D = 0$.

K

September 1, 1980

Consider a transmission line terminated by a tuned circuit:



A history of this system is given by functions I_L, I_C, [redacted] of t and function $I(x,t)$, $V(x,t)$ $x \geq 0$ satisfying the following equations.

$$\begin{cases} V(0,t) = L\dfrac{dI_L}{dt} \qquad I_C = C\dfrac{dV(0,t)}{dt} \\ I_L + I_C \; [redacted] = -I(0,t) \\ \dfrac{\partial V(x,t)}{\partial x} = -\dfrac{\partial I(x,t)}{\partial t} \qquad \dfrac{\partial I(x,t)}{\partial x} = -\dfrac{\partial V(x,t)}{\partial t} \end{cases}$$

Thus these histories form a [redacted] vector space with time evolution. We can eliminate I_C and replace the first three equations by the following two

$$(*) \quad \begin{cases} L\dfrac{dI_L}{dt} = V(0,t) \\ I_L + C\dfrac{dV(0,t)}{dt} = -I(0,t) \end{cases}$$

What I want now is a "basis" for the vector space of [redacted] these states, so I know exactly what can be assigned independently. When I have a basis I know exactly what [redacted] independent parameters are needed to describe my states.

But if $V(x,0)$, $I(x,0)$ are given for $x \geq 0$ then

$$\frac{\partial V(0,t)}{\partial t} = - \frac{\partial I(0,t)}{\partial x}$$

can be used with $(*)$ to solve for I_L and $\frac{dI_L}{dt}$ at $t=0$. In other words it appears that V, I can be assigned arbitrar(il)y at $t=0$. NO In more detail

$$I_L = -C \frac{\partial V}{\partial t}(0,t) - I(0,t)$$

$$= C \frac{\partial I}{\partial x}(0,t) - I(0,t)$$

$$\frac{dI_L}{\partial t} = C \frac{\partial}{\partial x}\left(\frac{\partial I}{\partial t}\right)(0,t) - \frac{\partial I}{\partial t}(0,t)$$

$$= \boxed{-C \frac{\partial V}{\partial x^2}(0,t) + \frac{\partial V}{\partial x}(0,t) \quad = \frac{1}{L} V(0,t)}$$

This last equation seems to show $V(x,t)$ cannot be arbitrarily assigned at $t=0$.

? ?

The box equation is a boundary condition required to pin down the solution of the transmission line equations. Simpler examples:

open termination : $I(0,t) = 0$

short-circuit " : $V(0,t) = 0$

Conclusion: This system doesn't have a bound state, i.e. a solution with $V = I = 0$ on the transmission line.

Let's compute the reflection coefficient carefully:

$$V(x,t) = A e^{-i\omega(x+t)} + B e^{i\omega(x-t)}$$

$$I(x,t) = -A e^{-i\omega(x+t)} + B e^{i\omega(x-t)}$$

$$V(0,t) = (A+B) e^{-i\omega t}$$

$$I(0,t) = (-A+B) e^{-i\omega t}$$

$$L \hat{I}_L (-i\omega) = \text{[redacted]} (A+B)$$ $\qquad I_L = \hat{I}_L e^{-i\omega t}$

[redacted]

$$\hat{I}_L + C(A+B)(-i\omega) = -(-A+B)$$

[redacted]

$$\hat{I}_L + C(-i\omega) L(-i\omega) \hat{I}_L = -(-A+B)$$

$$\underbrace{\hat{I}_L}_{\frac{A+B}{Ls}} (1 + CLs^2) = -(-A+B) \qquad s = -i\omega$$

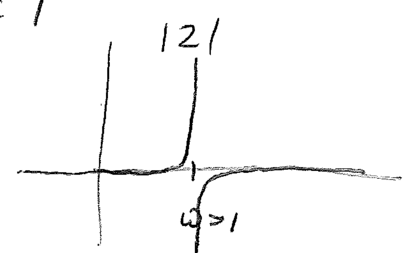
$$\frac{1}{Z} = \frac{1+CLs^2}{Ls} = \frac{A-B}{A+B}$$

which leads to familiar result:

$$\frac{B}{A} = \frac{Z-1}{Z+1} \quad \text{where} \quad Z = \frac{1}{\frac{1}{Ls} + Cs} = \frac{Ls}{CLs^2+1}$$

Interesting special case. Suppose L is very small and C is very large so that $LC = 1$, i.e. resonance occurs at [redacted] $\omega = \pm 1$

$$Z = \frac{-iL\omega}{1-\omega^2}$$



This means the reflection coefficient $\frac{B}{A}$ is close to -1 for all ω values away from ± 1. At these points it sharply shifts to $+1$.

Interpretation - small L means the inductance passes even medium frequencies, large C means the capacitance passes even medium frequencies, so one would think the circuit is like a short-circuit termination. This is true except at the resonant frequency where it suddenly converts to an open circuit.

Suppose we allow only outgoing waves on the line, or better, we want to regard the line as a load on the tuned circuit. The condition that only outgoing waves occur on the line is that $V(x,t) = I(x,t)$. Then from (*) we get

$$\begin{cases} CL\dfrac{d^2 I_L}{dt^2} + I_L = -I(0,t) \\ L\dfrac{dI_L}{dt} = V(0,t) \\ I(0,t) = V(0,t) \end{cases}$$

Eliminate I, V to get

$$CL\frac{d^2 I_L}{dt^2} + L\frac{dI_L}{dt} + I_L = 0$$

which has independent solutions e^{st} where

$$CLs^2 + Ls + 1 = 0$$

$$s = \frac{-L \pm \sqrt{L^2 - 4LC}}{2LC}$$

Recall we are assuming that $LC = 1$ and that L is small, hence C is large.

$$s = \frac{-L \pm \sqrt{L^2-4}}{2} = \frac{-L \pm 2i\sqrt{1-\frac{L^2}{4}}}{2}$$

$$\approx \frac{1}{2}\left(-L \pm 2i\left(1-\frac{L^2}{8}\right)\right)$$

$$\approx -\frac{L}{2} \pm i \quad + O(L^2)$$

This means the transmission line acts like a 1-ohm load on the tuned circuit, and we have damped oscillations decaying like $e^{-\frac{L}{2}t}$.

What we see on the transmission line is

$$V(x,t) = I(x,t) = Ls e^{s(t-x)} \qquad s = -\tfrac{L}{2} \pm i + O(L^2)$$

Such things are exponentially decaying in t for x fixed but exponentially increasing in x for fixed t. The idea is that what we see at x at time $t=0$ must come from something in the tuned circuit at $t=-x$.

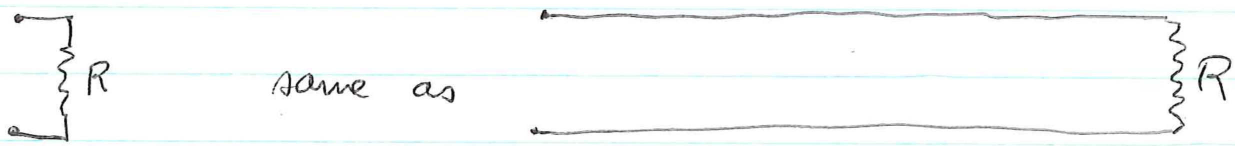
Similarly we could send in

$$V(x,t) = -I(x,t) = e^{-s(t+x)} \qquad s = -\tfrac{L}{2} \pm i + \dots$$

and all the power would be absorbed by the tuned circuit, so that there is no out-going wave.

September 2, 1980

Nyquist relation: This gives the frequency distribution of noise in a resistor at a given temperature T. The idea is that experimentally a resistor of resistance R behaves identically



as a length of transmission line of impedance R terminated by the resistance R, so there is no reflection. But the transmission line can store energy in its various modes; it behaves likes the ether in a cavity. Hence at the temperature T these modes must be excited like in black body radiation. Thus the resistor R has to act like a black-body absorbing + emitting energy into the transmission line.

Maybe a better way of expressing this is that a resistor at a given temperature is equivalent to a transmission line of the same impedance with all the modes excited like in black body radiation. What I want to compute is the intensity of a given frequency at the beginning of the transmission line. This is related to the problem of intensity for a black body which I never understood.

September 3, 1980

Let's consider a transmission line as a simplified model of the EM field. Suppose the line is periodic of length ℓ. Then we have modes:

$$V(x,t) = e^{i(kx-\omega t)} \qquad k \in \frac{2\pi}{\ell}\mathbb{Z}, \qquad \omega = c|k|$$

According to Planck, etc. if the transmission line is at temperature T, then each mode of frequency ω has the average energy

$$(n_\omega + \tfrac{1}{2})\hbar\omega \qquad n_\omega = \frac{1}{e^{\beta\hbar\omega}-1}$$

Let's compute the number of modes in the frequency range $(\omega, \omega+d\omega)$. It is the number of $n \in \mathbb{Z}$ with

$$\omega < c\left|\frac{2\pi}{\ell}n\right| < \omega + d\omega$$

$$\frac{\ell}{2\pi c}\omega < |n| < \frac{\ell}{2\pi c}(\omega+d\omega)$$

Hence

$$\text{no. of modes with freq. in } (\omega,\omega+d\omega) = \frac{\ell}{\pi c}d\omega$$

average [redacted] energy stored in modes with freq. in $(\omega,\omega+d\omega)$ $= (n_\omega+\frac{1}{2})\hbar\omega \cdot \frac{\ell}{\pi c}d\omega$

Notice that [redacted] these are proportional to ℓ, hence we can let $\ell \to \infty$ provided we work per unit length:

$$\text{no. of modes/unit length in range } \omega, \omega+d\omega = \frac{1}{\pi c}d\omega$$

$$\text{average energy in these modes} = (n_\omega+\tfrac{1}{2})\hbar\omega\frac{1}{\pi c}d\omega$$

(Review what happens in [redacted] 3 dims. – the number of modes per unit volume [redacted] with wave number in a [redacted] volume d^3k is

$$\frac{1}{l^3}\ \underset{\substack{\uparrow \\ \text{2 polar.} \\ \text{for light}}}{2}\ \frac{d^3k}{\underset{\substack{\uparrow \\ \text{vol. per} \\ \text{mode}}}{\left(\frac{2\pi}{l}\right)^3}} = 2\frac{d^3k}{(2\pi)^3}\ [redacted] = 2\cdot\frac{4\pi|k|^2 d|k|}{(2\pi)^3}$$

$$= [redacted]\ \frac{8\pi\ \omega^2 d\omega}{(2\pi c)^3}$$

so the energy density in the range $(\omega, \omega+d\omega)$ is

$$\frac{8\pi\omega^2 d\omega}{(2\pi c)^3}\left(\frac{\hbar\omega}{e^{\beta\hbar\omega}-1} + \frac{1}{2}\hbar\omega\right)$$

except that the $\frac{1}{2}\hbar\omega$ factor is usually omitted.)

Now what I want to do is to measure the voltage at a given point x on the transmission line assuming that it is being maintained at the temperature T. I am working classically, so there will be numbers $\hat{V}_k$ so that

$$(*)\quad \begin{cases} V(x,t) = \sum_k \hat{V}_k e^{i(kx-\omega t)} \\ I(x,t) = \sum_k \hat{I}_k e^{i(kx-\omega t)} \end{cases}$$

[redacted] For these to be a solution of

$$\frac{\partial V}{\partial x} = -L\frac{\partial I}{\partial t} \qquad \frac{\partial I}{\partial x} = -C\frac{\partial V}{\partial t}$$

means

$$\hat{V}_k = L\frac{\omega}{k}\hat{I}_k \qquad \hat{I}_k = C\frac{\omega}{k}\hat{V}_k$$

so

$$L\frac{\omega}{k}C\frac{\omega}{k} = 1 \qquad \omega = \underbrace{\frac{1}{\sqrt{LC}}}_{\text{speed}}|k|$$

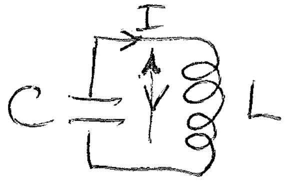
(Impedance is $\sqrt{\frac{L}{C}}$.)

What is the energy of the state (*)? It should be a sum of the energies of each of the modes. It is probably worth-while to go over the energy concept for electrical circuits. We know that the energy stored in a capacitor of capacitance C is $\frac{1}{2}CV^2$ where $V =$ voltage. This is because the work dW done is moving dQ thru a voltage drop of V is $V\,dQ$. Thus

$$dW = V\,dQ = CV\,dV$$

$$\text{or} \quad W = \tfrac{1}{2}CV^2$$

The energy stored in an inductor is $\frac{1}{2}LI^2$. To check this consider a tuned circuit



$$V = L\frac{dI}{dt} \qquad C\frac{dV}{dt} = -I$$

$$LC\frac{d^2V}{dt^2} = -L\frac{dI}{dt} = -V$$

Thus the resonant frequency is $LC\omega^2 = 1$ or $\omega = \frac{1}{\sqrt{LC}}$.

Solution $V = \mathrm{Re}(\hat{V}e^{i\omega t})$
$I = \mathrm{Re}(\hat{I}e^{i\omega t})$ $\qquad \hat{V} = Li\omega\hat{I}$

Assume $\hat{V}$ real

$$\frac{1}{2}CV^2 = \frac{1}{2}C\left(\mathrm{Re}\,\hat{V}e^{i\omega t}\right)^2 = \frac{1}{2}C\hat{V}^2\cos^2\omega t$$

$$\frac{1}{2}LI^2 = \frac{1}{2}L\left(\mathrm{Re}\,\frac{\hat{V}}{Li\omega}e^{i\omega t}\right)^2 = \frac{1}{2}L\frac{\hat{V}^2}{L^2\omega^2}\sin^2\omega t = \frac{1}{2}C\hat{V}^2\sin^2\omega t$$

The total energy is constant; this implies $\frac{1}{2}LI^2$ must be the magnetic energy in the inductance. Notice that the total energy is

$$\frac{1}{2}C|\hat{V}|^2 = \frac{1}{2}L\left|\hat{I}\right|^2$$

In the case of the transmission line the total energy is clearly

$$\frac{1}{2}\int C V^2 dx + \frac{1}{2}\int L I^2 dx.$$

We want to compute this when

$$V = \mathrm{Re}\, \underbrace{\sum_k \hat{V}_k e^{i(kx-\omega_k t)}}_{W} \qquad \omega_k = \frac{1}{\sqrt{LC}}|k|$$

$$I = \mathrm{Re}\, \underbrace{\sum_k \hat{I}_k e^{i(kx-\omega_k t)}}_{J} \qquad \hat{V}_k = \sqrt{\frac{L}{C}}\,\frac{|k|}{k}\,\hat{I}_k$$

Set $t=0$ to simplify

$$\frac{1}{2}\int_0^l C V^2 dx = l\,\frac{1}{8} C \int \left(W^2 + 2W\bar{W} + \bar{W}^2\right) dx$$

$$= \frac{l}{8} C \left\{ \sum_k \hat{V}_k \hat{V}_{-k} + 2\sum_k |\hat{V}_k|^2 + \sum_k \bar{\hat{V}}_k \bar{\hat{V}}_{-k} \right\}$$

$$\frac{1}{2}\int_0^l L I^2 dx = \frac{l}{8} L \int \left(J^2 + 2J\bar{J} + \bar{J}^2\right) dx$$

$$= \frac{l}{8} L \left\{ \sum_k \hat{I}_k \hat{I}_{-k} + 2\sum_k |\hat{I}_k|^2 + \sum_k \bar{\hat{I}}_k \bar{\hat{I}}_{-k} \right\}$$

since $\hat{V}_k = \pm\sqrt{\frac{L}{C}}\hat{I}_k$ with opposite signs for $\pm k$, when added only the middle terms survive to give

$$\frac{\text{energy}}{l} = \frac{1}{4}\sum C|\hat{V}_k|^2 + L|\hat{I}_k|^2$$

$$= \frac{1}{2} L \sum |\hat{I}_k|^2 \quad \text{or} \quad \frac{1}{2} C \sum |\hat{V}_k|^2$$

Conclusion: The energy is the sum of the energies of the different modes. The energy of the mode with $\quad V = \hat{V}_k e^{i(kx - \omega_k t)}$ is $\frac{1}{2} C |\hat{V}_k|^2 l$.

It seems like we are going to get infinite energy density as $l \to \infty$. Be careful.

Let's review the set-up. We have the transmission line of length l with all modes excited in the way dictated by the Boltzmann distribution. There are problems with this because the energy of a mode is $\hbar\omega$ by Planck and proportional to intensity classically. ??