January 29, 1979: interpreting $\det(1-G_k^+V)$ as a char. poly. ratio

Let try to understand $\mathcal{D}/\mathcal{D}'$ in the continuous case where the wave equation is

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2} - Vu$$

where V has support in $[-b,b]$. Start with smooth solutions of the wave equations whose support near $t=0$ is contained in $(-b,b)$. Closing this up suitably, possibly weakening the vanishing at $-b, b$ [redacted] should give us $\mathcal{D}/\mathcal{D}'$.

To simplify suppose $V=0$. Then we [redacted] know any solution is of the form

$$u(x,t) = f(x-t) + g(x+t)$$

To [redacted] solve the Cauchy problem we solve

$$\frac{\partial u}{\partial t}(x,0) = -f'(x) + g'(x)$$

$$\frac{\partial u}{\partial x}(x,0) = f'(x) + g'(x)$$

for f', g' and then we integrate to get f, g. This determines f, g up to arb. additive constants, but since the Cauchy data is compact we can fix the constants by requiring that f, g vanish to the [redacted] right

Recall the energy norm

$$\frac{1}{2}\int \left(\left|\frac{\partial u}{\partial t}\right|^2 + \left|\frac{\partial u}{\partial x}\right|^2 + V|u|^2 \right) dx$$

makes Cauchy data into a pre-Hilbert space (assuming

$-\frac{d^2}{dx^2}+V \geq 0$).

Consider a solution $u(x,t) = f(x-t)+g(x+t)$ which near $t=0$ is supported in $(-b,b)$. Then f, g are supported for $x<b$ and for $x \leq -b$ they are constant of opposite sign. Now let time evolve: $t>0$. After a while the wave begins to leave $[-b,b]$ and we have to somehow project it back into this interval. So it is clear that we want to truncate f' at $x=b$ and g' at $x=-b$, in some sense.

So let us consider solutions $u(x,t) = f(x-t)+g(x+t)$ such that f is smooth, defined on $[-b,b]$, and vanishes near $-b$; similarly g is smooth, defined on $[-b,b]$ and vanishes near b. These solutions obviously get mapped into themselves under time evolution. Moreover, one sees that $u \mapsto f', g'$ agrees with what we expect from the restriction of $\frac{d}{dx}$ to $L^2(-b,b)$ with vanishing endpoint conditions.

Somehow we are looking at the wave equation on $[-b,b]$ with the boundary conditions

$$\frac{\partial u}{\partial x}+\frac{\partial u}{\partial t}=0 \qquad \text{at} \qquad x=b$$

$$\frac{\partial u}{\partial x}-\frac{\partial u}{\partial t}=0 \qquad \text{at} \qquad x=-b$$

This makes sense when $V \neq 0$.

<u>Question</u>: Suppose we solve the wave equation with these boundary conditions, can we suppose f, g vanish at the appropriate endpoints?

It seems that $u(x,t) =$ constant is not in the right form. Here $f' = g' = 0$, [redacted] so $f = g = 0$ if we require vanishing at the ends.

[redacted] So we seem to get the following continuous model for D/D'. Take solutions of the wave equation on $[-b, b]$ with the above boundary conditions. Now it is clear that under time evolution any solution [redacted] becomes [redacted] constant. Observe that on D/D' we have time evolution defined for $t \geq 0$.

January 30, 1979

We are considering the wave equation

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2} - Vu \qquad \text{on } \mathbb{R}$$

where $\operatorname{Supp}(V) \subset (-b, b)$. We have an analogue of $\mathcal{D}/\mathcal{D}'$ consisting of solutions of the wave equation $u(x,t)$ defined for [redacted] $|x| \leq b$, $t \geq 0$ satisfying the boundary conditions

$$\frac{\partial u}{\partial x} + \frac{\partial u}{\partial t} = 0 \qquad x = b$$

$$\frac{\partial u}{\partial x} - \frac{\partial u}{\partial t} = 0 \qquad x = -b$$

[redacted] To understand such solutions, we use Laplace transform in time:

$$\int_0^\infty e^{ikt} u(x,t)\,dt = \psi(x,k)$$

Here $s = k/i$, so $-s = ik$. Then ψ satisfies

[redacted] $$\left(\frac{d^2}{dx^2} - V\right)\psi = -k^2\psi - (-ik)\,[redacted]\,u(x,0) - \frac{\partial u}{\partial t}(x,$$

$$\frac{d\psi}{dx} - ik\psi = 0 \qquad \text{at } x = b$$

$$\frac{d\psi}{dx} + ik\psi = 0 \qquad \text{at } x = -b.$$

and ψ should be analytic in a half-plane $\operatorname{Im} k \geq a$.

For example if $V = 0$, then we have

$$\left(\frac{d^2}{dx^2} + k^2\right)\psi = -s\,u(x,0) - u_t(x,0)$$

and the Green's function for $\frac{d^2}{dx^2}+k^2$ with the above outgoing boundary conditions is $G_k^+(x,x') = \frac{e^{ik|x-x'|}}{2ik}$, so that

$$\psi(x,k) = \int_{-b}^{b} \frac{e^{ik|x-x'|}}{2ik} \left(-s\,u(x',0) - u_t(x',0)\right) dx'$$

January 31, 1979:

Yesterday I decided that $\mathcal{D}/\mathcal{D}'$ [redacted] in the continuous case should be identified with solutions $u(x,t)$, defined for $|x| \le b$, $t \ge 0$, of

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2} - q u \qquad \mathrm{Supp}(q) \subset (-b,b)$$

with the boundary conditions

$$\frac{\partial u}{\partial x} + \frac{\partial u}{\partial t} = 0 \qquad \text{on } x = b$$

$$\frac{\partial u}{\partial x} - \frac{\partial u}{\partial t} = 0 \qquad \text{on } x = -b.$$

[redacted] This is a well-posed problem, whereas the same problem with $t \le 0$ isn't. The point is that given Cauchy data, that is, smooth functions $\binom{u}{\dot u}$ on $[-b,b]$ such that

$$(*) \qquad \begin{aligned} \frac{\partial u}{\partial x} + \dot u &= 0 \qquad \text{on } x=b \\ \frac{\partial u}{\partial x} - \dot u &= 0 \qquad \text{on } x=-b \end{aligned}$$

one can extend this to Cauchy data given on $\mathbb{R}$ having zero incoming component for $|x| \ge b$, and solve the resulting problem on $\mathbb{R}$, and restrict to $[-b,b]$ to get the desired solution.

From now on I think of $\mathcal{D}/\mathcal{D}'$ as pairs $\binom{u}{\dot u}$ satisfying

(∗). The infinitesimal generator of the time-evolution semi-group on $\mathcal{D}/\mathcal{D}'$ is

$$\frac{\partial}{\partial t}\begin{pmatrix} u \\ \dot{u} \end{pmatrix} = \begin{pmatrix} \dot{u} \\ \frac{\partial^2 u}{\partial x^2} - \gamma u \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ \frac{\partial^2}{\partial x^2} - \gamma & 0 \end{pmatrix}\begin{pmatrix} u \\ \dot{u} \end{pmatrix}$$

One can see that it preserves the boundary conditions, e.g.

$$\frac{\partial}{\partial x}\dot{u} + \left(\frac{\partial^2 u}{\partial x^2} - \gamma u\right) = \frac{\partial}{\partial x}\left(\dot{u} + \frac{\partial u}{\partial x}\right) = 0 \qquad \text{near } x = b.$$

(Note that if one were to start by defining $\mathcal{D}/\mathcal{D}'$ to consist of $\binom{u}{\dot{u}}$ satisfying (∗) with $\frac{\partial}{\partial t}$ as above, then it is not clear that $\frac{\partial}{\partial t}$ can be integrated. This requires some sort of existence thm. for PDE's).

Now the problem becomes to interpret the "characteristic polynomial of the Lax-Phillips semi-group" as being $\det(1 - G_k^+ q)$.

Recall what we did for

$$\frac{d\psi}{dx} = ik\psi + V\psi \qquad \text{on} \quad |x| \le b$$

$$\psi = 0 \qquad \text{near} \quad -b.$$

Here the wave equation was

$$-\frac{\partial u}{\partial t} = \frac{\partial u}{\partial x} - Vu$$

(because $u(x,t) = \int e^{-ikt}\psi(x,k)\, dk/2\pi$). Thus $\mathcal{D}/\mathcal{D}'$ is the functions on $[-b, b]$ vanishing near $x = -b$ with the infinitesimal generator

$$\frac{\partial}{\partial t} = -\frac{\partial}{\partial x} + V$$

which was denoted $-A$ before. Then

$$1 - G_k^+ V = (A_0 - ik)^{-1}(A - ik)$$

$$= (1 - ikA_0^{-1})^{-1}(A_0^{-1}A - ikA_0^{-1})$$

where both factors have determinants. If also $\det(A_0^{-1}A) \neq 0$, then we get

$$\det(1 - G_k^+ V) = \underbrace{\det(1 - ikA_0^{-1})^{-1}}_{=1} \cdot \det(A_0^{-1}A)\ \det(1 - ikA^{-1})$$

which in some sense interprets $\det(1 - G_k^+ V)$ [redacted] as "$\det(A - ik)$".

So the program is to do something similar for

$$+A = \begin{pmatrix} 0 & 1 \\ \frac{d^2}{dx^2} - g & 0 \end{pmatrix} \qquad \text{on pairs } \begin{pmatrix} u \\ \dot{u} \end{pmatrix}$$

satisfying $(*)$ on 523.

This time A is the actual infinitesimal generator. Begin by computing

$$(A_0 + ik)^{-1} = \begin{pmatrix} +ik & 1 \\ \frac{d^2}{dx^2} & +ik \end{pmatrix}^{-1}$$

on this space $\mathcal{D}/\mathcal{D}'$.

$$\begin{pmatrix} ik & 1 \\ \frac{d^2}{dx^2} & ik \end{pmatrix}\begin{pmatrix} u \\ \mathring{u} \end{pmatrix} = \begin{pmatrix} f \\ \mathring{f} \end{pmatrix}$$

$$\cdot ik \quad \begin{cases} iku + \mathring{u} = f \\ \frac{d^2}{dx^2}u + ik\mathring{u} = \mathring{f} \end{cases}$$

$$\left(\frac{d^2}{dx^2} + k^2\right)u = \mathring{f} - ikf$$

$$u = c_1 e^{ikx} + c_2 e^{-ikx} + \int G_k^+ (\mathring{f} - ikf)\, dx'$$

Now suppose $f, \mathring{f}$ supported inside $(-b, b)$. Then

$$iku + \mathring{u} = 0 \quad \text{at } x = b$$

$$\frac{\partial u}{\partial x} + \mathring{u} = 0 \quad \text{at } x = b$$

so near $x=b$.

$$\frac{\partial u}{\partial x} - iku = 0 \quad \text{at } x = b, \text{ so } u = \text{const } e^{ikx}$$

similarly

$$\frac{\partial u}{\partial x} = \mathring{u} = -iku \quad \text{at } x = b$$

so $u = \text{const.}\, e^{-ikx}$ near $x = b$. Thus

$$u = \int G_k^+ (\mathring{f} - ikf)\, dx'$$

$$\mathring{u} = f - iku = f - ik \int G_k^+ (\mathring{f} - ikf)\, dx'$$

or

$$\begin{pmatrix} u \\ \mathring{u} \end{pmatrix} = \begin{pmatrix} -ikG_k^+ & G_k^+ \\ 1 - k^2 G_k^+ & -ikG_k^+ \end{pmatrix}\begin{pmatrix} f \\ \mathring{f} \end{pmatrix}$$

$$= \begin{pmatrix} +ik & -1 \\ -\frac{d^2}{dx^2} & +ik \end{pmatrix}\left(-G_k^+\right)\begin{pmatrix} f \\ \mathring{f} \end{pmatrix}$$

(Cramer's Rule)

I ought to check that if $\begin{pmatrix} f \\ \mathring{f} \end{pmatrix}$ satisfies the boundary conditions, then so does $\begin{pmatrix} u \\ \mathring{u} \end{pmatrix}$. The reason this should be so, is that if we take the flow-line beginning with $\begin{pmatrix} f \\ \mathring{f} \end{pmatrix}$, and take Laplace transform:

$$\int_0^\infty e^{ikt}\, e^{tA_0} \begin{pmatrix} f \\ \mathring{f} \end{pmatrix} dt = -(A_0+ik)^{-1} \begin{pmatrix} f \\ \mathring{f} \end{pmatrix}$$

which is $\begin{pmatrix} u \\ \mathring{u} \end{pmatrix}$. We know this is the case when $\begin{pmatrix} f \\ \mathring{f} \end{pmatrix}$ has support inside $(-b, b)$, so it's enough to check when $\mathring{f} = -\frac{df}{dx}$ and f vanishes near $-b$, and when $\mathring{f} = \frac{df}{dx}$, f vanishes near $x = b$. Assume for the moment there is no problem – it's messy.

Next

$$(A+ik)(A_0+ik)^{-1} = \begin{pmatrix} ik & 1 \\ \frac{d^2}{dx^2} - q & ik \end{pmatrix} \begin{pmatrix} ik & -1 \\ -\frac{d^2}{dx^2} & ik \end{pmatrix} (-G_k^+)$$

$$= \begin{pmatrix} -k^2 - \frac{d^2}{dx^2} & 0 \\ -ikq & -\frac{d^2}{dx^2} + q - k^2 \end{pmatrix} (-G_k^+)$$

$$= \begin{pmatrix} 1 & 0 \\ +ikqG_k^+ & 1 - qG_k^+ \end{pmatrix}$$

So now we see that

$$\det\left((A+ik)(A_0+ik)^{-1}\right) = \det(1 - qG_k^+)$$

Consider the discrete case again. The model for $\mathcal{D}/\mathcal{D}'$ consists of $\binom{u_0}{u_1}$ defined on $[-N, N]$ where the support of the perturbation is [redacted] inside. Picture an element

To calculate $u(2)$, we adjoin $u(N+1, 1)$, $u(-N-1, 1)$ in the way indicated and use the wave equation

$$\frac{u(2) + u(0)}{2} = Hu(1)$$

Assuming that $(Hy)_N = \frac{1}{2} y_{N+1} + b_N y_N + a_{N-1} y_{N-1}$ and analogously at the other end, so that $H - H_0 = -V$ has its image inside $[-N, N]$, one sees that one gets the same result for $u(2)$ if one changes $u(N, 0) = u(N+1, 1)$ to zero, and similarly at the other end. Let τ be the operator [redacted] on sequences supported in $[-N, N]$ which kills the $-N, N$ components. Then we have

$$u_2 = 2Hu_1 - \tau u_0$$

where H is restricted to $[-N, N]$ by forcing other components to be zero, e.g.

$$H_0 = \frac{1}{2}(u_0 + u_0^*)$$

u_0 = right shift throwing away y_N setting 0 in $-N$.

[redacted] Thus time-evolution on $\mathcal{D}/\mathcal{D}'$ is given by the operator

$$\vec{\mathcal{U}}\begin{pmatrix} u_0 \\ u_1 \end{pmatrix} = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -\tau & 2H \end{pmatrix}\begin{pmatrix} u_0 \\ u_1 \end{pmatrix}$$

We want to compute $\det(I - z\vec{\mathcal{U}})$. Now

$$I - z\vec{\mathcal{U}} = \begin{pmatrix} 1 & -z \\ z\tau & 1-2zH \end{pmatrix}$$

$$(I - z\vec{\mathcal{U}})(I - z\vec{\mathcal{U}}_0)^{-1} = I + 2z\begin{pmatrix} 0 & 0 \\ 0 & V \end{pmatrix}(I - z\vec{\mathcal{U}}_0)^{-1}$$

$$= \begin{pmatrix} 1 & 0 \\ 2zV(*) & 1+2zV(*) \end{pmatrix}$$

Because of τ it is not immediately clear how to calculate $(1 - z\vec{\mathcal{U}}_0)^{-1}$.

However suppose one assumes $|z| < 1$ and computes the ℓ^2 inverse on $\mathbb{Z}$, not the finite-dimensional $\mathcal{D}/\mathcal{D}'$. In this case

$$I - z\vec{\mathcal{U}}_0 = \begin{pmatrix} 1 & -z \\ z & 1 - z\mathcal{U}_0 - z\mathcal{U}_0^{-1} \end{pmatrix}$$

and we can compute the inverse by Cramer:

$$\det = 1 - z\mathcal{U}_0 - z\mathcal{U}_0^{-1} + z^2 = (1 - z\mathcal{U}_0)(1 - z\mathcal{U}_0^{-1})$$

Thus

$$(I - z\vec{\mathcal{U}}_0)^{-1} = \begin{pmatrix} 1 - z\mathcal{U}_0 - z\mathcal{U}_0^{-1} & z \\ -z & 1 \end{pmatrix} \frac{1}{(1 - z\mathcal{U}_0)(1 - z\mathcal{U}_0^{-1})}$$

So it follows that

$$(I - z\vec{U})(I - z\vec{U}_0)^{-1} = I - \begin{pmatrix} 0 & 0 \\ -zV & V \end{pmatrix} \left(-2z\,(1-zU_0)^{-1}(1-zU_0)^{-1}\right)$$

But we saw that

$$H_0 - \lambda = -\frac{1}{2z}(1 - zU_0)(1 - zU_0^{-1}) \quad \text{[redacted]}$$

so that

$$(I - z\vec{U})(I - z\vec{U}_0)^{-1} = \text{[redacted]} \begin{pmatrix} 1 & 0 \\ -zVG_z^+ & 1 - VG_z^+ \end{pmatrix}$$

It is interesting to do this calculation on the line in the continuous case. [redacted]

$$\text{[redacted]} \quad \frac{\partial}{\partial t}\begin{pmatrix} u \\ \dot{u} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ \frac{d^2}{dx^2} - q & 0 \end{pmatrix}\begin{pmatrix} u \\ \dot{u} \end{pmatrix}$$

Thus

$$A = \begin{pmatrix} 0 & 1 \\ \frac{d^2}{dx^2} - q & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ \frac{d^2}{dx^2} & 0 \end{pmatrix} - \begin{pmatrix} 0 & 0 \\ q & 0 \end{pmatrix}$$

$$= \quad A_0 \quad - \quad V$$

Moreover

$$(A_0 + ik)^{-1} = \begin{pmatrix} ik & 1 \\ \frac{d^2}{dx^2} & ik \end{pmatrix}^{-1} = \begin{pmatrix} ik & -1 \\ -\frac{d^2}{dx^2} & ik \end{pmatrix} \underbrace{\left(-k^2 - \frac{d}{dx^2}\right)^{-1}}_{-G_k^+}$$

where the inverse is computed with outgoing boundary conditions (= inverse in L^2 when $\operatorname{Im} k > 0$). Thus

$$(A + ik)(A_0 + ik)^{-1} = I - \begin{pmatrix} 0 & 0 \\ q & 0 \end{pmatrix}\begin{pmatrix} ik & -1 \\ -\frac{d^2}{dx^2} & ik \end{pmatrix}(-G_k^+)$$

$$= \begin{pmatrix} 1 & 0 \\ ikqG_k^+ & 1 - qG_k^+ \end{pmatrix}$$

which shows that

$$\det\left((A+ik)(A_0+ik)^{-1}\right) = \det\left(1-g\,G_k^+\right)$$
$$= \det\left((H+k^2)(H_0+k^2)^{-1}\right)$$

Notice that this calculation doesn't yield anything about assigning a determinant to $(A_0+ik)^{-1}$. Nor do we have an interpretation of $\det\left((A+ik)(A_0+ik)^{-1}\right)$ as the characteristic polynomial of some endomorphism of cohomology.