October 12, 1977

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Instantons – Coleman's lectures.

Begin with Feynman's formula for $H = \frac{p^2}{2} + V(x)$:

$$\langle x | e^{-TH/\hbar} | x' \rangle = \int\limits_{\substack{x(0)=x' \\ x(T)=x}} \mathcal{D}x \; e^{-\frac{1}{\hbar}\int_0^T [\frac{1}{2}\dot{x}^2 + V(x)]\,dt}$$

The left side is $\sum_n e^{-TE_n/\hbar} \phi_n(x) \overline{\phi_n(x')}$

which has leading term $e^{-TE_0/\hbar} \phi_0(x) \overline{\phi_0(x')}$ [redacted] as $T \to +\infty$. Use stationary phase to get at the right side – think of $\hbar \to$ [redacted] 0. Critical points of the exponential are given by Euler DE

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{x}} - \frac{\partial L}{\partial x} = \ddot{x} - V'(x) = 0$$

which describes classical motion of a particle in the potential $-V(x)$. (Actually – one maybe should be suspicious because you want to minimize the exponential in the Feynman integral not just find stationary points).

Let's assume there is one classical path $\bar{x}(t)$ with boundary values x' at 0, x at T and expand around it: $x(t) = \bar{x}(t) + y(t)$

$$\frac{1}{2}\dot{x}^2 + V(x) = \frac{1}{2}\dot{\bar{x}}^2 + V(\bar{x}) + \dot{\bar{x}}\dot{y} + V'(\bar{x})y + \frac{1}{2}\dot{y}^2 + \frac{1}{2}V''(\bar{x})y^2 + O(y^3)$$

Then

$$\int_0^T [\tfrac{1}{2}\dot{x}^2 + V(x)]\,dt = \int_0^T [\tfrac{1}{2}\dot{\bar{x}}^2 + V(\bar{x})]\,dt + 0 + \int_0^T \tfrac{1}{2}(\dot{y}^2 + V''(\bar{x})y^2)\,dt + \dots$$

where the linear term disappears because the first variation around $\bar{x}$ is zero. So we get

$$\langle x| e^{-TH/\hbar} |x'\rangle \approx e^{-\frac{1}{\hbar}S(\bar{x})} \int_{\substack{y(0)=0 \\ y(T)=0}} \mathcal{D}y \; e^{-\frac{1}{\hbar}\int_0^T \frac{1}{2}\left(\dot{y}^2 + V''(\bar{x})y^2\right) dt}$$

and if this works like standard stationary phase problems, the error will be a multiplicative factor $(1+O(\hbar))$. The path integral on the right is the path integral belonging to the oscillator like Hamiltonian

$$H = \frac{1}{2}p^2 + \frac{1}{2}V''(\bar{x}(t))\, q^2$$

The path integral is a Gaussian integral and can be evaluated by inspection to be

$$N \det\big(\underbrace{-\partial_t^2 + V''(\bar{x})}_{\substack{\text{with 0 bdry conditions} \\ \text{at ends of } [0,T]}}\big)^{-1/2}$$

and the N has to be adjusted along with the normalization occurring in the $\mathcal{D}y$ so as to give the correct result for free motion.

$$\langle x| e^{-T\frac{p^2}{2}/\hbar} |x'\rangle = \int \frac{dp}{2\pi\hbar}\, e^{i\frac{p}{\hbar}(x'-x) - \frac{T}{\hbar}\frac{p^2}{2}}$$

$$= \int \frac{d\xi}{2\pi}\, e^{-i\xi(x'-x) - \hbar T\frac{\xi^2}{2}} = \frac{1}{\sqrt{2\pi\hbar T}}\, e^{-\frac{(x'-x)^2}{2\hbar T}}$$

Hence

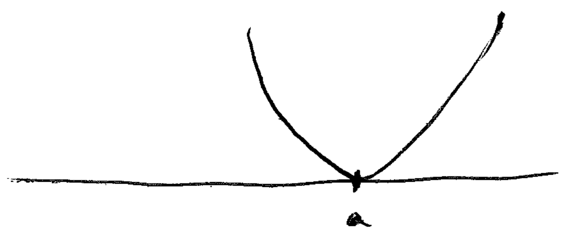
$$\langle x=0| e^{-T\frac{p^2}{2}\hbar} |x'=0\rangle = \frac{1}{\sqrt{2\pi\hbar T}} = \int_{\substack{y(0)=0 \\ y(T)=0}} \mathcal{D}y \; e^{-\frac{1}{\hbar}\int_0^T \frac{1}{2}\dot{y}^2 dt}$$

$$= N \det\left(-\partial_t^2 \text{ on } [0,T] \text{ with Dirichlet cond.}\right)^{-1/2}$$

so it seems that

$$(+) \quad \boxed{\langle x | e^{-TH/\hbar} | x' \rangle \approx e^{-\frac{1}{\hbar} S(\bar{x})} \frac{1}{\sqrt{2\pi\hbar T}} \det_{\text{on } [0,T]}\left(1 + (-\partial_t^2)^{-1} V''(\bar{x})\right)^{-1/2}}$$

Consider the case where V has a minimum value $= 0$ at $x = a$:



Then $\bar{x} = a$ and $S(\bar{x}) = 0$. Also $V''(\bar{x}) = V''(a) = \omega^2 > 0$. Let's compute eigenvalues for $-\partial_x^2$ and $-\partial_x^2 + \omega^2$ on $[0,T]$ with Dirichlet bdry conditions. The eigenfunctions are $\sin n\frac{\pi}{T}x$ and the eigenvalues are

$$\left(n\frac{\pi}{T}\right)^2 \quad \text{and} \quad \left(n\frac{\pi}{T}\right)^2 + \omega^2 \qquad n = 1, 2, \dots$$

respectively, so that

$$\frac{\det(-\partial_t^2 + \omega^2)}{\det(-\partial_t^2)} = \prod_{n=1}^{\infty} 1 + \left(\frac{\omega T}{n\pi}\right)^2 = \frac{\sinh(\omega T)}{\omega T}$$

Consider the oscillator $H_0 = \frac{p^2}{2} + \omega^2 \frac{q^2}{2}$.

$$H_0 = -\frac{1}{2}\hbar^2 \frac{d^2}{dx^2} + \frac{1}{2}\omega^2 x^2$$

has the ground state $\quad e^{-\frac{1}{2}\frac{\omega}{\hbar}x^2} / \text{norm}$

$$\int e^{-\frac{\omega}{\hbar}x^2}\,dx = \sqrt{\pi}\sqrt{\frac{\hbar}{\omega}} = \left(\frac{\pi\hbar}{\omega}\right)^{1/2}$$

so $$\phi_0(x) = \left(\frac{\omega}{\pi\hbar}\right)^{1/4} e^{-\frac{1}{2}\frac{\omega}{\hbar}x^2}$$

The ground energy is $E_0 = \frac{1}{2}\omega\hbar$. Thus

$$\langle x | e^{-TH_0/\hbar} | x' \rangle \approx e^{-\frac{1}{2}\omega T} \left(\frac{\omega}{\pi\hbar}\right)^{1/2} e^{-\frac{1}{2}\frac{\omega}{\hbar}(x^2+x'^2)}$$

Notice this checks with (+) on the previous page: Here $S(\bar{x}) = 0$ and

$$\frac{1}{\sqrt{2\pi\hbar T}}\left(\frac{\det(-\partial_t^2+\omega^2)}{\det(-\partial_t^2)}\right)^{-1/2} = \frac{1}{\sqrt{2\pi\hbar T}}\left(\frac{\sinh \omega T}{\omega T}\right)^{-1/2}$$

$$\sim \frac{\sqrt{\omega}}{\sqrt{\cancel{2}\pi\hbar}} \frac{1}{(e^{\omega T}/\cancel{2})^{1/2}} = \left(\frac{\omega}{\pi\hbar}\right)^{1/2} e^{-\frac{1}{2}\omega T}.$$

So the conclusion is that if the potential $V(x)$ has a unique abs. minimum at $x = a$ and $V(a) = 0$, $V''(a) = \omega^2 > 0$, then

$$\langle x{=}a | e^{-TH/\hbar} | x'{=}a \rangle = \left(\frac{\omega}{\pi\hbar}\right)^{1/2} e^{-\frac{1}{2}\omega T}(1 + O(\hbar))$$

and the ground energy is $\frac{1}{2}\omega\hbar\,(1 + O(\hbar))$.

Next project is the double well.

Consider the oscillator $H_0 = \frac{p^2}{2} + \frac{1}{2}\omega^2 q^2$. The formula

$$\langle x | e^{-iH_0 T/\hbar} | x' \rangle = \int_{\substack{q(0)=x' \\ q(T)=x}} \mathcal{D}q \; e^{i/\hbar \int_0^T \left[\frac{1}{2}\dot{q}^2 - \frac{1}{2}\omega^2 q^2\right] dt}$$

should allow the exact evaluation of the amplitude on the left, because the path integral is Gaussian. Since $V(x) = \frac{1}{2}\omega^2 x^2$ one has $V''(\bar{x}) \equiv \omega^2$, so that expanding around the classical solution

$$x(t) = \bar{x}(t) + y(t)$$

one sees the amplitude is $e^{i/\hbar S(\bar{x})}$ times a factor independent of x, x', namely the path integral over paths $y(t)$ vanishing at the ends $0, T$.

Let's compute $S(\bar{x})$.

$$\bar{x} = A \sin \omega t + B \cos \omega t$$

$$A \sin \omega T + B \cos \omega T = x$$

$$B = x'$$

$$\bar{x}(t) = \left(\frac{x - x' \cos \omega T}{\sin \omega T} \right) \sin \omega t + x' \cos \omega t$$

$$\dot{\bar{x}}(t)^2 = \omega^2 (A \cos \omega t - B \sin \omega t)^2$$

$$\omega^2 \bar{x}(t)^2 = \omega^2 (A \sin \omega t + B \cos \omega t)^2$$

$$\dot{\bar{x}}(t)^2 - \omega^2 \bar{x}(t)^2 = \omega^2 \left[A^2 \cos 2\omega t - B^2 \cos 2\omega t - 4AB \sin \omega t \cos \omega t \right]$$

$$\frac{1}{2}\int_0^T\left(\dot{\bar{x}}^2-\omega^2\bar{x}^2\right)dt = \frac{1}{2}\omega^2\left[(A^2-B^2)\frac{\sin 2\omega t}{2\omega} - 2AB\,\frac{\sin^2\omega t}{\omega}\right]_0^T$$

$$= \frac{1}{2}\omega\left[(A^2-B^2)\sin\omega T\cos\omega T - 2AB\sin^2\omega T\right]$$

$$= \frac{1}{2}\omega\sin\omega T\left[\left[\left(\frac{x-x'\cos}{\sin}\right)^2 - x'^2\right]\cos - 2\left(\frac{x-x'\cos}{\sin}\right)x'\sin\right]$$

$$= \frac{1}{2}\omega\sin\left\{\left(\frac{x^2}{\sin^2} - \frac{2xx'\cos}{\sin^2} + \frac{x'^2\cos^2}{\sin^2} - x'^2\right)\cos - 2xx' + 2x'^2\cos\right\}$$

$$= \frac{1}{2}\omega\sin\left\{\frac{x^2\cos}{\sin^2} - 2xx'\underbrace{\left(\frac{\cos^2}{\sin^2}+1\right)}_{1/\sin^2} + x'^2\underbrace{\left(\frac{\cos^3}{\sin^2}+\cos\right)}_{\cos/\sin^2}\right\}$$

$$\boxed{S(\bar{x}) = \frac{1}{2}\omega\frac{1}{\sin\omega T}\left(x^2\cos\omega T - 2xx' + x'^2\cos\omega T\right)}$$

Now

$$\int_{q(0)=q(T)=0}\mathcal{D}q\; e^{\frac{i}{\hbar}\int_0^T\left(\frac{1}{2}\dot{q}^2 - \frac{1}{2}\omega^2 q^2\right)dt} = \left[\frac{\det(-\partial_t^2-\omega^2)}{\det(-\partial_t^2)}\right]^{-1/2}\frac{1}{\sqrt{2\pi i\hbar T}}$$

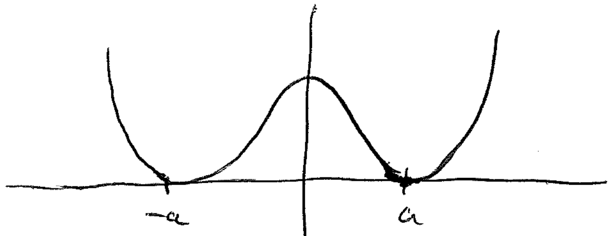
$$= \left[\prod_{n=1}^{\infty}\frac{\left(\frac{n\pi}{T}\right)^2-\omega^2}{\left(n\frac{\pi}{T}\right)^2}\right]^{-1/2}\frac{1}{\sqrt{}} = \left[\prod_{n=1}^{\infty}\left(1-\left(\frac{\omega T}{n\pi}\right)^2\right)\right]^{-1/2}\frac{1}{\sqrt{}}$$

$$= \left(\frac{\sin\omega T}{\omega T}\right)^{-1/2}\frac{1}{\sqrt{2\pi i\hbar T}}$$

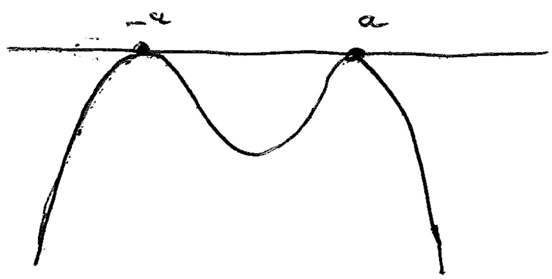
So these two things together one gets an explicit formula for $\langle x|e^{-iH_0T/\hbar}|x'\rangle$

$$= \frac{1}{\sqrt{2\pi i\hbar T}}\left(\frac{\sin\omega T}{\omega T}\right)^{-1/2} e^{\frac{i}{\hbar}S(\bar{x})}$$

Double well:



The idea is to compute $\langle a| e^{-TH/\hbar} |a\rangle$, $\langle +a| e^{-TH/\hbar} |-a\rangle$ approximately. Consider the latter with T very large. Recall in the Euclidean situation one looks at the classical motion in the potential $-V(x)$



and there is an obvious path $\bar{x}$ starting at $-a$ at $t=-T/2$ and ending at $+a$ at $t=T/2$. Let's compute its action

$$\ddot{\bar{x}} = V'(\bar{x})$$

$$\frac{1}{2}\dot{\bar{x}}^2 - V(\bar{x}) = \text{constant} = \text{initial \& final kin. energy.}$$

For T very large, we ignore the initial kin. energy and so $\frac{1}{2}\dot{\bar{x}}^2 - V(\bar{x}) = 0$

$$S(\bar{x}) = [redacted] \int \left[\tfrac{1}{2}\dot{\bar{x}}^2 + V(\bar{x})\right] dt = \int 2V(\bar{x})\, dt = \int_{-a}^{a} 2V(\bar{x}) \underbrace{\frac{dt}{d\bar{x}}}_{1/\sqrt{2V}} d\bar{x}$$

$$= \int_{-a}^{a} \sqrt{2V(x)}\, dx$$

[What would have happened if we took

$$\frac{1}{2}\dot{\bar{x}}^2 - V(\bar{x}) = E \; ?$$

Then $\quad \dfrac{d\bar{x}}{dt} = \sqrt{2(E+V(\bar{x}))} \quad$ and

$$S(\bar{x}) = \int \left[\frac{1}{2}\dot{\bar{x}}^2 + V(\bar{x})\right] dt = \int \left[E + 2V(\bar{x})\right] dt$$

$$= \int_{-a}^{a} \frac{E + 2V(x)}{\sqrt{2(E+V(x))}}\, dx \; . \;\;]$$

So the leading term as $\hbar \to 0$ in the $-a$ to a amplitude is

$$(*) \qquad e^{-\frac{1}{\hbar}\int_{-a}^{a}\sqrt{2V(x)}\,dx} \; N \det\left(-\partial_t^2 + V''(\bar{x})\right)^{-1/2}$$

except that a new phenomenon occurs with the case where one wants to let $T \to \infty$. One wants to consider [redacted] approximate classical motions which begin at $-a$ then jump to a [redacted] around a time t_1, then remain near a ~~until~~ t_2 jump back to $-a$, etc. For some reason, not clear at the moment, these other possibilities have to be included to get the good $T \to \infty$ limit.

Set $S_0 = \int_{-a}^{a} \sqrt{2V}\,dx$. I want to compute the contribution [redacted] to the $-a \mapsto a$ amplitude have n instantons ($=$ jumps [redacted] between $-a, a$). These instantons occur nearly infinitesimally when viewed on a large time scale. Take 3 instantons occurring at time $-T/2 < t_1 < t_2 < t_3 < T/2$ and calculate the amplitude contribution. I need the amplitude to stay at a_n^{-a} for a time interval $\Delta t = t_i - t_{i-1}$. We've computed this to be

$$\langle a | e^{-H\Delta t/\hbar} | a \rangle \approx \left(\frac{\omega}{\pi\hbar}\right)^{1/2} e^{-\frac{1}{2}\omega\Delta t}$$

where $\omega^2 = V''(a)$. The instanton contribution is

$$e^{-S_0/\hbar} K$$

where K is the determinantal factors in $(*)$. The 3-instanton contribution is

$$\left(e^{-S_0/\hbar} K\right)^3 \left(\left(\frac{\omega}{\pi\hbar}\right)^{1/2}\right)^4 \int\limits_{-\frac{T}{2}<t_1<t_2<t_3<\frac{T}{2}} dt_1\, dt_2\, dt_3\; e^{-\frac{\omega}{2}\sum_i \Delta t_i}$$

so the n-instanton contribution is

$$\left(e^{-S_0/\hbar} K\right)^n \left(\left(\frac{\omega}{\pi\hbar}\right)^{1/2}\right)^{n+1} e^{-\frac{1}{2}\omega T}\, \frac{T^n}{n!}$$

so we get

$$\langle -a | e^{-TH/\hbar} | a \rangle = \left(\frac{\omega}{\pi\hbar}\right)^{1/2} e^{-\frac{1}{2}\omega T} \sum_{n \text{ odd}} \frac{1}{n!} \left(e^{-S_0/\hbar} K \left(\frac{\omega}{\pi\hbar}\right)^{1/2} T\right)^n$$

A comparison with Coleman shows the above to be deficient mainly because the role of the $T \to \infty$ isn't understood. Let's concentrate on the reason multiple instantons occur (actually one has both instantons going from $-a$ to a and anti-instantons going backwards). It has something to do with the fact that the operator $-\partial_t^2 + V''(\bar{x})$ on $(-\infty, \infty)$ has the eigenvalue 0. What does this operator look like?

$$\frac{1}{2}\left(\frac{dx}{dt}\right)^2 = V(x)$$

$$\frac{dx}{dt} = \sqrt{2V} \qquad\qquad t = \int \frac{dx}{\sqrt{2V}}$$

Use symmetry and require $t=0$ where $x=0$, so that

$$t = \int_0^x \frac{dx}{\sqrt{2V}}$$

As $x \to a$ one has $V(x) = \frac{1}{2}\omega^2(x-a)^2 + \dots$

$$\sqrt{2V} \approx \omega(a-x)$$

$$t = \int \frac{dx}{\sqrt{2V}} = \int \frac{1}{\omega}\frac{dx}{a-x} = \frac{1}{\omega}\left(-\log(a-x)\right) + C$$

$$a - x = Ce^{-\omega t}$$

so that $x \to a$ exponentially as $t \to +\infty$. Similarly $x \to -a$ exponentially as $t \to -\infty$, with $x + a = O(e^{\omega t})$ as $t \to -\infty$.

Example: Take $V(x) = \frac{\omega^2}{8a^2}(x+a)^2(x-a)^2$

so that

$$t = \int \frac{dx}{\sqrt{2V}} = \int \frac{dx}{\frac{\omega}{2a}(x+a)(a-x)}$$

$$= \frac{2a}{\omega}\int\left[\frac{1}{a-x} + \frac{1}{a+x}\right]\frac{dx}{2a} = \frac{1}{\omega}\log\left(\frac{a+x}{a-x}\right)$$

$$\frac{a+x}{a-x} = e^{\omega t} \qquad \frac{2a}{a-x} = 1 + e^{\omega t} \qquad a - x = \frac{2a}{1+e^{\omega t}}$$

$$x = -\frac{2a}{1+e^{\omega t}} + a = a\,\frac{-1+e^{\omega t}}{1+e^{\omega t}} = a\tanh\frac{\omega}{2}t$$

So for this potential we get

$$\boxed{\bar{x}(t) = a\tanh\left(\frac{\omega t}{2}\right)}$$

since $\quad V''(x) = \omega^2 + O((x-a)^3) \quad$ as $x \to a$

one has $\quad V''(\bar{x}(t)) = \omega^2 + O(e^{-3\omega t}) \quad$ as $t \to \infty$.

Therefore the spectrum of the operator $-\partial_t^2 + V(\bar{x}(t))$ on $(-\infty, \infty)$ is continuous: It should be closely related to that of the operator $-\partial_t^2$ via scattering data.

Digression to review some solitons.

$$\left(\frac{d}{dx} + p\right)\left(-\frac{d}{dx} + p\right) = -\frac{d^2}{dx^2} + p^2 + p'$$

Hence $-\partial_x^2 + q$ can be factored in the above form when one has a non-vanishing solution ϕ of

$$(-\partial_x^2 + q)\phi = 0.$$

One then sets $\quad p = \frac{\phi'}{\phi}$. $\quad$ Better: if one has the factorization then $\quad \phi = e^{\int p}$ is killed by $-\partial_x^2 + q$.

Conversely

$$\left(\frac{\phi'}{\phi}\right)^2 + \left(\frac{\phi'}{\phi}\right)' = \frac{\phi'^2}{\phi^2} + \frac{\phi\phi'' - \phi'^2}{\phi^2} = \frac{\phi''}{\phi}.$$

Start with $-\partial_x^2 + V$ with $V \to 0$ rapidly at $|x| \to \infty$ and assume no bound states. Better pick $-\omega^2 <$ spectrum of $-\partial_x^2 + V$. Then we can find non-vanishing solutions of

$$(-\partial_x^2 + V + \omega^2)\phi = 0,$$

in fact the solution $\approx e^{-\omega x}$ as $x \to \infty$ won't vanish; otherwise one can show $-\omega^2$ is not below the spectrum. Then one factors

$$(-\partial_x^2 + V + \omega^2) = \underbrace{(\partial_x + p)}_{X^*}\underbrace{(-\partial_x + p)}_{X}$$

where $p = \frac{\phi'}{\phi}$. Then if we form [redacted]

$$XX^* = (-\partial_x + p)(\partial_x + p) = \text{[redacted]} = -\partial_x^2 + (p^2 - p')$$

we know the spectrum of X^*X and X^*X are the same except for 0 eigenvalue. The operator [redacted] XX^* kills

$$e^{-\int p} = e^{-\int \frac{\phi'}{\phi}} = \frac{1}{\phi}$$

One can arrange $\phi \approx C_{\pm} e^{\omega|x|}$ as [redacted] $x \to \pm\infty$, so that $\frac{1}{\phi}$ is square integrable. Also notice

$$XX^* = -\partial_x^2 + \underbrace{(p^2 - p' - \omega^2)}_{\underbrace{p^2 + p'}_{V+\omega^2} - 2p' - \omega^2 \;=\; V - 2p'} + \omega^2$$

Since $\phi \approx C_{\pm} e^{\omega|x|}$, $p \approx \pm\omega$, so $p' \approx 0$.

Typical example is $V = 0$

$(-\partial_x^2 + \omega^2)\phi = 0$ has soln $\phi =$ [redacted] $e^{\omega x} + c$[redacted]$e^{-\omega x}$

[redacted] which is non-vanishing for $c > 0$. Then

$$p' = \left(\frac{\omega e^{\omega x} - c\omega e^{-\omega x}}{e^{\omega x} + c e^{-\omega x}}\right)'$$

Better: Use [redacted] $\phi = \cosh(\omega x + \alpha)$

$\phi' = \omega \sinh(\omega x + \alpha)$

$$p = \omega \tanh(\omega x + \alpha) \qquad p' = \frac{\omega^2}{\cosh^2(\omega x + \alpha)}$$

and so the new potential is

$$\tilde{V} = -2p' = -2\omega^2 \frac{1}{\cosh^2(\omega x+\alpha)}$$

In the example on 337 we have

$$V''(x) = \frac{\omega^2}{8a^2}\left(12x^2 - 4a^2\right)$$

$$\bar{x}(t) = a\tanh\left(\tfrac{\omega}{2}t\right)$$

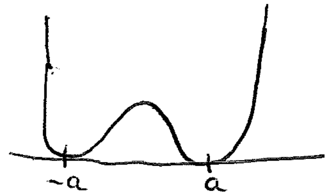
$$V''(\bar{x}) = \frac{\omega^2}{8}\left(12\tanh^2\left(\tfrac{\omega}{2}t\right) - 4\right)$$

$$V''(\bar{x}) - \omega^2 = \frac{3}{2}\omega^2\left(\tanh^2\left(\tfrac{\omega}{2}t\right) - 1\right)$$

$$= -\frac{3}{2}\omega^2 \operatorname{sech}^2\left(\tfrac{\omega}{2}t\right)$$

So this is close to the soliton example but not exactly the same.

October 14, 1979

Double-well potential V : 

has instanton solution $\bar{x}(t)$

obtained by solving $\ddot{x} = V'(x)$ with 0 energy:

$$\frac{1}{2}\dot{x}^2 = V(x) \quad \text{so} \quad dt = \frac{1}{\sqrt{2V}}\,dx$$

Then one looks at $-\partial_t^2 + V''(\bar{x}(t))$ on $(-\infty, \infty)$.

From
$$-\partial_t^2 \bar{x} + V'(\bar{x}) = 0$$
we get
$$-\partial_t^2(\dot{\bar{x}}) + V''(\bar{x})\,\dot{\bar{x}} = 0$$
so that $\dot{\bar{x}}(t)$ is an eigenfunction for $-\partial_t^2 + V''(\bar{x})$ with the eigenvalue zero. Moreover as $\dot{\bar{x}} = 1/\sqrt{2V}$ it is non-vanishing and it's also square-integrable because we [redacted] saw $\bar{x}(t) = a + O(e^{-\omega t})$ at $t \to \infty$ and similarly as $t \to -\infty$. Thus Sturm-theory says that the spectrum of $-\partial_t^2 + V''(\bar{x})$ is ≥ 0. In fact [redacted] since $V''(\bar{x}) \longrightarrow \omega^2$ as $|t| \to \infty$ we know [redacted] there [redacted] is continuous spectrum starting at ω^2 and possibly bound states between 0 and ω^2.

One can remove the 0 eigenvalue by factorization

$$V''(\bar{x}) = p^2 + p' \quad \text{where} \quad p = \frac{\ddot{\bar{x}}}{\dot{\bar{x}}} = V'(\bar{x})\Big/\sqrt{2V(\bar{x})}$$

New potential is

$$p^2 - p' = V''(\bar{x}) - 2p' = V''(\bar{x}) - 2\left(\frac{\dddot{\bar{x}}}{\dot{\bar{x}}} - \left(\frac{\ddot{\bar{x}}}{\dot{\bar{x}}}\right)^2\right)$$

$$= 2p^2 - V''(\bar{x}) = \text{[redacted]}$$

$$= \frac{V'(\bar{x})^2}{V(\bar{x})} - V''(\bar{x}) = \frac{V'(\bar{x})^2 - V(\bar{x})V''(\bar{x})}{V(\bar{x})}$$

But $\quad (\log V)'' = \left(\frac{V'}{V}\right)' = \frac{VV'' - (V')^2}{V^2}$

So
$$p^2 - p' = \left(-V(\log V)''\right)(\bar{x}(t))$$

For $\quad V(x) = \frac{\omega^2}{8a^2}(x-a)^2(x+a)^2$

$$(\log V)'' = \left(\frac{2}{x-a} + \frac{2}{x+a}\right)' = \frac{-2}{(x-a)^2} + \frac{-2}{(x+a)^2} < 0.$$

In fact

$$\begin{aligned} -V(\log V)'' &= \frac{\omega^2}{4a^2}\left((\bar{x}+a)^2 + (\bar{x}-a)^2\right) \\ &= \frac{\omega^2}{4a^2}\, 2\left(\bar{x}^2 + a^2\right) \\ &= \frac{\omega^2}{2a^2}\left(a^2 \tanh^2 \tfrac{\omega}{2}t + a^2\right) \\ &= \omega^2 - \tfrac{1}{2}\omega^2 \operatorname{sech}^2\left(\tfrac{\omega}{2}t\right) \end{aligned}$$

So $\quad p^2 - p' - \omega^2 = -\frac{1}{2}\omega^2 \operatorname{sech}^2\left(\frac{\omega}{2}t\right)$.
which seems to be in the good form at the top of page 340. So it seems that the basic potential

$$V''(\bar{x}) - \omega^2 = -\tfrac{3}{2}\omega^2 \operatorname{sech}^2\left(\tfrac{\omega}{2}t\right)$$

is obtained from 0 by introducing two bound states at energies $\left(\frac{1}{2}\omega\right)^2$ and (ω^2).

I just noticed that $\langle x | e^{-TH/\hbar} | x' \rangle$ depends only on $T/\hbar$ so that $T \to +\infty$ should be the same as $\hbar \to 0$ at first glance. But this isn't correct because when $H = \frac{p^2}{2}$, then

$$\langle x | e^{-\frac{T}{\hbar}\frac{p^2}{2}} | x' \rangle = \int \frac{dp}{2\pi\hbar} e^{-\frac{i}{\hbar}p(x-x') - \frac{T}{\hbar}\frac{p^2}{2}}$$

$$= \int \frac{d\xi}{2\pi} e^{-i\xi(x-x') - T\hbar\frac{\xi^2}{2}} = \frac{1}{\sqrt{2\pi\hbar T}} e^{-\frac{(x-x')^2}{2\hbar T}}$$

depends on $T\hbar$. In general you have

$$\langle x | e^{-\frac{T}{\hbar}H} | x' \rangle = \int_{\substack{q(0)=x' \\ q(\hbar T)=x}} \mathcal{D}q \; e^{-\int_0^{\hbar T}\left[\frac{1}{2}\left(\frac{dq}{dt}\right)^2 + \frac{1}{\hbar^2}V(q)\right]dt}$$

where $\mathcal{D}q$ has no $\hbar$ dependence.
Another way of writing this is

$$= \int_{\substack{q(0)=x' \\ q(\hbar T)=x}} d\mu \; e^{-\frac{1}{\hbar^2}\int_0^{\hbar T} V(q)\,dt}$$

$d\mu$ = Wiener measure on paths
gives $\frac{1}{\sqrt{2\pi\hbar T}} e^{-(x-x')^2/2\hbar T}$
when $V=0$

It's still not clear how to evaluate first the $T \to \infty$ limit and then the $\hbar \to 0$ limit, i.e. why multiple instantons occur.

October 16, 1979

Electrodynamics: Review:

force on a particle of charge e is

$$\vec{F} = e(\vec{E} + \vec{v} \times \vec{B})$$

non-relativistic equation of motion:

$$\frac{d}{dt}(mv) = e(\vec{E} + \vec{v} \times \vec{B})$$

To rewrite in Lagrangian + Hamiltonian form.

From Maxwell: $\vec{\nabla} \cdot \vec{B} = 0 \qquad \vec{\nabla} \times \vec{E} + \frac{\partial \vec{B}}{\partial t} = 0$

The former by Poincaré's lemma allows one to express B as

$$\vec{B} = \vec{\nabla} \times \vec{A}$$

whence $\vec{\nabla} \times \left(\vec{E} + \frac{\partial \vec{A}}{\partial t}\right) = 0$ whence

$$\vec{E} = -\vec{\nabla}\varphi - \frac{\partial \vec{A}}{\partial t}$$

for some function φ. Then

$$\begin{aligned} E + v \times B &= E + v \times (\nabla \times A) \\ &= -\nabla\varphi - \frac{\partial A}{\partial t} + \nabla(v \cdot A) - (v \cdot \nabla)A \\ &= -\nabla(\varphi - v \cdot A) - \frac{d}{dt}(\vec{A}) \end{aligned}$$

Thus if we put $L = \frac{1}{2}mv^2 - e(\varphi - v \cdot A)$ we have

$$\frac{\partial L}{\partial v} = mv + eA \qquad \frac{\partial L}{\partial x} = -e\nabla(\varphi - v \cdot A)$$

and so Lagrange's equation is

$$\frac{d}{dt}(mv + eA) = -e\nabla(\varphi - v \cdot A)$$

agreeing with the above.

Relativistic motion of a particle. Path of particle is $x(t) = (t, \vec{x}(t))$. It has unit tangent vector

$$u = \left(\frac{1}{\sqrt{1-v^2}}, \frac{\vec{v}}{\sqrt{1-v^2}} \right).$$

The energy-momentum vector is proportional to this

$$p = (E, \vec{p}) = [redacted] \left(\frac{m}{\sqrt{1-v^2}}, \frac{m\vec{v}}{\sqrt{1-v^2}} \right). \qquad m = \text{rest mass.}$$

(Two justifications are that $u \cdot p = m$ so

$$0 = u \cdot \frac{dp}{dt} \quad [redacted] \quad \text{hence} \quad \frac{dE}{dt} = \vec{v} \cdot \frac{d\vec{p}}{dt}$$

which expresses the fact that force $\cdot \vec{v}$ = rate of doing work. Other justification

$$E = \frac{m}{\sqrt{1-v^2}} \approx m + \frac{1}{2} m v^2 \quad \text{for small } v. \,)$$

For free motion we have $H = E$ and

$$L = \vec{p} \cdot v - E = \frac{mv^2}{\sqrt{1-v^2}} - \frac{m}{\sqrt{1-v^2}} = -m\sqrt{1-v^2}$$

Check: $\vec{p} = \dfrac{\partial L}{\partial v} = -m \frac{1}{2} (1-v^2)^{-1/2} (-2\vec{v}) = \dfrac{m\vec{v}}{\sqrt{1-v^2}}$ ✓

Thus the Lagrangian for a particle of mass m in an electromagnetic field is

$$\boxed{L = -m\sqrt{1-v^2} - e(\phi - \vec{v} \cdot \vec{A})}$$

because $\vec{p}_A = \dfrac{\partial L}{\partial v} = \dfrac{m\vec{v}}{\sqrt{1-v^2}} + e\vec{A} = \vec{p} + e\vec{A}$

and $\dfrac{d}{dt}(\vec{p} + e\vec{A}) = -e\nabla(\phi - \vec{v} \cdot \vec{A})$

is the correct relativistic equation.

$\vec{p}_A$ denotes the canonical momentum defined by this Lagrangian.
The Hamiltonian is

$$H = \vec{p}_A \cdot v - L = \frac{mv^2}{\sqrt{1-v^2}} + \cancel{e\vec{A}\cdot\vec{v}} + m\sqrt{1-v^2} + e\phi - \cancel{e\vec{v}\cdot\vec{A}}$$

$$= \frac{m}{\sqrt{1-v^2}} + e\phi$$

but we want it written with v [redacted] computed in terms of p_A.

$$m^2 + (p_A - eA)^2 = \frac{m^2v^2}{1-v^2} + m^2 = \frac{m^2}{1-v^2}$$

or

$$H = \sqrt{m^2 + (p_A - eA)^2} + e\phi$$

A truly invariant way to write this is to [redacted] write [redacted] a hypersurface in the cotangent bundle of space-time. Using $p = (E, \vec{p})$ to denote the fibre coordinates (so the form is

$$p\cdot dx = E\,dt - \vec{p}\cdot d\vec{x} \;)$$

(Thus $\vec{p}_A$ becomes $\vec{p}$)

the hypersurface is

$$E - e\phi = \sqrt{m^2 + (\vec{p} - e\vec{A})^2}$$

i.e. the covector [redacted] $(p - eA)\cdot dx$ has norm m. Here

$$A = (\phi, \vec{A}).$$

Actually the action form $L\,dt$ can be written invariantly

$$L\,dt = -m\,ds - eA\cdot dx$$

where $ds = \sqrt{1-v^2}\,dt$ is arc-length. This is not a true 1-form on space-time.

October 17, 1979:

Relativistic notation $p_\mu = (E, \vec{p})$ $dx_\mu = (dt, d\vec{x})$

$$p_\mu \cdot dx_\mu = E\,dt - \vec{p} \cdot d\vec{x}.$$

Gradient $\nabla_\mu f$ is defined so that

$$\nabla_\mu f \cdot dx_\mu = df$$

hence $\nabla_\mu = (\partial_t, -\vec{\nabla})$. Also

$$\nabla_\mu e^{i p_\mu \cdot x_\mu} = i p_\mu .$$

Dirac equation: Start with

$$E = \sqrt{p^2 + m^2}$$

and you want matrices $\alpha_x, \alpha_y, \alpha_z, \beta$ such that

$$p^2 + m^2 = (\alpha p + \beta m)^2$$

Thus $\alpha_i^2 = 1$ [redacted] $\alpha_i \alpha_j + \alpha_j \alpha_i = 0 \quad i \neq j$

$\beta^2 = 1 \qquad \alpha_i \beta + \beta \alpha_i = 0$

Then the Dirac equation is

$$i \partial_t \psi = \underbrace{\left(\alpha \tfrac{1}{i} \partial_x + \beta m\right)} \psi$$

So that the Hamiltonian H is self-adjoint you want α_i and β to be hermitian. Then $\psi^* \psi$ can be interpreted as probability of finding a particle, i.e. particle density, so the charge density is

$$\rho = e \psi^* \psi$$

Current density $\vec{j}$ is defined so that $\partial_t \rho + \vec{\nabla}\cdot\vec{j} = 0$.

$$\partial_t(\psi^*\psi) = [(-\alpha\partial_x - i\beta m)\psi]^*\psi + \psi^*[(-\alpha\partial_x - i\beta m)\psi]$$

$$= -(\partial_x\psi)^*\alpha\psi - \psi^*\alpha\partial_x\psi$$

$$= -\partial_x[\psi^*\alpha\psi]$$

hence

$$j_i = e\psi^*\alpha_i\psi \qquad \rho = e\psi^*\psi.$$

Thus customary way of writing the Dirac equation in order to show its Lorentz invariance is

$$(i\beta\partial_t + i\beta\alpha\partial_x - m)\psi = 0$$

and to put $\gamma^0 = \beta,\ \gamma^i = \beta\alpha_i$ so that

$$(\gamma^0)^2 = 1, \quad (\gamma^i)^2 = -1, \quad \gamma^j\gamma^i + \gamma^i\gamma^j = 0 \quad i \neq j$$

Then the Dirac equation becomes

$$\boxed{(i\gamma^\mu\partial_\mu - m)\psi = 0.}$$

Also one introduces

$$\bar{\psi} = \psi^*\gamma^0$$

whence $J^\mu = e\bar{\psi}\gamma^\mu\psi = e(\rho, \vec{j})$ is the charge-current density.

<u>Relativistic notation</u>: Because coordinates are given on space-time we have frames dx^i and ∂_i in T^* and T hence geometric quantities such as vectors + covectors are described by components. We might

[redacted] follow Feynman and think geometrically in terms of vectors. Then a vector is a gadget in the form

$$\sum_\mu a^\mu \partial_\mu$$

and should be described by superscripts. For example a displacement of position is a vector

$$\delta x^\mu = (\delta t, \delta \vec{x})$$

Normally I want to think of energy-momentum as a 1-form: $E dt - \vec{p} \cdot d\vec{x}$ hence

$$p_\mu dx^\mu = E dt - \vec{p} \cdot d\vec{x} \qquad \text{or} \qquad p_\mu = (E, -\vec{p})$$

The energy momentum vector is

$$p^\mu = (E, \vec{p})$$

Similarly charge-current density is a vector $J^\mu = (\rho, \vec{j})$, as well as $A^\mu = (\varphi, \vec{A})$ so that

$$A_\mu = (\varphi, -\vec{A})$$

Gradient of f is the vector corresponding to the 1-form $df = \partial_\mu f \, dx^\mu$, hence the gradient is

$$\partial^\mu f = (\partial_0 f, -\vec{\nabla} f)$$

Finally the 4-divergence is

$$\partial_\mu A^\mu = \partial_0 \varphi + \vec{\nabla} \cdot \vec{A}$$

and

$$\partial_\mu \partial^\mu f = (\partial_0^2 - \vec{\nabla}^2) f.$$

When we want to go to quantum mechanics

we make the replacements by operators as follows:

$$p_\mu \longmapsto i\hbar\partial_\mu \qquad \text{i.e.} \qquad E \mapsto i\hbar\partial_0 \,, \ \vec{p} \mapsto \frac{\hbar}{i}\vec{\nabla}.$$

Hence the Dirac equation can be written

$$(\underbrace{\gamma^\mu p_\mu}_{\not{p}} - m)\,\psi = 0$$

(Feyman notation).

(Memory: Use upper indices for vectors.

$$p^\mu = (E, \vec{p}) \qquad A^\mu = (\varphi, \vec{A}) \qquad J^\mu = (\rho, \vec{j})$$

so that equation of continuity is $\partial_\mu J^\mu = 0$)

The next project is to get the field equations for electrodynamics – EM field A interacting with Dirac field ψ. The Dirac [redacted] equation in the presence of a fixed A can be derived by analogy with the free case $A=0$ starting from

$$(E - e\varphi) = \sqrt{(\vec{p} - e\vec{A})^2 + m^2}\,.$$

This says $p^\mu - eA^\mu = (E - e\varphi, \vec{p} - e\vec{A})$ has norm m. All you do is replace p_μ by $p_\mu - eA_\mu$ and you get

$$\left[\gamma^\mu(p_\mu - eA_\mu) - m\right]\psi = 0$$

or

$$\boxed{\left[i\gamma^\mu(\partial_\mu + ieA_\mu) - m\right]\psi = 0}$$

To get the equation for A given ψ we need to use the rest of Maxwell's equations:

$$\vec{\nabla}\cdot\vec{E} = \rho \qquad \vec{\nabla}\times\vec{B} = \frac{\partial \vec{E}}{\partial t} + \vec{j}$$

$$\nabla\times(\nabla\times A) = \frac{\partial}{\partial t}\left(-\nabla\phi - \frac{\partial A}{\partial t}\right) + j$$

$$\nabla(\nabla\cdot A) - \nabla^2 A = -\partial_t^2 A - \nabla(\partial_t\phi) + j$$

$$(\partial_t^2 - \nabla^2)\, A + \nabla(\partial_t\phi + \nabla\cdot A) = j$$

Also $\rho = \nabla\cdot(-\nabla\phi - \partial_t A) = -\nabla^2\phi - \partial_t(\nabla\cdot A + \partial_t\phi) + \partial_t^2\phi$

or

$$(\partial_t^2 - \nabla^2)\phi - \partial_t(\partial_t\phi + \nabla\cdot A) = \rho$$

or finally

$$(\partial_t^2 - \vec{\nabla}^2)(\phi, \vec{A}) - (\partial_t, -\vec{\nabla})(\partial_t\phi + \vec{\nabla}\cdot\vec{A}) = (\rho, \vec{j})$$

or

$$\boxed{\Box A^\mu - \partial^\mu(\partial_\nu A^\nu) = J^\mu = e\bar{\psi}\gamma^\mu\psi}$$

Next we need a Lagrangian to yield these field equations. Notice the last one can be written

$$J_\mu = \Box A_\mu - \partial_\mu(\partial^\nu A_\nu) = \partial^\nu\partial_\nu A_\mu - \partial^\nu\partial_\mu A_\nu$$

$$= \partial^\nu(\partial_\nu A_\mu - \partial_\mu A_\nu)$$

or better $J^\mu = \partial_\nu\partial^\nu A^\mu - \partial^\mu(\partial_\nu A^\nu)$

$$= \boxed{\partial_\nu(\partial^\nu A^\mu - \partial^\mu A^\nu) = e\bar{\psi}\gamma^\mu\psi}$$

What is the Lagrangian? Start with

$$\mathcal{L} = i\bar{\psi}\gamma^\mu\partial_\mu\psi - m\bar{\psi}\psi .$$

$$\delta\mathcal{L} = i(\delta\bar{\psi}\gamma^\mu\partial_\mu\psi + \bar{\psi}\gamma^\mu\partial_\mu\delta\psi) - m(\delta\bar{\psi}\psi + \bar{\psi}\delta\psi)$$

and recall $\bar{\psi} = \psi^*\gamma^0$ so this is

$$\delta\mathcal{L} = (\delta\psi)^*(\gamma^0\gamma^\mu i\partial_\mu\psi) + \psi^*(\gamma^0\gamma^\mu i\partial_\mu\delta\psi)$$
$$+ (\delta\psi)^*(-m\gamma^0\psi) + \psi^*(-m\gamma^0\delta\psi)$$

Because $\gamma^0\gamma^\mu i\partial_\mu$ is self-adjoint when we integrate $\delta\mathcal{L}$ we can integrate by parts to get

$$\delta\int\mathcal{L}d^4x = \int d^4x\{\delta\psi^*(\gamma^0\gamma^\mu i\partial_\mu\psi) + (\gamma^0\gamma^\mu i\partial_\mu\psi)^*\delta\psi\}$$
$$+ \int d^4x\{(\delta\psi)^*(-m\gamma^0\psi) + (-m\gamma^0\psi)^*\delta\psi\}$$

Standard stuff on inner products says this vanishes for all $\delta\psi$ when

$$\gamma^0\gamma^\mu i\partial_\mu\psi - m\gamma^0\psi = 0$$

or

$$(i\gamma^\mu\partial_\mu - m)\psi = 0$$

which is the Dirac equation. Clearly we can replace ∂_μ by $D_\mu = \partial_\mu + ieA_\mu$ and get that

$$\mathcal{L} = i\bar{\psi}\gamma^\mu D_\mu\psi - m\bar{\psi}\psi = \bar{\psi}(i\gamma^\mu D_\mu - m)\psi$$

leads to the Dirac equation with an external field:

$$(i\gamma^\mu D_\mu - m)\psi = 0.$$

Notation: A determines a 1-form

$$\alpha = A_\mu \, dx^\mu = \varphi \, dt - \vec{A} \cdot d\vec{x}$$

so $\quad A_\mu = (\varphi, -A)$. This 1-form determines the 2-form

$$d\alpha = d(A_\mu \, dx^\mu) = \partial_\nu A_\mu \, dx^\nu dx^\mu = \frac{1}{2} \underbrace{(\partial_\mu A_\nu - \partial_\nu A_\mu)}_{F_{\mu\nu}} dx^\mu dx^\nu$$

Compute $d\alpha$:

$$\begin{aligned} d\alpha &= d\varphi \, dt - dA_x \, dx - dA_y \, dy - dA_z \, dz \\ &= (-\partial_x \varphi - \partial_t A_x) \, dt \, dx - (\partial_x A_y - \partial_y A_x) \, dx \, dy \\ &\quad\; (-\partial_y \varphi - \partial_t A_y) \, dt \, dy - (\partial_y A_z - \partial_z A_y) \, dy \, dz \\ &\quad\; (-\partial_z \varphi - \partial_t A_z) \, dt \, dz - (\partial_z A_x - \partial_x A_z) \, dz \, dx \\ &= E_x \, dt \, dx - B_z \, dx dy \\ &\quad\; E_y \, dt \, dy - B_x \, dy \, dz \\ &\quad\; E_z \, dt \, dz - B_y \, dz dx \end{aligned}$$

The other term in the Lagrangian density is

$$-\frac{1}{4} F_{\mu\nu} F^{\mu\nu} = -\frac{1}{2} \sum_{\mu<\nu}{}' (\partial_\mu A_\nu - \partial_\nu A_\mu)(\partial^\mu A^\nu - \partial^\nu A^\mu)$$

In forming $F^{\mu\nu}$ from $F_{\mu\nu}$ the space-like components dx, dy, dz change sign. Thus one gets

$$-\frac{1}{4} F_{\mu\nu} F^{\mu\nu} = \frac{1}{2}\left(|E|^2 - |B|^2\right)$$

as in Goldstein p 366. A more invariant way of writing it is

$$-\frac{1}{2}(d\alpha, d\alpha)$$

where $(\, , \,)$ is computed using the Minkowski inner product.

Combine the two densities to get

$$\mathcal{L} = i\bar{\psi}\gamma^\mu(\partial_\mu + ieA_\mu)\psi - \tfrac{1}{2}(d\alpha, d\alpha).$$

Now vary A keeping ψ fixed to get

$$\delta\mathcal{L} = e\bar{\psi}\gamma^\mu\psi\,\delta A_\mu - \tfrac{1}{2}\left[(d\delta\alpha, d\alpha) + (d\alpha, d\delta\alpha)\right]$$

After integration the [redacted] term in brackets becomes $2(d^* d\alpha, \delta\alpha)$.
Thus [redacted] the variational equation is

$$e\bar{\psi}\gamma^\mu\psi = d^* d\alpha$$

which is the Maxwell equation on the bottom of 351.

October 18, 1979

Yesterday I [redacted] saw that the Lagrangian for the EM field involves the term

$$-\frac{1}{4} F_{\mu\nu} F^{\mu\nu} = \frac{1}{2}\left(|E|^2 - |B|^2\right)$$

The Lagrangian for the EM field in the presence of a fixed current $J_\mu = (\rho, -\vec{j})$ is

(J,α)

$$\mathcal{L} = J^\mu A_\mu - \frac{1}{4} F_{\mu\nu} F^{\mu\nu} = \text{[redacted]} - \frac{1}{2}(d\alpha, d\alpha)$$

$$\delta\mathcal{L} = \text{[redacted]} \; (J, \delta\alpha) - \frac{1}{2}(d\delta\alpha, d\alpha) - \frac{1}{2}(d\alpha, d\delta\alpha)$$

$$= (J, \delta\alpha) - (d^* d\alpha, \delta\alpha) \qquad \text{mod divergence}$$

Question: Does the variational problem occurring in Hamilton's principle have a minimum or saddle point?

Calculate second variation around $\bar{x}$: $x = \bar{x} + y$

$$\int \left[\frac{1}{2}\dot{x}^2 - V(x)\right] dt = \int \left[\frac{1}{2}\dot{\bar{x}}^2 - V(\bar{x})\right] dt$$

$$+ \int \frac{1}{2}\left(\dot{y}^2 - V''(\bar{x})y^2\right) dt \; + O(y^3)$$

Thus we have a minimum when the operator

$$-\partial_t^2 - V''(\bar{x}(t)) \qquad \text{with Dirichlet conditions}$$

on the interval one is working has eigenvalues > 0. In any case there are only finitely many [redacted] negative eigenvalues.

Example: $V(x) = \frac{1}{2}\omega^2 x^2$ so that $V'' = \omega^2$

On the interval $[0,T]$ ∂_t^2 has eigenfunctions $\sin\left(n\frac{\pi}{T}t\right)$ and so $-\partial_t^2 - \omega^2$ has the spectrum

$$n^2\frac{\pi^2}{T^2} - \omega^2$$

so there can be negative eigenvalues. In fact the number of negative eigenvalues is given by the Morse index thm, i.e. you count the number of conjugate points along the curve.