May 28, 1979

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It's time to understand the Lagrangian approach and why a symmetry of the Lagrangian leads to a conservation law.

Let's begin with one independent variable t. The Lagrangian will be a function $L(q, \dot{q}, t)$. So we have a bundle over the t-line with fibre coordinates q and a connection so that we can speak of $\dot{q}$. Or else, we think of L as being a function on the 1 jets of the fibre bundle; this seems preferable.

Then we form the action integral

$$\int L(q, \dot{q}, t)\, dt$$

for any section $q = q(t)$ of the bundle. This requires a choice of dt. The variational equations are

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i} = 0$$

Let's look at the case of motion in a central force field:

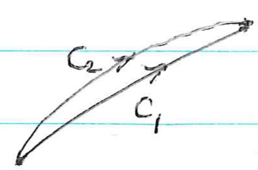
$$L = \tfrac{1}{2}|\dot{q}|^2 - V(|q|)$$

The rotation group $O(3)$ acts as symmetries of the Lagrangian. To simplify work in the plane, so that we have a 1-parameter group of rotations preserving L. How does this give rise to conservation of angular momentum?

<u>First approach:</u> Instead of $q, \dot{q}$ let's use the equivalent Hamiltonian formalism. Here the manifold we're working with has coordinates (t, q, p) and the action is $\int L\,dt$ where

$$L\,dt = p\,dq - H\,dt$$

with H a function of t, q, p. The variational equations can be put in the following form: If we have a curve and a variation of it



then by Stoke's thm

$$\left(\int_{C_2} - \int_{C_1}\right) L\,dt = \iint d(L\,dt).$$

If this is to vanish we need $d(L\,dt)$ to vanish on tangent 2 planes containing the tangent vector to C_1, or equivalently that

$$i(v)\, d(L\,dt) = 0$$

where v is the tangent vector to the integral curve.
Check

$$v = \frac{\partial}{\partial t} + \dot{q}\frac{\partial}{\partial q} + \dot{p}\frac{\partial}{\partial p}$$

$$i\left(\frac{\partial}{\partial t} + \dot{q}\frac{\partial}{\partial q} + \dot{p}\frac{\partial}{\partial p}\right)\left(dp\,dq - \frac{\partial H}{\partial q}dq\,dt - \frac{\partial H}{\partial p}dp\,dt\right)$$

$$0 = \left(-\frac{\partial H}{\partial q} + \dot{p}\right) dq + \left(-\frac{\partial H}{\partial p} - \dot{q}\right) dp + \left(-\dot{p}\frac{\partial H}{\partial p} - \dot{q}\frac{\partial H}{\partial q}\right) dt$$

which yields Hamilton's equations. (Recall $L\,dt$ gives a contact structure on our manifold)

Now suppose we have a vector field X on the manifold which leaves the form $L\,dt$ invariant:

$$\theta(X)(L\,dt) = 0$$

or

$$d\, i(X)(L\,dt) + i(X)\, d(L\,dt) = 0$$

Since $i(v)\, d(L\,dt) = 0$ for integral curves, it follows that the function $i(X)\, d(L\,dt)$ is constant along integral curves.

For example with a central force field in the plane take

$$X = \mathbb{k} \times R = x\hat{j} - y\hat{i} = -y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y}$$

$$L = \frac{1}{2}\left(\dot{x}^2 + \dot{y}^2\right) - V(x)$$

$$H = \frac{1}{2}\left(p_x^2 + p_y^2\right) + V(\vec{x})$$

$$L\,dt = p_x\, dx + p_y\, dy + \left(-\frac{1}{2}\left(p_x^2 + p_y^2\right) + V(\vec{x})\right) dt$$

$$i(X)L\,dt = i\left(-y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y} - p_y\frac{\partial}{\partial p_x} + p_x\frac{\partial}{\partial p_y}\right)(L\,dt)$$

$$= -y p_x + x p_y = \vec{R} \times p$$

2nd approach: Can we do the above without the Hamiltonian formalism. The idea is to work directly on the jet space.

We have to see what the form

$$L\,dt = p\,dq - H\,dt$$

on the (t, q, p) manifold pulls back to on the $(t, q, \dot{q})$ manifold. Recall that p, H are functions of $(t, q, \dot{q})$ given by:

$$p = \left(\frac{\partial L}{\partial \dot{q}}\right)_{t,q \text{ fixed}}$$

$$H = p\dot{q} - L = \left(\frac{\partial L}{\partial \dot{q}}\right)_{t,q} \dot{q} - L(q,\dot{q},t)$$

Thus the action form is

$$L(q,\dot{q},t)\,dt + \left(\frac{\partial L}{\partial \dot{q}}\right)_{t,q} (dq - \dot{q}\,dt)$$

and the integral curves for the Lagrange DE are the curves with tangent vector v such that $i(v)$ kills the d of this form.

The mysterious thing above is the coefficient $\left(\frac{\partial L}{\partial \dot{q}}\right)_{(q,t)}$ of $dq - \dot{q}\,dt$. Now it is clear that we want a stationary curve to satisfy $dq = \dot{q}\,dt$, so that it is appropriate to have some multiple of $dq - \dot{q}\,dt$ in the action. But why this multiple?

Maybe the idea is to look at the variation in $\dot{q}$.

$$\frac{\delta L}{\delta \dot{q}} = \frac{\partial L}{\partial \dot{q}}$$

Hence

$$\frac{\delta}{\delta \dot{q}}\left(L + \frac{\partial L}{\partial \dot{q}}\left(\frac{dq}{dt} - \dot{q}\right)\right) = \frac{\partial^2 L}{\partial \dot{q}^2}\left(\frac{dq}{dt} - \dot{q}\right)$$ so if this is zero we must have $\frac{dq}{dt} = \dot{q}$ on the stationary curve.

This is immensely confusing.

Next consider the situation with several independent variables x_i, say 3, and a single dependent variable ϕ. We suppose given a Lagrangian density

$$\mathcal{L}(\phi, d\phi, x)$$

and we want to make $L = \int \mathcal{L}\, d^3x$ stationary. The Euler-Lagrange equation is

$$\sum_\mu \frac{\partial}{\partial x_\mu}\left(\frac{\partial \mathcal{L}}{\partial \phi_\mu}\right) = \frac{\partial \mathcal{L}}{\partial \phi} \qquad \text{where } \phi_\mu = \frac{\partial \phi}{\partial x_\mu}$$

Is it possible to get this out of a functional on the jet space, i.e. where ϕ, ϕ_μ are allowed to vary independently?

[redacted] Proceeding as above we would start with $\mathcal{L} \cdot d^n x$. The variation of $\mathcal{L}$ with respect to ϕ_μ is

$$\frac{\delta \mathcal{L}}{\delta \phi_\mu} = \frac{\partial \mathcal{L}}{\partial \phi_\mu}$$

So I want to try something like

$$\mathcal{L} + \frac{\partial \mathcal{L}}{\partial \phi_\mu}\left(\frac{\partial \phi}{\partial x_\mu} - \phi_\mu\right)$$

because then if I vary ϕ_μ alone I get

$$\frac{\partial \mathcal{L}}{\partial \phi_\mu} + \frac{\partial^2 \mathcal{L}}{\partial \phi_\mu^2}\left(\frac{\partial \phi}{\partial x_\mu} - \phi_\mu\right) + \frac{\partial \mathcal{L}}{\partial \phi_\mu}(-1)$$

which I hope will force $\frac{\partial \phi}{\partial x_\mu} = \phi_\mu$ on the stationary field.

Try $n=2$. We seem to get

$$\left(\mathcal{L} - \phi_1 \frac{\partial \mathcal{L}}{\partial \phi_1} - \phi_2 \frac{\partial \mathcal{L}}{\partial \phi_2}\right) + \frac{\partial \mathcal{L}}{\partial \phi_1}\frac{\partial \phi}{\partial x_1} + \frac{\partial \mathcal{L}}{\partial \phi_2}\frac{\partial \phi}{\partial x_2}$$

Take the derivative with respect to ϕ_1:

$$-\phi_1 \frac{\partial^2 \mathcal{L}}{\partial \phi_1^2} - \phi_2 \frac{\partial^2 \mathcal{L}}{\partial \phi_1 \partial \phi_2} + \frac{\partial^2 \mathcal{L}}{\partial \phi_1^2}\frac{\partial \phi}{\partial x_1} + \frac{\partial^2 \mathcal{L}}{\partial \phi_1 \partial \phi_2}\frac{\partial \phi}{\partial x_2} \qquad ?$$

May 30, 1979

Given $\mathcal{L}(\phi, \phi_\mu, x_\mu)$ we consider the variational problem

$$\delta \int \mathcal{L}\left(\phi, \frac{\partial \phi}{\partial x_\mu}, x_\mu\right) dx^n = 0$$

which leads to the Euler-Lagrange equation

$$\underbrace{\sum_\mu \frac{\partial}{\partial x_\mu}\left(\frac{\partial \mathcal{L}}{\partial \phi_\mu}\right)} = \frac{\partial \mathcal{L}}{\partial \phi} .$$

$$= \sum_{\mu,\nu} \frac{\partial^2 \mathcal{L}}{\partial \phi_\nu \partial \phi_\mu} \frac{\partial^2 \phi}{\partial x_\mu \partial x_\nu} + \sum_\mu \frac{\partial^2 \mathcal{L}}{\partial \phi \partial \phi_\mu} \frac{\partial \phi}{\partial x_\mu} + \sum_\mu \frac{\partial^2 \mathcal{L}}{\partial x_\mu \partial \phi_\mu}$$

This is a second order PDE for $\phi = \phi(x_\mu)$. When can we solve it locally i.e. near a point (x_μ, ϕ, ϕ_μ)? Perhaps it is natural to require that the leading quadratic form be non-degenerate:

$$\det\left(\frac{\partial^2 \mathcal{L}}{\partial \phi_\nu \partial \phi_\mu}\right) \neq 0$$

(This includes $\frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial t^2}$ but excludes $\frac{\partial^2}{\partial x^2} - \frac{\partial}{\partial t}$.) [redacted]

In deriving the Euler-Lagrange DE one makes variations preserving the relation [redacted] $d\phi = \sum \phi_\mu dx_\mu$. Suppose instead of $\mathcal{L} dx^n$ we consider the form

$$\left\{\mathcal{L} + \sum_\mu \frac{\partial \mathcal{L}}{\partial \phi_\mu}\left(\frac{\partial \phi}{\partial x_\mu} - \phi_\mu\right)\right\} dx^n$$

and we treat ϕ, ϕ_μ as independent variables. Taking the variation with respect to ϕ_ν gives

$$\cancel{\frac{\partial \mathcal{L}}{\partial \phi_\nu}\delta\phi_\nu} + \sum_\mu \frac{\partial^2 \mathcal{L}}{\partial\phi_\nu \partial\phi_\mu}\delta\phi_\nu\left(\frac{\partial\phi}{\partial x_\mu} - \phi_\mu\right) + \cancel{-\frac{\partial\mathcal{L}}{\partial\phi_\nu}\delta\phi_\nu}$$

If we suppose $\dfrac{\partial^2\mathcal{L}}{\partial\phi_\nu\partial\phi_\mu}$ non-degenerate, this forces $\phi_\mu = \dfrac{\partial\phi}{\partial x_\mu}$ for a stationary section, and leads to the Euler-Lagrange DE.

We can introduce new functions on (x_μ, ϕ, ϕ_μ)-space [redacted]

$$p_\mu = \text{[redacted]}\frac{\partial\mathcal{L}}{\partial\phi_\mu}$$

$$\mathcal{H} = -\mathcal{L} + \sum \frac{\partial\mathcal{L}}{\partial\phi_\mu}\phi_\mu$$

whence the basic differential form becomes

$$\boxed{\eta = d\phi \sum_\mu p_\mu\, i\!\left(\frac{\partial}{\partial x_\mu}\right) dx^n - \mathcal{H}\, dx^n}$$

Consider a Lagrangian $\mathcal{L}(\phi, \phi_\mu)$ where $\phi = (\phi^i)$ is a vector function. Suppose we have a 1-parameter symmetry of the Lagrangian:

$$\phi \longmapsto e^{\varepsilon X}\phi$$

where X is a constant matrix. Then for infinitesimal ε, since $\mathcal{L}$ remains constant we have

$$0 = \frac{d}{d\varepsilon} \mathcal{L}\left(e^{\varepsilon X}\phi, e^{\varepsilon X}\phi_\mu\right)\Big|_{\varepsilon=0}$$

$$= \frac{\partial \mathcal{L}}{\partial \phi^i}(X\phi)^i + \frac{\partial \mathcal{L}}{\partial \phi^i_\mu}(X\phi_\mu)^i$$

If ϕ satisfies the Euler-Lagrange DE then

$$\frac{\partial \mathcal{L}}{\partial \phi^i} = \frac{\partial}{\partial x^\mu}\left(\frac{\partial \mathcal{L}}{\partial \phi^i_\mu}\right)$$

hence the above gives the conservation law

$$\boxed{0 = \frac{\partial}{\partial x^\mu}\left(\frac{\partial \mathcal{L}}{\partial \phi^i_\mu}(X\phi)^i\right)}$$

which says that the form

$$(*) \qquad \sum_{i,\mu} \frac{\partial \mathcal{L}}{\partial \phi^i_\mu}(X\phi)^i \; i\left(\frac{\partial}{\partial x_\mu}\right) dx^n$$

is closed when restricted to the section ϕ. The other way to obtain this is to consider the vector field defined by $e^{\varepsilon X}$ on ϕ, ϕ_μ space. This vector field is

$$\tilde{X} = \sum_i (X\phi)^i \frac{\partial}{\partial \phi^i} + \sum_{i,\mu} (X\phi)^i \frac{\partial}{\partial \phi^{i\mu}}$$

and just as for the 1-dimensional case $(*) = i(\tilde{X})\eta$ will be closed when restricted to a stationary section ϕ.

The next point is to understand how a

Lagrangian with a symmetry of the above type can be made invariant under local gauge transformations provided one introduces new fields called gauge bosons.

Simple example: Take complex scalar field ϕ with the Lagrangian

$$\mathcal{L} = \phi^*_\mu \phi_\mu - m^2 \phi^* \phi$$

Then we have the symmetry $\phi_\mu \mapsto e^{\varepsilon} \phi_\mu$, $\phi^*_\mu \mapsto e^{-\varepsilon} \phi^*_\mu$.

EL DE: $$\begin{cases} \dfrac{\partial}{\partial x_\mu} \phi^*_\mu + m^2 \phi^* = 0 \\[2ex] \dfrac{\partial}{\partial x_\mu} \phi_\mu + m^2 \phi = 0. \end{cases}$$

Conservation Law: $$\frac{\partial}{\partial x^\mu}\left(\phi^*_\mu \phi - \phi_\mu \phi^* \right) = 0$$

May 31, 1979

Let's consider a complex scalar field ϕ equipped with the Lagrangian

$$\mathcal{L}_m = \phi^*_\mu \phi_\mu - m^2 \phi^* \phi \qquad \phi_\mu = \frac{\partial \phi}{\partial x^\mu} = \partial_\mu \phi$$

The standard way to introduce the electromagnetic field $A = (A_\mu)$ is via the replacement

$$\partial_\mu \phi \longmapsto (\partial_\mu - iqA_\mu)\phi$$

whence we get

$$\mathcal{L}_m = (\partial_\mu + iqA_\mu)\phi^* (\partial_\mu - iqA_\mu)\phi - m^2 \phi^* \phi$$

where the subscript m stands for "matter"

Construct Lagrangian for a charged particle moving in a fixed electromagnetic field.

Lorentz force $\quad F = qE + qv \times B$

so

$$\frac{d}{dt}(mv) = qE + qv \times B$$

is the equation of motion. Recall $B = \nabla \times A$, $E = -\nabla\phi - \frac{\partial A}{\partial t}$.

$$v \times B = v \times (\nabla \times A) = \nabla(v \cdot A) - (v \cdot \nabla)A$$

so

$$\frac{d}{dt}(mv) = -q\nabla\phi + q\nabla(v \cdot A) \underbrace{- q\frac{\partial A}{\partial t} - q(v \cdot \nabla)A}_{-q\left(\frac{dA}{dt}\right)}$$

where we recall that A is a

function of position x and time t. Thus

$$\frac{d}{dt}(m\vec{v} + q\vec{A}) = \nabla(-q\phi + qv \cdot A)$$

which is the Euler diff. equation with

$$L(t,x,v) = \frac{1}{2}mv^2 - q\phi + qv \cdot A$$

In terms of this Lagrangian, the momentum of the particle is

$$\vec{p} = \frac{\partial L}{\partial \vec{v}} = m\vec{v} + q\vec{A}$$

June 1, 1979

It seems that the relation between Hamilton-Jacobi PDE and Hamilton's equations gives the complete story about solving first order PDE's with one dependent variable.

Let us begin with a first order PDE with independent variables $x^1, \ldots, x^n$ and dep. variable y:

$$(*) \qquad F(x, \nabla y) = 0.$$

not the most general since y doesn't appear.

We want to solve the Cauchy problem which means to prescribe y on a hypersurface. If we change independent variables we can suppose the hypersurface is $x^1 = 0$. Put $t = x^1$ and let q_i denote the remaining independent variables. Assume $(*)$ can be solved for $\frac{\partial y}{\partial t}$, so it becomes a Hamilton-Jacobi PDE:

$$\frac{\partial y}{\partial t} + H\left(t, q, \frac{\partial y}{\partial q}\right) = 0$$

We want to solve this with y prescribed on $t = 0$.

Associated to H is the 1-form on (t, q, p)-space

$$\eta = p\,dq - H\,dt$$

whose stationary curves are obtained by solving Hamilton's equations

$$\dot{q} = \frac{\partial H}{\partial p} \qquad \dot{p} = -\frac{\partial H}{\partial q}$$

Let's recall that if we fix q', t' and let $S(q, t)$ denote the action $(= \int \eta)$ along the stationary curve starting at (t', q') ending at (t, q), then S satisfies the Hamilton-Jacobi DE.

In effect we know that

$$\delta\int_{t_1}^{t_2}\eta = \int_{t_1}^{t_2}\delta p\, dq + p\,d\delta q - \frac{\partial H}{\partial q}\delta q\, dt - \frac{\partial H}{\partial p}\delta p\, dt$$

$$= \Big[p\,\delta q\Big]_{t_1}^{t_2} + \int_{t_1}^{t_2}\delta p\left(dq - \frac{\partial H}{\partial p}dt\right) + \delta q\left(-dp - \frac{\partial H}{\partial q}dt\right)$$

The integral vanishes for stationary curves, so for $S(t,q) = S(t,q,t',q')$ we have

$$\frac{\partial S}{\partial q} = p$$

Also

$$\frac{d}{dt}S = \frac{\partial S}{\partial q}\frac{dq}{dt} + \frac{\partial S}{\partial t}$$

$$\parallel$$

$$p\frac{dq}{dt} - H$$

so we have $\dfrac{\partial S}{\partial t} + H\left(q, \dfrac{\partial S}{\partial q}, t\right) = 0$ as claimed.

The next point is to generalize so that paths don't have to start at a fixed point (t',q'). Let us suppose S is given along $t=0$. Then at each point q we take the stationary curve with $p = \dfrac{\partial S}{\partial q}$. We define

$$S(t,q) = S(0,q') + \int_0^t \eta$$

where among the stationary curve family selected we choose the one going through (t,q) and take $\int_0^t \eta$ of it; $(0,q')$ is the initial point of this curve. Now varying q but keeping t fixed we get a corresponding variation of q' such that

$$\delta\int_0^t \eta = p\,\delta q - p'\delta q'$$

Thus
$$\delta S(t,q) = \delta S(0,q') + \delta\int_0^t \eta$$
$$= \cancel{\frac{\partial S}{\partial q}(0,q')\,\delta q'} + p\,\delta q - \cancel{p'\,\delta q'}$$

and so we see
$$\frac{\partial S}{\partial q} = p$$

But it is also true that $\dfrac{dS}{dt} = \dfrac{\partial S}{\partial q}\dfrac{dq}{dt} + \dfrac{\partial S}{dt} = p\dfrac{dq}{dt} - H$

so we have solved the Cauchy problem for the Hamilton-Jacobi equation.

June 2, 1979

Consider a Hamilton-Jacobi PDE

$$(1) \qquad \frac{\partial y}{\partial t} + H\left(t, q, \frac{\partial y}{\partial q}\right) = 0$$

where $q = (q_1, \ldots, q_n)$ and $\frac{\partial y}{\partial q} = \left(\frac{\partial y}{\partial q_i}\right)$ in general. The idea is to understand what solutions of this look like. Introduce the 1-form on (t,q,p)-space:

$$\eta = p\,dq - H(t,q,p)\,dt$$

Then $d\eta = dp\,dq - dH\,dt$ has rank $2n$ since it is non-degenerate when restricted to $t =$ constant. The kernel of $d\eta$ is a tangent line at each point of (t,q,p)-space which is transversal to $t =$ const. So there is the Hamilton vector field

$$X = \frac{\partial}{\partial t} + \dot{q}\frac{\partial}{\partial q} + \dot{p}\frac{\partial}{\partial p}$$

characterized by

$$i(X)\,d\eta = 0 \qquad \text{and} \qquad i(X)\,dt = 1.$$

What I want to show next is that solutions of (1) are essentially the same thing as Lagrangian submanifolds of (t,q,p)-space for the form $d\eta$, by which I mean $(n+1)$-dimensional submanifolds whose tangent spaces are isotropic for $d\eta$. The first point is that if we restrict $d\eta$ to $t =$ const t_0, then $d\eta$ becomes $dp\,dq$ and we know that Lagrangian submanifolds for $dp\,dq$ are given by closed 1-forms: (Review this: if $\omega = \sum a_i\,dq_i$ is a 1-form, then as a section of T^*

one has $\quad p_i(\omega) = a_i$, hence

$$\omega^*\Big(\sum_i p_i dq_i\Big) = \omega$$

It follows that $\quad \omega^*(\sum dp_i dq_i) = d\omega$, so $d\omega = 0$
$\Leftrightarrow$ the section ω is Lagrangian.)

Let's take a solution $S(t,q)$ of (1). Then I want to show that $d\eta$ is killed by pulling back via the map

$$\psi : t, q \longmapsto \Big(t, q, \frac{\partial S}{\partial q}\Big).$$

But

$$\psi^*\eta = \frac{\partial S}{\partial q} dq - H\Big(t, q, \frac{\partial S}{\partial q}\Big) dt$$

$$= \frac{\partial S}{\partial q} dq + \frac{\partial S}{\partial t} dt = dS$$

so $\quad \psi^*(d\eta) = 0.$ Conversely given a map

$$\psi : t, q \longmapsto (t, q, p_\psi)$$

whose image is Lagrangian for $d\eta$, we have

$$d\,\psi^*\eta = \psi^*(d\eta) = 0$$

so $\psi^*\eta = dS$ where S is unique up to an additive constant. Then

$$\psi^*\eta = p_\psi dq - H(\cancel{t}, q, p_\psi)\, dt$$

$$dS = \frac{\partial S}{\partial q} dq + \frac{\partial S}{\partial t} dt$$

and so $\quad p_\psi = \dfrac{\partial S}{\partial q}$ and S ~~satisfies~~ $\dfrac{\partial S}{\partial t} + H\Big(t, q, \dfrac{\partial S}{\partial q}\Big) = 0$

I see now that a Hamilton-Jacobi DE doesn't involve S. Thus we have to do more work in order to handle the general case.

First let's look at the equation

$$H\left(q, \frac{\partial S}{\partial q}\right) = 0$$

where a time coordinate hasn't been singled out. To solve this we use the Hamilton flow in (q,p)-space associated to H. To solve the Cauchy problem where S is given along $q_1 = 0$, we take the image of $\{q_1 = 0\} \to \left(q, \frac{\partial S}{\partial q}\right)$ and let it move under the Hamilton flow. This requires that we can solve for $\frac{\partial S}{\partial q_1}$ near our starting point, so we want to assume that

$$\frac{\partial H}{\partial p_1} \neq 0$$

Since $\dot{q}_1 = \frac{\partial H}{\partial p_1}$, the flow moves transversally to $q_1 = 0$, and we know H stays constant. The only question is why the manifold swept out is Lagrangian for $dp\,dq$.

June 3, 1979

To understand solutions of general first order PDE with one dependent variable (from Ford's book):

$$f\left(q, S, \frac{\partial S}{\partial q}\right) = 0$$

~~[illegible]~~

Let us fix a point (q_0, S_0). The above equation gives us the possible tangent planes to solutions passing thru (q_0, S_0). The tangent plane is

$$dS = p\,dq \quad \text{where} \quad f(q_0, S_0, p) = 0.$$

Let us fix one of these tangent planes: p_0. An infinitesimally nearby tangent plane is given by

$$dS = (p_0 + \delta p)\,dq \quad \text{where} \quad \underbrace{f(q_0, S_0, p_0 + \delta p)}_{= \left.\frac{\partial f}{\partial p}\right|_0 \delta p} = 0$$

Assuming $\frac{\partial f}{\partial p} \neq 0$ at points of interest, the possible δp form an $(n-1)$ diml vector space, so we have $(n-1)$ independent nearby tangent planes, and they intersect in a line. This line ~~[illegible]~~ contains vectors dq, dS such that

$$dS = (p_0 + \delta p)\,dq \quad \text{for all } \delta p \text{ with } \left.\frac{\partial f}{\partial p}\right|_0 \delta p = 0$$

hence dq must be a multiple of $\left.\frac{\partial f}{\partial p}\right|_0$:

$$dq = \left.\frac{\partial f}{\partial p}\right|_0 dt$$

and then $dS = p_0\,dq$.

Next suppose we have a solution of the given

DE: $S = S(q)$. The preceding procedure gives us a tangent line inside each tangent plane to the solution, namely the line in

$$dS = p\,dq \qquad p = \frac{\partial S}{\partial q}$$

$$\text{with} \quad dq = \frac{\partial f}{\partial p}\,dt$$

Thus the solution hypersurface has a natural vector field on it. The miracle is that integral curves of the vector field can be be determined directly. All we have to do is give a point q, S and the tangent plane of the solution at this point. This determines the integral curve independent of the rest of the solution. To see this we compute for our solution $S(q)$:

$$\frac{dp_i}{dt} = \frac{d}{dt}\left(\frac{\partial S}{\partial q_i}\right) = \sum_j \frac{\partial^2 S}{\partial q_j \partial q_i} \underbrace{\frac{dq_j}{dt}}_{\frac{\partial f}{\partial p_j}}$$

But

$$0 = \frac{\partial}{\partial q_i}\left(f\left(q, S, \frac{\partial S}{\partial q}\right)\right) = \frac{\partial f}{\partial q_i} + \frac{\partial f}{\partial S}\frac{\partial S}{\partial q_i} + \sum_j \frac{\partial f}{\partial p_j}\frac{\partial^2 S}{\partial q_i \partial q_j}$$

so we conclude

$$\frac{dp_i}{dt} = -\frac{\partial f}{\partial q_i} - p_i \frac{\partial f}{\partial S}$$

So the equations of the bicharacteristic strip are:

$$\frac{dq}{dt} = \frac{\partial f}{\partial p} \qquad \frac{dS}{dt} = p\frac{\partial f}{\partial p} \qquad \frac{dp}{dt} = -\frac{\partial f}{\partial q} - p\frac{\partial f}{\partial S}$$

June 4, 1979

Why $f(q, S, \frac{\partial S}{\partial q}) = 0$ is a special case of $H(x, \frac{\partial \varphi}{\partial x}) = 0$

Set $x = (q, S)$ and notice that [redacted] the former equation gives a family of hyperplanes $dS = p\,dq$ [redacted] at each $x = (q, S)$ for each p such that $f(q, S, p) = 0$. The latter gives a family of hyperplanes $\xi\,dx = 0$ for each ξ such that $H(x, \xi) = 0$.

$$dS - p\,dq = 0 \quad \text{same as} \quad dx_{n+1} + \frac{\xi_1 dx_1 + \dots + \xi_n dx_n}{\xi_{n+1}} = 0$$

so you want $p = -\frac{\vec{\xi}}{\xi_{n+1}}$, and so put

$$H(x, \xi) = f\left(q, S, -\frac{\vec{\xi}}{\xi_{n+1}}\right).$$

It turns out better to put

$$H(x, \xi) = -\xi_{n+1} f\left(q, S, -\frac{\vec{\xi}}{\xi_{n+1}}\right)$$

for then f linear $\Rightarrow$ H linear. Also the formulas below are simpler.

Bicharacteristic flow:

$$\frac{dq_i}{dt} = \frac{dx_i}{dt} = \frac{\partial H}{\partial \xi_i} = -\xi_{n+1} \frac{\partial f}{\partial p_i}(q, S, p) - \frac{1}{\xi_{n+1}} = \frac{\partial f}{\partial p_i}$$

[redacted] $$\frac{dS}{dt} = \frac{dx_{n+1}}{dt} = \frac{\partial H}{\partial \xi_{n+1}} = -f - \xi_{n+1} \sum \frac{\partial f}{\partial p_j} \frac{\xi_j}{\xi_{n+1}^2}$$

$$= \sum_j p_j \frac{\partial f}{\partial p_j} \quad \text{because } f = 0 \text{ on the interesting lines}$$

$$\frac{dp_i}{dt} = \frac{d}{dt}\left(-\frac{\xi_i}{\xi_{n+1}}\right) = -\frac{1}{\xi_{n+1}}\left(-\frac{\partial H}{\partial x_i}\right) + \frac{\xi_i}{\xi_{n+1}^2}\left(\frac{-\partial H}{\partial x_{n+1}}\right) = -\frac{\partial f}{\partial q_i} - p_i \frac{\partial f}{\partial S}$$

June 5, 1979

Consider $H(q, \frac{\partial S}{\partial q}) = 0$ where $H(q,p)$ is a homogeneous function of p of degree r, so that Euler's theorem gives

$$p\frac{\partial H}{\partial p} = rH.$$

Let S be a solution of the DE. The hypersurface $S =$ constant has tangent plane given by

$$p\,dq = 0 \quad \text{where} \quad p = \frac{\partial S}{\partial q}.$$

Suppose we fix q but vary p infinitesimally to $p + \delta p$ so that we have

$$0 = H(q, p+\delta p) = H(q,p) + \frac{\partial H}{\partial p}\delta p$$

or

$$\frac{\partial H}{\partial p}\delta p = 0.$$

Then the hyperplanes

$$(p+\delta p)\,dq = 0 \quad \text{or} \quad \delta p \cdot dq = 0$$

intersect in the line

$$dq = \text{multiple of } \frac{\partial H}{\partial p}$$

(Check: Clearly $\delta p \cdot dq = 0$ for all δp satisfying $\frac{\partial H}{\partial p}\delta p = 0$ implies $dq = c\frac{\partial H}{\partial p}$. Conversely using Euler:

$$(p+\delta p)\frac{\partial H}{\partial p} = p\frac{\partial H}{\partial p} = rH = 0.)$$

Therefore each tangent plane to $S =$ const contains a distinguished line and so we can define a flow by

$$\frac{dq}{dt} = \frac{\partial H}{\partial p}.$$

To integrate this we need to know $p = \frac{\partial S}{\partial q}$. The miracle is that we can determine $\frac{dp}{dt}$ directly from the values of p, q on the curve:

$$0 = \frac{\partial}{\partial q} H\left(q, \frac{\partial S}{\partial q}\right) = \frac{\partial H}{\partial q} + \frac{\partial H}{\partial p}\frac{\partial^2 S}{\partial q^2}$$

Hence

$$\frac{dp}{dt} = \frac{d}{dt}\left(\frac{\partial S}{\partial q}\right) = \frac{\partial^2 S}{\partial q^2}\frac{dq}{dt} = -\frac{\partial H}{\partial q}$$

To solve the Cauchy problem for $H\left(q, \frac{\partial S}{\partial q}\right) = 0$ suppose S given on $q_1 = 0$ and that at $q = 0$ we have given p° with $H(0, p^\circ) = 0$ and $p_i^\circ = \frac{\partial S}{\partial q_i}(0)$, $i > 1$. Assume $\frac{\partial H}{\partial p_1} \neq 0$ at $(0, p^\circ)$ so that the implicit fn. thm can be used to construct p over $q_1 = 0$, with $H(q,p) = 0$ and $p_i = \frac{\partial S}{\partial q_i}$, $i > 1$. Then one solves the Hamilton's equations, and since $\dot q_1 = \frac{\partial H}{\partial p_1} \neq 0$, the flow is transversal to $q_1 = 0$. Then S is constant on the trajectories because

$$\frac{dS}{dt} = \frac{\partial S}{\partial q}\frac{dq}{dt} = p\frac{\partial H}{\partial p} \overset{\text{Euler}}{=} rH = 0$$

because H is constant.

The interesting thing about the trajectories in this case is that the action S is constant because

$$\frac{dS}{dt} = p\frac{dq}{dt} - H = p\frac{\partial H}{\partial p} - H = rH - H = 0$$

If $r = 1$, the action is constant for all trajectories.

June 8, 1979

Program: To understand the calculus of variations well-enough so I can develop Feynman integrals. The intriguing idea is there might be some kind of diffraction or quantization belonging to any variational problem. The key ideas are amplitude, interference, diffraction, stationary phase, steepest descent.

One might hope that to any variational problem belongs a diffraction problem. So suppose we take the simplest case of a non-degenerate fn.

$$f : M \longrightarrow \mathbb{R}$$

Example: M is embedded in Euclidean space and f is distance^2 from a fixed point.

Now the idea is to assign to each $m \in M$ the amplitude $e^{ikf(m)}$ and to average over M:

$$\psi(k) = \int e^{ikf(m)}\, d\mu$$

This gives us the characteristic function of the measure $d\mu$ pushed forward by f.

The method of stationary phase says that for large k this depends only on the critical points of f. Specifically use a partition of unity in order to worry only about $d\mu$ supported in a ball. If $df \neq 0$ on this ball, then as $d\mu$ is supposed to be smooth, we see by the extended Riemann-Lebesgue lemma that the above integral is $O(k^{-N})$ for all k.

So let's look at a critical point, assumed to be non-degenerate. By Morse's lemma the function f can be expressed in local coordinates

$$f = x^2 - y^2 \qquad \text{if } f(\text{point}) = 0$$

so

$$\psi(k) = \int e^{ikx^2 - iky^2} g(x,y)\, d^p x\, dy^{n-p} \qquad g \in C_0^\infty$$

The problem now is to find the asymptotic development of $\psi(k)$. But

$$\psi(k) = \int e^{ix^2 - iy^2} g\left(\frac{x}{\sqrt{k}}, \frac{y}{\sqrt{k}}\right) \frac{d^p x\, d_y^{n-p}}{k^{n/2}}$$

so it looks as if

$$\psi(k) \sim k^{-n/2} g(0,0) \int e^{ix^2} e^{-iy^2} d^p x\, dy^{n-p}.$$

The last integral is easily evaluated, since

$$\int_{-\infty}^{\infty} e^{ix^2} dx = \int_{-\infty}^{\infty} e^{-u^2} e^{\frac{i\pi}{4}} du = \sqrt{\pi}\, e^{i\frac{\pi}{4}}$$

The conclusion seems to be that if f is non-degenerate with critical points p_i then

$$\psi(k) \sim \sum_i e^{ikf(p_i)} \alpha(p_i) k^{-n/2}$$

where $\alpha(p_i)$ is computed as follows. At a critical point one has the Hessian of f which is a non-degenerate quadratic form on the tangent space and so defines a volume element dV at the critical point. Comparing $d\mu$ with this volume gives us a number $h = \frac{d\mu}{dV}(p_i)$.

[redacted] Then $\alpha(p_i) = (\sqrt{\pi})^n \left(e^{\frac{i\pi}{4}}\right)^{2p-n} h$ where p is the number of positive eigenvalues of the Hessian. (Better maybe instead of p to use $q = n - p =$ [redacted] the number of negative eigenvalues; [redacted] then $2p - n = 2(n-q) - n = n - 2q$. Notice that if the indices are even then the arguments of $\alpha(p_i)$ are the same, assuming $d\mu \geqslant 0$.

$$\delta L = \frac{\partial L}{\partial q}\delta q + \frac{\partial L}{\partial \dot q}\frac{d}{dt}(\delta q) = \left(\frac{\partial L}{\partial q} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot q}\right)\right)\delta q + \frac{d}{dt}\left(\frac{\partial L}{\partial \dot q}\delta q\right)$$

hence if $q = q(t)$ satisfies Euler's DE one has

$$\delta L = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot q}\delta q\right)$$

For example suppose $L = L(q, \dot q)$ and we consider the variation $(q + \delta q)(t) = q(t + \delta t)$, i.e. δq [redacted] $= \frac{dq}{dt}\delta t$. Then

$$\delta L = \frac{d}{dt}(L) \cdot \delta [redacted] t$$

so combining with the above formula for δL we get

$$\frac{d}{dt}\Big(\underbrace{\frac{\partial L}{\partial \dot q}\frac{dq}{dt} - L}_{H}\Big) = 0$$

i.e. that the Hamiltonian is time-independent.

June 9, 1979

Let me try next to understand geometric optics from the wave equation viewpoint. Let's derive the eikonal equation. The scalar wave equation is

$$\frac{\partial^2 \psi}{\partial t^2} = \Delta \psi .$$

Assume one has a solution of the form

$$\psi = A e^{iS}$$

where the argument S is rapidly varying and the amplitude A is slowly varying. Substitute:

$$\nabla \psi = (\nabla A) e^{iS} + A e^{iS} i \nabla S$$

$$\nabla^2 \psi = \nabla \cdot \left((\nabla A) e^{iS} + A e^{iS} i \nabla S \right)$$

$$= (\nabla^2 A) e^{iS} + 2(\nabla A \cdot \nabla S) i e^{iS} + A e^{iS} (i \nabla S)^2 + A e^{iS} i (\nabla^2 S).$$

$$\partial_t^2 \psi = (\partial_t^2 A) e^{iS} + 2(\partial_t A \, \partial_t S) i e^{iS} + A e^{iS} (i \partial_t S)^2 + A e^{iS} i (\partial_t^2 S)$$

The leading terms are the ones involving $(\nabla S)^2, (\partial_t S)^2$; think of S as $k\varphi$ where $k \to \infty$. Thus we see S should satisfy the eikonal equation

$$(\nabla S)^2 = (\partial_t S)^2$$

and then we want

$$\partial_t^2 A + 2i\, \partial_t A\, \partial_t S + iA\,(\partial_t^2 S)$$
$$= \nabla^2 A + 2i\, \nabla A \cdot \nabla S + iA(\nabla^2 S).$$

The idea is to solve this for A by an asymptotic series

$$A = A_0 + \frac{A_1}{k} + \frac{A_2}{k^2} + \cdots$$

where $S = k\varphi$. The equations satisfied by the A_n are

$$\underbrace{\Big[2i(\partial_t S)\,\partial_t - 2i(\nabla S)\cdot\nabla\Big]A + i\big(\partial_t^2 S - \nabla^2 S\big)A}_{\text{first order operator for } A_n} = \underbrace{\nabla^2 A - \partial_t^2 A}_{\text{depends on } A_{n-1}}$$

There are a number of curious coincidences involving square roots of operators. I propose to list these with the hope of eventually sorting them out.

1. Geodesics: These curves can be obtained as the stationary curves for $\int ds$ or for the Kinetic Energy Lagrangian. The latter method is better because the curves are parameterized by arc length.

2. When studying the wave equation on a Riemannian manifold, Duistermaat + Guillemin, following Hörmander work with

$$\frac{\partial u}{\partial t} = -i\sqrt{-\Delta}\, u$$

The symbol of $\sqrt{-\Delta}$ is a homogeneous function of order 1 on T^*, and we have seen that for such a Hamilton the action is constant along all trajectories.

3. From the physics viewpoint, the natural thing to look at is the Schroedinger equation

$$\frac{\partial u}{\partial t} = -i(-\Delta)u$$

instead of the wave equation. But relativity seems to force one to look at the wave equation + mass term, instead of the Schroedinger equation.

June 10, 1979

Let us be given a solution $S = S(t,q)$ of the Hamilton-Jacobi DE:

$$\frac{\partial S}{\partial t} + H\left(t, q, \frac{\partial S}{\partial q}\right) = 0$$

Consider the graph of S in (t,q,S)-space; at the point $(t, q, S(t,q))$ the tangent plane is given by

$$dS = \frac{\partial S}{\partial t}dt + \frac{\partial S}{\partial q}dq = p\,dq - H\,dt$$

where we put $p = \frac{\partial S}{\partial q}$. Next consider an infinitesimal variation of the tangent plane lying in the family of tangent planes specified by the Hamilton-Jacobi DE. Such a infinitesimal close plane will be given by

$$dS = (p+\delta p)\,dq - \underbrace{H(t,q,p+\delta p)}_{H + \frac{\partial H}{\partial p}\delta p}dt$$

Intersecting these infinitesimally near planes gives a line containing (dt, dq, dS) when

$$dS = p\,dq - H\,dt$$

$$\delta p\,dq = \frac{\partial H}{\partial p}\delta p \qquad \text{for all } \delta p$$

so we get the line described by

$$dq = \frac{\partial H}{\partial p}dt \qquad dS = \left(p\frac{\partial H}{\partial p} - H\right)dt.$$

This line lies in the tangent plane to the graph of S, so the graph has a canonical vector field.

Next taking an integral curve $q(t)$ of this vector field we find:

$$\frac{d}{dt}(p) = \frac{d}{dt}\frac{\partial S}{\partial q}(t, q(t)) = \frac{\partial^2 S}{\partial t \partial q} + \frac{\partial^2 S}{\partial q^2}\frac{dq}{dt}$$

Differentiating the Ham. Jac. DE wrt q gives

$$\frac{\partial^2 S}{\partial q \partial t} + \frac{\partial H}{\partial q} + \frac{\partial H}{\partial p}\frac{\partial^2 S}{\partial q^2} = 0$$

and so we see that on the integral curve $q = q(t)$ we have

$$\frac{dp}{dt} = -\frac{\partial H}{\partial q} \qquad \text{where } p = \frac{\partial S}{\partial q}(t, q(t)).$$

Next suppose that we are given a family of solutions $S(t, q, \alpha)$ of the Hamilton-Jacobi D.E. Then we have

$$\frac{\partial}{\partial t} S(t, q(t), \alpha) = p\frac{dq}{dt} - H(t, q, p)$$

with $\quad p = \frac{\partial S}{\partial q}(t, q, \alpha) \qquad q = q(t)$

for each α. Differentiating wrt α we get

$$\frac{\partial}{\partial t}\frac{\partial S}{\partial \alpha}(t, q(t), \alpha)) = \frac{\partial p}{\partial \alpha}\frac{dq}{dt} - \frac{\partial H}{\partial p}\frac{\partial p}{\partial \alpha} = 0$$

which shows that $\frac{\partial S}{\partial \alpha}(t, q, \alpha)$ will be constant on each of the trajectories associated to $S(t, q, \alpha)$.

Recall $q = (q_1, \dots, q_n)$. Suppose the parameter α is n-fold $\alpha = (\alpha_1, \dots, \alpha_n)$ and that the equation

$$p = \frac{\partial S}{\partial q}(0, q, \alpha)$$

can be inverted so that α is a function of p. Then in order to find the trajectory with initial values (q_0, p_0) at $t=0$, we choose α_0 in this way. Then [redacted] the trajectory is given by the equation

$$\frac{\partial S}{\partial \alpha}(t, q, \alpha) = \frac{\partial S}{\partial \alpha}(0, q_0, \alpha_0)$$

The solution $S(t, q, \alpha)$ depending on the n-constants α is called a <u>complete</u> solutions of the Hamilton-Jacobi DE.

Let's try to understand solutions of the Hamilton-Jacobi equation where

$$L(q, \dot q) = |\dot q| = (\dot q^2)^{1/2}$$

$$p = \frac{\partial L}{\partial \dot q} = \text{[redacted]} \; \frac{1}{2}(\dot q^2)^{-1/2}(2\dot q) = \frac{\dot q}{|\dot q|}$$

Unfortunately this situation is degenerate. We can't make a change of variable, because the momenta p_i are [redacted] not independent variables. In any case we can form

$$H = p\dot q - L = \frac{\dot q}{|\dot q|}\dot q - |\dot q| = 0.$$

(Curious: If L is homogeneous in $\dot q$ of degree r, then Euler's thm. says

$$H = \dot q \frac{\partial L}{\partial \dot q} - L = (r-1)L.)$$

Next consider the Hamilton-Jacobi DE:

$$\frac{\partial S}{\partial t} + |\nabla S| = 0$$

where the Hamiltonian is $H(q,p) = |p|$. Hamilton's equations are

$$\dot{q} = \frac{\partial H}{\partial p} = \frac{p}{|p|} \qquad \dot{p} = -\frac{\partial H}{\partial q} = 0.$$

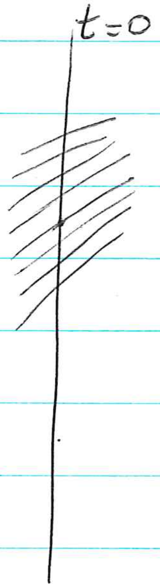
Because H is homogeneous of degree 1 one has

$$\frac{dS}{dt} = p\dot{q} - H = \frac{p^2}{|p|} - |p| = 0.$$

so S is constant on the trajectories. The trajectories are straight lines

$$q = \frac{p}{|p|}t + \text{const}, \qquad p = \text{const}.$$

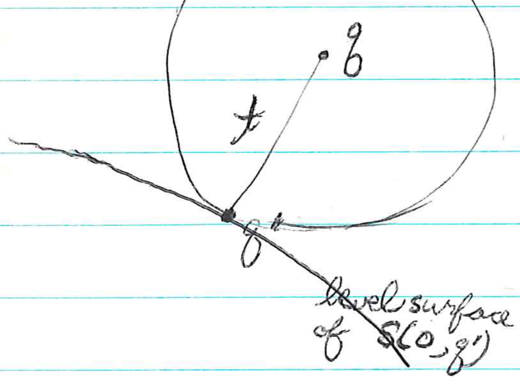
To solve for S we suppose $S(0,q)$ is given such that $p = \frac{\partial S}{\partial q} \neq 0$ around where we are working. Then for each q we have a direction $\frac{p}{|p|}$ for a line and so one gets a family of lines



The way to think. $S(0,q)$ gives us a family of level surfaces – surfaces of constant phase. The assumption $p = \frac{\partial S}{\partial q} \neq 0$ means these surfaces are non-singular and then $\frac{p}{|p|}$ is the unit vector normal to the surface in the direction of increasing S. Then the fact $S(t, \frac{p}{|p|}t + q)$ is constant means these constant phase surfaces propagate in time via Huyghen's principle at unit speed.

Thus to compute $S(t,q)$, you look at the sphere of radius t around q and look on this sphere for the minimum value of $S(0,q')$. Then

$$S(t,q) = \min \{ S(0,q') \mid |q - q'| = t \}.$$



It's clear geometrically that $S(t,q)$ is constant as q varies $\perp$ to the line $q'q$. If q varies δq so as to extend the line $q'q$, then clearly $S(t, q+\delta q)$ increases, hence $\frac{\partial S}{\partial q}$ points in the direction of $q'q$. But if t changes by $\delta t = |\delta q|$, then S doesn't change so

$$0 = \delta S = \frac{\partial S}{\partial t}\delta t + \frac{\partial S}{\partial q}\delta q$$

$$= \left(\frac{\partial S}{\partial t} + \left| \frac{\partial S}{\partial q} \right| \right) \delta t$$

because δq is in the same direction as $\frac{\partial S}{\partial q}$. So S does satisfy the Hamilton-Jacobi DE.

June 11, 1979

Phase & Group Velocity:

Standard example (from French or Furry-Purcell-Street): Superimpose two sinusoidal waves of same amplitudes. Need trig identity:

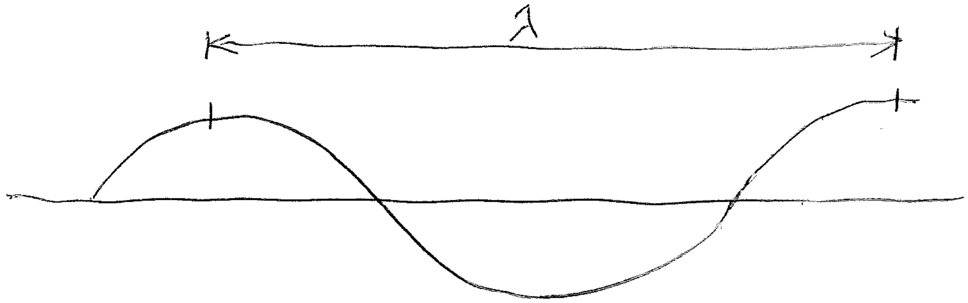
$$\sin(A+B) + \sin(A-B) = 2\sin A \cos B$$

$$\sin\alpha + \sin\beta = 2\sin\left(\frac{\alpha+\beta}{2}\right)\cos\left(\frac{\alpha-\beta}{2}\right)$$

Take the wave

$$u(x,t) = \sin(kx - \omega t)$$

Here ω is the <u>frequency</u>, the thing you see when you fix x and look at the vibration. Here k is the <u>wave number</u>; it's essentially the wave-length, i.e. the distance between crests in the wave:



$$k\lambda = 2\pi \qquad \boxed{k = \frac{2\pi}{\lambda}}$$

The <u>phase velocity</u> of the wave is the speed at which the crests move, which is

$$\boxed{v = \frac{\omega}{k} = \frac{\omega\lambda}{2\pi}}$$

For light waves v is always c, so that the relation between frequency and wave-length is

$$\lambda = 2\pi \frac{c}{\omega}$$

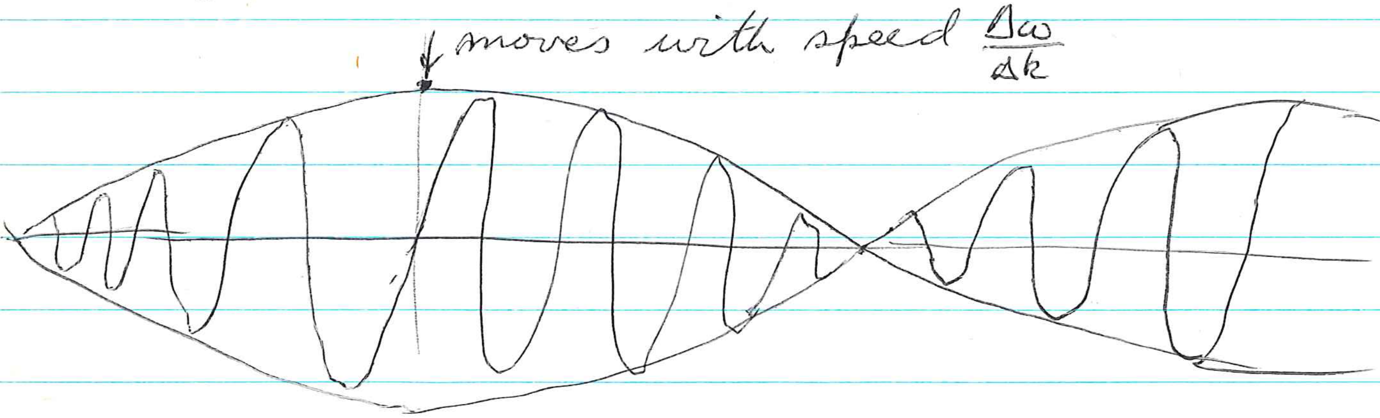
Now let's superimpose two sinusoidal waves of the same amplitude but slightly different frequencies + wave numbers:

$$\sin(kx - \omega t) + \sin(k'x - \omega' t)$$

$$= 2\cos\left(\frac{k'-k}{2}x - \frac{\omega'-\omega}{2}t\right)\sin\left(\frac{k'+k}{2}x - \frac{\omega'+\omega}{2}t\right)$$

$$\approx 2\cos\left(\frac{\Delta k}{2}x - \frac{\Delta\omega}{2}t\right)\sin(kx - \omega t)$$

so what we get is a modulated wave of the same frequency and wave number, with amplitude moving with speed $\frac{\Delta\omega}{\Delta k}$.



The speed of the modulated signal is called the group velocity. So we get the formulas

$$\text{phase velocity} = \frac{\omega}{k}$$

$$\text{group velocity} = \frac{d\omega}{dk}.$$

Let's now consider the Hamilton Jacobi DE

$$(1) \qquad \frac{\partial S}{\partial t} + H\left(q, \frac{\partial S}{\partial q}\right) = 0$$

where H does not depend on time. Also it will be useful to think of

$$(2) \qquad H(q,p) = [redacted] \frac{p^2}{2m} + V(q)$$

Let's recall how solutions of (1) are obtained. Starting with $S(0,q)$ given, for each q we consider the trajectory (= solution of Hamilton's equations) starting at q with $p = \frac{\partial S}{\partial q}(0,q)$. [redacted] Along this trajectory one has

$$\frac{dS}{dt} = p\frac{\partial H}{\partial p} - H = [redacted]$$

$$= \frac{p^2}{2m} - V(q)$$

which we can integrate to find S.

Special case occurs where all the trajectories have the same energy E. This means

$$H\left(q, \frac{\partial S}{\partial q}(0,q)\right) = E$$

and it implies that the solution $S(t,q)$ is given by

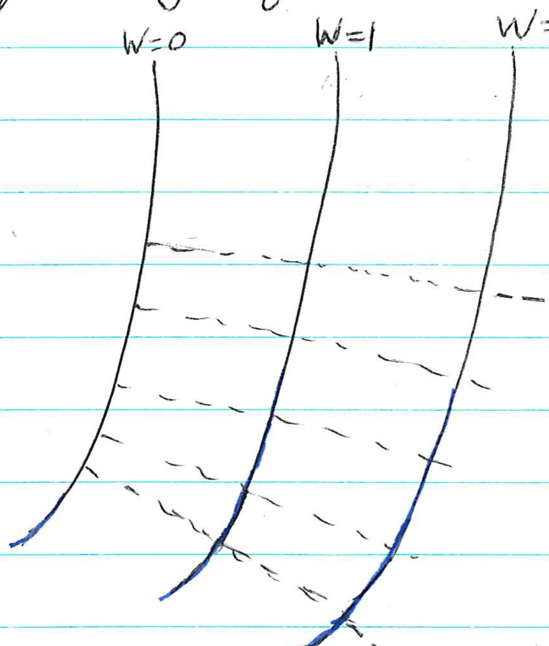
$$S(t,q) = S(0,q) - Et$$

(Check: Given $W(q)$ satisfying $H(q, \frac{\partial W}{\partial q}) = E$ constant, if we put $S(t,q) = W(q) - Et$

then $\frac{\partial S}{\partial t} = -E = -H(q, \frac{\partial W}{\partial q})$ and $\frac{\partial S}{\partial q} = \frac{\partial W}{\partial q}$

so we get a solution of the H-J. equation. Thus $S(t,q) = W(q) - Et$ is the unique solution with $S(0,q) = W(q)$.)

So now what we want to do is to plot a picture of the surfaces [redacted] $S =$ const, which recall represent the surfaces of constant phases. Do this for $S(t,q) = W(q) - Et$. At time $t = 0$ we get a family of level surfaces.

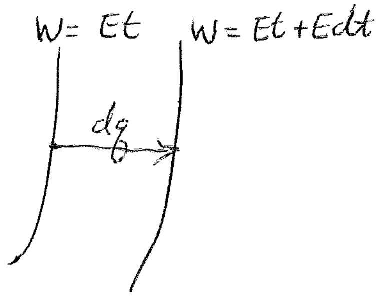


The trajectories are given by

$$\dot{q} = \frac{\partial H}{\partial p} = \frac{p}{m} \qquad p = \frac{\partial W}{\partial q} = \nabla W$$

so they are [redacted] perpendicular to these level surfaces

Next suppose I compute the phase velocity thinking of the surface $S(t,q) = \text{const}$ as being a wave front. If $S(t,q) = W - Et$, then as t goes from t to $t+dt$, the surface $S(t,q) = 0$ in q-space moves to $S(t+dt,q) = 0$; i.e. we go from $W = Et$ to $W = E(t+dt)$. Let dq be the change in q in the perpendicular direction, so that



$$dW = E\,dt = \underbrace{|\nabla W|}_{=|p|}\cdot|dq|$$

Thus the speed of the wave front is

$$\underline{\text{phase velocity}} = \frac{|dq|}{dt} = \frac{E}{|\nabla W|} = \frac{E}{|p|}$$

which is not the same as the speed of the particle.

Next we want to look at group velocity, and it turns out we can proceed more generally. Thinking of $S(t,q)$ as the phase in wave motion, we can linearize around a point:

$$S = S_0 + \frac{\partial S}{\partial q}\,dq + \frac{\partial S}{\partial t}\,dt$$

$$= S_0 + p\,dq - H\,dt$$

From this we see that the frequency of the wave is H and the wave number is $|p|$. Therefore

$$\text{phase velocity} = \frac{H}{|p|}$$

$$\text{group velocity} = \frac{dH}{d|p|} = \frac{d}{d|p|}\left(\frac{|p|^2}{2m} + V(q)\right) = \frac{|p|}{m} = |\dot{q}|.$$

Therefore we see that the group velocity of the wave motion is just the velocity of the particles.

June 12, 1979

~~Yesterday I understood how to formulate a "geometric optics" version of classical mechanics~~ [redacted] ~~More precisely, we replace the particles by wavefronts.~~

Yesterday, I began to understand the Hayghen's wave-front approach to classical mechanics. Let's review:

Begin with [redacted] particle mechanics described in Hamilton's form. [redacted] Particle positions are given by coordinates q_i; configuration space is a manifold with coords q. Phase space is the cotangent bundle of configuration space, so there is on phase space the canonical 1-form $p\,dq$; a state of the system is a triple (t, q, p). Then one [redacted] is given a Hamiltonian $H(t, q, p)$ function on phase space, and the motion is given by Hamilton's equns.

Now the wavefront description uses a function $S(t, q)$ whose level surfaces are the surfaces of constant phase for the wave. The ~~[illegible]~~ S-function satisfies the Hamilton-Jacobi PDE

$$\frac{\partial S}{\partial t} + H\left(t, q, \frac{\partial S}{\partial q}\right) = 0$$

June 13, 1979

Let us consider relativistic motion of a single particle. Space-time is $\mathbb{R}^4$ equipped with the form $t^2 - |\mathbf{x}|^2$. A state consists of a point x of $\mathbb{R}^4$ and a unit tangent vector $u = (u_\mu)$ with $u_0 > 0$. Thus states form the forward half of the unit tangent vector hyperboloid bundle. To describe trajectories, which are curves in space-time with time-like tangent lines, we need to give the acceleration of the particle at each point. This means that for each x, u we need to give a vector $F(x,u)$ which is perpendicular to u, and hence is space-like. The equation of motion is then

$$\frac{d^2x}{ds^2} = F\left(x, \frac{dx}{ds}\right).$$

Mathematically an analogous situation is obtained by considering ordinary motion in $\mathbb{R}^3$ where the force is perpendicular to the direction of motion, and where one looks only at trajectories with unit speed. The force function at each point is a section of the tangent bundle of the sphere of unit tangent vectors. In the case of $S^2 \subset \mathbb{R}^3$ the simplest way to get vector fields on S^2 is to use the vector fields obtained from rotations, i.e. $F = v \times B$.

Hence relativistic motion of a single particle in a given EM field should mathematically understandable once we understand motion of a charged particle in a fixed (time-ind), magnetic fields, always of unit speed.

The next project will be to get a wavefront description of this motion. Let's look at the motion of a charged particle in a stationary magnetic field. Although we are ultimately interested in the case where $|v| = 1$, we can look at the Lagrangian which handles general v. This Lagrangian is

$$L(x, v) = \frac{1}{2} m v^2 + e\, v \cdot A$$

(Check:

$$p = \frac{\partial L}{\partial v} = mv + eA$$

$$\frac{dp}{dt} = m\frac{dv}{dt} + e\underbrace{\frac{dA}{dt}}_{(v\cdot\nabla)A} = \frac{\partial L}{\partial x} = e\nabla(v\cdot A)$$

$$\therefore \quad m\dot{v} = e\Big(\nabla(v\cdot A) - (v\cdot\nabla)A\Big) = e\Big(v\times(\nabla\times A)\Big)$$
$$= e(v\times B).\,)$$

The Hamiltonian is

$$H = pv - L = (mv + eA)\cdot v - \left(\tfrac{1}{2}mv^2 + e v\cdot A\right) = \tfrac{1}{2}mv^2$$
$$= \frac{1}{2m}(p - eA)^2.$$

and so the Hamilton-Jacobi equation is

$$\frac{1}{2m}\left(\frac{\partial W}{\partial q} - eA\right)^2 = E \qquad \text{or}$$

$$\frac{\partial S}{\partial t} + \frac{1}{2m}\left(\frac{\partial S}{\partial q} - eA\right)^2 = 0$$

Now since we are interested in particle motions with

$v = 1$, it seems that we only want to look at the [redacted] first equation with $E = \frac{m}{2}$:

$$\left| \frac{\partial W}{\partial q} - eA \right| = m$$

Let's do the same thing relativistically, paying attention to the signs. The unit velocity four vector is (u^μ) and the kinetic term of the Lagrangian is

$$\frac{m}{2} u_\mu u^\mu = \frac{m}{2}\left(u_0^2 - u_1^2 - u_2^2 - u_3^2\right)$$

The other term of the Lagrangian is

$$e\, u_\mu A^\mu$$

Check

$$L = \frac{m}{2} u_\mu u^\mu + e u_\mu A^\mu$$

$$\frac{\partial L}{\partial x^\mu} = e u_\nu \partial_\nu A^\mu$$

$$p_\mu = \frac{\partial L}{\partial u^\mu} = m u_\mu + e A_\mu$$

$$H = p_\mu u^\mu - L = \frac{1}{2} m u_\mu u^\mu$$

Thus it seems we get the Hamilton-Jacobi equation

$$\sum_\mu g^{\mu\mu} \left(\frac{\partial W}{\partial x^\mu} - e A_\mu \right)^2 = m^2$$

June 15, 1979

The problem is to find a wavefront description for classical relativistic mechanics. Consider a single particle - its path in space-time is a curve whose tangent line at each point is time-like, hence one can define its unit tangent vector

$$u^\mu = \frac{dx^\mu}{ds}$$

at each point, so that $u^0 > 0$ and $u_\mu u^\mu = 1$. At each point there is an acceleration $\frac{d^2x^\mu}{ds^2}$ which is a vector perpendicular to $u = \frac{dx}{ds}$. So the equations of motion can be written

$$m\frac{d^2x}{ds^2} = F\left(x, \frac{dx}{ds}\right)$$

where [redacted] F is arbitrary except for

$$u \cdot F(x,u) = 0$$

One can interpret F as a vector field on the hyperboloid bundle over space-time consisting of (x,u) with $u \cdot u = 1$, $u^0 > 0$.

The simpler situation to look at first is [redacted] ordinary motion in space where the force is orthogonal to the direction of motion:

$$\frac{d^2x}{dt^2} = F\left(x, \frac{dx}{dt}\right)$$

F is a vector field on the unit tangent bundle over space.

For example if $F=0$, then you have geodesic flow.

The simplest vector fields on spheres come from infinitesimal rotations, i.e skew-symmetric matrices $F_{\mu\nu}$. Such a thing can be identified with a 2-form over space, but there is no reason in general for the 2-form to be closed.

For example in 3-space the equation of motion can be written

$$\frac{d^2x}{dt^2} = \frac{dx}{dt} \times B$$

where B is a vector field which is identified with the 2-form

$$\beta = B_x\, dy\, dz + B_y\, dz\, dx + B_z\, dx\, dy .$$

One has $d\beta = 0 \iff \nabla \cdot B = 0$.

We now want a wave-front description of this motion. It seems this means we want to single out a class of functions $W(x)$ such that integral curves for ∇W are particle motions.

So we can pose the more general problem: Given a motion described by Newton's Law with some force $F(x, \frac{dx}{dt})$, can we find a Hamilton-Jacobi equation

$$H\left(x, \frac{\partial W}{\partial x}\right) = E$$

with the same trajectories?

June 16, 1979

Consider on R^3 the motion given by

$$\frac{d^2x}{dt^2} = \frac{dx}{dt} \times \vec{B}$$

where $\vec{B} = \vec{B}(x)$ is a given vector field. Because the force is $\perp \frac{dx}{dt}$, the speed $|v|$ remains constant. We are ultimately interested only in motions having speed $= 1$, i.e. in the flow on the unit tangent bundle to $\mathbb{R}^3$.

I want to find a wavefront theory consistent with the above motions. This means that I want a first order PDE

$$H\left(x, \frac{\partial W}{\partial x}\right) = 0$$

whose bicharacteristic flow:

$$\frac{dx}{dt} = \frac{\partial H}{\partial \xi} \qquad \frac{d\xi}{dt} = -\frac{\partial H}{\partial x}$$

(2nd equation is $\frac{d\xi}{dt} = -\frac{\partial H}{\partial x} - \xi\frac{\partial H}{\partial W}$ in general)

can be identified with the given motion. Recall that along a bicharacteristic one has

$$\frac{dW}{dt} = \frac{\partial W}{\partial x}\frac{dx}{dt} = \xi\frac{\partial H}{\partial \xi}.$$

and H is constant, but not necessarily zero.

Now let us consider the customary solution. We assume $\nabla \cdot B = 0$ and choose A with $B = \nabla \times A$. Then

$$L = \frac{1}{2}v^2 + v \cdot A$$

is a Lagrangian for the motion:

$$p = \frac{\partial L}{\partial v} = v + A \qquad \frac{dp}{dt} = \frac{dv}{dt} + \overbrace{(v \cdot \nabla)A}^{\frac{dA}{dt}} = \overbrace{\nabla(v \cdot A)}^{\frac{\partial L}{\partial x}}$$

$$\frac{dv}{dt} = \nabla(v \cdot A) - (v \cdot \nabla)A = v \times (\nabla \times A)$$

The corresponding Hamiltonian is

$$H = p \cdot v - L = (v + A) \cdot v - \frac{1}{2}v^2 - v \cdot A = \frac{1}{2}v^2$$

$$H = \frac{1}{2}(p - A)^2$$

and so the Hamilton PDE is

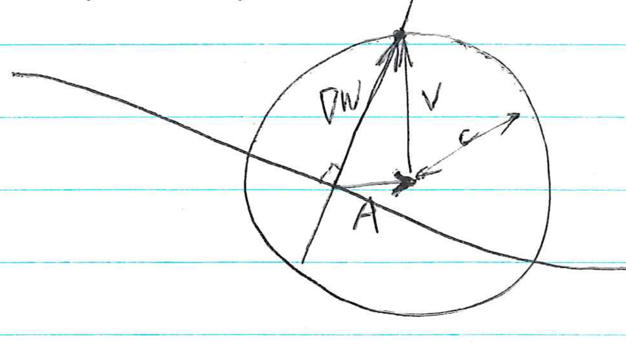
$$\frac{1}{2}\left(\frac{\partial W}{\partial x} - A\right)^2 = E \qquad \text{const.}$$

It appears difficult to visualize W as the phase for some wave, and especially to understand the significance of A. It might help to work in 2-dimensions where B is a scalar function giving an infinitesimal rotation about each point. A is then a vector field in the plane whose circulation is B.

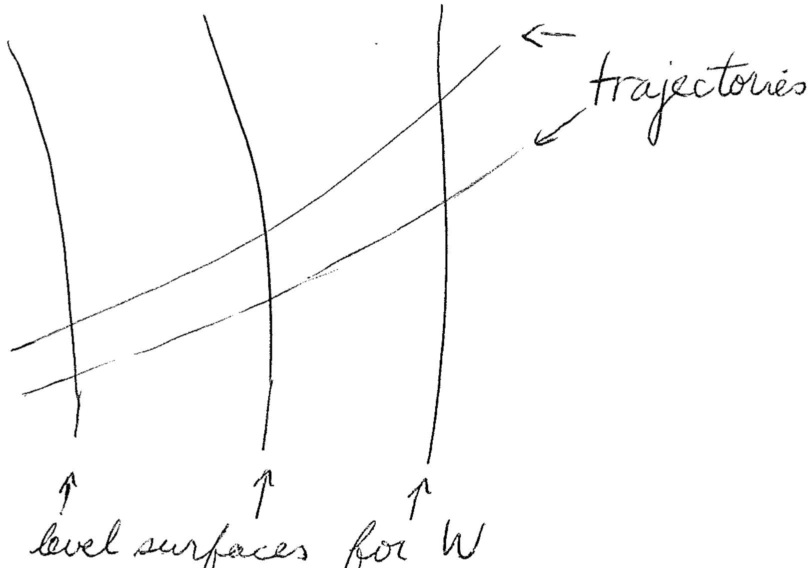
Suppose we try to integrate

$$|\nabla W - A| = c$$

and let's write $\nabla W = v + A$ and think of v as the velocity of our particles. Assume $|A| \ll c$ so that ∇W and v are close. If we start with a level surface for W, then we can locate ∇W by drawing a circle of radius c around the head of the vector A and seeing where it intersects the normal line.



Picture:



Since $\frac{dW}{dt} = \frac{\partial W}{\partial x} \cdot \frac{dx}{dt} = (v+A)\cdot v = v^2 + v\cdot A = c^2 + v\cdot A$

it seems there is no correlation between the motion of the constant phase surfaces and the trajectories. The point is somehow that the particle motion belongs to the group velocity - variation of the energy (frequency) H with respect to the momentum (wave number) p.