March 20, 1979

fermions 704-720

New idea: Think thru Morse theory from the viewpoint of quantum mechanics. The model here is that the critical points are classical trajectories. Somehow from gauge theories one gets the picture of a quantum state as being an average of classical states (?). Related questions: KAC'S version of semi-classical QM & Morse's theory. Take the building of a compact Lie group - does it have a QM interpretation with the complexification as symmetry group?

Gauge theory - review Bott's lectures. One starts with a principal G-bundle P over M. The gauge group is the group of all autos of P over M:

$$\mathcal{G} = \mathrm{Aut}(P/M)$$

$$= \Gamma(M, \underline{\mathrm{Aut}}(P))$$

where $\underline{\mathrm{Aut}}(P)$ is the bundle over M with fibres = group of autos. of the fibres of P:

$$\underline{\mathrm{Aut}}(P)_x = \mathrm{Aut}(P_x)$$

$$= P_x \times^G (G)$$

where G acts on itself via conjugation. To see this fix $y \in P_x$ so that one has a right G-isom.

$$G \xrightarrow{\sim} P_x$$

$$g \longmapsto yg$$

Then because the autos. of G commuting with right mult. are left mults. we get $Aut(P_x) \simeq G$, the isom given by

$$\begin{aligned} G &\xrightarrow{\sim} Aut(P_x) \\ g &\longmapsto (yg' \longmapsto ygg') \end{aligned}$$

So we can define

$$\begin{aligned} P_x \times G &\longrightarrow Aut(P_x) \\ (y, g) &\longmapsto (yg' \longmapsto ygg') \\ & \qquad\qquad\qquad \parallel \\ (yh, h^{-1}gh) &\longmapsto ((yh)g'' \longmapsto (yh)h^{-1}ghg'') \\ &\quad = (yg' \longmapsto ygg') \end{aligned}$$

and so
$$P_x \times^G (G) \xrightarrow{\sim} Aut(P_x).$$

One lets the gauge group $\mathcal{G}$ act on the space C of all connections in the principal bundle P. C is contractible, hence [redacted]

$$B\mathcal{G} \sim B\left(\begin{matrix}\text{top category formed by } \mathcal{G} \\ \text{acting on } C\end{matrix}\right)$$

Then one analyzes the latter by using [redacted] critical point theory on the space of connections.

It seems that the key Feynman idea is to attach an amplitude to each connection, then average, and conclude by stationary phase that only contributions from the critical points matter.

Example: Consider the building X belonging to the group $U(n)$ i.e. the space of $n \times n$ hermitian matrices. Fix an element H_0 which has distinct eigenvalues and consider the induced flow on X. Maybe X should be one orbit in the building, so that the[redacted] fixpoints are isolated and their indices are understood. Problem: What is the quantum mechanical situation corresponding to [redacted] X ?

A basic thing for me to understand [redacted] is where unitary representations of the gauge group come into the picture.

March 22, 1979   Carl is 14

To quantize the Dirac equation.

Recall the basic algebra. One is given an even-dim'l Euclidean space W and lets[redacted] $\mathcal{H}$ be the Clifford module associated to W. Then $\mathcal{H}$ is a complex Hilbert space equipped [redacted] with operators $\rho(\lambda)$ for each $\lambda : W \to \mathbb{C}$ satisfying

$$\rho(\lambda)^* = \rho(\bar{\lambda})$$

$$\{\rho(\lambda_1), \rho(\lambda_2)\} = (\lambda_1, \lambda_2)$$

where $(\ ,\ )$ denotes the natural extension of the inner product on W to $W^* = \mathrm{Hom}_{\mathbb{R}}(W, \mathbb{C})$.

Next one is given a Hamiltonian in W which is essentially a skew-adjoint transformation which is non-singular. Under the action of this skew-adjoint transformation W is a direct sum of orthogonal 2-planes, and [redacted] in each W generates a 1-parameter rotation group. There is a unique complex structure in W so that the skew-adjoint transf. is linear and such that its eigenvalues are $-i\lambda$[redacted] with $\lambda > 0$. Then the skew-adjoint transf. can be written $\frac{1}{i} H$ where H is a positive hermitian operator. Another way of getting the same complex structure is to consider the effect of the Hamiltonian on $\mathcal{H}$ and take the ground ~~state~~. The annihilator of the ground state in $\rho(W^*)$ is an isotropic subspace of W^* which corresponds to a complex structure in W.

Now consider the Dirac[redacted] equation

$$i\frac{\partial \phi}{\partial t} = \left(\alpha \frac{1}{i}\frac{\partial}{\partial x} + \beta m\right)\phi$$

Here W will be the space of solns. of this equation with the usual inner product $\int|\phi|^2 dx$. However it's important to notice that we don't want to use the given complex structure on the space of solutions, because the operator $(\alpha \frac{1}{i}\frac{\partial}{\partial x} + \beta m)$ has both positive and negative spectrum.

We will have operators associated to the linear functions

$$\psi_i(x): \quad \phi \longmapsto \phi_i(x,0)$$

$$\overline{\psi_i}(x): \quad \phi \longmapsto \overline{\phi_i(x,0)}$$

on W, and these operators will be adjoints of each other. Note that $\psi_i(x)$ will not be a creation or annihilation operator, even though it is a $\mathbb{C}$-linear function, because the good $\mathbb{C}$-structure on W is not the obvious one.

Next stage is to decompose into positive & negative frequency parts. ~~Proceed abstractly first.~~ Here $H = (\alpha \frac{1}{i}\frac{\partial}{\partial x} + \beta m)$ acting on vector functions $\phi(x)$ of x. One has

$$\phi = P^+\phi + P^-\phi$$

where P^+ is the projection onto the positive eigenspaces and P^- the projection ~~onto~~ the negative eigenspaces.

Proceed abstractly. Suppose W is equipped with a complex structure, but that H has both positive and negative spectrum. Then we can write

$$W = W^+ \oplus W^-$$

Choose an orthonormal basis w_j^+ for W^+ and an orthonormal basis w_j^- for W^-.

The philosophy is that we want to work with $\mathcal{H} = \Lambda(W^\vee)$ for the given complex structure on W, but that the ground state will be the one with "all negative energy states filled."

Basic formulas: Initially H is a given self-adjoint operator on $\mathcal{H}$ and we want to solve the Schrod DE

$$\frac{\partial \psi}{\partial t} = -iH\psi \qquad \text{in } \mathcal{H}.$$

Let the solution be $\psi(t) = U(t)\psi(0)$. On the other hand elts. of $W^* = \mathrm{Hom}_{\mathbb{R}}(W, \mathbb{C})$ are interpreted as operators on $\mathcal{H}$: $\lambda \mapsto \rho(\lambda)$. Since the Hamiltonian is quadratic, time evolution preserves $\rho(W^*)$. Think of $\rho(\lambda)$ as a position operator and $\rho(\lambda_t)$ as the position at time t operator. Then

$$U(t)\,\rho(\lambda_t)\,\psi(0) = \rho(\lambda)\,\psi(t)$$

or

$$\rho(\lambda_t) = U(+t)^{-1}\rho(\lambda)\,U(t)$$

or

$$\rho\left(\frac{d}{dt}\lambda_t\right) = +[iH, \rho(\lambda)]$$

which is the Heisenberg equation of motion. Now define iH in W^* by

$$\rho(iH\cdot\lambda) = [iH, \rho(\lambda)]$$

and then define the motion in W to be contragrediented. What does this all mean? I need a model.

[redacted] The things to keep straight are the signs of the eigenvalues. So let us start with W a complex vector space with inner product, fix an orthonormal basis w_j, and let

$$a_j = i(w_j) \qquad a_j^* = e(w_j^*) \quad \text{on} \quad \Lambda(W^\vee).$$

Now suppose you have a Hamiltonian (nonsingular)

$$H = \sum \lambda_j a_j^* a_j \quad \text{on} \quad \Lambda(W^\vee)$$

[redacted] We have the basis $a_{j_1}^* \cdots a_{j_r}^* \cdot 1$ for $\Lambda W^\vee$ and

$$H\, a_{j_1}^* \cdots a_{j_r}^* 1 = (\lambda_{j_1} + \cdots + \lambda_{j_r}) \cdot a_{j_1}^* \cdots a_{j_r}^* 1$$

The ground state [redacted] (= state of least energy) occurs when $j_1, \ldots, j_r$ is the subset of j for which $\lambda_j < 0$.

(Again as a practical matter it seems to be easiest ~~to~~ work with a fixed presentation for $\mathcal{H}$ using creation + ann. ops.)

Notice that the ground state energy is the sum of the negative λ_j. Another way to see this is to introduce operators $b_j = a_j^*$ for [redacted] $\lambda_j < 0$. Then

$$H = \sum \lambda_j a_j^* a_j = \sum_{\lambda_j > 0} \lambda_j a_j^* a_j + \sum_{\lambda_j < 0} (-\lambda_j) b_j^* b_j + \sum_{\lambda_j < 0} \lambda_j$$

In this form, the operators a_j for $\lambda_j > 0$, b_j for $\lambda_j < 0$ are the true destruction operators because they kill the ground state.

Now consider the case of the Dirac eqn. Here W is the space of vector-valued functions $\phi(x) = (\phi_i(x))$ with usual norm $\|\phi\|^2 = \int \sum_i |\phi_i(x)|^2 dx$. H is the operator on W given by

$$H = \alpha \frac{1}{i} \frac{\partial}{\partial x} + \beta m$$

To quantize this, I can proceed as above and choose an orthonormal basis for [redacted] W, call it w_j, in which H is diagonal: $Hw_j = \lambda_j w_j$. Use Fourier transform to locate the eigenfunctions for H

$$\phi(x) = \int \frac{d\xi}{2\pi} e^{+ix\xi} \hat{\phi}(\xi)$$

$$(H\phi)(x) = \int \frac{d\xi}{2\pi} e^{+ix\xi} (-\alpha\xi + \beta m) \hat{\phi}(\xi) = k\phi(x)$$

means

$$(+\alpha\xi + \beta m)\hat{\phi}(\xi) = k\hat{\phi}(\xi)$$

which means k is an eigenvalue for $\alpha\xi + \beta m$. Recall that

$$\alpha = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \qquad \beta = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

so

$$\alpha\xi + \beta m = \begin{pmatrix} \xi & m \\ m & -\xi \end{pmatrix}.$$

This has the eigenvalues $k = \pm\sqrt{\xi^2 + m^2}$. Therefore for each $\xi \in \mathbb{R}$ we get two eigenfunctions for H

with eigenvalues $\pm\sqrt{\xi^2+m^2} = \pm\,\omega(\xi)$

March 23, 1979

So far I have been very confused by the linear algebra. Let's try working directly with $\mathcal{H}$, a_j, a_j^* and to work in W later. A Hamiltonian

$$H = \sum{}' \lambda_j a_j^* a_j$$

is given on $\mathcal{H}$ and gives a Schroedinger equation with solutions [redacted] $\psi(t) = e^{-itH}\psi(0)$.

[redacted] One then gets the Heisenberg motion on the space $\tilde{W} = \text{span } a_j, a_j^*$ by

$$e^{-itH}\underset{\substack{\uparrow\\ \text{Heisenberg}\\ \lambda\text{ meas.}\\ \text{at time } t}}{\lambda(t)}\underset{\substack{\uparrow\\ \text{Heisenberg}\\ \text{state vector}}}{\psi(0)} = \lambda\, e^{-itH}\psi(0)$$

or

$$\lambda(t) = e^{itH}\lambda e^{-itH}$$

so

$$\frac{d}{dt}\lambda(t) = [iH, \lambda]$$

For the above Hamiltonian
$$[H, a_j] = [\lambda_j a_j^* a_j, a_j] = -\lambda_j a_j$$
$$[H, a_j^*] = \lambda_j a_j^*$$

so

$$\begin{cases} a_j(t) = e^{-it\lambda_j} a_j \\ a_j^*(t) = e^{+it\lambda_j} a_j^* \end{cases}$$

So the next point is to fit this model to the

Dirac equation. This time one is presented with a non-singular hermitian operator A on a Hilbert space W and one looks at the equation

$$\frac{d}{dt} w(t) = -i A\, w(t)$$

Choose an orth basis for W diagonalizing A:

$$A w_j = \lambda_j w_j$$

Now we can set up an isomorphism between $W^* = \mathrm{Hom}_{\mathbb{R}}(W, \mathbb{C})$ and $\tilde{W}$ compatible with time evolution. We want to interpret a_j as an elt $w \longmapsto a_j(w)$ of W^* so that

$$a_j(t)(w(t)) = a_j(w)$$

$$w_j(t) = e^{-i\lambda_j t} w_j$$

$$a_j(t) = e^{-i\lambda_j t} a_j$$

so it seems that $a_j(c w_j) = \bar{c}$ and $a_j^*(c w_j) = c$.

([redacted] The creation operator a_j^* increases energy:

$$e^{-itH} a_j^* e^{+itH} = e^{-it\lambda_j} .$$

So $w_j \longmapsto a_j^* |0\rangle$ is an isomorphism of (W, A) with 1-particle states. Thus maybe the good way to think is

$$\Lambda W \xrightarrow{\sim} \mathcal{H} .)$$

March 5, 1979

Let W be a fin. diml. complex vector space with inner product, and form $\mathcal{H} = \hat{\Lambda} W$ with the operators $e(w)$, $i(\lambda)$. Let H be a hermitian operator on W and extend it to $\mathcal{H}$ and consider the time evolution e^{-iHt} on W and $\mathcal{H}$. This induces an action

$$A \longmapsto e^{-iHt} A e^{iHt}$$

on $\mathrm{End}(\mathcal{H}) \xhookleftarrow{\alpha} W \oplus W^\vee$, where $\alpha(w \oplus \lambda) = e(w) + i(\lambda)$. Now using the inner product on W, any $w \in W$ gives an element w^* of $W^\vee$, and

$$\begin{tikzcd} W \arrow[r] \arrow[rr, bend left, "\rho"] & W \oplus W^\vee \arrow[r, hook] & \mathrm{End}(\mathcal{H}) \\ w \arrow[r, mapsto] & w \oplus w^* \arrow[r, mapsto] & e(w) + i(w^*) \end{tikzcd}$$

is an isomorphism of the underlying real vector space of W with the self-adjoint operators in $\alpha(W \oplus W^\vee)$. Also

$$\{\rho(w), \rho(w')\} = \{e(w) + i(w^*), e(w') + i(w'^*)\}$$
$$= (w, w') + (w', w)$$

better: $\rho(w)^2 = (w, w)$.

Now because everything is functorial we must have

$$e^{-itH} \rho(w)\, e^{itH} = \rho(e^{-itH} w)$$

The next point is to assume H non-singular and diagonalize it $Hw_j = \lambda_j w_j$ with w_j an orth.

basis for W. If $a_j = i(w_j^*)$, $a_j^* = e(w_j)$, then on $\mathcal{H}$ we have

$$H = \sum{}' \lambda_j a_j^* a_j$$

The "ground state" for H is the eigenvector of lowest energy, i.e. smallest eigenvalue. We know

$$H(a_{j_1}^* \cdots a_{j_r}^* 1) = (\lambda_{j_1} + \cdots + \lambda_{j_r})\, a_{j_1}^* \cdots a_{j_r}^* 1$$

so that the ground state $|0\rangle$ occurs when $\lambda_{j_1}, \ldots, \lambda_{j_r}$ are all the negative eigenvalues of H. The ground state energy is the sum of the negative λ_j.

Next we fix a Hamiltonian H_0 as above and consider a time-dependent perturbation of it $H = H_0 - V(t)$, where $V(t)$ has compact support. Let $U(t,t')$ be the time-evolution operator for $H(t)$. We have a scattering operator

$$S = e^{iH_0 t_f}\, U(t_f, t_i)\, e^{-iH_0 t_i}$$

where $t_i < \operatorname{Supp} V < t_f$, and we are interested in computing the "vacuum-vacuum" amplitude

$$\begin{aligned}\langle 0|S|0\rangle &= \langle 0|e^{iH_0 t_f}\, U(t_f, t_i)\, e^{-iH_0 t_i}|0\rangle \\ &= e^{-iE_0(t_f - t_i)}\, \langle 0|\, U(t_f, t_i)|0\rangle\end{aligned}$$

Then if H is subjected to a variation δH we have

$$\delta U(t_2, t_1) = \int_{t_1}^{t_2} dt\; U(t_2, t)\, \frac{1}{i}\, \delta H(t)\, U(t, t_1)$$

so
$$\delta\langle 0|S|0\rangle = e^{-iE_0(t_f-t_i)}\langle 0|\delta U(t_f,t_i)|0\rangle$$
where
$$\langle 0|\delta U(t_f,t_i)|0\rangle = \int_{t_i}^{t_f} dt\ \underbrace{\langle 0|U(t_f,t)\tfrac{1}{i}\delta H(t)\,U(t,t_i)|0\rangle}.$$

The goal is to interpret as a trace. It has the form
$$\langle \chi(t)|\tfrac{1}{i}\delta H(t)|\psi(t)\rangle$$

where $\psi(t) = U(t,t_i)|0\rangle$, $\chi(t) = U(t,t_f)|0\rangle$. On the other hand $\delta H(t)$ is a linear combination of the operators $a_j^* a_k$.

Let's look at things more algebraically. We are assuming H on $\mathcal{H} = \Lambda W$ comes from an H on W. Thus H preserves the grading on ΛW. Moreover if we define $U(t,t')$ on both W and ΛW by
$$\frac{\partial}{\partial t}U(t,t') = \frac{1}{i}H(t)\,U(t,t') \qquad U(t',t') = I$$
then on ΛW we must have $\Lambda U(t,t')$.

Let p be the number of negative eigenvalues for H_0, so that $|0\rangle$ is a unit vector in $\Lambda^p W$. Then $\langle 0|U(t_f,t_i)|0\rangle$ is just a matrix element for $\Lambda^p(\theta)$, where $\theta = U(t_f,t_i)$ on W. In fact recall that $|0\rangle$ denotes a unit vector in $\Lambda^p(W)$ in the line belonging to the negative eigenvalue subspace W^-:
$$|0\rangle = a^*_1 \cdots a^*_n 1$$

where $w_1, \dots, w_n$ are the negative eigenvectors. The number

$\langle 0 | U(t_f, t_i) | 0 \rangle$ should be the determinant of the map $U(t_f, t_i)$ restricted to W^- followed by projection onto W^-.

On the other hand $\langle 0 | U(t_f, t)$ [redacted] $\delta H(t)\, U(t, t_i) | 0 \rangle$ should represent [redacted] a kind of trace. Specifically we take two subspaces $U(t, t_i) W^-$ and $U(t, t_f) W^-$ and the endo. $\delta H(t)$ and we have to get out a number. At this point it would seem profitable to understand the [redacted] kinds of Green's functions for

$$\frac{\partial w}{\partial t} = \frac{1}{i} H w$$

which there are.

March 26, 1979

I was considering a DE

$$\frac{\partial w}{\partial t} = -iHw$$

on a f.d. Hilbert space W, where $H(t)$ is hermitian, hence the solution operator $U(t,t')$ is [redacted] unitary. Lets consider this on an interval $[t_i, t_f]$ [redacted] with boundary conditions

$$w \in W^- \quad \text{at} \quad t_i$$
$$w \in W^+ \quad \text{at} \quad t_f$$

where W^-, W^+ are subspaces of complementary dimensions. In order that the Green's function $G(t,t')$ exist, it is n. and s. that the homog. equation have only zero soln:

$$U(t_f, t_i)W^- \cap W^+ = 0$$

The Green's function solves the inhomog. equation

$$\left(\frac{d}{dt} + iH\right) w = f$$
$$w = \int_{t_i}^{t_f} G(t,t') f(t') dt'$$

hence $G(t,t') \in \mathrm{End}(W)$ for each t, t'. The forward Green's fn. is

$$G_f(t,t') = \begin{cases} U(t,t') & t > t' \\ 0 & t < t' \end{cases}$$

To get G we add a solution of homog. DE, just the one with initial values in W^- to kill the part of G_f outside W^+.

A better way is to put

$$G(t,t') = \begin{cases} U(t,t')(I+A(t')) & t>t' \\ U(t,t')\,A(t') & t<t' \end{cases}$$

This satisfies the boundary conditions when

$$\operatorname{Im}\ U(t_i,t')A(t') \subset W^- \qquad \text{or} \qquad \operatorname{Im} A(t') \subset U(t',t_i)W^-$$

$$\operatorname{Im}\ U(t_f,t')(I+A(t')) \subset W^+ \qquad \text{or} \qquad \operatorname{Im}(I+A(t')) \subset U(t',t_f)W^+$$

But

$$W = U(t',t_f)W^+ + U(t',t_i)W^-$$

$$w = (w + A(t')w) \quad - A(t')w$$

hence

$$-A(t') = \text{projection onto } U(t',t_i)W^- \text{ with kernel } U(t',t_f)W^+.$$

The next point is [redacted] how, given an endo $\delta H(t')$ of W, to express

$$\langle 0 | U(t_f,t')\ \delta H(t')\ U(t',t_i) | 0 \rangle$$

in terms of the Green's function. Recall $\delta H(t')$ is a derivation of ΛW.

To simplify suppose that $p = \dim W^- = 1$. Let

$$\phi_t = U(t,t_i)|0\rangle$$

where $|0\rangle$ spans W^-. Now W^+ is the kernel of the linear functional $\langle 0|$, so $U(t,t_f)W^+$ is the kernel of

$$\lambda_t = \langle 0 | U(t_f,t)$$

Hence

$$U(t,t_i)W^- = \text{span } \phi_t \quad , \quad U(t,t_f)W^+ = \operatorname{Ker} \lambda_t$$

and $\lambda_t(\phi_t) = \langle 0| U(t_f,t)U(t,t_i)|0\rangle = \langle 0|U(t_f,t_i)|0\rangle$ is assumed to be $\neq 0$. It is constant; call it σ. Then

$$-A(t) = \frac{1}{\sigma}\phi_t\lambda_t \quad = \begin{array}{l}\text{proj on } \; U(t,t_i)W^- \\ \text{with kernel } \; U(t,t_f)W^+\end{array}$$

because $\phi_t\lambda_t\phi_t\lambda_t = \sigma\,\phi_t\lambda_t$. So the Green's function is

$$G(t,t') = \begin{cases} U(t,t')\left(I - \frac{1}{\sigma}\phi_{t'}\lambda_{t'}\right) & t > t' \\ U(t,t')\left(-\frac{1}{\sigma}\phi_{t'}\lambda_{t'}\right) & t < t' \end{cases}$$

Now I can consider the quantity of interest:

$$\langle 0|U(t_f,t)\,\delta H(t)\,U(t,t_i)|0\rangle = \lambda_t(\delta H(t)\phi_t)$$
$$= tr\left((\phi_t\otimes\lambda_t)\,\delta H(t)\right)$$

One has

$$\frac{1}{2}\left\{G(t'^+,t') + G(t'^-,t')\right\} = \frac{1}{2}I - \frac{1}{\sigma}\phi_{t'}\otimes\lambda_{t'}$$

$$= \text{operator with value } \tfrac{1}{2} \text{ on } U(t',t_f)W^+$$
$$-\tfrac{1}{2} \text{ on } U(t',t_i)W^-$$

It seems to be desirable to require $tr(\delta H(t)) = 0$. In this case the jump in G doesn't contribute to the trace, and we have

$$\boxed{\frac{\langle 0|U(t_f,t)\,\delta H(t)\,U(t,t_i)|0\rangle}{\langle 0|U(t_f,t_i)|0\rangle} = -\,tr\left(G(t,t)\,\delta H(t)\right)}$$

General case

$$\phi_{1t}\wedge\cdots\wedge\phi_{pt} = u(t,t_i)|0\rangle$$

$$\lambda_{1t}\wedge\cdots\wedge\lambda_{pt} = c\langle 0|u(t_f,t) .$$

Here $\phi_{it} = u(t,t_i)\phi_i$ where ϕ_i is an orth. basis for W^-. $\lambda_1,\dots,\lambda_p$ is a basis for the linear functionals vanishing on $u(t_i,t_f)W^+$ and $\lambda_{it} = u(t,t_i)\lambda_i$. Since W^- is complementary to $u(t_i,t_f)W^+$, we have

$$W^- \xrightarrow{\sim} W/u(t_i,t_f)W^+$$

and so I can arrange $\lambda_i(\phi_j) = \delta_{ij}$.

Then

$$1 = \lambda_{1t}\wedge\cdots\wedge\lambda_{pt}(\phi_{1t}\wedge\cdots\wedge\phi_{pt}) = c\langle 0|u(t_f,t_i)|0\rangle.$$

Now

$$\lambda_{1t}\wedge\cdots\wedge\lambda_{pt}(\delta \overset{(t)}{H}_{\wedge}\phi_{1t}\wedge\cdots\wedge\phi_{pt})$$

$$= \lambda_{1t}\wedge\cdots\wedge\lambda_{pt}\Big(\sum_j{}' \underbrace{\phi_{1t}\wedge\cdots\wedge\delta H(t)\phi_{jt}\wedge\cdots\phi_{pt}}\Big) \quad \det\begin{bmatrix} 1 & & & * & & \\ & 1 & & * & & \\ & & 1 & * & & \\ & & & \circledast & & \\ & & & * & 1 & \\ & & & * & & 1 \end{bmatrix}$$

$$= \sum_j{}' \lambda_{jt}(\delta H(t)\phi_{jt})$$

$$= \mathrm{tr}\Big(\delta H(t)\cdot \underbrace{\sum_j{}' \phi_{jt}\lambda_{jt}}_{\substack{\text{projection onto } u(t,t_i)W^- \\ \text{with kernel } u(t_i,t_f)W^+}}\Big) \qquad = -A(t)$$

$$= -\mathrm{tr}(G(t,t)\,\delta H(t)) \qquad \text{assuming } \mathrm{tr}(\delta H(t))=0.$$

So can we put this all together? We start with H_0 which gives us the space W^- and the Greens function $G_0(t,t')$:

$$G_0(t,t') = \begin{cases} e^{-iH_0(t-t')}P_+ & t>t' \\ -e^{-iH_0(t-t')}P_- & t<t' \end{cases}$$

where P_- is the orthogonal projection on W^-, and $p^+ = 1 - P_- =$ projection on $W^+ = W \ominus W^-$. Then we have a perturbed Hamiltonian

$$H(t) = H_0 + V(t)$$

where $V(t)$ has compact support and also trace $= 0$. On W we look at the DE

$$\frac{d\psi}{dt} = -iH\psi \quad \text{or}$$

$$\left(\frac{d}{dt} + iH_0\right)\psi = -iV(t)\psi$$

and [redacted] using the Green's function we replace it by an [redacted] integral equation:

$$\psi(t) = \psi_0(t) + \int G_0(t,t')(-i)V(t')\psi(t')\,dt'$$

The latter should have a determinant

$$\det(1 - G_0(-iV))$$

although there might be problems because the kernel has a discontinuity on the diagonal.

Let's vary V infinitesimally.

$$\Big(\delta \log \det A = \operatorname{tr} \delta \log A = \operatorname{tr}(A^{-1}\delta A)$$

$$\delta \log \det(1 - G_0(-iV)) = \operatorname{tr}\left((1+G_0 iV)^{-1}(iG_0\,\delta V)\right)$$

Now from

$$\left(\frac{d}{dt} + iH\right) G = 1$$

$$\left(\frac{d}{dt} + iH_0\right) G = 1 - iVG$$

$$G = G_0 - G_0 iVG$$

$$(1 + G_0 iV) G = G_0$$

$$G = (1 + G_0 iV)^{-1} G_0$$

we get

$$\delta \log \det(1 + G_0 iV) = \operatorname{tr}(iG\delta V).$$

But we have seen that

$$\frac{\delta \langle 0 | U(t_f, t_i) | 0 \rangle}{\langle 0 | U(t_f, t_i) | 0 \rangle} = \frac{\int \langle 0 | U(t_f, t) \frac{1}{i}\delta H(t) U(t, t_i) | 0 \rangle \, dt}{\langle \qquad \rangle}$$

$$= \frac{1}{i} \int - \operatorname{tr}(G(t,t)\,\delta H(t))\, dt$$

$$= i \operatorname{tr}(G\delta V).$$

So the conclusion is

$$\langle 0 | U(t_f, t_i) | 0 \rangle = \langle 0 | e^{-iH(t_f - t_i)} | 0 \rangle \det(1 + G_0 iV)$$

But notice that this ~~[illegible]~~ should be nothing more than our old method for computing the Fredholm determinant using Wronskians.

Next problem: Understand the forced harmonic oscillator. Newton's equation is

$$\ddot{x} = -\omega^2 x + f(t) = -\frac{\partial}{\partial x}\underbrace{\left\{\tfrac{1}{2}\omega^2 x^2 - f(t)\,x\right\}}_{V(x,t)}$$

This can be obtained from the Lagrangian

$$L(q,\dot{q},t) = \frac{1}{2}\dot{q}^2 - \frac{1}{2}\omega^2 q^2 + f(t)q$$

Then

$$p = \frac{\partial L}{\partial \dot{q}} = \dot{q}$$

so

$$H(q,p,t) = p\dot{q} - L = \frac{1}{2}(p^2 + \omega^2 q^2) - f(t)q$$

and the Schroedinger equation is

$$i\frac{\partial \psi}{\partial t} = \frac{1}{2}\left(-\frac{\partial^2}{\partial x^2} + \omega^2 x^2\right)\psi - f(t)x\psi$$

What sort of problems does one do with the forced harmonic oscillator? 1) Suppose $f(t)$ has compact support and $x=0$ for $t<0$. 2) $f(t) = \sin \nu t$, look at steady-state response.

March 27, 1979

To understand Schwinger's "sources". Consider a harmonic oscillator with a compact support forcing term:

$$H = \frac{1}{2}(p^2+q^2) - \varepsilon(t)q$$

and let's compute $\langle 0|S|0\rangle = \dfrac{\langle 0|U(t_f,t_i)|0\rangle}{\langle 0|U_0(t_f,t_i)|0\rangle}$

(here $U_0(t,t') = e^{-iH_0(t-t')}$, $H_0 = \frac{1}{2}(p^2+q^2)$). One has

$$\delta\langle 0|U(t_f,t_i)|0\rangle = \int_{t_i}^{t_f} \langle 0|U(t_f,t)\underbrace{\tfrac{1}{i}\delta H(t)}_{i\delta\varepsilon(t)q}U(t,t_i)|0\rangle\,dt$$

Put

$$\langle q\rangle = \frac{\langle 0|U(t_f,t)\,q\,U(t,t_i)|0\rangle}{\langle 0|U(t_f,t_i)|0\rangle}$$

Then I have seen that

$$\frac{d^2}{dt^2}\langle q\rangle = \frac{\langle 0|U(t_f,t)[iH,[iH,q]]\,U(t,t_i)|0\rangle}{\langle 0|U(t_f,t_i)|0\rangle}$$

$$= -\langle q\rangle + \varepsilon$$

since $[iH,[iH,q]] = [iH,p] = \left[i\left(\frac{q^2}{2}-\varepsilon q\right),p\right] = q+\varepsilon$

Moreover

$$\frac{d}{dt}\langle q\rangle - i\langle q\rangle = \langle p-iq\rangle = -i\langle q+ip\rangle \quad = 0 \text{ at } t_i$$

so $\langle q\rangle = e^{+it}\cdot\text{const}$ $\qquad t < \text{Supp}\,\varepsilon$

and similarly at the other end, so that outgoing bdry conditions hold for $\langle q\rangle$. Thus

$$\langle q\rangle = \int G_0(t,t')\,\varepsilon(t')dt' \qquad G_0(t,t') = \frac{e^{-i|t-t'|}}{-2i}$$

and

$$\frac{\delta\langle 0|U(t_f,t_i)|0\rangle}{\langle 0|U(t_f,t_i)|0\rangle} = i\int dt\,\delta\varepsilon(t)\int G_0(t,t')\,\varepsilon(t')dt'$$

Integrating from $\varepsilon = 0$ to ε we get

$$\langle 0|S|0\rangle = \exp\left(\frac{1}{2}i\iint dt\,dt'\,\varepsilon(t)\,G_0(t,t')\,\varepsilon(t')\right)$$

A consequence of this formula is that the preceding exponential has to have abs. value ≤ 1, so

$$\mathrm{Re}\iint dt\,dt'\,\varepsilon(t)\overset{e^{-i|t-t'|}}{\text{[redacted]}}\varepsilon(t') \geqslant 0$$

and hence the kernel $\cos|t-t'|$ is positive. [redacted] We can see this directly for $\cos|t-t'| = \cos(t-t')$, so the above is the same as the real part without the $|\;|$, which is

$$\iint dt\,dt'\,\varepsilon(t)e^{-it+it'}\varepsilon(t') = \left|\int dt\,\varepsilon(t)e^{-it}\right|^2$$

(It may be necessary for R.H. to prove some hermitian form is positive, and this one might do by [redacted] something like the above, by using the fact that a transition probability is ≤ 1.)

Let's see if we can get the Dirac picture together. The equation is

$$i\frac{\partial\phi}{\partial t} = H\phi = \left\{\alpha\frac{1}{i}\frac{\partial}{\partial x} + \beta m\right\}\phi .$$

Let W be Hilbert space of Cauchy data for this DE, that is, W is the Hilbert space of vector functions of x: $\phi = (\phi_i(x))$, in the L^2 norm. Let w_j be an orthonormal basis for W diagonalizing H:

$$H w_j = \lambda_j w_j$$

(In practice the spectrum of H is continuous, but pretend!). This means the w_j are orthonormal:

$$\int \sum_i w_{ji}(x)\,\overline{w_{j'i}(x)}\,dx = \delta_{jj'}$$

and that they are complete which means for any ϕ in W:

$$\phi = \sum_j w_j \langle w_j | \phi \rangle$$

$$\phi_i(x) = \sum_j w_{ji}(x) \int \sum_{i'} \overline{w_{ji'}(x')}\,\phi_{i'}(x')\,dx'$$

or

$$\sum_j w_{ji}(x)\,w_{ji'}(x') = \delta_{ii'}\,\delta(x-x')$$

The Hilbert space $\mathcal{H}$ is going to be a modification of ΛW. ΛW contains finite products $w_{j_1}\wedge\cdots\wedge w_{j_r}$; what we want is an infinite wedge product with almost all negative energy states filled. Thus what one does is split: $W = W^- \oplus W^+$ and form

$$\mathcal{H} = \Lambda(W^-)^\vee \otimes \Lambda W^+ .$$

The idea here is that if W^- were of dim. n, then

$$\Lambda^{n-p} W^- \cong \Lambda^n W^- \otimes \Lambda^p (W^-)^\vee$$

However this is all window-dressing. The real point is that $\mathcal{H}$ comes with a ground state $|0\rangle$ and operators a_j, a_j^* satisfying the canonical commutation relations $\{a_j, a_{j'}^*\} = \delta_{jj'}$ and

$$a_j |0\rangle = 0 \qquad \lambda_j > 0$$
$$a_j^* |0\rangle = 0 \qquad \lambda_j < 0.$$

In this sense a_j is a creation op for $\lambda_j < 0$ and a destruction op. for $\lambda_j > 0$.

The next point is construct operators $\psi_i(x)$ corresponding to the linear function

$$\phi \longmapsto \phi_i(x)$$

on W. This is an elt of $W^\vee$ which acts on $\mathcal{H}$ via the a_j. Since

$$\phi = \sum_j w_j \langle w_j | \phi \rangle$$

one has $\quad \phi_i(x) = \sum_j w_{ji}(x) \langle w_j | \phi \rangle$

so $\quad \big(\phi \mapsto \phi_i(x)\big) = \sum_j w_{ji}(x) \langle w_j |$. To $\langle w_j |$ belongs $a_j = i(\langle w_j|)$, so we have

$$\underline{\psi_i}(x) = \sum_j w_{ji}(x)\, a_j$$

To the conjugate linear fn: $\phi \longmapsto \overline{\phi_i(x)}$ we associate the adjoint operator

$$\underline{\psi_i(x)}^* = \sum_j \overline{w_{ji}(x)}\, a_j^*$$

Then

$$\{\underline{\psi_i(x)}, \underline{\psi_{i'}(x')}^*\} = \sum_{j,j'}{}' w_{ji}(x)\,\overline{w_{j'i'}(x')}\,\{a_j, a_{j'}^*\}$$

$$= \sum_j{}' w_{ji}(x)\,\overline{w_{ji'}(x')} = \delta_{ii'}\,\delta(x-x')$$

In practice we leave out the underlines $\underline{\quad}$.

Notice that

$$\psi_i(x) = \underbrace{\sum_{\lambda_j>0}{}' w_{ji}(x)\,a_j}_{\substack{\psi_i^+(x) \\ \text{destroys electron}}} + \underbrace{\sum_{\lambda_j<0}{}' w_{ji}(x)\,a_j}_{\substack{\psi_i^-(x) \\ \text{creates positron}}}$$

$$\overline{\psi}_i(x) = \psi_i(x)^* = \underbrace{\sum_{\lambda_j>0}{}' \overline{w_{ji}(x)}\,a_j^*}_{\substack{\overline{\psi}_i(x)^- \\ \text{creates electron.}}} + \underbrace{\sum_{\lambda_j<0}{}' \overline{w_{ji}(x)}\,a_j^*}_{\substack{\overline{\psi}_i(x)^+ \\ \text{destroys pos.}}}$$

Summary: Starting with $i\frac{\partial\phi}{\partial t} = H_0\phi$ in the Hilbert space W, we decompose W: $W = W^- \oplus W^+$ according to the sign of the eigenvalues of H_0. Then one finds a representation $\mathcal{H}$ of the CCR such that there is a vacuum state $|0\rangle$ killed by "positive frequency parts of the field components"

I should think of $\mathcal{H}$ as a kind of modification of $\Lambda(W)$ required because [redacted] H_0 has infinitely many negative eigenvalues. $|0\rangle$ [redacted] spans the line line in ΛW which is the highest exterior power of W^-. We [redacted] have to renormalize H_0 [redacted] so that it [redacted] is defined on the modified ΛW; this means we require $H_0|0\rangle = 0$. Then

$$H_0 = \sum_{\lambda_j>0}' \lambda_j a_j^* a_j - \sum_{\lambda_j<0}' \lambda_j a_j a_j^*$$

Recall that if we have a finite support perturbation $H = H_0 + V(t)$, then [redacted] we saw that the vacuum-vacuum scattering amplitude is a determinant:

$$\langle 0|S|0\rangle = \det(1 + iG_0V).$$

However the determinant on the right needs to be made precise. The way I compute it is to take independent solutions $\phi_1, \dots, \phi_p$ of $\left(\frac{d}{dt} + iH\right)\phi = 0$ which satisfy the boundary condition at t_i, i.e. $\phi_i(t) \in W^-$ for $t \ll 0$. I also [redacted] find solutions $\psi_1, \dots, \psi_p$ which satisfy the boundary condition $\psi_j(t) \in W^-$ for $t \gg 0$. Then we can form the Wronskian

$$(\phi_1 \wedge \cdots \wedge \phi_p, \psi_1 \wedge \cdots \wedge \psi_p) = \det\langle \psi_i | \phi_j \rangle$$

and divide this by the corresponding gadget with H replaced by H_0. Thus we compute

$$\frac{\langle 0|U(t_f, t_i)|0\rangle}{\langle 0|U_0(t_f, t_i)|0\rangle} = \langle 0|S|0\rangle$$

It seems that an interesting question is how to compute this determinant, say when V is infinitesimal, so that we want [redacted] $\mathrm{Tr}(G_0 V)$. What I want to understand is the condition $\mathrm{tr}(V(t)) = 0$ occurring above, which I seem to need to make sense out of $\mathrm{tr}(G_0(t,t)V(t))$. It seems I ought to be able to understand this whole business in a simple case.

So take W to be $\mathbb{C}^2$ with $H_0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$.

$$\frac{d\phi}{dt} = \frac{1}{i}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\phi$$

Then

$$G_0(t,t') = \begin{pmatrix} e^{-i(t-t')}\eta(t-t') & 0 \\ 0 & -e^{i(t-t')}\eta(t'-t) \end{pmatrix}$$

Take V ~~to~~ to be εI, where [redacted] $\varepsilon(t)$ has compact support. In this case H differs from H_0 by a scalar, so $U(t,t')$ will be a scalar times $U_0(t,t')$.

$$U(t,t') = e^{-i\int_{t'}^{t}\{H_0 + \varepsilon(\hat{t})\}d\hat{t}} = e^{-iH_0(t-t')}\, e^{-i\int_{t'}^{t}\varepsilon}$$

March 28, 1979

Setup - W is a finite diml Hilbert space with a Schroedinger DE: $i\frac{\partial w}{\partial t} = Hw$ which I quantize by extending H to a derivation on $\mathcal{H} = \Lambda W$. I suppose $H(t) = H_0 + V(t)$ where H_0 is non-singular and V has compact support. Then $W = W^- \oplus W^+$ where $H_0 > 0$ on W^+ and $H_0 < 0$ on W^-. The ground state of H_0 on $\mathcal{H}$ is a unit vector in $(\Lambda^p W^-) \subset \Lambda^p W$ where $p = \dim(W^-)$. I have found that

$$\langle 0|S|0\rangle = \det(1 + G_0 iV)$$

and the goal is to understand the determinant a bit better. I recall that this formula gets established by multiplicative character of both sides. What this means is ~~$\langle 0|U(t_f,t_i)|0\rangle$~~ that if $H_2 = H_1 + \delta H$, then we have the more general formula

$$\frac{\langle 0|U_2(t_f,t_i)|0\rangle}{\langle 0|U_1(t_f,t_i)|0\rangle} =$$

$$= \det\Big(\underbrace{\left(\frac{d}{dt} + iH_1\right)^{-1}}_{\downarrow}\left(\frac{d}{dt} + iH_2\right)\Big)$$

$$= \det(1 + G_1 i\delta H)$$

The important case occurs when δH is infinitesimal in which case the thing to prove is

$$-i\int_{t_i}^{t_f} \underbrace{\frac{\langle 0|U(t_f,t)\,\delta H(t)\,U(t,t_i)|0\rangle}{\langle 0|U(t_f,t_i)|0\rangle}} dt = i\,\mathrm{Tr}(G\,\delta H)$$

However calculation shows the integrand is

$$tr(\delta H(t) P_-(t))$$

where $P_-(t)$ = the projection onto $U(t,t_i)W^-$ with kernel $U(t,t_f)W^+$. Also

$$G(t,t') = \begin{cases} U(t,t')\,P_+(t') & t > t' \\ -U(t,t')\,P_-(t') & t < t' \end{cases}$$

Therefore it appears that

$$\boxed{\mathrm{Tr}(G\,\delta H) = \int tr\left(G(t^-,t)\,\delta H(t)\right) dt}$$

As a check suppose that $V(t) = \varepsilon(t)\cdot Id$ is scalar. Then [redacted] on W we have

$$U(t,t') = U_0(t,t')\,e^{-i\int_{t'}^{t}\varepsilon}$$

so that on $\Lambda^p W$ [redacted] we get p-fold product, so

$$\frac{\langle 0|U(t_f,t_i)|0\rangle}{\langle 0|U_0(t_f,t_i)|0\rangle} = \left(e^{-i\int_{t_i}^{t_f}\varepsilon}\right)^p$$

So infinitesimally

$$\mathrm{Tr}(G i\delta\varepsilon) = -pi\int\delta\varepsilon$$

$$\mathrm{Tr}(G\delta\varepsilon) = -p\int\delta\varepsilon$$ which agrees.

The problem now is to understand a time-independent perturbation:

$$H = H_0 + V$$

where V is constant in time. If we suppose V is small then the number of negative eigenvalues for H is the same as for [redacted] H_0. In fact this will be true [redacted] whenever H is obtained from H_0 through non-singular Hamiltonians.

I should look at the following linear algebra problem: Given H a self-adjoint non-singular operator on W, decompose W into positive and negative eigenspaces:

$$W = W^- \oplus W^+$$

and compute the sum E of the negative eigenvalues of H. Now let H be subjected to an infinitesimal variation δH and compute how W^- and E change.

Review simple perturbation theory.

$$H = H_0 + \varepsilon H_1 + \cdots$$
$$\psi = \psi_0 + \varepsilon \psi_1 + \cdots$$
$$\lambda = \lambda_0 + \varepsilon \lambda_1 + \cdots$$

$$H\psi = H_0\psi_0 + \varepsilon(H_1\psi_0 + H_0\psi_1) + \varepsilon^2(H_2\psi_0 + H_1\psi_1 + H_0\psi_2)$$
$$\lambda\psi = \lambda_0\psi_0 + \varepsilon(\lambda_1\psi_0 + \lambda_0\psi_1) + \varepsilon^2(\lambda_2\psi_0 + \lambda_1\psi_1 + \lambda_0\psi_2)$$

Assume ψ_0, λ_0 are given. Only the line $\mathrm{Ker}(H-\lambda)$ is determined, assuming λ_0 simple eigenvalue. Thus to get a well-defined problem we shall want $\psi_1, \psi_2, \cdots$ to be orthogonal to ψ_0. Because λ_0 is a simple eigenvalue for H_0, the equation

$$(H_0 - \lambda_0)\beta = \alpha$$

has a unique soln. β with $(\beta, \psi_0) = 0$ for each α with $(\alpha, \psi_0) = 0$. So to solve

$$(H_0 - \lambda_0)\psi_1 + (H_1 - \lambda_1)\psi_0 = 0$$

we must have $\quad ((H_1 - \lambda_1)\psi_0, \psi_0) = 0$

or

$$\lambda_1 = \frac{(H_1 \psi_0, \psi_0)}{(\psi_0, \psi_0)}$$

This determines λ_1 and then we can solve for ψ_1. The process obviously iterates.

Let's return to $W = W^- \oplus W^+$ and pick an orthonormal basis for W diagonalizing H: $Hw_j = \lambda_j w_j$. Assuming λ_j a simple eigenvalue we find corresponding to δH in H a change

$$\delta \lambda_j = (\delta H w_j, w_j)$$

and hence a change in the ground energy on ΛW:

$$\delta E(0) = \sum_{\lambda_j < 0}' \delta \lambda_j = \sum_{\lambda_j < 0}' (\delta H w_j, w_j) = tr(\delta H \cdot P^-)$$

where P^- is the orthogonal projection onto W^-.

The problem now is to integrate the above formula to find $E(0)$. Note that P^- is a function of H. Thus we take a path $H_t = H_0 + tV$ and try to integrate

$$\int_0^1 tr(V \cdot P^-(H_t))\, dt$$

March 29, 1979

Recall that we have a Schroedinger equation

$$i\frac{\partial\psi}{\partial t} = H\psi$$

in a Hilbert space W, which we quantize by considering the extension of H to $\mathcal{H} = \Lambda W$. [redacted] Assuming H [redacted] is non-singular, the ground state energy on $\mathcal{H}$ is the sum of the negative eigenvalues of H: (All this makes sense for W finite-dimensional.)

$$E(0) = \sum_{\lambda_j<0}\lambda_j = \operatorname{tr}(H\cdot P^-) \qquad P^- = \text{proj on } W^-.$$

$$= \frac{1}{2\pi i}\oint_{\substack{\text{negative}\\\text{eigenvalues}}} \lambda \operatorname{tr}(\lambda - H)^{-1}\, d\lambda$$

$$= \frac{1}{2\pi i}\oint_{\text{neg. eigen.}} \lambda \frac{d}{d\lambda}\log(\det(\lambda - H))\, d\lambda$$

But one really wants to apply this to the Dirac equation, where [redacted] W is infinite-dimensional and $H = H_0 + V$ is a perturbation of [redacted] an operator H_0 we know. In this case the quantization [redacted] for H_0 is rigged, so that the ground state energy is 0, and then one wants to compute the ground state energy change, which will be

$$E(0) = \frac{1}{2\pi i}\oint_{\substack{\text{neg.}\\\text{spec}}} \lambda \operatorname{tr}\{(\lambda - H)^{-1} - (\lambda - H_0)^{-1}\}\, d\lambda$$

$$= \frac{1}{2\pi i} \oint_{\substack{\text{neg.}\\ \text{spec.}}} \lambda \frac{d}{d\lambda} \log \det\left((\lambda - H_0)^{-1}(\lambda - H)\right) d\lambda$$

Review Green's function:

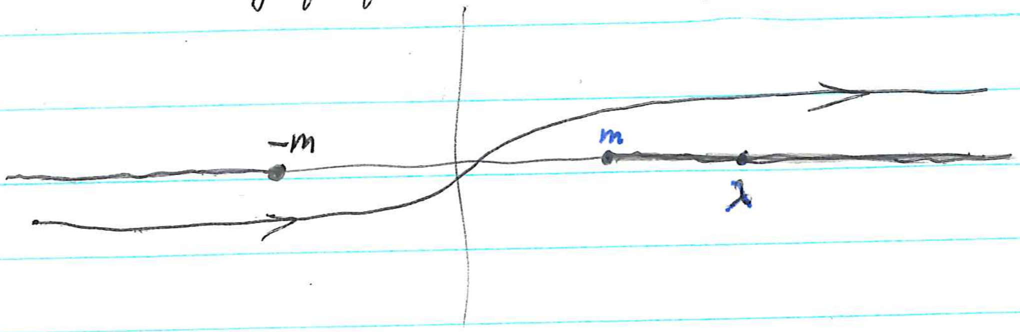
$$\left(\frac{d}{dt} + iH\right) G = \delta \qquad G(t) = \int \frac{dk}{2\pi} e^{-ikt} \hat{G}(k)$$

$$(-ik + iH)\hat{G} = 1$$

$$\hat{G} = \frac{1}{-ik + iH} = \frac{i}{k - H}$$

$$\therefore \quad [redacted] G(t) = \int_{-\infty}^{\infty} \frac{dk}{2\pi} \frac{e^{-ikt}\, i}{k - H} \qquad e^{-ikt} \text{ decays in LHP for } t > 0$$

To get the Green's function with pos. frequencies for positive time and neg. frequencies for neg. time, you use contour



Next to understand

$$(\lambda - H_0)^{-1}(\lambda - H_{[redacted]}) = 1 - G_0(\lambda) V$$

where H_0 is a Dirac operator.

$$\left(\alpha \frac{1}{i}\frac{d}{dx} + \beta m + V\right)\psi = \lambda\psi$$

$$(H_0 + V)\psi = \lambda\psi$$

$$(\lambda - H_0)\psi = V\psi$$

$$\psi = \psi_0 + G_0(\lambda)V\psi \qquad \text{with } \psi_0 \in \mathrm{Ker}(\lambda - H_0).$$

Thus the zeroes of $\det(1-G_0(\lambda)V)$ occur for those λ for which we have an eigenfunction of H satisfying the "outgoing" boundary conditions. What does this mean?

March 30, 1979:

$$(\lambda - H_0)^{-1}(\lambda - H) = 1 - G_0(\lambda)V$$

The operator $G_0^+(k)$ for k real is obtained by approaching the real axis from above for $k>0$ and from below for $k<0$. Now if we deform the contour

into the Feynman contour via a large circle in the UHP we get

$$E(0) = \frac{1}{2\pi i}\oint_{\substack{\text{neg.}\\ \text{spec.}}} \lambda \frac{d}{d\lambda}\log\det\left(1-G_0(\lambda)V\right)d\lambda$$

$$= \frac{1}{2\pi i}\int_{\substack{\text{Feynman}\\ \text{contour}}} \lambda \frac{d}{d\lambda}\log\det\left(1-G_0(\lambda)V\right)d\lambda$$

$$= +\frac{1}{2\pi i}\int_{-\infty}^{\infty} k \frac{d}{dk}\log\det\left(1-G_0^+(k)V\right)dk$$

$$= \frac{i}{2\pi} \int_{-\infty}^{\infty} \log\det\left(1 - G_0^+(k) V\right) dk$$

which is Schwinger's formula VI 50). It seems that he has a way of [redacted] seeing this formula [redacted] as a giant trace or determinant: The idea first is that

$$G_0^+(t,t') = \int_{-\infty}^{\infty} \frac{dk}{2\pi} e^{-ik(t-t')} G_0^+(k)$$

so that
$$G_0^+(t,t) = \int_{-\infty}^{\infty} \frac{dk}{2\pi} G_0^+(k)$$

The next point

$$e^{-iE(0)} = \exp\left\{ \frac{1}{2\pi} \int_{-\infty}^{\infty} dk \log\det\left(1 - G_0^+(k) V\right) \right\}$$

which is incredibly reminiscent of the Szego formula page 726 missing

April 7, 1979

Recall the Lax-Phillips semi-groups. Consider the Schroedinger equation

$$\left(-\frac{d^2}{dx^2}+q\right)u = k^2u$$

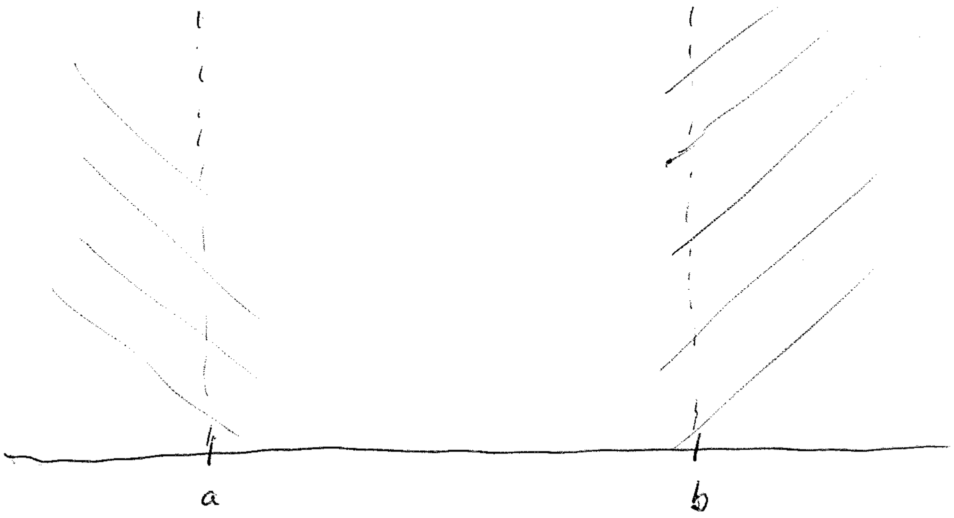
on the line with q of compact support contained in (a,b). Consider the wave equation

$$\frac{\partial^2}{\partial t^2}\psi = \left(\frac{\partial^2}{\partial x^2} - q\right)\psi$$

and look at solutions $\psi(x,t)$ supported in (a,b) at time 0. Since we know solutions of the wave equation with $q=0$ are of the form $f(x-t)+g(x+t)$, one sees, for $t>0$ and x outside (a,b), that $\psi(x,t)$ is an outgoing wave, hence

$$\frac{\partial\psi}{\partial t}+\frac{\partial\psi}{\partial x}=0 \qquad \begin{matrix} t\geq 0 \\ x\geq b \end{matrix}$$

$$\frac{\partial\psi}{\partial t}-\frac{\partial\psi}{\partial x}=0 \qquad \begin{matrix} t\geq 0 \\ x\leq a \end{matrix}$$



So we look at the problem

$$\frac{\partial^2 \psi}{\partial t^2} = \frac{\partial^2 \psi}{\partial x^2} - q\psi \quad \text{for } t \geq 0, \quad a \leq x \leq b$$

$$\frac{\partial \psi}{\partial t} + \frac{\partial \psi}{\partial x} = 0 \quad x = b \quad , \quad \frac{\partial \psi}{\partial t} - \frac{\partial \psi}{\partial x} = 0 \quad x = a$$

$$\psi(x,0) = \psi_0(x) \; , \; \frac{\partial \psi}{\partial t}(x,0) = \dot{\psi}_0(x)$$

and to solve it we use the Laplace transform in t:
Recall

$$\mathcal{L}(\psi_{tt}) = s^2 \mathcal{L}(\psi) - \dot{\psi}_0 - s\psi_0$$

and

$$\mathcal{L}(\psi) = \int_0^\infty e^{-st} \psi(x,t)\, dt .$$

Also $s = -ik$ [redacted] converts to the Fourier transform [redacted]

[redacted] $$u(x,k) = \int e^{ikt} \psi(x,t)\, dt$$

$$\psi(x,t) = \int \frac{dk}{2\pi} e^{-ikt} u(x,k)$$ [redacted]

So

$$s^2 u - \dot{\psi}_0 - s\psi_0 = u_{xx} - qu$$

or

$$(*) \quad \left(k^2 + \frac{d^2}{dx^2} - q\right) u = -\dot{\psi}_0 + ik\psi_0$$

Next

$$su + \frac{du}{dx} = 0 \quad \text{at } b \quad \text{or} \quad \frac{du}{dx} = iku$$

$$su - \frac{du}{dx} = 0 \quad \text{at } a \quad \text{or} \quad \frac{du}{dx} = -iku$$

which means $u(x,k)$ satisfies outgoing boundary conditions, and satisfies $(*)$. So if we denote by G_k^+ the Green's function for $(*)$ with these bdry conditions, we obtain

$$u(x,k) = G_k^+(-\dot{\psi}_0 + ik\psi_0)$$

hence

$$\psi(x,t) = \int_{i\alpha-\infty}^{i\alpha+\infty} \frac{dk}{2\pi}\, e^{-ikt}\, G_k^+(-\dot{\psi}_0 + ik\psi_0)$$

where α is positive enough so the singularities are below $i\alpha + \mathbb{R}$.

In the Lax-Phillips case, [redacted] one assumes there are no bound states, so G_k^+ is analytic on the UHP, so $\alpha = 0$. The decay of ψ depends on the singularities of G_k^+ in the LHP.

Now I think the above formula for $\psi(x,t)$ in terms of [redacted] Cauchy data $\psi_0, \dot{\psi}_0$ can be interpreted by [redacted] a formula of the form

$$e^{tA} = \frac{-1}{2\pi i} \int_{-\infty+i\alpha}^{\infty+i\alpha} e^{-ikt}\, \frac{1}{k - iA}\, dk$$

where A is the infinitesimal generator of the semi-group e^{tA}.

$$e^{tA} = \frac{1}{2\pi} \int_{-\infty+i\alpha}^{\infty+i\alpha} e^{-ikt}\, \frac{1}{-ik - A}\, dk$$

Answer is yes: Rewrite the equation on Cauchy data as

$$\frac{\partial}{\partial t}\begin{pmatrix}\psi \\ \dot{\psi}\end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -H & 0 \end{pmatrix}\begin{pmatrix}\psi \\ \dot{\psi}\end{pmatrix} \qquad H = -\frac{\partial^2}{\partial x^2} + q$$

Then A is the operator

$$A : \begin{pmatrix}\psi\\ \dot\psi\end{pmatrix} \longmapsto \begin{pmatrix}0 & 1\\ -H & 0\end{pmatrix}\begin{pmatrix}\psi\\ \dot\psi\end{pmatrix}$$

equipped with the boundary conditions

$$\dot\psi + \frac{d\psi}{dx} = 0 \quad \text{at } b$$

$$\dot\psi - \frac{d\psi}{dx} = 0 \quad \text{at } a$$

Then when we compute the resolvent for A:

$$\underbrace{(-ik - A)}_{\begin{pmatrix}-ik & -1\\ H & -ik\end{pmatrix}}\begin{pmatrix}\varphi\\ \dot\varphi\end{pmatrix} = \begin{pmatrix}\psi\\ \dot\psi\end{pmatrix}$$

$ik.$ $\quad -ik\varphi - \dot\varphi = \psi$

or $-1.$ $\quad H\varphi - ik\dot\varphi = \dot\psi$

$$(k^2 - H)\varphi = ik\psi - \dot\psi$$

we end up solving

$$\left(k^2 + \frac{d^2}{dx^2} - q\right)\varphi = ik\psi - \dot\psi$$

with the outgoing boundary conditions on φ. Hence

$$G_k^+(-\dot\psi + ik\psi)$$

is the first component of $(-ik-A)^{-1}\begin{pmatrix}\psi\\ \dot\psi\end{pmatrix}$.

Pages 731 — 780 missing because I misread 730 as 780