February 5, 1978 attempt at auton. Dirac equation
modular fn. $\Theta(\tau)$ for subgroup of $SL_2(\mathbb{Z})$. Guillemin's scattering fmla 795

Maxwells equations with no charges or currents:

$$\operatorname{div}\begin{pmatrix}E\\H\end{pmatrix} = 0 \qquad \operatorname{curl}\begin{pmatrix}E\\H\end{pmatrix} = \frac{\partial}{\partial t}\begin{pmatrix}-\mu H\\ \varepsilon E\end{pmatrix}$$

Consider a plane wave solution

$$\begin{pmatrix}E\\H\end{pmatrix} = \begin{pmatrix}E^\circ\\H^\circ\end{pmatrix} e^{i\omega t + i(x,\xi)} \qquad x = \vec{x} \quad \xi = \vec{\xi}$$

This represents a wave with equi-phase surfaces

$$\omega t + (x,\xi) = \text{const}$$

$$\frac{\omega}{|\xi|}t + \left(x, \frac{\xi}{|\xi|}\right) = \text{const}$$

Hence it represents a wave with speed $\frac{\omega}{|\xi|}$ travelling in the direction of the unit vector $-\frac{\xi}{|\xi|}$. The velocity of the wave is $-\frac{\omega\xi}{|\xi|^2}$.

[redacted] Maxwell equations become:

$$0 = \nabla\cdot E = \nabla\cdot\left(E^\circ e^{i\omega t + i(x\cdot\xi)}\right) = i\left(\xi\cdot E^\circ\right)e^{i\omega t + i(x\cdot\xi)}$$

hence the divergence equations become

$$\xi\cdot E^\circ = 0$$

$$\xi\cdot H^\circ = 0.$$

In other words E, H are $\perp$ the direction of propagation. The curl equations become

$$\xi \times E^\circ = -\mu\omega H^\circ$$

$$\xi \times H^\circ = \varepsilon\omega E^\circ$$

If we use the identity

$$A\times(B\times C) = (A\cdot C)B - (A\cdot B)C$$

then we get

$$\xi \times (\xi \times E^0) = -\mu\omega(\xi \times H^0) = -\mu\varepsilon\,\omega^2 E^0$$

$$\parallel$$

$$(\xi \cdot E^0)\xi - (\xi \cdot \xi)E^0 = -|\xi|^2 E^0$$

so $$|\xi|^2 = \mu\varepsilon\,\omega^2$$

hence $$\text{speed of propagation} = \frac{\omega}{|\xi|} = \frac{1}{\sqrt{\mu\varepsilon}}$$

Maxwell's equations with time dependence $e^{i\omega t}$ are

$$\operatorname{div}\begin{pmatrix}E\\H\end{pmatrix} = 0 \qquad \operatorname{curl}\begin{pmatrix}E\\H\end{pmatrix} = \begin{pmatrix}-i\omega\mu H\\ i\omega\varepsilon E\end{pmatrix}$$

or written out:

$$\frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} + \frac{\partial E_z}{\partial z} = \frac{\partial H_x}{\partial x} + \frac{\partial H_y}{\partial y} + \frac{\partial H_z}{\partial z} = 0$$

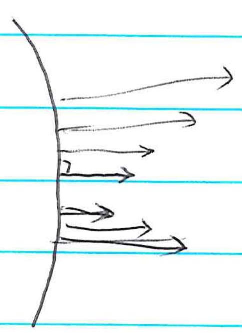
$$\frac{\partial E_z}{\partial y} - \frac{\partial E_y}{\partial z} = -i\omega\mu H_x \qquad \frac{\partial H_z}{\partial y} - \frac{\partial H_y}{\partial z} = i\omega\varepsilon E_x$$

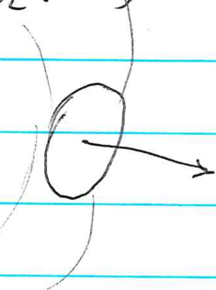
permute cyclically.

A metal wall ÷ i.e. conducting surface puts the following boundary conditions: E must be normal to the surface. From $\operatorname{div}(E) = 0$ one sees that

$$\frac{\partial}{\partial n}(E_n) = 0$$

where E_n is the normal component. Also from $\operatorname{curl} E = -i\omega\mu H$ one sees that $H_n = 0$ (for $\omega \neq 0$)





To see the curl use a paddle wheel.

Take two parallel conducting planes and work out the possible plane waves.

Idea: $SL_2(\mathbb{R})$ is a 3-manifold on which Maxwell's DE makes sense. A cusp on $\Gamma \backslash SL_2(\mathbb{R})$ resembles a rectangular wave guide except that one imposes periodic [redacted] conditions on the rectangle.

Dirac-style version of scattering in the automorphic setting:

Consider the system

$$(*) \qquad y\frac{d}{dy}\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} i\lambda & -i\xi y \\ i\xi y & -i\lambda \end{pmatrix}\begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$$

as being analogous to the system

$$\left[y^2\left(\frac{d}{dy}\right)^2 - \lambda - \xi^2 y^2 \right] u = 0$$

encountered before. We can put this into the form

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} y\frac{d}{dy}u + \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} i\xi y u = i\lambda u$$

which shows the system is self-adjoint and hence λ is real if one has a square-integrable eigenfunction. This equation comes from the [redacted] equation:

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} y\frac{\partial u}{\partial y} + \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} y\frac{\partial u}{\partial x} = i\lambda u$$

by assuming $e^{i\xi x}$ dependence in x.

February 6, 1978

I want to construct an invariant self-adjoint operator on the UHP starting from

$$\bar\partial : 1 \longrightarrow T^{01}$$

These are G-line bundles with invariant metric, hence together with the G-invariant volume on UHP we can form the adjoint $\bar\partial^*$. Then $\bar\partial + \bar\partial^*$ on $1 \oplus T^{01}$ is self-adjoint.

Sections of T^{10} are things of the form $f(z)dz$. Under $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ they pull back as follows:

$$A^*(f\,dz) = f\left(\frac{az+b}{cz+d}\right)\frac{dz}{(cz+d)^2}$$

I want the norm of $f(z)dz$ which is a positive-valued function. Recall that $y\frac{\partial}{\partial x}, x\frac{\partial}{\partial y}$ is an orthonormal frame for T, hence $\frac{dx}{y}, \frac{dy}{y}$ is an orth. frame for T^*. Hence

$$\frac{dz}{y} = \frac{dx}{y} + i\frac{dy}{y}$$

has norm $\sqrt{2}$. Thus an orth. frame for T^{10} is $\frac{dz}{\sqrt{2}y}$.

Check:

$$A^*\left(\frac{dz}{\sqrt{2}y}\right) = \frac{1}{\sqrt{2}}\frac{dz}{(cz+d)^2}\frac{1}{\frac{y}{|cz+d|^2}} = \frac{c\bar z+d}{cz+d}\frac{dz}{\sqrt{2}y}$$

abs. value 1.

What is $\bar\partial$:

$$df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy = \frac{\partial f}{\partial z}(dx+idy) + \frac{\partial f}{\partial \bar z}(dx - idy)$$

$$\frac{\partial f}{\partial x} = \frac{\partial f}{\partial z} + \frac{\partial f}{\partial \bar z} \qquad \frac{1}{i}\frac{\partial f}{\partial y} = \frac{\partial f}{\partial z} - \frac{\partial f}{\partial \bar z}$$

so
$$\frac{\partial}{\partial \bar z} = \frac{1}{2}\left(\frac{\partial}{\partial x} - \frac{1}{i}\frac{\partial}{\partial y}\right).$$

So
$\bar\partial f(z) =$ [redacted] $\frac{\partial f}{\partial \bar z}\, d\bar z = \sqrt{2}\, y \frac{\partial f}{\partial \bar z} \cdot \frac{d\bar z}{\sqrt{2}\, y}$

Its adjoint relative to $dV = \frac{dx\,dy}{y^2}$ is
$$\bar\partial^* = y^2\, \frac{1}{2}\left(-\frac{\partial}{\partial x} - \frac{1}{i}\frac{\partial}{\partial y}\right) \sqrt{2}\, y\, \frac{1}{y^2}$$

or
$-\bar\partial^*\left(g(z)\cdot\frac{d\bar z}{\sqrt{2}\, y}\right) =$ [redacted] $\sqrt{2}\, y^2 \frac{\partial}{\partial z}\left(\frac{1}{y} g\right)$

Check:
$$-\bar\partial^*\bar\partial = \sqrt{2}\, y^2 \frac{\partial}{\partial z}\frac{1}{y}\sqrt{2}\, y \frac{\partial}{\partial \bar z} = y^2\, 2 \frac{\partial}{\partial z}\frac{\partial}{\partial \bar z}$$
$$= y^2\, 2\left(\frac{1}{4}\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)\right) = \frac{1}{2}\, y^2\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)$$

In order to remove some of the asymmetry between ∂ and $\bar\partial$ I should tensor the sequence $1 \xrightarrow{\bar\partial} T^{01}$ with a suitable holomorphic line bundle.

So I consider the G-holomorphic line bundle L on UHP whose holomorphic sections are of the form $f(z)\, dz^{1/2}$ with G-action
$$A^*\left(f\, dz^{1/2}\right) = (A^* f)(z)\cdot \frac{dz^{1/2}}{cz+d}$$

Here G has to be $SL_2(\mathbb{R})$ for [redacted] this to make sense.
Define a norm on L by requiring $\frac{dz^{1/2}}{y^{1/2}}$ be of norm 1.
Then
$$\bar\partial : L \longrightarrow L \otimes T^{01}$$
$$f\, dz^{1/2} \longmapsto \frac{\partial}{\partial \bar z} f \cdot dz^{1/2}\, d\bar z$$

or

$$\bar\partial\left(f(z)\, y^{1/2} \cdot \frac{dz^{1/2}}{y^{1/2}}\right) = \sqrt{2}\, y^{3/2} \frac{\partial f}{\partial \bar z} \; \frac{dz^{1/2}}{y^{1/2}} \, \frac{d\bar z}{\sqrt{2}\, y}$$

or

$$\bar\partial\left(f(z) \cdot \frac{dz^{1/2}}{y^{1/2}}\right) = \sqrt{2}\, y^{3/2} \frac{\partial}{\partial \bar z}\left(y^{-1/2} f\right) \cdot \frac{dz^{1/2}}{y^{1/2}} \cdot \frac{d\bar z}{\sqrt{2}\, y}$$

So in terms of the orthonormal frames chosen for $L, T^{0,1}$ we get

$$\bar\partial = \sqrt{2}\, y^{3/2} \frac{\partial}{\partial \bar z}\, y^{-1/2}$$

$$-\bar\partial^* = y^2\, y^{-1/2} \frac{\partial}{\partial z}\, y^{+3/2} \sqrt{2}\, \frac{1}{y^2} = \sqrt{2}\, y^{3/2} \frac{\partial}{\partial z}\, y^{-1/2}$$

The self-adjoint operator I am interested in is $\bar\partial + \bar\partial^*$ which in terms of the above orthonormal frames leads to the self-adjoint system

$$1) \quad \begin{aligned} \sqrt{2}\, y^{3/2} \frac{\partial}{\partial \bar z}\, y^{-1/2}\, w_1 &= \lambda w_2 \\ -\sqrt{2}\, y^{3/2} \frac{\partial}{\partial z}\, y^{-1/2}\, w_2 &= \lambda w_1 \end{aligned}$$

Incorporate $\sqrt{2}$ into λ, and notice that if $v_i = \frac{w_i}{y^{1/2}}$ this becomes

$$2) \quad \begin{aligned} y \frac{\partial}{\partial \bar z} v_1 &= \lambda v_2 \\ -y \frac{\partial}{\partial z} v_2 &= \lambda v_1 \end{aligned}$$

which should be essentially the system on page 779. So 1) is self-adjoint, but 2) isn't unless you use the suitable inner product.

So how do I set this all up? One way is to consider the system 2 on pairs of functions. Define the action of $SL_2(\mathbb{R})$ on pairs $\begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$ by

$$A^* \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} A^* v_1 \cdot \frac{1}{cz+d} \\ A^* v_2 \cdot \frac{1}{c\bar{z}+d} \end{pmatrix}$$

Notice that this agrees with

$$A^*\left(v_1(z)\, dz^{1/2}\right) = (A^* v_1) \cdot \frac{dz^{1/2}}{cz+d}$$

$$A^*\left(v_2(z)\, dz^{1/2} \frac{d\bar{z}}{y}\right) = (A^* v_2)\, \frac{dz^{1/2}}{cz+d}\, \frac{d\bar{z}}{(c\bar{z}+d)^2}\, \frac{|cz+d|^2}{y} = (A^* v_2)\, dz^{1/2} \frac{1}{c\bar{z}+d}\, \frac{d\bar{z}}{y}$$

Define the global norm of $\begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$ to be

$$\left\| \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \right\|_{gl}^2 = \int \left(|v_1|^2 + |v_2|^2\right) y \, \frac{dx\, dy}{y^2}$$

In other words we use the Riemannian metric given by

$$\| v_1\, dz^{1/2} \| = |v_1|\, y^{1/2}$$

$$\| v_2\, dz^{1/2} \frac{d\bar{z}}{y} \| = |v_2|\, y^{1/2}$$

Check: $\| A^*(v_1\, dz^{1/2}) \| = \| A^* v_1 \cdot (cz+d)^{-1} dz^{1/2} \| = |A^* v_1| \dfrac{y^{1/2}}{|cz+d|}$

$$= A^*\left(|v_1|\, y\right) = A^* \| v_1\, dz^{1/2} \|$$

Start again:

On the upper half-plane we consider the vector space of pairs of sections

$$\begin{pmatrix} v_1\, dz^{1/2} \\ v_2\, dz^{1/2} \frac{d\bar{z}}{dy} \end{pmatrix}$$

of the line bundles L, $L \otimes T^{01}$ respectively. The action of $G = SL_2(\mathbb{R})$ is given by

$$A^*\begin{pmatrix} v_1 \, dz^{1/2} \\ v_2 \, dz^{1/2} \frac{d\bar{z}}{y} \end{pmatrix} = \begin{pmatrix} v_1 \frac{1}{cz+d} \, dz^{1/2} \\ v_2 \frac{1}{c\bar{z}+d} \, dz^{1/2} \frac{d\bar{z}}{y} \end{pmatrix}$$

The operator $\bar{\partial} : L \to L \otimes T^{01}$ is given by

$$\bar{\partial}\left(v_1 \, dz^{1/2}\right) = \left(y \frac{\partial v_1}{\partial \bar{z}}\right) dz^{1/2} \frac{d\bar{z}}{y}$$

Equip $L, L \otimes T^{01}$ with the inner products given by

$$\| v_1 \, dz^{1/2} \| = |v_1| \, y^{1/2}$$

$$\left\| v_2 \, dz^{1/2} \frac{d\bar{z}}{y} \right\| = |v_2| \, y^{1/2}$$

These inner products are G-invariant. Compute $\bar{\partial}^*$:

$$\int \left(\bar{\partial}(v_1 \, dz^{1/2}), \, v_2 \, dz^{1/2} \frac{d\bar{z}}{y} \right) \frac{dx\,dy}{y^2} = \int \left(y \frac{\partial v_1}{\partial \bar{z}} \, dz^{1/2} \frac{d\bar{z}}{y}, \, v_2 \, dz^{1/2} \frac{d\bar{z}}{y} \right) \frac{dx\,dy}{y^2}$$

$$= \int y \frac{\partial v_1}{\partial \bar{z}} \, \bar{v}_2 \, y \, \frac{dx\,dy}{y^2} = \int \frac{\partial v_1}{\partial \bar{z}} \, \bar{v}_2 \, dx\,dy = -\int v_1 \overline{\frac{\partial}{\partial z} v_2} \, dx\,dy$$

$$= -\int \left(v_1 \, dz^{1/2}, \, y \frac{\partial v_2}{\partial z} \, dz^{1/2} \right) \frac{dx\,dy}{y^2}.$$

Hence

$$-\bar{\partial}^* \left(v_2 \, dz^{1/2} \frac{d\bar{z}}{y} \right) = y \frac{\partial v_2}{\partial z} \, dz^{1/2}$$

The eigenvector equations for $\bar{\partial} + \bar{\partial}^*$ become:

$$y \frac{\partial v_1}{\partial \bar{z}} = \lambda v_2 \qquad y \cdot \frac{1}{2}\left(\frac{\partial}{\partial x} + i \frac{\partial}{\partial y}\right) v_1 = \lambda v_2$$

$$-y \frac{\partial v_2}{\partial z} = \lambda v_1 \qquad y \cdot \frac{1}{2}\left(\frac{-\partial}{\partial x} + i \frac{\partial}{\partial y}\right) v_2 = \lambda v_1$$

Change λ to -2λ and assume x independent.

$$y \frac{\partial}{\partial y} v_1 = i\lambda v_2$$

$$y \frac{\partial}{\partial y} v_2 = i\lambda v_1$$

which has the solutions

$$\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = c_1 \begin{pmatrix} y^{i\lambda} \\ y^{i\lambda} \end{pmatrix} + c_2 \begin{pmatrix} y^{-i\lambda} \\ -y^{-i\lambda} \end{pmatrix}$$

In order to determine the scattering operator I need to have an eigenfunction such as an Eisenstein series. [redacted]

However there appears to be an interesting problem here when one tries to construct the Eisenstein series in the usual way:

$$\sum_{A^{-1}\in\Gamma/\Gamma_0}' (A^{-1})^* \begin{pmatrix} y^{i\lambda} \\ y^{i\lambda} \end{pmatrix} = \sum_{\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}(\infty)}' \begin{pmatrix} \dfrac{y^{i\lambda}}{|cz+d|^{2i\lambda}} \dfrac{1}{(cz+d)} \\ \dfrac{y^{i\lambda}}{|cz+d|^{2i\lambda}} \dfrac{1}{c\bar{z}+d} \end{pmatrix}$$

for this sum is zero. The problem stems from the fact that there are no Γ-invariant pairs $\begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$ because the center acts as -1.

The way to handle this, maybe is to introduce a representation Θ of [redacted] Γ on which $-I$ acts as -1 and then to tensor with Θ and take Γ-invariants. The simplest such representation appears to be the standard representation on $\mathbb{R}^2$, except $\begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix}$ acts non-trivially.

February 7, 1978:

Eisenstein series of weight k.

Consider the line bundle on the UHP with sections $f(z)\,dz^k$, i.e. $\Omega^{\otimes k}$. We have

$$\bar\partial : \Omega^{\otimes k} \longrightarrow \Omega^{\otimes k} \otimes T^{0,1}$$

$$f\,dz^k \longmapsto \frac{\partial f}{\partial \bar z}\,dz^k\,d\bar z$$

Assign dz the norm y (ignore $\sqrt{2}$) and compute $\bar\partial^*$:

$$\int \left(\bar\partial(f\,dz^k),\, g\,dz^k d\bar z\right) \frac{dx\,dy}{y^2} = \int \left(\frac{\partial f}{\partial \bar z}\,dz^k d\bar z,\, g\,dz^k d\bar z\right) \frac{dx\,dy}{y^2}$$

$$= \int \frac{\partial f}{\partial \bar z}\,\bar g\, y^{2k}\,dx\,dy = -\int f\, \overline{\frac{\partial}{\partial z}(y^{2k} g)}\,dx\,dy$$

$$= -\int \left(f\,dz^k,\, \frac{\partial}{\partial z}(y^{2k} g)\,dz^k\right) \frac{1}{y^{2k}}\,\frac{dx\,dy}{y^2}\cdot y^2$$

$$\therefore\ -\bar\partial^*\left(g\,dz^k d\bar z\right) = y^{2-2k}\frac{\partial}{\partial z}(y^{2k} g)\cdot dz^k$$

So the Laplacian for this line bundle $\Omega^{\otimes k}$ is

$$\Delta = 4\cdot y^{2-2k}\frac{\partial}{\partial z}\, y^{2k}\frac{\partial}{\partial \bar z} = y^{2-2k}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right) y^{2k}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)$$

On a form $f(y)\,dz^k$ independent of x one has

$$\Delta(f\,dz^k) = \left(y^{2-2k}\frac{d}{dy}\, y^{2k}\frac{df}{dy}\right) dz^k$$

If $f = y^s$, then $\Delta(y^s dz^k) = \left(y^{2-2k}\frac{d}{dy}\left(s\, y^{2k+s-1}\right)\right) dz^k$

$$= s(s-1+2k)\, y^s dz^k$$

so the eigenvalue is $\lambda = s(s-1+2k)$. We can form the

Eistenstein series

$$E = \frac{1}{2}\sum_{c,d\in\mathbb{Z}\times\mathbb{Z}}{}' \frac{y^s dz^k}{|cz+d|^{2s}(cz+d)^{2k}} = \zeta(2s+2k)\cdot\sum_{-\frac{d}{c}\in\mathbb{Q}\cup\{\infty\}} \begin{pmatrix} a & b \\ c & d\end{pmatrix}^*(y^s dz^k)$$

Assume the analogous thing holds that did for $k=0$:

$$\hat{E} = \pi^{-s-k}\Gamma(s+k)\,E \sim \hat{\zeta}(2s+2k)\cdot y^s dz^k + \text{symmetrical term}$$

Now $y^s dz^k$ is not square-integrable when

$$\int \|y^s dz^k\|^2 \frac{dy}{y^2} = \int y^{2s+2k-2}\,dy = \infty$$

or $\quad 2s+2k-2 > -1 \quad$ or $\quad 2s+2k > 1$. Thus the only thing we will get is that $2s+2k > 1 \Rightarrow \hat{\zeta}(2s+2k) \neq 0$.

Recall that any rational number with $0 < q < 1$ has a unique continued fraction expansion

$$q = \frac{1}{n_0+}\,\frac{1}{n_1+}\cdots\cdots\frac{1}{n_k}$$

where $n_0, \ldots, n_k$ are positive integers (≥ 1) and $n_k \geq 2$. The question is whether this has anything to do with the $PSL_2(\mathbb{Z})$ tree.

Maybe it would be more natural to consider an expansion

$$q = n_0 - \frac{1}{n_1-}\,\frac{1}{n_2-}\cdots\frac{1}{n_k}$$

where n_0 is the greatest integer $\geq q$. This corresponds to choosing n_0 so that $T^{-n_0}q \in (-1,0]$, then n_1 so that

$$T^{-n_1} S T^{-n_0} q \in (-1,0)$$

until the process stops with 0. This process doesn't seem to be too canonical as one could use $(0,1)$ instead of $(-1,0)$.

However if we work with the group $\Gamma' \subset \Gamma = PSL_2(\mathbb{Z})$ generated by S, T^2 then there appears to be a canonical procedure for reducing any rational number to either 0 or 1. First apply a power of T to get into $(-1,1]$; if you reach 0 or 1 you're done, otherwise apply S and repeat.

The group Γ' has two cusps represented by $1, \infty$. A cusp is a Γ' orbit on $\mathbb{P}^1(\mathbb{Q})$. Call a rational number $\frac{m}{n}$ odd if both m, n are odd and even if one is even, where we suppose $\frac{m}{n}$ in lowest terms. Note that

$$S\left(\frac{m}{n}\right) = -\frac{n}{m}$$

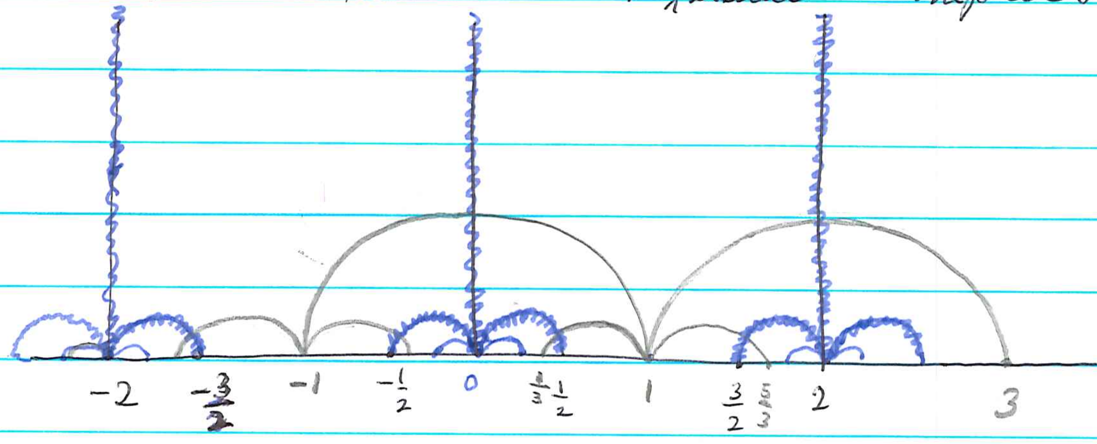
$$T^2\left(\frac{m}{n}\right) = \frac{m+2n}{n}$$

preserves this odd-even character, hence this invariant characterizes the cusp.

Digression: From $\frac{1}{2}\hat{E}(z,s)y^s \sim \hat{\zeta}(2s)y^s + \hat{\zeta}(2-2s)y^{1-s}$ on p. 776. Suppose $\hat{\zeta}(2s) = 0$ with $s = \frac{1}{2} + i\lambda$ λ real. Then because $\hat{\zeta}(s) = \overline{\hat{\zeta}(\bar{s})}$ one also has $\hat{\zeta}(2-2s) = \hat{\zeta}(1-2i\lambda) = \overline{\hat{\zeta}(1+2i\lambda)} = 0$. Hence one should have a square-integrable eigenfunction for Δ. Thus one sees that $\hat{\zeta}(s) \neq 0$ for $\operatorname{Re} s \geq 1$ which is a key step in the prime number theorem.

Proposition: $\Gamma' = (\mathbb{Z}/2) * \mathbb{Z}$ with the generators S, T.

Observe that in any simplex in the triangulation of the UHP with vertices $\mathbb{Q} \cup \{\infty\}$ there are exactly 2 even and one odd vertex. To see this we use the fact that any point in the UHP is Γ' conjugate to a point in the triangle $\{0, 1, \infty\}$ and hence you check for this triangle. So now take the triangulation and remove all the odd vertices which are not joined by any edge and whose links are contractible. One is left with a tree whose vertices are the even rational numbers with the usual 2 simplices



Subdivide the tree. The stabilizer of ∞ is $\langle T^2 \rangle$ and the stabilizer of i is $\langle S \rangle$, so by Serre's course $\Gamma' = (\mathbb{Z}/2) * \mathbb{Z}$.

The odd vertices also form a tree. Any edge of even vertices belongs to two triangles so you form a transversal edge out of the two odd vertices in these triangles. Then two odd rational numbers form an edge when

$$\begin{vmatrix} a & c \\ b & d \end{vmatrix} = \pm 2$$

Tomorrow review the boundary values of Θ function at the rational points.

February 8, 1978

Define the distance between two elements of $P_1(\mathbb{Q})$ to be the absolute value of $\begin{vmatrix} a & b \\ c & d \end{vmatrix}$ if the two elements are $\frac{a}{c}, \frac{b}{d}$ in lowest terms. This notion of distance is invariant under $PSL_2(\mathbb{Z})$. Triangle inequality? NO. The distance of $\frac{p}{q}$ to $\infty = \frac{1}{0}$ is $|q|$. So given $\frac{p}{q}, \frac{r}{s}$ a triangle inequality would give

$$|ps - qr| \leq |q| + |s|$$

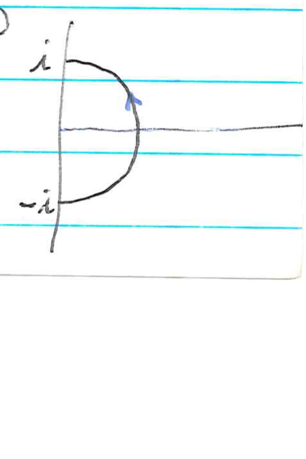
for all r relatively prime to s. There isn't even an estimate $|ps - qr| \leq$ function of $|q|, |s|$.

Return to basic formula $\hat{\zeta}(s) = \int_0^\infty \frac{\theta(t)-1}{2} t^{s/2} \frac{dt}{t}$. This hold for $Re(s) > 1$. If we use analytic continuation, this formula could be written

$$\hat{\zeta}(s) = \frac{1}{2}\int_0^\infty \theta(t)\, t^{s/2} \frac{dt}{t}$$

in some sense. It seems that I ought to understand similar integrals over different strategic contours in the upper half-plane. For example: I recall that θ dies fast as one approaches an odd rational numbers. Thus the integral along a geodesic circle joining two odd rational numbers should be an entire function of s, hopefully $\neq 0$. Question: Find the significance of

$$\int_{-i}^{+i} \theta(t)\, t^{s/2} \frac{dt}{t}$$



$$\int_{-i}^{i} \theta(t)\,t^{1/4}\, t^{(s-\frac{1}{2})/2}\,\frac{dt}{t} \underset{t=e^{i\varphi}}{=} \int_{-\pi/2}^{\pi/2} \theta(e^{i\varphi})\,e^{i\varphi/4}\, e^{\frac{i\varphi}{2}(s-\frac{1}{2})}\, i\,d\varphi$$

Now the function $\theta(e^{i\varphi})e^{i\varphi/4}$ is C^∞ on the interval $-\frac{\pi}{2} \le \varphi \le \pi/2$ and it vanishes to all orders at the ends, so we see this function is non-zero, entire in s, of exponential type. It would be nice to evaluate this in terms of Γ.

Recall $\theta(\tau) = \sum_{n\in\mathbb{Z}} e^{i\pi n^2\tau}$ satisfies

$$\theta(\tau+2) = \theta(\tau)$$
$$\theta\left(-\frac{1}{\tau}\right) = (-i\tau)^{1/2}\,\theta(\tau)$$

It follows that θ^8 is a modular form for the group Γ' of weight 2. Specifically $\theta^8(\tau)d\tau^2$ is invariant under T^2 and under S:

$$S^*\left(\theta^8(\tau)d\tau^2\right) = \theta^8\left(-\tfrac{1}{\tau}\right) d\left(-\tfrac{1}{\tau}\right)^2 = \theta^8(\tau)(-i\tau)^4\left(\frac{d\tau}{\tau^2}\right)^2 = \theta^8 d\tau^2$$

hence it is invariant under the full group Γ'. It follows for any $\begin{pmatrix} a & b \\ c & d\end{pmatrix} \in \Gamma'$ that we have

$$\theta\left(\frac{a\tau+b}{c\tau+d}\right) = \varepsilon\,\left(-i(c\tau+d)\right)^{1/2}\theta(\tau)$$

where ε is a ~~fourth~~ 8th root of unity. Suppose we require $c>0$, so as to specify the argument of $(-i(c\tau+d))^{1/2}$; (it should be in the right-half-plane). It is more or less clear that ε depends only on the rational number $-\frac{d}{c}$.

Observation: Γ' consists of $\begin{pmatrix} a & b \\ c & d\end{pmatrix}$ where $\frac{a}{c}, \frac{b}{d}$ are even.

To evaluate ε let $\tau = it$, $t \to +\infty$ and use that $\theta(\tau) = \sum' e^{-\pi n^2 t} \longrightarrow 1$ very fast. One can also take $\tau = it + x$, x fixed chosen judiciously with the same behavior for $\theta(\tau)$.

$$\left(-i(c\tau+d)\right)^{1/2} = \left(\frac{cit+cx+d}{i}\right)^{1/2} = c^{1/2}t^{1/2}$$

if we take $x = -\frac{d}{c}$. Then

$$\frac{a\tau+b}{c\tau+d} = \frac{ax+b+ait}{\cancel{cx+d}+cit} = \frac{a}{c} + \frac{a(-\frac{d}{c})+b}{cit}$$

$$= \frac{a}{c} + \frac{i}{c^2 t}$$

Put $\eta = \frac{1}{c^2 t}$. Then

$$\theta\left(\frac{a}{c}+i\eta\right) = \sum_{n\in\mathbb{Z}}' e^{i\pi n^2\left(\frac{a}{c}+i\eta\right)} = \sum e^{-\pi n^2\eta + i\pi n^2\frac{a}{c}}$$

$\frac{a}{c}(n+c)^2 = \frac{a}{c}n^2 + 2an + ac$, hence since $\frac{a}{c}$ is even the function

$$n \longmapsto e^{i\pi n^2\frac{a}{c}}$$

has the period c. So

$$\theta\left(\frac{a}{c}+i\eta\right) = \sum_{r=0}^{c-1}' e^{i\pi r^2\frac{a}{c}} \sum_{m\in\mathbb{Z}} e^{-\pi(mc+r)^2\eta}$$

$$-\pi m^2(c^2\eta) - 2\pi mcr\eta - \pi r^2\eta$$

Recall the θ-transformation formula:

$$4\pi\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} \qquad \frac{1}{\sqrt{t}}\sum_{n\in\mathbb{Z}}' e^{-\frac{\pi}{t}(x-n)^2} = \sum_{m\in\mathbb{Z}}' e^{-\pi m^2 t} e^{2\pi i m x}$$

So

$$e^{-\pi r^2\eta}\sum_{m\in\mathbb{Z}} e^{-\pi m^2\underbrace{(c^2\eta)}_{t}+2\pi i m\underbrace{(icr\eta)}_{x}} = \frac{e^{-\pi r^2\eta}}{\sqrt{c^2\eta}}\sum_n e^{-\frac{\pi}{c^2\eta}(n-icr\eta)^2}$$

hence

$$\sum_{m\in\mathbb{Z}} e^{-\pi(mc+r)^2\eta} \sim \frac{1}{c\sqrt{\eta}} \quad \text{as } \eta\downarrow 0$$

Thus

$$\theta\left(\frac{a}{c}+i\eta\right) \sim \frac{1}{c\sqrt{\eta}}\sum_{r=0}^{c-1} e^{i\pi r^2\frac{a}{c}}$$

so

$$\theta\left(\frac{a\tau+b}{c\tau+d}\right) \sim \sqrt{t}\sum_{r=0}^{c-1} e^{i\pi r^2\frac{a}{c}}$$

and also

$$\text{"} \quad \sim \varepsilon\, c^{1/2}t^{1/2}$$

Therefore $\varepsilon = \frac{1}{\sqrt{c}}\sum_{r=0}^{c-1} e^{i\pi r^2\frac{a}{c}}$. So we have the formula

$$\boxed{\theta\left(\begin{pmatrix}a & b\\ c & d\end{pmatrix}\tau\right) = \left(\frac{1}{\sqrt{c}}\sum_{r=0}^{c-1} e^{i\pi r^2\frac{a}{c}}\right)\left(-i(c\tau+d)\right)^{+\frac{1}{2}}\theta(\tau)}$$

provided $\begin{pmatrix}a & b\\ c & d\end{pmatrix}\in\Gamma'$ i.e. both $\frac{a}{c}, \frac{b}{d}$ even and $c>0$.

I want to check that θ dies exponentially at $\tau\to 1$ along geodesics. Instead look at $\theta(\tau+1)$ as $\tau\to 0$

$$\theta(\tau+1) = \sum e^{i\pi n^2(\tau+1)} \qquad e^{i\pi n^2} = (-1)^{n^2} = (-1)^n$$

$$= \sum e^{-\pi n^2 t + i\pi n} = \sum e^{-\pi n^2 t + 2\pi n\cdot\frac{1}{2}}$$

$$= \frac{1}{\sqrt{t}}\sum_n e^{-\frac{\pi}{t}(\frac{1}{2}-n)^2}$$

$$\left|e^{-\frac{\pi}{t}(n-\frac{1}{2})^2}\right| = e^{-\pi(n-\frac{1}{2})^2 Re(\frac{1}{t})}$$

This decays exponentially provided $Re(\frac{1}{t}) = Re(\frac{i}{\tau}) = Im(-\frac{1}{\tau}) \to 0$

Hence $\theta(\tau)$ is uniformly small in a circle of the form

If $a, c > 0$, then [redacted] letting $\tau \to i\infty$ in

$$\theta\left(-\frac{a\tau+d}{a\tau+b}\right) = \left(-i\,\frac{a\tau+b}{c\tau+d}\right)^{1/2} \theta\left(\frac{a\tau+b}{c\tau+d}\right)$$

we get

$$\sum_{r=0}^{a-1} e^{-i\pi r^2 \frac{c}{a}} = \left(-i\,\frac{a}{c}\right)^{1/2} \sum_{r=0}^{c-1} e^{i\pi r^2 \frac{a}{c}}$$

or the formula for Gaussian sums

$$\boxed{\frac{1}{\sqrt{a}} \sum_{r=0}^{a-1} e^{-i\pi r^2 \frac{c}{a}} = e^{-i\pi/4} \frac{1}{\sqrt{c}} \sum_{r=0}^{c-1} e^{i\pi r^2 \frac{a}{c}}}$$

a, c positive, rel prime, [redacted] and one is even.

Check: $a = 3$, $c = 2$.

$$\frac{1}{\sqrt{3}}\left(1 + e^{-i\pi\frac{2}{3}} + e^{-i\pi\frac{2}{3}}\right) = \frac{1}{\sqrt{3}}\left(1 + 2\left(-\frac{1}{2} - i\frac{\sqrt{3}}{2}\right)\right) = -i$$

$$e^{-i\pi/4} \frac{1}{\sqrt{2}}\left(1 + e^{i\pi\frac{3}{2}}\right) = \frac{1-i}{\sqrt{2}}\,\frac{1}{\sqrt{2}}(1-i) = \frac{(1-i)^2}{2} = \frac{1-1-2i}{2} = -i$$

Let's evaluate $\sum'_{\substack{-\frac{d}{c}\in Q \\ 0\le -\frac{d}{c}<1}} \frac{1}{c^s}$ where as usual $-\frac{d}{c}$ is in lowest terms.

This is clearly the same as

$$\sum_{n=1}^{\infty} \frac{\varphi(n)}{n^s} = \prod_p \left(\sum_{k=0}^{\infty} \frac{\varphi(p^k)}{p^{ks}} \right)$$

Now

$$\sum_{k=0}^{\infty} \frac{\varphi(p^k)}{p^{ks}} = 1 + \sum_{k=1}^{\infty} \frac{(p-1)p^{k-1}}{p^{ks}} = 1 + \frac{p-1}{p^s} \sum_{k=1}^{\infty} \frac{1}{p^{(k-1)(s-1)}}$$

$$= 1 + \frac{p-1}{p^s} \frac{1}{1-\frac{1}{p^{s-1}}} = 1 + \frac{p-1}{p^s - p} = \frac{p^s-1}{p^s-p}$$

$$= \frac{1-\frac{1}{p^s}}{1-\frac{1}{p^{s-1}}}$$

So $\displaystyle\sum_{n=1}^{\infty} \frac{\varphi(n)}{n^s} = \frac{\zeta(s-1)}{\zeta(s)}$. Actually a simpler proof could be based on $n = \sum'_{d|n} \varphi(d)$

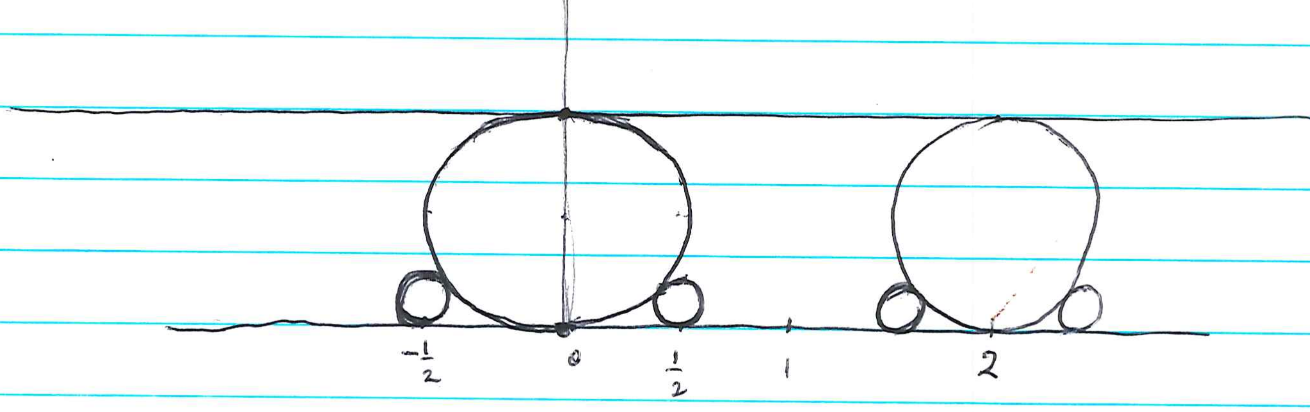
This arises when one computes Guillemin's formula for the scattering matrix

$$\frac{\hat{\zeta}(2-2s)}{\hat{\zeta}(s)} = \frac{\hat{\zeta}(2s-1)}{\hat{\zeta}(2s)} = \sum_{\substack{-\frac{d}{c}\in Q \\ 0\le -\frac{d}{c}<1}} \frac{1}{c^{2s}} \cdot \int_{-\infty}^{\infty} \frac{dq}{(1+q^2)^s}$$

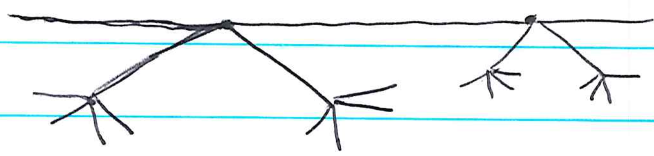
The first factor is just $\frac{\zeta(2s-1)}{\zeta(2s)}$ as I have seen; the second must come from the Γ-factors.

February 16, 1978

Here is another version of the Γ'-tree. The idea is to replace the cusp points by a Ford circle. Thus the union of the Ford circles associated to the even rational numbers is a tree:



which with straight edges looks like this:



so it has four edges per vertex

Question: What is $\int \theta(\tau)\,(-i\tau)^s \frac{d\tau}{\tau}$ over the Ford circle belonging to a rational number? The circle can be shrunk, so this depends only on the behavior near the rational no.

Suppose the rational number $\frac{a}{c}$ is even, with $c>0$, whence

$$\theta\left(\frac{a\tau+b}{c\tau+d}\right) = \varepsilon(a,c)\,\big(-i(c\tau+d)\big)^{1/2}\,\theta(\tau)$$

The Ford circle is the locus of $\frac{a\tau+b}{c\tau+d}$ where $\tau = x+iy$, $y=1$ and x runs over $\mathbb{R}$. We want to let $y \to +\infty$ to shrink the circle. Then $\theta(\tau) \to 1$ very fast so

$$\theta\left(\frac{a\tau+b}{c\tau+d}\right) \sim \varepsilon(a,c)\,(cy)^{1/2}$$

The diameter of the circle is $\frac{1}{c^2y}$ hence its circumference is $\frac{\pi}{c^2y}$ and so

$$\left|\int \Theta(\tau)(-i\tau)^{s/2}\frac{d\tau}{\tau}\right| = O\left[(cy)^{1/2}\cdot\frac{\pi}{c^2y}\right] \longrightarrow 0$$

If the rational number $\frac{a}{c}$ is odd, then I have seen that its size in a circular neighborhood (4) can be estimated by $e^{-\pi d}$ hence there is zero integral for such a circle.

February 12, 1978

Return to the $PSL_2(\mathbb{Z})$ tree. According to Borel the cohomology with compact support embeds in the ℓ^2-cohomology. Recall that

$$H^1_c(X) = \varinjlim_K H^1(X, X-K)$$

where K ranges over all finite subtrees. Also

$$\underset{\mathbb{Z}}{\overset{\|}{H^0(X)}} \longrightarrow H^0(X-K) \longrightarrow H^1(X, X-K) \longrightarrow \underset{0}{\overset{\|}{H^1(X)}}$$

hence $H^1(X, X-K)$ can be indentified with locally constant functions on $X-K$. Now $\varprojlim \pi_0(X-K)$ is a Cantor set called the set of ends of the graph, so we see that $H^1_c(X)$ is the set of locally constant function on the ends modulo constants. Put ∂X for the space of ends, so

$$H^1_c(X, A) = \{\text{loc. const. maps } \partial X \longrightarrow A\}/A$$

What is ℓ^2-cohomology. You put the obvious inner product on the space of cochains with compact support and look at the square integrable ones. One has

$$C^0_{\ell^2}(X) \underset{\delta^*}{\overset{\delta}{\rightleftarrows}} C^1_{\ell^2}(X)$$

These are bounded operators. For example

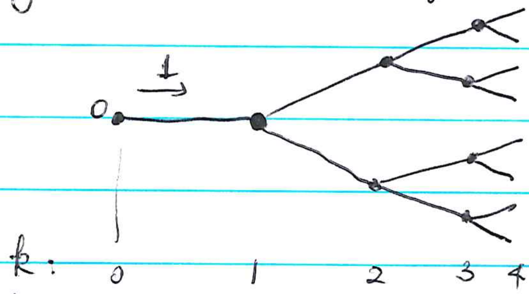
$$\|\delta f\|^2 = \sum_{\sigma \text{ 1-simplex}} |\delta f(\sigma)|^2 \le 2 \sum_{\substack{\sigma \text{ 1-simplex} \\ \sigma = \{x,y\}}} \{|f(x)|^2 + |f(y)|^2\}$$

$$\le 2 \Big\{ \sum_{x \text{ 0-simplex}} |f(x)|^2 \Big(\text{number of } \sigma \text{ containing } x \Big) \Big\} \le 6 \|f\|^2$$

The good situation occurs when the image of δ is [redacted] closed. In fact the good situation is when [redacted] $\|\delta f\|^2 \geq \text{const} \|f\|^2$ for then $\Delta u = f$ is uniquely solvable. We know that $\Delta = \delta^* \delta \geq 0$ and that it is bounded. What is the spectrum of Δ?

I think Borel can show that Δ is bounded away from zero. Then $\operatorname{Im} \delta$ is closed and so $\operatorname{Coker} \delta = \operatorname{Ker}(\delta^*)$ is a Hilbert space defined to be $H^1_{\ell^2}(X)$.

Why the resistance of $\frac{1}{2}$ the tree is 2.



Suppose 1 amp flows in at the left and that V is a voltage distribution on the network which is square integrable. Let k denote the distance from zero. I want to compare the end of maximum current flow with the end of minimum current flow. [redacted] To be specific for the maximum current flow you start at zero and reach the vertex at $k=1$. Here the current $I_0^{max} = 1$ divides and you take a [redacted] branch of maximum current flow, call it I_1^{max}. Then

$$I_1^{max} \geq \frac{1}{2} I_0^{max} = \frac{1}{2}$$

So repeat and you get $I_k^{max} \geq \frac{1}{2} I_{k-1}^{max}$. Similarly there is the minimum current path. Note that

$$I_1^{max} \geq \frac{1}{2} I_0^{max} = \frac{1}{2} I_0^{min} \geq I_1^{min}$$

and hence by induction

$$I_k^{max} \geq \frac{1}{2} I_{k-1}^{max} \geq \frac{1}{2} I_{k-1}^{min} \geq I_k^{min}$$

So [redacted] then we have

$$V_0 = \sum_{k=0}^{n-1} I_k^{max} + [redacted]\ V_n^{max}$$

where V_n^{max} is the voltage ~~at~~ at $k=n$ on the maximum path.

[redacted] So

$$V_0 \geqslant \sum_{k=0}^{n-1} \frac{1}{2^k} + V_n^{max} = \left(2 - \frac{1}{2^{n-1}}\right) + V_n^{max}$$

and similarly

$$V_0 \leqslant [redacted] \left(2 - \frac{1}{2^{n-1}}\right) + V_n^{min}$$

By assumption $V \to 0$ as $k \to \infty$ so that all the inequalities have to be equalities.

Better proof: Take the [redacted] voltage distribution and remove from it the good one which divides the current evenly. Then you apply the maximum principle to conclude the voltage must be constant, hence zero.

~~Same argument shows that the only harmonic functions~~
~~which are bounded are the constants.~~

February 13, 1978:

Δ is the Laplacian on the $PSL_2(\mathbb{Z})$-tree. The goal will be to determine its spectrum.

If f is a function on the vertices, then δf is the alternating function on oriented edges given by

$$(\delta f)(x,y) = f(y) - f(x)$$

if (x,y) is a 1-simplex. Think of δf as being like the gradient of f. Hence if f is a voltage, then δf is the negative of the current flow. Compute δ^*:

$$(f, \delta^* g) = (\delta f, g) = \tfrac{1}{2}\sum_{(x,y)}' (f(y) - f(x))\, g(x,y)$$

sum over pairs of vertices joinable by an edge

$$= \tfrac{1}{2}\sum_x' f(x) \sum_y' -g(x,y) + \tfrac{1}{2}\sum_y' f(y) \sum_x' g(x,y)$$

$$= \sum_x' f(x) \sum_y' g(y,x)$$

(Here suppose g vanishes when x, y are not connected by an edge). Thus

$$(\delta^* g)(x) = \sum_y' g(y,x)$$

(think of δ^* as $-$ div)

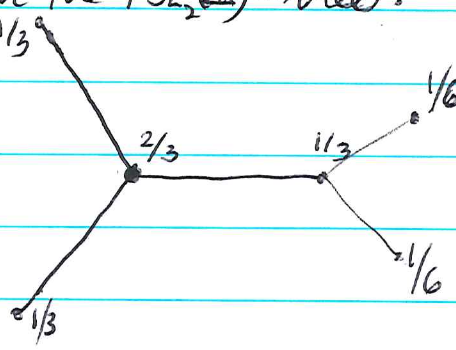
is the sum of the flows coming into the vertex x. So

$$-(\delta^*\delta f)(x) = -\sum_{y \ni (y,x) \text{ is an edge}}' \{f(x) - f(y)\} = \sum_{\substack{y \ni \\ (y,x) \text{ is an edge}}}' (f(y) - f(x)).$$

Thus $-\delta^*\delta$ is what we should call the Laplacian Δ. Notice that the equation giving the voltage distribution for 1 amp pumped into the tree at y is

$$\Delta g = -\delta_y$$

for example on the $PSL_2(\mathbb{Z})$-tree:

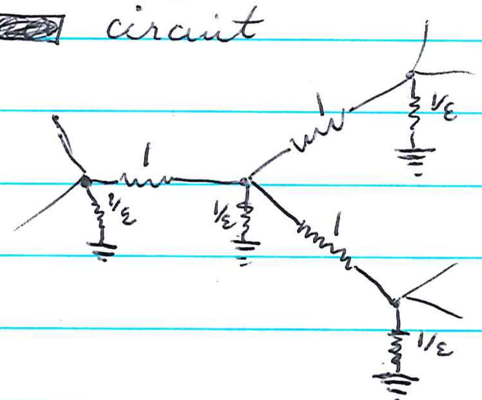


$$\frac{1}{3}+\frac{1}{3}+\frac{1}{3}-3\cdot\frac{2}{3}=-1$$

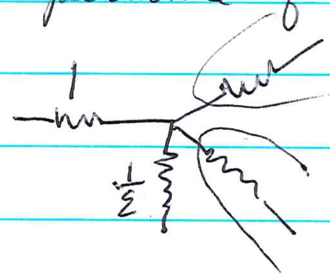
Consider now the equation

$$(\Delta-\varepsilon)g = -\delta_y .$$

Think of g as the voltage, then at a vertex $x \neq y$ we want a current loss of $\varepsilon g(x)$, i.e. we want to connect x to ground via a resistance $\frac{1}{\varepsilon}$. Hence we have the circuit



Calculate the impedance of one of the branches:



$$Z = 1 + \frac{1}{\varepsilon + \frac{1}{2} + \frac{1}{Z}} = 1 + \frac{Z}{\varepsilon Z + 2}$$

$$\varepsilon Z^2 + 2Z = \varepsilon Z + 2 + Z$$

$$\varepsilon Z^2 + (1-\varepsilon)Z - 2 = 0$$

$$Z = \frac{-(1-\varepsilon) \pm \sqrt{(1-\varepsilon)^2 + 8\varepsilon}}{2\varepsilon}$$

Observe that what's inside the radical

$$(1-\varepsilon)^2 + 8\varepsilon = \varepsilon^2 + 6\varepsilon + 1$$

is negative in the interval between the roots

$$\varepsilon = -3 \pm \sqrt{9-1} = -3 \pm \sqrt{8}$$

My guess is that this gives the spectrum of the operator Δ, namely the interval $[-3-\sqrt{8}, -3+\sqrt{8}]$. At least it agrees with the earlier bound

$$-(\Delta f, f) = \|\delta f\|^2 \leq 6\|f\|^2$$

hence $\Delta \geq -6$.