June 6, 1978

Goal: to understand Deift-Trubowitz trace formula.

Let's begin with the Feynman-Kac formula in the discrete case. Following Kac-Case we take the discrete Schroedinger equation to be

$$1)\qquad \frac{1}{2}(\psi_{n+1} + \psi_{n-1}) = \lambda e^{+v(n)} \psi_n$$

should be $+v(n)$

where $v(n)$ is the "potential". To get the ordinary Schroedinger DE as a limiting case we put

$$\psi_n = \psi(nh), \qquad v(n) = +\frac{1}{2}h^2 q(nh), \qquad \lambda = e^{-\frac{1}{2}h^2 E}.$$

whence we have as $h \to 0$, $nh \to x$

$$\frac{1}{2}\frac{\psi((n+1)h) - 2\psi(nh) + \psi((n-1)h)}{h^2} = \frac{e^{\frac{1}{2}h^2(-E+q(nh))} - 1}{h^2}\psi(nh)$$

$$\downarrow \qquad\qquad\qquad \downarrow$$

$$\frac{1}{2}\frac{d^2\psi}{dx^2} = \frac{1}{2}(-E + q)\psi$$

In order to get 1) in J-matrix form, we put

$$\psi_n = e^{-\frac{1}{2}v(n)} y_n$$

whence 1) becomes

$$\frac{1}{2} e^{-\frac{1}{2}v(n+1)} y_{n+1} + \frac{1}{2} e^{-\frac{1}{2}v(n-1)} y_{n-1} = \lambda e^{+\frac{1}{2}v(n)} y_n$$

or

$$2)\qquad \underbrace{\frac{1}{2} e^{-\frac{1}{2}[v(n)+v(n+1)]}}_{a_n} y_{n+1} + \underbrace{\frac{1}{2} e^{-\frac{1}{2}[v(n-1)+v(n)]}}_{a_{n-1}} y_{n-1} = \lambda y_n$$

The heat equation belonging to $\mathbb{Z}$ is

3) $$u(t+1) = Lu(t) \qquad (Ly)_n = a_n y_{n+1} + a_{n-1} y_{n-1}$$

and its solution is

$$u(t) = L^t u(0).$$

An entry of L^t can be represented as a sum of terms, each associated to a walk of length t on $\mathbb{Z}$. Better $L = aT + T^{-1}a$ so

$$L^t = (aT + T^{-1}a)^t$$

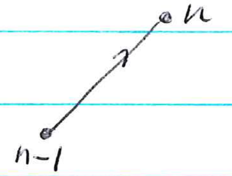
will be a sum of 2^t terms each labeled by a sequence of ± 1 steps. Thus

$$(L^t y)_n = \sum_{\substack{\text{paths} \\ \text{ending at} \\ n \text{ of length } t}} \underset{\text{on the path}}{\text{coeff depending}} \; y_{\substack{\text{beginning} \\ \text{of path}}}$$

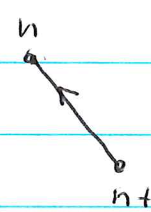
To determine the coefficient which is a t-fold product of certain of the a_i note that

$$(L^t y)_n = a_n (L^{t-1} y)_{n+1} + a_{n-1} (L^{t-1} y)_{n-1}$$

so over a bdy



factor $a_{n-1} = e^{-\frac{1}{2}[v(n-1) + v(n)]}$



factor $a_n = e^{-\frac{1}{2}[v(n) + v(n-1)]}$

So the factor belonging to a path $p_0, p_1, \dots, p_t$ is

$$\frac{1}{2^t}\, e^{-\left\{\frac{1}{2} v(p_0) + v(p_1) + \cdots + v(p_{t-1}) + \frac{1}{2} v(p_t)\right\}}$$

Let's denote the exponent $-\int_{p} v$

where $p = \{p_0, \dots, p_t\}$ is the path. Then we have

$$(L^t y)_n = \frac{1}{2^t} \sum_{\substack{p \text{ of length} \\ t \text{ ending} \\ \text{at } n}} e^{-\int_p v}\, y_{p(0)}$$

The above is the Feynman-Kac formula in this situation.

Observation: Recall that $\|L\| \le 1$ in the discrete case without bound states, hence $u(t) = L^t u(0)$ decays as $t \to \infty$.

I SHOULD REPLACE v by $-v$ in the above so as to get the right Schroedinger equation. Thus

$$a_n = \frac{1}{2} e^{-\frac{1}{2}[v(n) + v(n+1)]}$$

June 8, 1978

Remarks: 1) In passing from

$$\tfrac{1}{2}(\psi_{n+1}+\psi_{n-1}) = \lambda e^{v(n)}\psi_n \qquad \text{to Schroedinger}$$

one maybe should think of λ as $\cos(kh) = 1-\frac{1}{2}h^2k^2+\cdots$

2) Symmetry $\lambda \longleftrightarrow -\lambda$ is reflected in the extra symmetry of the measure $d\nu$ on S^1 under $z \mapsto -z$. Note this symmetry implies

$$p_n(-z) = (-1)^n p_n(z)$$

and hence $h_n = (p_n, z^n p_n^*) = \int z^{-n} p_n^2\, d\nu = (-1)^n \int z^{-n} p_n^2\, d\nu$

$= (-1)^n h_n$ Also $c_n = \int z^n d\nu = (-1)^n c_n$ so $c_n = 0$ for n odd.

Thus $h_n = 0$ if n is odd, and hence from

$$\begin{pmatrix} p_n \\ z^{+n}p_n^* \end{pmatrix} = R(h_n)\begin{pmatrix} z & 0 \\ 0 & 1\end{pmatrix}\begin{pmatrix} p_{n-1} \\ z^{n-1}p_{n-1}^*\end{pmatrix}$$

we see that $\boxed{p_{2n+1} = z\, p_{2n}}$

With only the symmetry $z \mapsto z^{-1}$ for $d\nu$ we get that $c_n = \int z^n d\nu = \int z^{-n} d\nu = c_{-n} = \bar{c}_n$ so the c_n are real, so the $\tilde{p}_n = z^n + \sum_{i=0}^{n-1} k_{ni} z^i$ given by

$$c_{n-m} + \sum_{i=0}^{n-1} k_{ni} c_{i-m} = 0 \qquad 0 \le m < n$$

have to be real, and so $h_n = k_{n0}$ is real.

We also know that in the space $F_{-n,n}$ of Laurent polys $\sum_{i=-n}^{n} a_i z^i$ there is a unique line consisting of odd polys (under $z \mapsto z^{-1}$) $\perp$ to $F_{-n+1,n-1}$. Hence we know that as $z^{-n}p_{2n}$ is perpendicular to $z^{-n}, \dots, z^{n-1}$ that

$$z^{-n}p_{2n} - z^{n}p_{2n}^{*} \perp \{z^{-n+1}, \dots, z^{n-1}\}.$$

Also $z^{-n+1}p_{2n-1} \perp \{z^{-n+1}, \dots, z^{n-1}\}$ so also

$$z^{-n+1}p_{2n-1} - z^{n-1}p_{2n-1}^{*} \perp \{ \quad " \quad \}.$$

Thus

$$z^{-n}p_{2n} - z^{n}p_{2n}^{*}, \quad z^{-n+1}p_{2n-1} - z^{n-1}p_{2n-1}^{*}$$

are proportional. Observe that coeff. of z^n in

$$z^{-n}\tilde{p}_{2n} - z^{n}\tilde{p}_{2n}^{*} \quad \text{is} \quad 1 - p_{2n}(0)^{*} = 1 - h_{2n}$$

$$z^{-n+1}\tilde{p}_{2n-1} - z^{n-1}\tilde{p}_{2n-1}^{*} \quad \text{is} \quad 1$$

hence

$$\boxed{z^{-n}\tilde{p}_{2n} - z^{n}\tilde{p}_{2n}^{*} = (1-h_{2n})\left(z^{-n+1}\tilde{p}_{2n-1} - z^{n-1}\tilde{p}_{2n-1}^{*}\right)}$$

To remove $\sim$ recall $\quad p_n = \dfrac{\tilde{p}_n}{\|\tilde{p}_n\|} = \dfrac{\tilde{p}_n}{(1-|h_n|^2)^{1/2} \cdots (1-|h_0|^2)^{1/2}}$

Let us now define

$$\tilde{\phi}_\lambda(n) = \frac{z^{-n+1}\tilde{p}_{2n-1} - z^{n-1}\tilde{p}_{2n-1}^{*}}{z - z^{-1}} = \frac{1}{1-h_{2n}} \, \frac{z^{-n}\tilde{p}_{2n} - z^{n}\tilde{p}_{2n}^{*}}{z - z^{-1}}$$

which has highest degree terms $\equiv \dfrac{z^n - z^{-n}}{z - z^{-1}} \equiv (z+z^{-1})^{n-1} = (2\lambda)^{n-1}$

Thus $\quad \tilde{\phi}_\lambda(n) = (2\lambda)^{n-1} + \text{lower terms}$

so that it satisfies a recursion relation

$$\frac{1}{2}\tilde{\phi}_\lambda(n+1) + (\text{coeff})\,\tilde{\phi}_\lambda(n-1) = \lambda\,\tilde{\phi}_\lambda(n)$$

assuming that $d\nu$ has $z \mapsto z^{-1}$, $z \mapsto -z$ symmetries.

$$\tilde{\phi}_\lambda(n+1) = \frac{z^{-n}\tilde{p}_{2n+1} - z^{n}\tilde{p}^*_{2n+1}}{z-z^{-1}} = \frac{z^{-n+1}\tilde{p}_{2n} - z^{n-1}\tilde{p}^*_{2n}}{z-z^{-1}}$$

I have used $z\tilde{p}_{2n} = \tilde{p}_{2n+1}$ which is a consequence of $h_{2n+1}=0$.

$$(1-h_{2n})\tilde{\phi}_\lambda(n) = \frac{z^{-n}\tilde{p}_{2n} - z^{n}\tilde{p}^*_{2n}}{z-z^{-1}}$$

Hence we have

$$\begin{pmatrix}\tilde{\phi}_\lambda(n+1)\\ (1-h_{2n})\tilde{\phi}_\lambda(n)\end{pmatrix} = \frac{1}{z-z^{-1}}\begin{pmatrix} z & -z^{-1}\\ 1 & -1\end{pmatrix}\begin{pmatrix} z^{-n}\tilde{p}_{2n}\\ z^{n}\tilde{p}^*_{2n}\end{pmatrix}$$

or

$$\begin{pmatrix} z^{-n}\tilde{p}_{2n}\\ z^{n}\tilde{p}^*_{2n}\end{pmatrix} = \begin{pmatrix} 1 & -z^{-1}\\ 1 & -z\end{pmatrix}\begin{pmatrix}\tilde{\phi}_\lambda(n+1)\\ (1-h_{2n})\tilde{\phi}_\lambda(n)\end{pmatrix}$$

So

$$\begin{pmatrix}\tilde{\phi}_\lambda(n+1)\\ (1-h_{2n})\tilde{\phi}_\lambda(n)\end{pmatrix} = \frac{1}{z-z^{-1}}\begin{pmatrix} z & -z^{-1}\\ 1 & -1\end{pmatrix}\begin{pmatrix} 1 & h_{2n}\\ h_{2n} & 1\end{pmatrix}\begin{pmatrix} z & 0\\ 0 & z^{-1}\end{pmatrix}\begin{pmatrix} z^{-n+1}\tilde{p}_{2n-2}\\ z^{n-1}\tilde{p}^*_{2n-2}\end{pmatrix}$$

$h=h_{2n}$

$$= \frac{1}{z-z^{-1}}\begin{pmatrix} z-z^{-1}h & zh-z^{-1}\\ 1-h & h-1\end{pmatrix}\begin{pmatrix} z & -1\\ z^{-1} & -1\end{pmatrix}\begin{pmatrix}\tilde{\phi}_\lambda(n)\\ (1-h_{2n-2})\phi_\lambda(n-1)\end{pmatrix}$$

$$= \frac{1}{z-z^{-1}}\begin{pmatrix} z^2-z^{-2} & -z+z^{-1}h-zh+z^{-1}\\ z-hz+z^{-1}h-z^{-1} & 0\end{pmatrix}$$

$$= \begin{pmatrix} z+z^{-1} & -1-h_{2n}\\ 1-h_{2n} & 0\end{pmatrix}\begin{pmatrix}\tilde{\phi}_\lambda(n)\\ (1-h_{2n-2})\tilde{\phi}_\lambda(n-1)\end{pmatrix}$$

So we have

$$\tilde{\phi}_\lambda(n+1) = 2\lambda\tilde{\phi}_\lambda(n) - (1+h_{2n})(1-h_{2n-2})\tilde{\phi}_\lambda(n-1)$$

or

$$\boxed{\tilde{\phi}_\lambda(n+1) + (1+h_{2n})(1-h_{2n-2})\tilde{\phi}_\lambda(n-1) = 2\lambda\tilde{\phi}_\lambda(n)}$$

Now I want to get this in a form where I can see the "potential." Put $\alpha_{n-1} = (1+h_{2n})(1-h_{2n-2})$ and put this in Jacobi form by $\phi(n) = \rho_n y_n$:

$$\underbrace{\frac{\rho_{n+1}}{\rho_n}}_{2a_n} y_{n+1} + \underbrace{\frac{\alpha_{n-1}\rho_{n-1}}{\rho_n}}_{a_{n-1}} y_{n-1} = 2\lambda y_n$$

$$\frac{\rho_{n+1}}{\rho_n} = 2a_n = \alpha_n \frac{\rho_n}{\rho_{n+1}} \quad \text{or} \quad \left(\frac{\rho_{n+1}}{\rho_n}\right)^2 = \alpha_n$$

or

$$\rho_{n+1}^2 = \alpha_n\alpha_{n-1}\cdots\alpha_1\rho_1^2 = (1+h_{2n+2})(1-h_{2n}^2)(1-h_{2n-2}^2)\cdots(1-h_2^2)\rho_1^2$$

I can check this by noting that ρ_n is essentially the norm of $\tilde{\phi}(n)$ computed for the measure $|z-z^{-1}|^2 d\nu$, so

$$\rho_n^2 = \int |\tilde{\phi}_\lambda(n)|^2 |z-z^{-1}|^2 d\nu = \int \left| \frac{1}{1-h_{2n}} \left(z^{-n}\tilde{p}_{2n} - z^n \tilde{p}_{2n}^*\right)\right|^2 d\nu$$

$$= \frac{1}{(1-h_{2n})^2}(2-2h_{2n})\|\tilde{p}_{2n}\|^2$$

$$= \frac{2}{(1-h_{2n})^2}(1-h_{2n})(1-h_{2n}^2)\cdots(1-h_2^2)$$

$$= 2(1+h_{2n})(1-h_{2n-2}^2)\cdots\cdots(1-h_2^2)$$

So now recall that if we are given the discrete Schroedinger á la Kac-Case

$$\frac{1}{2}(\psi_{n+1} + \psi_{n-1}) = \lambda e^{v_n} \psi_n$$

then $y_n = e^{+\frac{1}{2}v_n}\psi_n$, $\psi_n = e^{-\frac{1}{2}v_n} y_n$ converts it to Jacobi form:

$$\underbrace{\frac{1}{2}e^{\frac{1}{2}v_{n+1}+\frac{1}{2}v_n}}_{a_n} y_{n+1} + \underbrace{\frac{1}{2}e^{\frac{1}{2}v_n+\frac{1}{2}v_{n-1}}}_{a_{n-1}} y_{n-1} = \lambda y_n$$

So

$$e^{\frac{1}{2}(v_{n+1}+v_n)} = 2a_n = \frac{\rho_{n+1}}{\rho_n} = \left(\frac{2(1+h_{2n+2})(1-h_{2n}^2)(1-h_{2n-2}^2)\cdots}{2(1+h_{2n})(1-h_{2n-2})^2\cdots}\right)^{1/2}$$

$$= \left[(1+h_{2n+2})(1-h_{2n})\right]^{1/2}$$

Thus we get the formula

$$\boxed{e^{v_n+v_{n+1}} = (1-h_{2n})(1+h_{2n+2})}$$

relating the potential and the Schur parameters. Assuming $v_n = 0$ and $h_n = 0$ for n large we get by iterating

$$\boxed{e^{v_n+2v_{n+1}+2v_{n+2}+\cdots} = (1-h_{2n})(1-h_{2n+2}^2)(1-h_{2n+4}^2)\cdots}$$

$$\boxed{e^{v_n+2v_{n-1}+2v_{n-2}+\cdots} = (1+h_{2n})(1-h_{2n-2}^2)(1-h_{2n-4}^2)\cdots}$$

June 9, 1978

Suppose we consider a 1-port $(\mathcal{H}, V)$ with scattering [redacted] fn. $S(z) = ((1-zT^*)^{-1}u_i, u_{-i})$. I've seen that if $h_0 = (u_i, u_{-i})$ then the 1-port given by $\mathcal{H}_{-1} = \mathcal{D}_V$, $\mathcal{D}_{V_1} = \mathcal{D}_{V^2}$ has scattering function

$$S_1(z) = \frac{(S(z)-h_0)/z}{1-\bar{h}_0 S(z)}$$

or

$$S(z) = \begin{pmatrix} 1 & h_0 \\ \bar{h}_0 & 1 \end{pmatrix} (z\, S_{-1}(z))$$

So we get a chain of 1-ports $\mathcal{H} = \mathcal{H}_0 \supset \mathcal{H}_{-1} \supset \mathcal{H}_{-2} \supset \cdots\cdots$ associated to the Schur development

$$S_0 = \begin{pmatrix} 1 & h_0 \\ \bar{h}_0 & 1 \end{pmatrix}\begin{pmatrix} z & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & h_{-1} \\ \bar{h}_{-1} & 1 \end{pmatrix}\begin{pmatrix} z & 0 \\ 0 & 1 \end{pmatrix}\cdots\cdots\begin{pmatrix} 1 & h_{-n+1} \\ \bar{h}_{-n+1} & 1 \end{pmatrix}\begin{pmatrix} z & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} S_{-n} \end{pmatrix}$$

If I assume $S(z)$ is rational and inner, then this process stops with S_{-n} a constant of modulus 1, in which case $\mathcal{H}_{-n}$ is 1-dimensional and S_{-n} gives the ratio between the choices for u_i^{-n}, u_{-i}^{-n} in $\mathcal{H}_{-n}$.

If $S(z)$ is rational, then does the process [redacted] stop?

Example: Suppose S is a constant h_0 of modulus <1. Then

$$S = \begin{pmatrix} 1 & h_0 \\ \bar{h}_0 & 1 \end{pmatrix}(zS_{-1})$$

with $S_{-1} = 0$ and the Schur process continues with $h_{-1} = h_{-2} = \cdots\cdots = 0$. In fact, [redacted] the case $S=0$ corresponds to the [redacted] case when $V^{-n}u_i$ is defined for all n, so that we have the following picture of $\tilde{\mathcal{H}}$

$$\langle \cdots, z^{-2}u_{-i}, z^{-1}u_{-i}\rangle \oplus \overbrace{\langle u_{-i}, zu_{-i}, \cdots\rangle \oplus \langle \cdots\cdots, z^{-1}u_i, u_i\rangle}^{\mathcal{H}} \oplus \langle zu_i, z^2u_i, \cdots\cdots\rangle$$

I want to synthesize the following 2 examples:

1) S analytic on $|z|=1$ and of modulus 1 there. Suppose to simplify that S has degree zero; [redacted] then we have

$$S = \frac{g}{g^*}$$

with g analytic non-vanishing on $|z| \leq 1$, normalized so that $|g|^2 \frac{d\theta}{2\pi}$ is a prob. measure. Then we know how to a sequence of Schur parameters $1 = h_0, h_1, h_2, \dots$ from this measure. We have $\frac{1}{g} = \lim_{n\to\infty} z^n p_n^*$, so

$$S = \lim_{n\to\infty} \frac{z^{-n} p_n}{z^n p_n^*} = \lim_{n\to\infty} z^{-n} \begin{pmatrix} 1 & h_n \\ \bar{h}_n & 1 \end{pmatrix} \begin{pmatrix} z & 0 \\ 0 & 1 \end{pmatrix} \cdots \begin{pmatrix} 1 & h_1 \\ \bar{h}_1 & 1 \end{pmatrix} \begin{pmatrix} z & 0 \\ 0 & 1 \end{pmatrix} (1)$$

2) S analytic on $|z| \leq 1$ and of modulus ≤ 1, and to simplify assume $|S| < 1$ at some point of S^1. Then we get a Schur development

$$S = \begin{pmatrix} 1 & h_0 \\ \bar{h}_0 & 1 \end{pmatrix} \begin{pmatrix} z & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & h_{-1} \\ \bar{h}_{-1} & 1 \end{pmatrix} \cdots\cdots\cdots$$

The goal will be to make a doubly-infinite Schur system for an arbitrary S analytic on $|z| \leq 1$ and of modulus ≤ 1 there.

Suppose S arises from a $(\mathcal{H}, V)$ and let $(\tilde{\mathcal{H}}, z)$ be the unitary dilation. We have incoming + outgoing pictures of $\tilde{\mathcal{H}}$:

$$L^2(S^1) \xleftarrow{\ \text{out}\ } \tilde{\mathcal{H}} \xrightarrow{\ \text{in}\ } L^2(S^1)$$

$$\sum_n (f, z^n u_{-i}) \bar{z}^n u_{-i} \longleftarrow\!\shortmid\ f \ \longmapsto \sum_n (f, z^n u_i) z^n u_i$$

$$\text{in}(\text{out}(1)) = \text{in}(u_{-i}) = \sum_{n \geq 0} (u_{-i}, \bar{z}^n u_i) \bar{z}^n u_i$$

$$\| $$

$$\sum_{n \geq 0} (u_{-i}, T^{*n} u_i) z^{-n} u_i = \overline{S(z)}\, u_i$$

Also

$$\text{out}(\text{in}(1)) = \text{out}(u_i) = \sum_{n \geq 0} (u_i, \bar{z}^n u_{-i}) \bar{z}^n u_{-i} = S(z) u_i$$

$$\| $$

$$(T^{*n} u_i, u_{-i})$$

Claim $\tilde{\mathcal{H}} \hookrightarrow L^2(S^1) \times L^2(S^1)$. In effect, we know that $\{z^n u_i, z^n u_{-i}, n \in \mathbb{Z}\}$ span $\tilde{\mathcal{H}}$ for otherwise, the orthogonal complement would be a nonzero unitary component of V, so we can identify $\tilde{\mathcal{H}}$ with the subspace of $L^2(S^1) \times L^2(S^1)$ stable under z, z^{-1} containing the elements

$$\text{Im}(u_i) = (1, S(z))$$

$$\text{Im}(u_{-i}) = (\underset{\substack{\uparrow \\ \text{in}}}{\overline{S(z)}}, \underset{\substack{\uparrow \\ \text{out}}}{1})$$

If $S\bar{S} = 1$, then $\tilde{\mathcal{H}}$ becomes isomorphic to $L^2(S^1)$ via either representation.

June 10, 1978

$\tilde{\mathcal{H}}$ is obtained by taking two copies of $L^2(S^1)$ and glueing them together using the scattering function S. More precisely elements of $\tilde{\mathcal{H}}$ are of the form $f(z)u_i + g(z)u_{-i}$ with norm determined by

$$\left(z^k u_i, z^l u_{-i}\right) = \left(T^{*(l-k)}u_i, u_{-i}\right)$$
$$= \int S(z) z^{k-l} \frac{d\theta}{2\pi} = (z^k S, z^l)$$

Thus

$$(*) \qquad \|fu_i + gu_{-i}\|^2 = \|f\|^2 + 2\,\mathrm{Re}\,(fS, g) + \|g\|^2$$

Now we want to use this formula to define $\tilde{\mathcal{H}}$ starting from an S defined on $|z| = 1$, and say analytic there. Then it is necessary to know the above is ≥ 0 for all f, g. This by replacing g by cg is seen to be equivalent to

$$|(fS, g)| \leq \|f\| \cdot \|g\|$$

for all f, g. Thus $(*) \geq 0 \iff \|S\| \leq 1$. Moreover we have

$$\|fu_i + gu_{-i}\|^2 = \|f\|^2 + 2\,\mathrm{Re}\,(fS, g) + \|g\|^2$$
$$= \|f\|^2 - \|Sf\|^2 + \|fS + g\|^2$$
$$= \|f + \bar{S}g\|^2 + \|g\|^2 - \|\bar{S}g\|^2$$

showing that $\|fu_i + gu_{-i}\| = 0 \iff \|Sf\| = \|f\|$ and $g = -Sf$

So we can think of $\tilde{\mathcal{H}}$ as consisting of $fu_i + gu_{-i}$

with [redacted] the above norm. We have

$$f+\bar{S}g \xleftarrow{\text{in}} fu_i + gu_{-i} \xrightarrow{\text{out}} Sf+g.$$

In the case where $S\bar{S}=1$ we obtained a natural filtration of $\tilde{\mathcal{H}}$ by analyticity requirements. One puts

$$F_0\tilde{\mathcal{H}} = \left\{ \text{those } fu_i+gu_{-i} \in \tilde{\mathcal{H}} \;\middle|\; \begin{array}{lll} Sf+g & \text{anal: } & |z|\le 1 \\ f+\bar{S}g & \text{"} & |z|\ge 1 \end{array} \right\}$$

Since

$$S(f+\bar{S}g) = Sf+g$$

we have

$$F_0\tilde{\mathcal{H}} \cong H_+ \cap SH_- .$$

As a check note that if $Sf+g \in H^+$ then

$$fu_i+gu_{-i} = (Sf+g)u_{-i} = \sum a_n z^n u_{-i}$$

doesn't appear in the $\langle \dots, z^{-2}u_{-i}, z^{-1}u_{-i}\rangle$ wing of $\tilde{\mathcal{H}}$.

In general we put

$$F_n\tilde{\mathcal{H}} = \left\{ fu_i+gu_{-i} \in \tilde{\mathcal{H}} \;\middle|\; \begin{array}{l} Sf+g \in H_+ \\ f+\bar{S}g \in z^{n}H_- \end{array} \right\}$$

Then $F_n\tilde{\mathcal{H}},\ zF_n\tilde{\mathcal{H}} \subset F_{n+1}\tilde{\mathcal{H}}$. Question is $F_{n-1}\tilde{\mathcal{H}}$ of codim 1 in $\tilde{\mathcal{H}}$.

Suppose we consider the 'generic' case when $\|S\|<1$. In this case we have

$$\|fu_i+gu_{-i}\|^2 \geq \|f\|^2-\|Sf\|^2,\ \|g\|^2-\|\bar{S}g\|^2$$

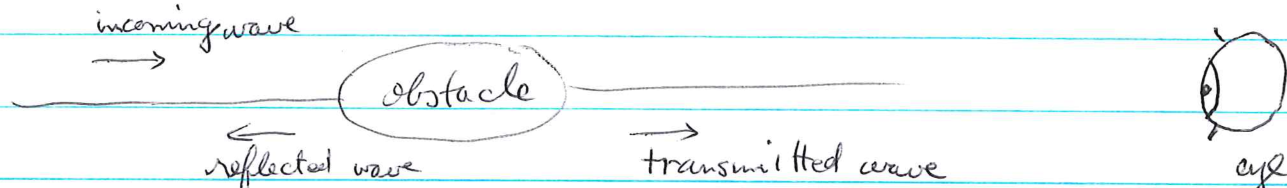
and so this norm is equivalent to $\|f\|^2+\|g\|^2$. Hence $\tilde{\mathcal{H}} = L^2(S^1)\oplus L^2(S^1)$ with the isomorphism given by the in and out representations. It's clear in this case that

$$F_n \tilde{\mathcal{H}} \xrightarrow{\sim} H_+ \times z^n H_-$$

and hence that $F_{n-1}\tilde{\mathcal{H}}$, $zF_{n-1}\tilde{\mathcal{H}}$ are both of codim 1 in $F_n\tilde{\mathcal{H}}$.

Consequently for each n we get a 1-port on $F_n\tilde{\mathcal{H}}$ with $V=$ mult. by z from $F_{n-1}\tilde{\mathcal{H}}$ to $zF_{n-1}\tilde{\mathcal{H}}$. [redacted]

I want to get at the transmission coefficient [redacted]. To do this I want to think of a [redacted] wave



coming from the left as never giving rise to an incoming wave on the right of the obstacle

R [redacted] $e^{-ikx} + e^{ikx} \longleftrightarrow$ [redacted] Te^{ikx}

In the model I am dealing with the incoming and outgoing representations are associated to the waves on the right, hence a left incoming wave always give zero incoming on the right.

Thus it seems that I should define the left incoming representation using the orthogonal complement to the right incoming one, i.e. to the space $\langle z^n u_i \rangle$. [redacted] Call this orthogonal complement $\mathcal{H}$. It consists of $fu_i + gu_{-i}$ with $\mathrm{in}(fu_i + gu_{-i}) = f + \bar{S}g = 0$ or $f = -\bar{S}g$. The norm is

$$\|g(u_i - \bar{S}u_{+i})\| = \|g\|^2 - \|\bar{S}g\|^2 = \left((1-|S|^2)g, g\right)$$

Let $1-|S|^2 = |h|^2$ with h outer. Then we get an isomorphism

$$\begin{aligned} L^2(S^1) &\xrightarrow{\ \sim\ } \mathcal{K} \\ g\bar{h} &\longleftarrow\!\!\shortmid\ g(u_i - \bar{S}u_{+i}) \\ g_1 &\longmapsto g_1 \frac{u_i - \bar{S}u_{+i}}{\bar{h}} \end{aligned}$$

Moreover if one composes this with

$$\begin{aligned} \mathcal{K} \subset \mathcal{H} &\xrightarrow{\text{out}} L^2 \\ fu_i + gu_{-i} &\longmapsto Sf+g \end{aligned}$$

one gets $$1 \longmapsto \frac{u_i - \bar{S}u_i}{\bar{h}} = \frac{1-\bar{S}S}{\bar{h}} = h$$

It follows because h is outer that the subspace H_+ of $L^2(S^1) \xrightarrow{\sim} \mathcal{K}$ gets mapped isomorphically onto H_+ under the out representation.

Similarly we should consider the orth. complement $\mathcal{K}'$ of $\langle z^n u_{-i} \rangle$ which consists of $fu_i + gu_{-i}$ with

$$\text{out}(fu_i + gu_{-i}) = Sf + g = 0 \qquad \text{or} \qquad g = -Sf$$

with the norm $\|fu_i + gu_{-i}\|^2 = \left((1-|S|^2)f, f\right)$. We have an isom

$$\begin{aligned} L^2(S^1) &\longrightarrow \mathcal{K}' \\ 1 &\longmapsto \frac{u_i - Su_{-i}}{h} \\ fh &\longleftarrow\!\!\shortmid\ f(u_i - Su_{-i}) \end{aligned}$$

whose composition with in is $1 \longmapsto \dfrac{u_i - Su_{-i}}{h} \longmapsto \dfrac{1-S\bar{S}}{h} = \bar{h}$.
Because h is outer, H_- gets mapped isomorphically onto H_-

under the in representation.

Now consider the composition

$$L^2(S^1)\oplus L^2(S^1) \xrightarrow{\ \sim\ } \tilde{\mathcal{H}} \xleftarrow{\ \sim\ } L^2(S^1)\oplus L^2(S^1)$$

$$\begin{pmatrix} f_1 \\ f_2 \end{pmatrix} \longmapsto f_1 \frac{u_{-i} - \bar{S} u_i}{\bar{h}} + f_2 u_i$$

$$g_1 u_{-i} + g_2 \frac{u_i - S u_{-i}}{h} \longleftarrow\!\!\shortmid \begin{pmatrix} g_1 \\ g_2 \end{pmatrix}$$

$$g_1 - \frac{S}{h} g_2 = \frac{1}{\bar{h}} f_1$$

$$\cdot S\left(\frac{1}{h} g_2 = f_2 - \frac{\bar{S}}{\bar{h}} f_1 \right)$$

$$g_1 = S f_2 + \frac{1 - S\bar{S}}{\bar{h}} f_1 = S f_2 + h f_1$$

$$g_2 = h f_2 - \frac{\bar{S} h}{\bar{h}} f_1$$

Hence the matrix going from $\begin{pmatrix} f_2 \\ f_1 \end{pmatrix}$ to $\begin{pmatrix} g_1 \\ g_2 \end{pmatrix}$ is

$$\begin{pmatrix} g_1 \\ g_2 \end{pmatrix} = \begin{pmatrix} S & h \\ h & -\frac{\bar{S} h}{\bar{h}} \end{pmatrix} \begin{pmatrix} f_2 \\ f_1 \end{pmatrix}$$

This 2×2 matrix is unitary on S^1 and the "transmission" coefficient h is analytic and non-vanishing for $|z| \le 1$

June 11, 1978

Construction of Schur system. Lets suppose S analytic on $|z|=1$ and of modulus < 1 there so that $\tilde{\mathcal{H}}$ consists of $fu_i + gu_{-i}$ with $f, g \in L^2(S^1)$ with norm

$$\|fu_i + gu_{-i}\|^2 = \|f\|^2 + 2\,\mathrm{Re}\,(Sf, g) + \|g\|^2 .$$

Recall we put

$$F_n \tilde{\mathcal{H}} = \left\{ fu_i + gu_{-i} \;\middle|\; \begin{array}{l} \mathrm{out}\,(fu_i + gu_{-i}) = Sf + g \in H_+ \\ \mathrm{in}\;(fu_i + gu_{-i}) = f + \bar{S} g \in z^n H_- \end{array} \right\}$$

Then I get a partial isometry on $F_n \tilde{\mathcal{H}}$ given by multiplication by z with domain F_{n-1}.

Let's pick $p_n \in F_n \ominus F_{n-1}$, $q_n \in F_n \ominus zF_{n-1}$ to be of norm 1. From the fact that

$$F_n \mathcal{H} \simeq H_+ \times z^n H_-$$

under the vector space isom $\mathcal{H} \xrightarrow{(\mathrm{out},\, \mathrm{in})} L^2(S^1) \times L^2(S^1)$, we know that $F_{n-1} + zF_{n-1} = F_n$ and $F_{n-1} \cap zF_{n-1} = zF_{n-2}$. Thus zp_{n-1} spans F_n / F_{n-1} and so there is a non-zero k_n such that

$$zp_{n-1} - k_n p_n \in F_{n-1}$$

But this element is clearly $\perp$ to zF_{n-2}, hence for some h_n we have

$$zp_{n-1} = k_n p_n - h_n q_{n-1} .$$

Because $p_n \perp q_{n-1}$ we have $1 = |k_n|^2 + |h_n|^2$. Moreover

$$(zp_{n-1}, q_{n-1}) = -h_n .$$

On the other hand, $q_{n-1} \in F_{n-1} \ominus zF_{n-2} \subset F_n \ominus zF_{n-2}$ and to make it orthogonal to zF_{n-1} all we have to do is remove its projection on $zF_{n-1} \ominus zF_{n-2} = \langle zp_{n-1} \rangle$. Thus

$$q_{n-1} - (q_{n-1}, zp_{n-1}) zp_{n-1} = q_{n-1} + \bar{h}_n zp_{n-1} \in F_n \ominus zF_{n-1}$$

and so must be a multiple of q_n:

$$q_{n-1} = \bar{h}_n zp_{n-1} + k_n' q_n$$

and since this is an orthogonal sum $|h_n|^2 + |k_n'|^2 = 1$. So if we had arranged our choices of p_n, q_n so that $k_n, k_n' > 0$, then we get

$$\begin{pmatrix} p_n \\ q_n \end{pmatrix} = \frac{1}{k_n} \begin{pmatrix} 1 & h_n \\ \bar{h}_n & 1 \end{pmatrix} \begin{pmatrix} z & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} p_{n-1} \\ q_{n-1} \end{pmatrix}$$

$$k_n = \sqrt{1 - |h_n|^2}$$

Here's how to choose p_n, q_n. Observe that

$$(u_{-i}, fu_i + gu_{-i}) = (1, \text{out}(fu_i + gu_{-i})) = (1, Sf + g)$$

Hence if $fu_i + gu_{-i} \in F_{n-1}$ we have

$$(u_{-i}, z(fu_i + gu_{-i})) = (1, z(Sf + g)) \subset (1, zH_+) = 0$$

and yet since F_n maps onto H_+ we have

$$(u_{-i}, F_n) = (1, H_+) = \mathbb{C}.$$

Thus the projection of u_{-i} onto F_n can be normalized to be q_n. If we make this choice then because

$$\overline{UF_n} \simeq H_+ \times \overline{Uz^n H_-} = H_+ \times L^2(S^1)$$

contains $\quad u_{-i} \longmapsto (1, S)$

we must have $\quad \lim_{n\to\infty} q_n = u_{-i}$

similarly

$$(z^n u_i, F_m) = (z^n \text{[redacted]}, in(F_m)) = (z^n, z^m H_-)$$

$$= \begin{cases} 0 & m < n \\ \mathbb{C} & m = n \end{cases}$$

hence we can obtain p_n by projecting $z^n u_i$ onto F_n and [redacted] normalizing. Clearly because

$$\overline{Uz^{-n}F_n} \simeq \overline{Uz^{-n}H_+} \times H_- = L^2(S^1) \times H_-$$

contains $\quad u_i \longmapsto (\bar{S}, 1)$

we have

$$\lim_{n\to\infty} z^{-n} p_n = u_i$$

$$\begin{cases} k_n p_n = z p_{n-1} + h_n q_{n-1} \\ k_n q_n = \bar{h}_n z p_{n-1} + q_{n-1} \end{cases}$$

Take inner product with p_n.

$$k_n = (p_n, k_n p_n) = (p_n, z p_{n-1})$$

$$k_n (p_n, q_n) = h_n (p_n, z p_{n-1})$$ and since $k_n \neq 0$

we get

$$\boxed{(p_n, q_n) = h_n}$$

June 12, 1978:

Suppose $|S|<1$ and S is analytic on $|z|=1$. New notation $u_{in} = u_i$, $u_{out} = u_{-i}$. We define $p_n \in F_n \ominus F_{n-1}$ $q_n \in F_n \ominus zF_{n-1}$ by

$$z^{-n} p_n = pr_{z^{-n}F_n}(u_{in}) \Big/ \text{normal. const.}$$

$$q_n = pr_{F_n}(u_{out}) \Big/ \text{normal. const.}$$

so that we have from yesterday the relations

$$\lim_{n\to\infty} z^{-n} p_n = u_{in}$$

$$\lim_{n\to\infty} q_n = u_{out}$$

$$(1) \qquad \begin{pmatrix} p_n \\ q_n \end{pmatrix} = R(h_n) \begin{pmatrix} z & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} p_{n-1} \\ q_{n-1} \end{pmatrix}$$

where $h_n = (p_n, q_n)$.

What I am now going to be interested in are solutions $\binom{x_n(z)}{y_n(z)}$ of the recursion relations

$$(2) \qquad \begin{pmatrix} x_n \\ y_n \end{pmatrix} = R(h_n) \begin{pmatrix} z & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x_{n-1} \\ y_{n-1} \end{pmatrix}$$

(Note that z is a constant in (2) if one thinks of $\binom{x_n}{y_n}$ as in $\mathbb{C}^2$, whereas in (1) it denotes the basic unitary operator on $\tilde{\mathcal{H}}$. We can obtain solutions of (2) by applying any map $\tilde{\mathcal{H}} \longrightarrow L^2(S^1)$ commuting with z and then evaluating at the chosen point z of S^1. (Assumes nice behavior.)

Apply $in: \tilde{\mathcal{H}} \longrightarrow L^2(S^1)$ and we get a solution of 2) with

$$\begin{pmatrix} in(p_n) \\ in(q_n) \end{pmatrix} \sim \begin{pmatrix} z^n \\ \bar{S}(z) \end{pmatrix}$$

Similarly

$$\begin{pmatrix} out(p_n) \\ out(q_n) \end{pmatrix} \sim \begin{pmatrix} z^n S(z) \\ 1 \end{pmatrix}$$

Now we also have the left-incoming repn.

$$L^2(S^1) \xleftarrow{\ in^-\ } \tilde{\mathcal{H}}$$

$\bar{h} g$ [redacted] $\longleftarrow\!\shortmid\ f u_{in} + g u_{out}$

which is adjoint to the embedding

$$g_1 \longmapsto g_1 \text{ [redacted] } \left(\frac{-\bar{S} u_{in} + u_{out}}{h} \right)$$

~~In effect in^- is out followed by dividing by h.~~
~~So therefore~~

$$\cancel{\begin{pmatrix} in^-(p_n) \\ in^-(q_n) \end{pmatrix} \sim \begin{pmatrix} \\ \frac{1}{h} \end{pmatrix}}$$

In effect, one has

$$\left(z^n \frac{-\bar{S} u_{in} + u_{out}}{h}, f u_{in} + g u_{out} \right)$$

$$= \left(-z^n \frac{\bar{S}}{h}, f\right) + \left(-z^n \frac{\bar{S}}{h} S, g\right) + \left(z^n \frac{1}{h}, fS\right) + \left(z^n \frac{1}{h}, g\right)$$

$$= (z^n h, g) = (z^n, \text{[redacted]}\ \bar{h} g)$$

Thus

$$\begin{pmatrix} in^- p_n \\ in^- q_n \end{pmatrix} \sim \begin{pmatrix} z^n \; in^-(u_{in}) \\ in^-(u_{out}) \end{pmatrix} = \begin{pmatrix} 0 \\ \bar{h} \end{pmatrix} \qquad \text{as } n \to \infty$$

Similarly we have

$$\begin{array}{ccc} L^2(S^1) & \xleftarrow{\;out^-\;} & \tilde{\mathcal{H}} \\ hf & \longleftarrow\!\shortmid & fu_{in} + gu_{out} \end{array}$$

adjoint to $1 \longmapsto \dfrac{u_{in} - Su_{out}}{h}$ and so

$$\begin{pmatrix} out^-(p_n) \\ out^-(q_n) \end{pmatrix} \sim \begin{pmatrix} z^n \; out^-(u_{in}) \\ out^-(u_{out}) \end{pmatrix} = \begin{pmatrix} z^n h \\ 0 \end{pmatrix}$$

Let's work out the asymptotic behavior of p_n, q_n as $n \to -\infty$. We ought to be able to compute $\lim_{n\to-\infty} q_n$ by projecting u_{-i} onto $\cap F_n$ and normalizing. Since

$$\cap F_n \xrightarrow{\sim} \bigcap H_+ \times z^n H_- = H_+ \times 0$$

we can identify $\cap F_n$ with the subspace spanned by the elements

$$z^n \, \frac{-\bar{S}u_{in} + u_{out}}{\bar{h}} \qquad n \geq 0$$

(which goes to $(z^n h, 0) \in H_+ \times 0$). The projection of u_{-i} on $\cap F_n$ is therefore obtained by first projecting onto the complement of $L^2(S^1)u_i$ and then ~~onto~~ $H_+ \dfrac{-\bar{S}u_{in} + u_{out}}{\bar{h}}$, however one can do this by applying in^-. We get

$$pr_{\beta F_n}(u_{-i}) = pr_{H_+}\left(\frac{in^- u_{-i}}{\bar h}\right) \cdot \frac{-\bar S u_{in} + u_{out}}{\bar h}$$

$$= \bar h(0) \cdot \frac{-S u_{in} + u_{out}}{\bar h}$$

and since it's natural to put $h(0) > 0$ we obviously get

$$\boxed{\begin{aligned} \lim_{n\to-\infty} q_n &= \frac{-\bar S u_{in} + u_{out}}{\bar h} \\ \lim_{n\to-\infty} z^{-n} p_n &= \frac{u_{in} - S u_{out}}{h} \end{aligned}}$$

Similarly

So

$$in \begin{pmatrix} p_n \\ q_n \end{pmatrix} \sim \begin{pmatrix} z^n \bar h \\ 0 \end{pmatrix} \qquad in^- \begin{pmatrix} p_n \\ q_n \end{pmatrix} \sim \begin{pmatrix} z^n \\ -S\frac{\bar h}{h} \end{pmatrix}$$

$$out \begin{pmatrix} p_n \\ q_n \end{pmatrix} \sim \begin{pmatrix} 0 \\ h \end{pmatrix} \qquad out^- \begin{pmatrix} p_n \\ q_n \end{pmatrix} \sim \begin{pmatrix} z^n(-\bar S \frac{h}{\bar h}) \\ 1 \end{pmatrix}$$

as $n \to -\infty$.