May 24, 1978:

The goal: I have a nice Hilbert space picture which explains (or should explain) one-ports in terms of the scattering or reflection coefficient $S(k)$ which is an analytic function in the UHP with unitary values on the real line. The goal is to understand how to incorporate bound states into this theory.

Start with a Schroedinger equation on $0 \le x < \infty$ and boundary condition at $x=0$

$$\left(-\frac{d^2}{dx^2}+q\right)u = \lambda u$$

$$u'(0) = h\,u(0)$$

where q decays fast. Define $\phi(x,\lambda) = \phi_\lambda(x)$ to be the solution of the Sch. D.E. with $\phi_\lambda(0)=1$, $\phi_\lambda'(0)=h$; and define scattering data:

$$\phi(x,k^2) \cong A(k)e^{ikx} + B(k)e^{-ikx} \qquad B(k) = A(-k).$$

so that

$$S(k) = \frac{A(k)}{B(k)} .$$

$B(k)$ is analytic in the UHP and its zeroes there correspond to bound states. But notice that if we assume there are no bound states, then this doesn't imply in an obvious way that S is analytic in the UHP because $A(k)$ might have poles in the UHP. Such poles would be due to the behavior of q at ∞. Perhaps the Yukawa potential puts such singularities in.

May 25, 1978

It is possibly for $S(k)$ to have poles in the UHP even when there are no bound states.

Example: $\left(-\frac{d^2}{dx^2}+e^{2x}\right)u = k^2 u$ on $\mathbb{R}$ has the solution

$K_{ik}(e^x)$ decaying as $x \longrightarrow +\infty$. Also see p.644
one has

$$K_{ik}(e^x) \sim 2^{ik}\Gamma(ik)e^{-ikx} + 2^{-ik}\Gamma(-ik)e^{ikx}$$

as $x \longrightarrow -\infty$. Hence $A(k) = 2^{ik}\Gamma(ik)$ (allow for $x \mapsto -x$ shift), and $A(-k) = 2^{-ik}\Gamma(-ik)$. So $A(-k)$ has no zeroes hence there are no bound states. But

$$S(k) = \frac{2^{ik}\Gamma(ik)}{2^{-ik}\Gamma(-ik)}$$

has poles at the points $k = in$, $n = 1, 2, \ldots\ldots$ in the UHP.

The problem now is to understand why there can be poles for S in the UHP without bound states. The port picture is somewhat destroyed. There is a lack of orthogonality between incoming and outgoing spaces.

Suppose given $-u'' + qu = k^2$ on $0 \le x < \infty$ with initial condition $u'(0) = hu(0)$ so that we get $\phi(x,k^2)$ with asymptotic behavior

$$\phi(x,k^2) \sim A(k)e^{ikx} + B(k)e^{-ikx} \qquad B(k) = A(-k).$$

We consider the wave equation:

$$-\frac{\partial^2 u}{\partial t^2} = Lu \qquad \frac{\partial u}{\partial x}' = hu \quad \text{on} \quad x=0$$

and form a space $\mathcal{E}$ from its global solutions of finite energy norm:

$$E(u) = \left\|\frac{\partial u}{\partial t}\right\|^2 + (Lu, u) \qquad \| \|, (,) \text{ in } L^2(\mathbb{R}_{\geq 0})$$

which is invariant under time. If $\alpha \in C_0^\infty(\mathbb{R})$, then

$$u(x,t) = \int e^{-ikt}\phi(x,k^2)\alpha(k)\,dk$$

should be in the space $\mathcal{E}$. To prove this seems to require estimates. If the potential has compact support, then for $x \gg 0$ we have

$$u(x,t) = \int e^{-ikt}\left(A(k)e^{ikx} + B(k)e^{-ikx}\right)\alpha(k)\,dk$$

$$= \widehat{A\alpha}(x-t) + \widehat{B\alpha}(-x-t)$$

and so the fact that $u(x,t)$ is L^2 in x is clear from Riemann-Lebesgue. Moreover RL implies $u(x,t) \to 0$ as $t \to \pm\infty$ uniformly for x in a fixed compact set

May 28, 1978:

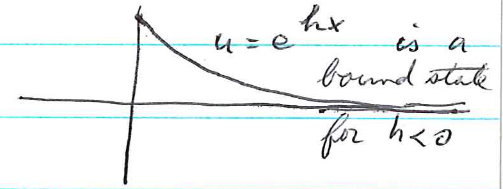
Example: Consider $-u'' = k^2 u$ in $0 \le x < \infty$ with boundary condition $u'(0) = h u(0)$. If

$$\phi_\lambda(x) = A e^{ikx} + B e^{-ikx}$$

$$(1 = A + B) \cdot ik$$

$$h = ikA - ikB$$

$$A = \frac{ik+h}{2ik} \qquad B = \frac{ik-h}{2ik}$$



so there is a bound state for $h < 0$ at $k = -ih$. Thus as we vary h from negative to positive we make disappear the bound state.

Put $p = \frac{u'}{u}$ for $u = e^{hx}$, i.e. $p = h$. Then if $L = -\frac{d^2}{dx^2}$ we have

$$L + h^2 = \left(-\frac{d}{dx} + h\right)\left(\frac{d}{dx} + h\right)$$

so if $\tilde{L} + h^2 = \left(\frac{d}{dx} + h\right)\left(-\frac{d}{dx} + h\right)$, then also $\tilde{L} = -\frac{d^2}{dx^2}$. But the operator

$$-\frac{d}{dx} + h$$

takes the boundary condition $u'(0) = h u(0)$ into the Dirichlet boundary condition $u(0) = 0$, and transforms

$$\phi_\lambda(x) = \cos kx + h \frac{\sin kx}{k} \quad \text{into} \quad (k^2 + h^2) \frac{\sin kx}{k}.$$

Better

$$\left(\frac{d}{dx} + h\right) \frac{\sin kx}{k} = \cos kx + h \frac{\sin kx}{k} = \phi_\lambda(x)$$

May 31, 1978

Program: At the moment I have some understanding of how the following are related:

1-ports $(\mathcal{H}, V)$
abstract scattering $(\tilde{\mathcal{H}}, U, \mathcal{D}^-, \mathcal{D}^+)$
inner functions $S(z)$
orthogonal polys on S^1
Schur development

The goal is somehow to incorporate bound states into the theory, and I propose to do this using J-matrix á la Kac-Case.

Let's start with an inner rational function

$$S(z) = \zeta \prod_{i=1}^{n} \frac{z - a_i}{1 - \bar{a}_i z} \qquad |a_i| < 1, \quad |\zeta| = 1$$

Modify S so that $\zeta = 1$, whence we can write

$$S(z) = \frac{p_n}{z^n p_n^*}$$

where p_n is a poly of degree n with positive leading coefficient normalized so that

$$d\mu = \frac{d\theta}{|p_n|^2 \, 2\pi}$$

is a probability measure. For any 1-port $(\mathcal{H}, V)$ with scattering matrix S we have the following concrete model (say n even)

$$L^2(S^1) \xrightarrow{\text{in}} L^2(d\mu) \xleftarrow{\text{out}} L^2(S^1)$$

$$1 \longmapsto u_{-i} = z^{n/2} p_n^* \quad , \quad z^{-n/2} p_n = u_i \longleftarrow\!\shortmid 1$$

$$f \longmapsto \frac{f}{z^{-n/2} p_n}$$

for the scattering situation:

$$\tilde{\mathcal{H}} = \langle \cdots\cdots, z^{-2}u_{-i}, z^{-1}u_{-i} \rangle \oplus \mathcal{H} \oplus \langle z u_i, z^2 u_i, \cdots\cdots \rangle$$

Now suppose we have a J-matrix $\lambda y_n = a_n y_{n+1} + b_n y_n + a_{n-1} y_{n-1}$ and define $\phi_\lambda(n)$ to be the eigenfunction with initial conditions $\phi_\lambda(0) = h_0$, $\phi_\lambda(1) = h_1$. Then $\phi_\lambda(n)$ is a polynomial of degree $n-1$ in λ for $n \geq 1$. Suppose $a_n = \frac{1}{2}$, $b_n = 0$ for large n so that we have

$$\phi_\lambda(n) = A(z) z^n + B(z) z^{-n} \qquad n \geq d$$

where $\frac{1}{2}(z + z^{-1}) = \lambda$. (Note that above seems to implies that $a_d = a_{d+1} = \cdots = \frac{1}{2}$ and $b_{d+1} = b_{d+2} = \cdots = 0$ and conversely.) Then we have

$$\phi_\lambda(d) = A(z) z^d + B(z) z^{-d}$$

$$\phi_\lambda(d+1) = (A(z) z^d) z + (B(z) z^{-d}) z^{-1}$$

$A(z^{-1}) = B(z)$

$\overline{A(z)} = B(z)$ for $|z| = 1$

$\overline{A(\bar{z})} = B(\bar{z}) = A(\bar{z}^{-1})$ $= A(z)$

so A is a real rational function

so that

$$A(z) z^d = \frac{\phi_\lambda(d+1) - z^{-1}\phi_\lambda(d)}{z - z^{-1}}$$

$$B(z) z^{-d} = \frac{\phi_\lambda(d+1) - z\,\phi_\lambda(d)}{z^{-1} - z}$$

It follows that $(z - z^{-1}) z^{2d} A(z)$, $(z^{-1} - z) B(z)$ are polynomials of degree $\leq 2d$ in z. If we put

$$S(z) = \frac{(z - z^{-1}) z^{2d} A(z)}{(z^{-1} - z) B(z)} = \frac{\phi_\lambda(d+1) - z^{-1}\phi_\lambda(d)}{\phi_\lambda(d+1) - z\,\phi_\lambda(d)}$$

this ignores roots on S^1 see p.53 below

then $|S(z)| = 1$ for $|z| = 1$. Furthermore if there are no bound states then $B(z)$ doesn't vanish for $|z| < 1$ and so $S(z)$ is analytic in the disk. So I have a 1-port, $\tilde{\mathcal{H}}$ etc belonging to S. The problem is now to identify this [redacted] $\tilde{\mathcal{H}}$ space with the solutions of the wave equation.

Simple example: $\quad \phi_\lambda(n) = \dfrac{z^n - z^{-n}}{z - z^{-1}} \qquad n \geq 0 = d$

Here $\quad z^d A(z) = \dfrac{1}{z - z^{-1}}, \quad z^{-d} B(z) = \dfrac{1}{z^{-1} - z} \quad$ so $\quad S(z) = +1$.

The wave equation is ~~$\frac{u(t+1)+u(t-1)}{2}$~~ [redacted] ~~$\frac{u(n+1)+u(n-1)}{2}$~~

$$\frac{u(n,t+1)+u(n,t-1)}{2} = \frac{u(n+1,t)+u(n-1,t)}{2}$$

and it has solutions of the form

$$\int z^{-t} \phi_\lambda(n)\, \alpha(z) \frac{d\theta}{2\pi} = \int z^{-t} \frac{z^n - z^{-n}}{z - z^{-1}}\, \alpha \frac{d\theta}{2\pi}$$

$$= \hat{A\alpha}(n-t) - \hat{A\alpha}(-n-t)$$

The point seems to be that the wave equation has the general solution $f(n-t) + g(n+t)$ and if we want the boundary condition $u(0,t) = 0$, then $g(t) = -f(-t)$ so that the general solution of the wave equation with boundary condition is

$$f(n-t) - f(-n-t)$$

So $\tilde{\mathcal{H}}$ will be some completion of such solutions with f of compact support.

The obvious candidate for [redacted] $u_i =$ [redacted] u_{-i} in this example is for $f(n) = \delta(n) = \begin{cases} 0 & n \neq 0 \\ 1 & n = 0 \end{cases}$. The E-norm of this solution is

$$\left\| \frac{u(t+1) - u(t-1)}{2i} \right\|^2 + \|u(t)\|^2 - \|Lu(t)\|^2$$

Take $t=0$,
$$u(n,0) = \delta(n) - \delta(-n) = 0$$
$$u(n,1) = \delta(n-1) - \delta(-n-1) = \delta_1$$
$$u(n,-1) = \delta(n+1) - \delta(-n+1) = -\delta_1$$

so that the E-norm is 1.

Recall the general formulas. If
$$u(n,t) = \int z^{-t} \phi_{\lambda}(n)\, \alpha(z) \frac{d\theta}{2\pi} \underset{n \geq d}{=} \hat{A\alpha}(n-t) + \hat{B\alpha}(-n-t)$$

then the E-norm of can be computed by letting $t \to +\infty$ whence the $\hat{B\alpha}(-n-t)$ disappears and putting $n+t$ in for n. Thus

$$E(u_\alpha) = \sum_{n \in \mathbb{Z}} \left| \frac{\hat{A\alpha}(n-1) - \hat{A\alpha}(n+1)}{2i} \right|^2 + \sum_n \left|\hat{A\alpha}(n)\right|^2 - \sum_n \left| \frac{\hat{A\alpha}(n+1) + \hat{A\alpha}(n-1)}{2} \right|^2$$

$$= \int \left\{ \left| \frac{z^{-1} - z}{2i} A\alpha \right|^2 + |A\alpha|^2 - \left| \frac{z + z^{-1}}{2} A\alpha \right|^2 \right\} \frac{d\theta}{2\pi}$$

$$= \int \left(\sin^2\theta + 1 - \cos^2\theta \right) |A\alpha|^2 \frac{d\theta}{2\pi} = 2 \int \sin^2\theta\, |A\alpha|^2 \frac{d\theta}{2\pi}$$

Check: When $A = \frac{1}{z - z^{-1}}$ if $\alpha = z - z^{-1}$ then
$$u_\alpha = \int z^{-t} \frac{z^n - z^{-n}}{z - z^{-1}} (z - z^{-1}) \frac{d\theta}{2\pi} = \delta(n-t) - \delta(-n-t)$$

and the energy norm is
$$2 \int \sin^2\theta \frac{d\theta}{2\pi} = 1$$

However we have $E(u_\alpha) = 2 \int \sin^2\theta \left| \frac{\alpha}{2i \sin\theta} \right|^2 \frac{d\theta}{2\pi}$
$$= \frac{1}{2} \int |\alpha|^2 \frac{d\theta}{2\pi}$$
in general, hence

$$E(u_\alpha, u_\beta) = \frac{1}{2}\int \alpha\bar{\beta}\,\frac{d\theta}{2\pi}$$

This means that if $\beta = z - z^{-1}$ and $\alpha = z^n\cdot\beta$ we get

$$E(u_{z^n\beta}, u_\beta) = \frac{1}{2}\int z^n\, 4\sin^2\theta\, d\theta$$

which is definitely not zero for $n = \pm 2$. So I conclude therefore that the energy norm might not be the good norm to put on the space of solutions of the wave equations.

Check: The wave equation has the solutions $f(n-t) - f(-n-t)$. I want to make a Hilbert space out of these which will be $\tilde{\mathcal{H}}$, so there has to be subspaces $\mathcal{D}^\pm$. So it's pretty clear we want $\mathcal{D}^+$ to consist of f with support in $[0,\infty)$ and $\mathcal{D}^-$ to consists of f with support in $(-\infty, 0]$. The intersection is spanned by $u_i = u_{-i} = \delta$ which should be perpendicular to $z\mathcal{D}^+$ and $z^{-1}\mathcal{D}^-$. So it's clear that the good norm on $\tilde{\mathcal{H}}$ must make $z^n\delta$ an orthonormal basis, and therefore it cannot be the energy norm which I recall is

$$\left\|\frac{u(t+1) - u(t-1)}{2}\right\|^2 + \|u(t)\|^2 - \|Lu(t)\|^2$$

$$= \|u(t)\|^2 - \frac{1}{2}(u(t-1), u(t+1)) - \frac{1}{2}(u(t+1), u(t-1))$$

Let's now consider the general case. Concentrate on solutions of the wave equation and see if we can locate u_i, u_{-i} before we put on a norm. Solutions can be represented

$$u_\alpha(n,t) = \int z^{-t}\phi_\lambda(n)\,\alpha(z)\,\frac{d\theta}{2\pi} \overset{n\geq d}{=} \hat{A}\alpha(n-t) + \hat{B}\alpha(-n-t)$$

If $\alpha = \frac{1}{z^{-d}B}$, then $\hat{B}\alpha(-n) = \int z^{-n} z^{d}\,\frac{d\theta}{2\pi} = \delta(d-n)$, hence $\hat{B}\alpha(-n-t) = \delta(d-n-t)$ represents an unit impulse travelling to the left reaching the positive $n=d$ at time $t=0$.

$$\hat{A}\alpha(n-t) = \int_0^{2\pi} z^{n-t}\,\frac{A}{z^{-d}B}\,\frac{d\theta}{2\pi} = \int_0^{2\pi} z^{n-d-t}\,\underbrace{\frac{z^{2d}A}{B}}_{S(z)}\,\frac{d\theta}{2\pi}$$

Since $S(z)$ is analytic $\hat{A}\alpha(n-t) = 0$ for $n > d+t$, so $\hat{A}\alpha(n-t)$ represents a wave travelling to the right whose front is at $n=d$ when $t=0$.

By convention multiplication by z corresponds to $t \mapsto t-1$ hence it transforms $f(n-t)$ to $f(n-t+1)$ which moves the wave backward one step. So multiplying by z means moving time backwards. If I take u_i to be the trajectory $\delta(d-n-t)$ for $t \ll 0$ and u_{-i} to be the trajectory $\delta(n-d-t)$ for $t \gg 0$ which means

u_i corresp. to $\alpha = \dfrac{1}{z^{-d}B}$

u_{-i} corresp. to $\alpha = \dfrac{1}{z^{d}A}$

then clearly $u_i = S(z)\,u_{-i}$ as it should be. So the good cyclic vector seems to be

$$\frac{u_i}{z^{-m/2}p_n} = \frac{1}{z^dA}\,\frac{1}{z^{-d}B} = \frac{1}{AB} = \frac{1}{|B|^2} = \frac{1}{|A|^2}$$

NO $z^{-m/2}p_n = z^dA\,(z-z^{-1})$

and the good representation is to associate to a function $f(z)$ the solution of the wave equation

$$u(n,t) = \int z^{-t}\phi_\lambda(n)\, f(z)\, \frac{d\theta}{2\pi|B|^2} \overset{n\geq d}{=} \left(\widehat{\frac{f}{B}}\right)(n-t) + \left(\widehat{\frac{f}{A}}\right)(-n-t)$$

The behavior as $t \to \infty$ is asymptotic to $\left(\widehat{\frac{f}{B}}\right)(n-t)$ which has L^2 norm

$$\int |f|^2 \frac{d\theta}{2\pi|B|^2}.$$

Therefore we see that the good spectral measure for the wave equation is $d\mu = \dfrac{d\theta}{2\pi|B|^2}$ in the sense that we get an isometry

$$L^2(d\mu) \xrightarrow{\sim} \tilde{\mathcal{H}}$$

$$f \longmapsto \int z^{-t}\phi_\lambda(n) f \frac{d\theta}{2\pi|B|^2}.$$

(Notice by putting $t=0$ that we get the spectral measure for the $\mathcal{J}$ matrix – maybe?)

June 1, 1978

Starting from a 1-port $(\mathcal{H}, V)$ I can form its unitary dilation

$$\tilde{\mathcal{H}} : \langle \cdots, z^{-2}u_{-i}, z^{-1}u_{-i} \rangle \oplus \mathcal{H} \oplus \langle zu_i, z^2 u_i, \cdots\cdots \rangle$$

and define $S(z)$ by $S(z)\, u_{-i} = u_i$. When $\mathcal{H}$ is finite-dimensional we have up to a scalar of modulus 1

$$S(z) = \frac{p_n}{z^n p_n^*} \qquad p_n \text{ poly in } z$$

and so if p_n is normalized so $d\mu = \dfrac{d\theta}{|p_n|^2\, 2\pi}$ then we get an isom.

$$\begin{array}{ccc} L^2(d\mu) & \xrightarrow{\ \sim\ } & \tilde{\mathcal{H}} \\ z^{-n/2} p_n & \longleftrightarrow & u_i \\ z^{n/2} p_n^* & \longleftrightarrow & u_{-i} \end{array}$$

Now suppose given a J-matrix situation

$$\phi_\lambda(n) = A(z) z^n + B(z) z^{-n} \qquad n \geq d$$

$$z^d A(z)(z - z^{-1}) = \phi_\lambda(d+1) - z^{-1}\phi_\lambda(d)$$

$$z^{-d} B(z)(z^{-1} - z) = \phi_\lambda(d+1) - z\,\phi_\lambda(d)$$

so that

$$z^{2d} A(z)(z - z^{-1}) = p_{2d} \qquad\qquad S(z) = \pm \frac{z^{2d} A(z)}{B(z)}$$

$$B(z)(z^{-1} - z) = z^{2d} p_{2d}^*$$

you are assuming that p_{2d} has its roots inside $\mathcal{S}$!

Now let's consider the wave equation $\dfrac{u(t+1) + u(t-1)}{2} = Lu(t)$ whose solutions are in the form

$$u(n,t) = \int z^{-t} \phi_\lambda(n)\, \alpha(z) \frac{d\theta}{2\pi} \overset{n \geq d}{=} \hat{A\alpha}(n-t) + \hat{B\alpha}(-n-t).$$

If $\alpha = \dfrac{1}{z^{-d} B}$, then $\hat{B\alpha}(-n-t) = \displaystyle\int z^{-n-t+d} \frac{d\theta}{2\pi} = \delta(d-t-n)$

represents a unit impulse at $n = d-t$ moving to the left which reaches $n=d$ when $t=0$. The reflected wave

$$\hat{A_\alpha}(n-t) = \int z^{n-t-d}\, z^{2d} \frac{A}{B} \frac{d\theta}{2\pi} = \hat{S}(n-d-t)$$

since S is analytic vanishes for $n-t-d>0$, so it is a wave moving to the right with leading edge at $n=t+d$. It seems clear to me that I want u_i to correspond to $\alpha = \frac{1}{z^{-d}B}$ and for u_{-i} to " to $\alpha = \frac{1}{z^d A}$.

Check:
$$\hat{A}_{\frac{1}{z^d A}}(n-t) = \int z^{n-t-d} \frac{d\theta}{2\pi} = \delta(n-d-t)$$

which is a unit impuls at $n=d+t$. (Recall that multiplying α by z corresponds to changing $u(t)$ to $u(t-1)$).

So then I want the good cyclic vector in the space of wave equation solutions to be proportional to

$$\frac{u_i}{z^{-d}P_{2d}} = \frac{1}{z^{-d}B}\,\frac{1}{z^d A(z-z^{-1})} = \frac{1}{BA(z-z^{-1})}$$

In other words solutions should be represented

$$u(n,t) = \int z^{-t}\phi(n,\lambda)\frac{f(z)}{z-z^{-1}}\frac{d\theta}{BA\,2\pi}$$
$$\overset{n\geq d}{=} \widehat{\frac{f}{(z-z^{-1})B}}(n-t) + \widehat{\frac{f}{(z-z^{-1})A}}(-n-t)$$

By the choice of u_i, u_{-i} it should be true that the norm in the space of wave equation solutions should coincide with ordinary L^2 norm for the asymptotic trajectories, i.e.

$$\int \frac{|f|^2}{|(z-z^{-1})B|^2} \frac{d\theta}{2\pi}$$

Recall that S determines a holomorphic line bundle over $\mathbb{P}^1$ whose [redacted] global sections are pairs (f,g) with f holomorphic for $|z|\le 1$ and g holomorphic for $|z|\ge 1$ such that

$$Sg = f$$

In other words the space of holom. sections is

$$\Gamma = H_+^2 \cap S H_-^2$$

[redacted] But recall that $H_-^2 = (zH_+^2)^\perp$, [redacted] and multiplication by S is unitary, hence

$$\Gamma = H_+^2 \cap (zSH_+^2)^\perp = H_+^2 \ominus zSH_+^2$$

Hence there might be some relation between line bundles and the Schur development. What is Yang-Mills for line bundles?

June 2, 1978

Suppose given $S(z)$ we can consider the line bundle on $\mathbb{P}^1$ associated to S. Its global sections are

$$\Gamma(\mathcal{L}_S) = \{(f,g) \mid Sg = f,\ f \text{ analytic for } |z|\leq 1,\ g \text{ anal. } |z|\geq 1\}$$
$$= H^2_+ \cap SH^2_- = H^2_+ \cap S(zH^2_+)^\perp$$
$$= H^2_+ \ominus zSH^2_+$$
$$= \text{Hilbert space } \mathcal{H}$$

Up to now I have been thinking of S as being analytic in $|z|\leq 1$, but the above makes sense more generally.

Suppose S is analytic on $|z|=1$ with modulus 1 there. Then $\Gamma(\mathcal{L}_S) = H^2_+ \cap SH^2_-$ will have dimension $d+1$ if the degree of S is $d\geq 0$. Suppose $d=1$ and let (f,g) be the unique-up-to-a-scalar non-vanishing section. Note $\mathcal{L}_S \simeq \mathcal{O}$ so f,g do not vanish.

$$S(z)g(z) = f(z)$$

$$\overline{g(z^*)} = S(z)\cdot\overline{S(z^*)g(z^*)} = \overline{f(z^*)}\cdot S(z)$$

But $f^*(z) = \overline{f(z^*)}$ is analytic outside, so we have $f^* = cg$, $g^* = cf$ for some scalar c. Then $g^* = cf = c\bar{c}g^* \Rightarrow |c| = 1$ so by altering f we can suppose $f^* = g$. So therefore we see that

$$S = \frac{f}{f^*}$$

where f is analytic for $|z|\leq 1$ and non-vanishing there. Thus f is an outer function. $\leftarrow$?

Another proof. Because S is of degree 0 $\log S(z)$ is a well-defined analytic function on $|z|=1$ with real part zero. Thus if we expand in Fourier series

$$\log S(z) = \sum{}' c_n z^n$$

we have $c_{-n} = -\bar{c}_n$ so we have

$$\log S(z) = h(z) - h^*(z)$$

with $h(z)$ analytic for $|z| \leq 1$ unique up to an additive real constant. Then $S = f/f^*$ where $f = e^h$. f can be varied by a positive real constant, but because the $\log S(z)$ is determined up to $2\pi i n$ constant, h is determined up to $+\pi i n$ + real, so f varies by $\pm e^{\text{real}}$ = arb. non-zero real constant.

June 3, 1978:

Let S be analytic on $|z|=1$ and of modulus 1. Adjust its degree to be zero, whence we've seen

$$S = \frac{g}{g^*}$$

where g is analytic on $|z| \leq 1$ and non-vanishing. Use S to form a holomorphic line bundle $\mathcal{L}_S$ on $\mathbb{P}^1$. We have

$$\begin{array}{ccc} \Gamma(\mathcal{L}_S(n)) = \Gamma(\mathcal{L}_{z^n S}) = & H_+ \cap z^n S H_- & \\ & \cong \big\downarrow \quad \begin{matrix} f \\ \downarrow \\ fg \end{matrix} & \qquad H_+ = H^2 \\ & gH_+ \cap z^n g^* H_- & \end{array}$$

Because g is analytic on $|z| \leq 1$ and non-vanishing one knows $gH_+ = H_+$; also $g^* H_- = H_-$. Hence the last space is $H_+ \cap z^n H_-$ which has the basis $1, \dots, z^n$. Thus we see that $\Gamma(\mathcal{L}_S(n))$ has the basis $\frac{1}{g}, \frac{z}{g}, \dots, \frac{z^n}{g}$.

As a Hilbert space $\Gamma(\mathcal{L}_S(n))$ can be identified with the space of polys in z of degree $\leq n$ with the norm

$$\int |f|^2 \frac{d\theta}{2\pi |g|^2}.$$

Digression: What happens if S is piecewise smooth but not continuous? This is the problem (Riemann-Hilbert problem) considered by Hilbert, for more general curves in the plane.

Take the case $S = z$ and see what happens when you try to write it as g/g^*. $\frac{1}{i}\log z$ is discontinuous. Put the break at $\theta = 0$, and calculate the Fourier series of θ for $0 \leq \theta < 2\pi$

$$\theta = \sum c_n e^{in\theta}$$

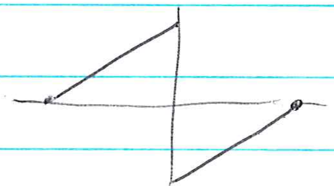
$$c_n = \frac{1}{2\pi}\int_0^{2\pi} \theta e^{-in\theta}\, d\theta = \frac{1}{2\pi}\left[\theta \frac{e^{-in\theta}}{-in}\right]_0^{2\pi} - \frac{1}{2\pi}\underbrace{\int_0^{2\pi} \frac{e^{-in\theta}}{-in}\, d\theta}_{0}$$

$\frac{1}{2\pi}\frac{(2\pi)^2}{2}$

$$= \begin{cases} \dfrac{i}{n} & \text{for } n \neq 0 \\ \pi & \text{for } n = 0 \end{cases}$$

Hence

$$\sum_{n\neq 0} \frac{i}{n} e^{in\theta} = \begin{cases} \theta - \pi & 0 < \theta < \pi \\ \theta + \pi & -\pi < \theta < 0 \end{cases}$$



So we put $\frac{1}{i}h = \sum_{n=1}^{\infty} \frac{i}{n} e^{in\theta} = -i\log(1-z)$ [redacted]

$$\frac{1}{i}\log z = \frac{1}{i}(h - h^*) \qquad \text{so} \qquad h = \log(1-z)$$

and $g = e^h = 1 - z$. Thus the factorization becomes

$$\frac{g}{g^*} = \frac{1-z}{1-z^{-1}} = z\,\frac{1-z}{z-1} = -z$$

(minus comes from π-shift)

Suppose $S : S^1 \to S^1$ measurable and that $H_+ \cap SH_-$ is 1-dimensional. [redacted]. Then I've seen that $S = g/g^*$ where g is a generator of $H_+ \cap SH_-$ unique up to a real scalar. In effect if [redacted]

$$\frac{1}{S}g \in H_- \Rightarrow \underset{\overset{\|}{S}}{\frac{1}{S^*}}g^* \in H_+$$

$$\Rightarrow Sg^* \in H_+ \cap SH_-$$

so $g^* = cg$, so $|c| = 1$, so we can arrange $g^* = g$.

<u>Question:</u> Does g have to be an outer function?

Suppose g is an outer function (this means $\overline{gH_+} = H_+$ or that

$$g = c\cdot\exp\left(\int \frac{\zeta + z}{\zeta - z}\log|g(z)|\frac{d\theta}{2\pi}\right) \qquad \text{with } |c| = 1.)$$

Suppose $S = g/g^*$. Then we have

$$g \in H_+ \cap \frac{g}{g^*}H_- \simeq \frac{1}{g}H_+ \cap \frac{1}{g^*}H_- = H_+ \cap H_-$$

provided $gH_+ = H_+$. Can it happen that $\frac{1}{g} \notin L^2$?

Examples: $z - 1$ is an outer function because it is in H_+ + $\log|z-1| = \log 2\left|\frac{e^{i\theta}-1}{2}\right| = \log\left|2\sin\frac{\theta}{2}\right| \sim \log|\theta|$ which is L^1.

But $S(z) = z-1/z^{-1}-1 = -z$, and $H_+ \cap SH_-$ is 2-dimensional.

June 4, 1978:

Problem: Given $S(z)$ a measurable function of modulus 1 on S^1 we can consider $H_+ \cap z^n S H_-$. This is an increasing family of closed subspaces, each at most of codim 1 in the succeeding one. If $H_+ \cap z^n S H_- = H_+ \cap z^{n+1} S H_-$ then this intersection is stable under z, hence 0 because the only z-stable subspace of H_- is 0. Hence if for some n, $H_+ \cap z^n S H_-$ is finite-dimensional and $\neq 0$, then we can arrange $H_+ \cap S H_-$ to be one-dimensional whence $S = g/g^*$ where g is a suitable generator of this intersection.

Recall that if T is a bounded measurable function on S^1 we get a Toeplitz (or Wiener-Hopf) operator

$$f \longmapsto pr\, Tf$$

on H_+ where pr denotes orthogonal projection on H_+. If $T(z) = \sum c_m z^m$ is the Fourier series expansion of T, then

$$z^n \longmapsto pr\left(\sum_{m \in \mathbb{Z}} c_m z^{m+n}\right) = \sum_{m \geq 0} c_{m-n} z^m$$

so the matrix of the Toeplitz operator is c_{m-n}. The kernel of the Toeplitz op is

$$\{f \in H_+ \mid Tf \in z^{-1} H_-\} = H_+ \cap (zT)^{-1} H_-$$

hence $H_+ \cap S H_-$ is the kernel of the Toeplitz operator belonging to $(zT)^{-1} = S$ or

$$T = z^{-1} S^{-1}.$$

The cokernel of the Toeplitz operator is isom. to

$$L^2 / z^{-1}H_- + TH_+$$

When $T = z^{-1}S^{-1}$ this is isomorphic to

$$L^2 / H_+ + SH_- \quad .$$

For T to be a Fredholm operator means that $H_+ + SH_-$ is closed of finite codimension, and $H_+ \cap SH_-$ is finite-dimensional.

Suppose that $S = q/q^*$ where q is a unit in H^∞_+, that q and q^{-1} are analytic and bounded. Then multiplication by q, q^{-1} gives bounded linear operators on L^2 mutually inverse to each other. Hence

$$H_+ + SH_- = H_+ + \frac{q}{q^*}H_- \simeq \frac{1}{q}H_+ + \frac{1}{q^*}H_- = H^+ + H^- = L^2$$

$$H_+ \cap SH_- \simeq \frac{1}{q}H_+ \cap \frac{1}{q^*}H_- = H_+ \cap H_- = \langle 1 \rangle$$

and so T is Fredholm with index 1.

Here's something I missed yesterday. Suppose $\dim(H_+ \cap SH_-) = 1$ and let $0 \neq q \in H_+ \cap SH_-$ satisfying $q = S\bar{q}$ claim q is outer. Indeed let $q = q_i q_0$ with q_i inner. Then

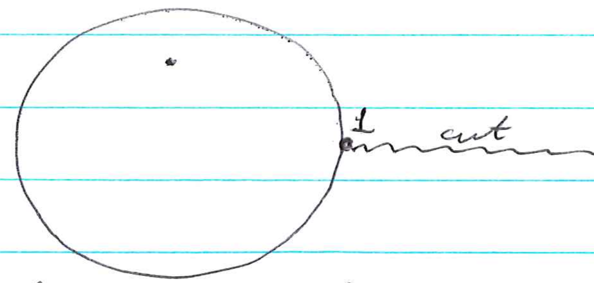
$$q_0 = \bar{q}_i q_i q_0 = \bar{q}_i q = \bar{q}_i S \bar{q} = S(\overline{q_i q})$$

with $q_0 \in H_+$ and $\overline{q_i q} \in \overline{H_+} = H_-$. Thus $q_0 \in H_+ \cap SH_-$

and we conclude that g_i has to be a constant of modulus 1.

Example: Take an analytic function h in the disk whose real part is bounded but whose imaginary part isn't, for example

$$h = \frac{1}{i}\log(1-z)$$



$$\operatorname{Re} h = \arg(1-z)$$

varies between $\pm\frac{\pi}{2}$.
Then $g = e^h$ will be a bounded analytic function with $g^{-1} = e^{-h}$ bounded also. But

$$S = g/\bar{g} = e^{h-\bar{h}} = e^{2i\operatorname{Im} h}$$

will oscillate and won't be analytic or even continuous on S^1. So this gives an example where the Toeplitz operator is Fredholm, but where S is not continuous.

How the above discussion arose: Consider a J-matrix scattering situation, say $a_n = \frac{1}{2}$, $b_{n+1} = 0$ for $n \geq 0$ so that

$$\phi_\lambda(n) = Az^n + Bz^{-n}, \quad n \geq 0$$

Assume no bound states, or better consider only solutions of the wave equation of the form

$$u(n,t) = \int z^{-t}\phi_\lambda(n)\,\alpha(z)\,\frac{d\theta}{2\pi}$$

Then $\qquad u(n,t) = \hat{A\alpha}(n-t) + \hat{B\alpha}(-n-t) \quad$ for $n \geq 0$.

I equip solutions of the wave equation [redacted] with the norm coming from the obvious ℓ^2 norm for the asymptotic behavior as either $t \to \pm\infty$:

$$\|u\|^2 = \int |A\alpha|^2 \frac{d\theta}{2\pi} = \int |B\alpha|^2 \frac{d\theta}{2\pi}$$

The idea is to look for those solutions of the wave equation $u(n,t)$ supported for $n \le |t|$. These are given by α such that $\widehat{A\alpha}(n) = 0$ for $n > 0$ and $\widehat{B\alpha}(n) = 0$ for $n \not< 0$, in other words such that $A\alpha$ [redacted] is analytic for $|z| \le 1$, ~~[illegible]~~ and $B\alpha$ is analytic for $|z| \geq 1$. Thus we want $f = A\alpha$ analytic for $|z| \le 1$ such that $S^{-1}f = \frac{B}{A} \cdot A\alpha = B\alpha$ is analytic for $|z| \geq 1$. Hence the solutions supported for $n \le |t|$ are exactly $H_+ \cap SH_-$. Clean this up.

June 5, 1978:

It is now possible to organize a little the discrete version of the half-line inverse scattering problem.

Begin with what corresponds to the Gelfand-Levitan paper. On one hand one has a probability measure $d\mu$ on S^1 and on the other a sequence of Schur parameters $h_1, h_2, h_3, \ldots\ldots$ which are complex numbers of modulus <1. We suppose $d\mu$ has infinite support. If $p_0, p_1, \ldots$ is the sequence of orthonormal polys associated to $d\mu$ we have the recursion formula

$$z p_n = k_{n+1} p_{n+1} + h_{n+1} z^n p_n^* \qquad k_{n+1} = \sqrt{1-|h_{n+1}|^2}$$

or

$$\begin{pmatrix} p_n \\ z^n p_n^* \end{pmatrix} = \frac{1}{k_n} \begin{pmatrix} 1 & h_n \\ \bar{h}_n & 1 \end{pmatrix} \begin{pmatrix} z & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} p_{n-1} \\ z^{n-1} p_{n-1}^* \end{pmatrix}$$

The Gelfand-Levitan formulas relate the h_n to the moments of $d\mu$. The Schur sequence is the analogue of the potential function.

So far one has made no assumption about the sequence h_n tending to zero sufficiently fast that one has scattering. The condition we want is that in $L^2(d\mu)$ the subspace $H_+(d\mu)$ spanned by $1, z, z^2, \ldots$ be outgoing, so that we get an isomorphism

$$\begin{array}{ccc} L^2(d\mu) & \xrightarrow{\sim} & L^2(S^1) \\ f & \longmapsto & fg \\ H_+(d\mu) & \xrightarrow{\sim} & H_+ \end{array} \qquad d\mu = |g|^2 \frac{d\theta}{2\pi}$$

where g is an outer function of norm 1. It should be that $\frac{1}{g} = \lim_{n\to\infty} z^n p_n^*$ in $L^2(d\mu)$, and that

$$g(0) = \lim \frac{1}{(z^n p_n^*)(0)} = \left\{\prod (1-|h_n|^2)\right\}^{-1/2}.$$

Hence the correct condition on the sequence h_n is that $\sum |h_n|^2 < \infty$ in order that one has a scattering situation. In this case the scattering operator is

NO, this doesn't force $d\mu$ to be abs. cont.

$$S = g/g^*$$

The above constitutes the forward scattering problem, namely, going from the "potential" h_n to the scattering operator S. We see that it amounts essentially to going from an outer function g to its phase.

When does S determine g? Note that $H_+ \cap SH_-$ has a natural conjugation $f \longmapsto S\bar{f}$ for

$$\overline{H_+ \cap SH_-} = H_- \cap S^{-1}H_+ \xrightarrow{\cdot S} H_+ \cap SH_-$$

and $f \longmapsto S\bar{f} \longmapsto S\overline{(S\bar{f})} = S\bar{S}f = f$. Hence it is spanned by its real elements f, i.e. those with $f = S\bar{f}$. So if $\dim(H_+ \cap SH_-) = 1$, then there is only one non-zero function f with $S = f/\bar{f}$ up to a real scalar. (I don't see the iff part claimed ~~on~~ McKean p. 100).

Example: $g = 1-z$ $\quad S = \frac{1-z}{1-z^{-1}} = -z$ but also

$g_1 = i(1+z)$ $\quad \frac{i+iz}{-i-iz} = \frac{i(1+z)}{-i(1+z^{-1})} = i^2 z = -z$

So there are lots of outer functions with phase $-z$.