May 13, 1978

Suppose $T \in GL_2(\mathbb{C}[z,z^{-1}])$ is such that

1) $$|z| \lessgtr 1 \implies T^*PT \lessgtr P$$

and suppose also that T is a polynomial function of z. One has $\det T = cz^n$ $n \geq 0$ with $c \in S^1$. If $n=0$, i.e. $T \in GL_2(\mathbb{C}[z])$, then from

$$T^*PT = P$$

on S^1 we have $T^{-1} = PT^*P$. This shows T^{-1} is analytic for $|z| \geq 1$ and $|z| \leq 1$ including $0, \infty$, hence T must be constant, hence an element of $U(1,1)$.

(Actually this last argument shows that any $T \in GL_2(\mathbb{C}[z])$ with $T^*PT = P$ on S^1 is constant and consequently gives us an embedding

$$\{T \in GL_2(\mathbb{C}[z,z^{-1}]) \mid T^*PT = P\}/U(1,1) \hookrightarrow GL_2(\mathbb{C}[z,z^{-1}])/GL_2(\mathbb{C}[z]).$$

This allows one to associate lattices to T with $T^*PT = P$.)

Next suppose $\det T = cz^n$ with $n \geq 1$. From 1) we have $T(0)^*PT(0) \leq P$, hence $T(0) \neq 0$. Since $\det T(0) = 0$, $T(0)$ has rank 1. Hence the effect of $T(0)$ on $\mathbb{C}P^1$ is to map all lines $\neq$ its kernel to the line given by its image. So for all but one α we have

$$h = \frac{A(0)\alpha + B(0)}{C(0)\alpha + D(0)}$$

where h is independent of α and also $|h| < 1$ by 1). More precisely $\binom{\alpha}{1}^* T(0)^* P T(0) \binom{\alpha}{1} = |A(0)\alpha + B(0)|^2 - |C(0)\alpha + D(0)|^2$

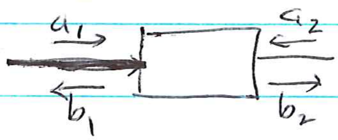
$\leq |\alpha|^2 - 1$, so for $|\alpha| < 1$ one has $C(0)\alpha + D(0) \neq 0$

If we form $\frac{1}{|1-|h|^2}\begin{pmatrix} 1 & -h \\ -h & 1 \end{pmatrix} T(z)$, then this transformation for $z=0$ maps almost all lines to the line $\mathbb{C}\binom{0}{1}$, so replacing T by this we get $A(0) = B(0) = 0$ and so

$$T(z) = \begin{pmatrix} zA_1 & zB_1 \\ C & D \end{pmatrix} = \begin{pmatrix} z & 0 \\ 0 & 1 \end{pmatrix} \underbrace{\begin{pmatrix} A_1 & B_1 \\ C & D \end{pmatrix}}_{T_1}$$

I'd like now to show that $T_1^* P T_1 \leq P$ for $|z| \leq 1$ etc.

The first method seems to be to calculate the scattering matrix belonging to T_1 and show it is analytic for $|z| \leq 1$. We need new formula because we've changed notation slightly for now we have



$$\begin{pmatrix} b_1 \\ b_2 \end{pmatrix} = S \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} \qquad T\begin{pmatrix} a_1 \\ b_1 \end{pmatrix} = \begin{pmatrix} b_2 \\ a_2 \end{pmatrix}$$

so that $|a_1|^2 - |b_1|^2 \geq |b_2|^2 - |a_2|^2$ for $|z| \leq 1$

means $P \geq T^* P T$ for $|z| \leq 1$.

New formulas are

$$S = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \Longrightarrow T = \begin{pmatrix} \frac{\gamma\beta - \alpha\delta}{\beta} & \frac{\delta}{\beta} \\ -\frac{\alpha}{\beta} & \frac{1}{\beta} \end{pmatrix}$$

$$T = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \Longrightarrow S = \begin{pmatrix} -\frac{C}{D} & \frac{1}{D} \\ \frac{AD-BC}{D} & \frac{B}{D} \end{pmatrix}$$

From this last formula we see that if S_1 belongs to $\begin{pmatrix} A_1 & B_1 \\ C & D \end{pmatrix}$ then

$$S_1 = \begin{pmatrix} -\frac{C}{D} & \frac{1}{D} \\ \frac{A_1 D - B_1 C}{D} & \frac{B_1}{D} \end{pmatrix}$$

which is again analytic.

Direct proof based on maximum modulus?

We therefore have proved:

Prop: Let $T(z)$ be a 2×2 matrix of polynomials in z such that $T^*PT \leq P$ for $|z| \leq 1$ and $\geq P$ for $|z| \geq 1$. Then $T(z)$ admits a factorization

$$T(z) = R(h_1)\begin{pmatrix} z & 0 \\ 0 & 1 \end{pmatrix} \cdots\cdots R(h_n)\begin{pmatrix} z & 0 \\ 0 & 1 \end{pmatrix} T_0$$

where $|h_1|, \ldots, |h_n| < 1$ and $T_0 \in U(1,1)$. This factorization is unique.

The uniqueness comes from Schur's results.

So next consider the case where $T(z) \in GL_2(\mathbb{C}[z, z^{-1}])$ and it has the positivity property 1). We want to show that $T(z)$ is a finite product of factors

$$R(h), \quad \begin{pmatrix} z & 0 \\ 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 \\ 0 & z^{-1} \end{pmatrix}, \quad \theta \in U(1,1).$$

If so, then $z^k T(z)$ should be in the form of the above proposition where k is the number of $\begin{pmatrix} 1 & 0 \\ 0 & z^{-1} \end{pmatrix}$ factors. As

$$S = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \Rightarrow T = \begin{pmatrix} \frac{\gamma\beta - \alpha\delta}{\beta} & \frac{\delta}{\beta} \\ -\frac{\alpha}{\beta} & \frac{1}{\beta} \end{pmatrix} \qquad \det T = \frac{\gamma}{\beta}$$

one sees that multiplying T by z corresponds to

$$S = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \longmapsto \begin{pmatrix} \alpha & \frac{1}{z}\beta \\ z\gamma & \delta \end{pmatrix}$$

The poles of T for $|z| \leq 1$ are due to the zeroes of β. Hence we can multiply T by z^p, where p is the order of β at $z=0$, without destroying analyticity of S at 0 hence preserving positivity. But if β doesn't vanish at 0, then T has no poles at 0, so T is a polynomial function of z.

General remarks: Given $S = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$ a rational scattering matrix in general, we've seen that multiplying T by a rational function ρ of modulus 1 on S^1 changes S to

$$\begin{pmatrix} \alpha & \rho^{-1}\beta \\ \rho\gamma & \delta \end{pmatrix}.$$

So one can see that for each S we have a range of ρ such that ρT is still a transfer matrix, because we are allowed to eliminate zeroes in β with $|z|<1$ and transfer them to γ and conversely. We can thereby arrange that β doesn't vanish for $|z|<1$, hence $T(z)$ will be analytic for $|z|<1$. Since $\det T(z) = \frac{\gamma}{\beta}$, $T(z)$ will be non-invertible at those $|z|<1$ which are roots of γ. Since $T^*PT \leq P$ for $|z|<1$ we know that $T(z)$ can't vanish, hence $T(z)$ has rank 1 at roots of γ.

Fix a root of γ in the disk and change z by a $R(h)$ so this root is moved to zero. Thus $T(0)$ has rank 1 and so for a suitable h we have $R(-h)T(0)$ has image the line $\mathbb{C}\binom{0}{1}$. Change T to this, so then I know that if $T = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$ then $A(0) = B(0) = 0$ and so

$$T(z) = \begin{pmatrix} z & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} A_1 & B_1 \\ C & D \end{pmatrix}$$

But dividing A, B by z and keeping C, D fixed changes S to (see formula for S on p. 969)

$$\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \longmapsto \begin{pmatrix} \alpha & \beta \\ \frac{1}{z}\gamma & \frac{1}{z}\delta \end{pmatrix}$$

Better:
$$S = \begin{pmatrix} -\frac{C}{D} & \frac{1}{D} \\ \frac{AD-BC}{D} & \frac{B}{D} \end{pmatrix} \qquad S_1 = \begin{pmatrix} -\frac{C}{D} & \frac{1}{D} \\ \frac{A_1 D - B_1 C}{D} & \frac{B_1}{D} \end{pmatrix}$$ remains analytic at ∞.

(If T is analytic near z_0 [redacted] with $|z_0| < 1$ (i.e. $\beta(z_0) \neq 0$) and if $\det T(z_0) = 0$ (i.e. $\delta(z_0) = 0$), [redacted] and if $\operatorname{Im} T(z_0) = \mathbb{C}\binom{0}{1}$, then $\delta(z_0) = 0$.)

So we see that if T is analytic for $|z| < 1$ and $\det T(z_0) = 0$, then we can [redacted] factor T:

$$T(z) = R(h) \begin{pmatrix} \frac{z - z_0}{1 - \bar{z}_0 z} & 0 \\ 0 & 1 \end{pmatrix} T_1(z)$$

where T_1 is a transfer matrix analytic for $|z| < 1$.

The conclusion of the above that I more or less understand rational matrices $T(z)$ satisfying the positivity condition equivalent to the corresponding $S(z)$ being analytic for $|z| \leq 1$ and unitary on the boundary. But I still have to look at singularities on $|z| = 1$. Let's defer this.

Let's consider a rational $T(z)$ with $U(1,1)$ values for $|z| = 1$ such that for $|z| \leq 1$, $T(z)$ shrinks the disk and for $|z| \geq 1$ expands the disk. Let $S(z)$ be the corresponding rational matrix with $U(2)$ values for $|z| = 1$. The first thing to note is that

$$T(z)(0) = \frac{B}{D} = \delta$$

is a bounded rational function for $|z| \le 1$, hence it is analytic. Similarly because $T(z)$ shrinks the disk, $T(z)^{-1}$ expands the disk so that

$$T(z)^{-1}(\infty) = \begin{pmatrix} D & -B \\ -C & A \end{pmatrix}(\infty) = -\frac{D}{C}$$

is outside the unit disk for $|z| < 1$, hence

$$\alpha = -\frac{C}{D}$$

is analytic for $|z| \le 1$. By multiplying T by a rational scalar function ξ of modulus 1 on $|z| = 1$ we can arrange that β is analytic and non-vanishing for $|z| < 1$. It follows that T will be the transfer matrix of a 2-port iff γ has no poles for $|z| < 1$. Now because $\beta = \frac{1}{D}$ we know that D is analytic and non-vanishing for $|z| < 1$, so the same is true for B, C. But if $|\varepsilon| < 1$ we know

$$T(z)(\alpha) = \frac{A\varepsilon + B}{C\varepsilon + D}$$

is a rational function with values in the disk, hence without poles. Hence $A\varepsilon + B$ is analytic in $|z| < 1$ and so taking $\varepsilon \ne 0$ we see that A is without poles for $|z| < 1$. But then

$$\alpha = \frac{AD - BC}{D}$$

is without poles in $|z| < 1$. Thus $S(z)$ will be without poles for $|z| < 1$, and so T is a transfer matrix. So the original T will be a transfer matrix times a scalar function of modulus 1 on the circle.

May 14, 1978.

I want to under transfer matrices $T(z)$ whose only singularity is at $z=1$. Change from z to $\lambda = \frac{1}{i}\frac{z+1}{z-1}$, $z = \frac{\lambda - i}{\lambda + i}$, where $T(\lambda) \in GL_2(\mathbb{C}[\lambda])$ and we might as well suppose that $\det T = 1$.

Let us now [redacted] change from $P = \begin{pmatrix} 1 & \\ & -1 \end{pmatrix}$ whose P-negative disk is $|z|<1$ to the $\tilde{P}$ whose $\tilde{P}$-positive disk is the UHP:

$$\begin{pmatrix} x \\ 1 \end{pmatrix}^* \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix} \begin{pmatrix} x \\ 1 \end{pmatrix} = i\bar{x} - ix = \frac{1}{i}(x - \bar{x}) = 2\,\mathrm{Im}\, x$$

If we do this, then the basic positivity condition on a transfer matrix becomes

$$\mathrm{Im}\,\lambda \underset{\leq}{\geq} 0 \implies [redacted]\ T^*\tilde{P}T \underset{\leq}{\geq} \tilde{P} \implies T \text{ shrinks (expands) UHP.}$$

Also because $\det T = 1$ we know shrinking UHP is equivalent to $T^*\tilde{P}T \geq \tilde{P}$, and we know that

$$T^*\tilde{P}T = \tilde{P} \iff T \in SL_2(\mathbb{R})$$

So it is clear that if we make these changes, then a tranfer matrix becomes simply a polynomial Nevanlinna matrix.

So suppose $T(\lambda) = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$ is a polynomial Nevanlinna matrix. Then for any x in $\mathbb{R}$ such that $Cx + D \neq 0$ one has

$$\frac{Ax+B}{Cx+D} = p\lambda + c + \sum_1 \frac{r_i}{\lambda_i - \lambda} \qquad p \geq 0,\ r_i > 0 \qquad c \in \mathbb{R}$$

is in the Riesz-Herglotz class. (For $Cx + D = 0$ it is ∞.

interpreted as a real constant.)

Let's suppose x chosen generically on the real axis so that $\deg(Cx+D) = \max(\deg C, \deg D)$. If $p>0$ for this choice of x, then $\deg(Ax+B) = 1 + \deg(Cx+D) =$ [redacted] $r \geq 0$. Because $AD - BC = 1$, one has for these highest coeffs.

[redacted] $a_{r+1} d_r - b_{r+1} c_r = 0.$

[redacted]

This [redacted] implies that for some h

$$(A, B) - h(C, D) = (A_1, B_1)$$

has degree $\leq r$, whence necessarily $h = p$. Thus one sees that p is independent of x. We have

$$\frac{Ax+B}{Cx+D} = p\lambda + \frac{A_1 x + B_1}{Cx+D}$$

for any x. It follows from the Riesz-Herglotz expansion that $\begin{pmatrix} A_1 & B_1 \\ C & D \end{pmatrix}$ has to be a Nevanlinna matrix of lower degree.

~~So now it's clear from the fact that we can multiply by an element of $SL_2(\mathbb{R})$ until the~~ [redacted]

If in general $T(\lambda)$ has degree [redacted] $r \geq 1$, then we know its highest coeff is of rank 1, and hence by multiplying by an elt. of $SL_2(\mathbb{R})$ we can arrange the bottom row of this coeff. matrix to be zero so we get in the above case. Thus we see that any polynomial Nevanlinna matrix can be factored into a [redacted] product of elements of $SL_2(\mathbb{R})$ and matrices

$$\begin{pmatrix} 1 & p\lambda \\ 0 & 1 \end{pmatrix}$$

with $p > 0$.

May 15, 1978

$\exists$ relation between factorizing matrices and Nagao's thm.

$$SL_2(k[\lambda]) = SL_2(k) \underset{\begin{pmatrix} 1 & k \\ 0 & 1 \end{pmatrix}}{*} \begin{pmatrix} 1 & k[\lambda] \\ 0 & 1 \end{pmatrix}$$

How to simplify a matrix: $\begin{pmatrix} A & B \\ C & D \end{pmatrix} \in SL_2(k[\lambda])$. Multiply by $\theta \in SL_2(k)$ so that $\deg C < \deg A$. Then divide $\frac{A}{C} = q + \frac{A_1}{C}$,

$$\begin{pmatrix} 1 & -q \\ 0 & 1 \end{pmatrix}\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} A-qC & B-qD \\ C & D \end{pmatrix} = \begin{pmatrix} A_1 & B_1 \\ C & D \end{pmatrix}$$

until $\deg(A_1) < \deg C$. Now interchange rows and repeat. Process stops with $C = 0$, whence $D \in k^*$ and by doing another division one can arrange $B = 0$. What this does is to express any $T \in SL_2(k[\lambda])$ as a product

$$T = \theta \begin{pmatrix} 1 & q_1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} 1 & q_2 \\ 0 & 1 \end{pmatrix} \cdots\cdots \begin{pmatrix} 1 & q_n \\ 0 & 1 \end{pmatrix}\begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix}$$

and by changing the q's we can suppose the last factor is I.

[redacted]

The above decomposition corresponds to a continued fraction expansion

$$\theta^{-1}\left(\frac{A}{C}\right) = q_1 - \frac{1}{q_2 -} \; \frac{1}{q_3 -} \cdots\cdots \frac{1}{q_{n-1}}$$

<u>May 18, 1978</u> bound states.

Suppose we can factor a Schroedinger D.E.

$$L = -\frac{d^2}{dx^2} + q = \left(\frac{d}{dx} + p\right)\left(-\frac{d}{dx} + p\right) \quad \Rightarrow \quad q = p' + p^2$$

We've seen this implies the spectrum of L is ≥ 0 because

$$(Lu, u) = (X^* X u, u) = \|Xu\|^2 \qquad \text{where } X = -\frac{d}{dx} + p$$

Consider the operator

$$\tilde{L} = XX^* = \left(-\frac{d}{dx} + p\right)\left(\frac{d}{dx} + p\right) = -\frac{d^2}{dx^2} + \underbrace{p^2 - p'}_{q - 2p'}$$

[redacted] Given [redacted] a solution (local or [redacted] global) of

$$Lu = X^* X [redacted] u = \lambda u$$

[redacted] then $v = Xu$ is a solution of

$$\tilde{L} v = XX^* X u = \lambda X u = \lambda v$$

It follows that $L, \tilde{L}$ have the same spectrum except possibly for the eigenvalue $\lambda = 0$. In effect $Lu = 0$ with $u \in l^2 \Longrightarrow Xu = 0$ so $v = Xu = 0$.

<u>Example 1</u>: Recall that

$$L = -\frac{d^2}{dx^2} + x^2 - 1 = \left(\frac{d}{dx} - x\right)\left(-\frac{d}{dx} - x\right)$$

has discrete eigenvalues $2n$, n integer [redacted] ≥ 0. [redacted] Also

$$\tilde{L} = [redacted] \left(-\frac{d}{dx} - x\right)\left(\frac{d}{dx} - x\right) = -\frac{d^2}{dx^2} + x^2 + 1$$

has discrete eigenvalues $2(n+1)$, $n \geq 0$.

<u>Example 2:</u> $L = -\frac{d^2}{dx^2} + \beta^2$

The general solution of $Lu = 0$ is $u = c_1 e^{\beta x} + c_2 e^{-\beta x}$ and the possible p's are $\frac{u'}{u} = \frac{c\beta e^{\beta x} - \beta e^{-\beta x}}{c e^{\beta x} + e^{-\beta x}}$ where $c \geq 0$ in order that $u \neq 0$.

Such a u is not square-integrable. Note that

$$\tilde{L} = \left(-\frac{d}{dx} + p\right)\left(\frac{d}{dx} + p\right)$$

has in its kernel solutions v of $\left(\frac{d}{dx} + p\right) v = 0$

$$v = e^{-\int p\,dx}$$

$$= e^{-\int \frac{u'}{u} dx} = e^{-\ln u} = \frac{1}{u}$$

which is l^2.

$$p = \beta \frac{e^{\beta x} c - e^{-\beta x}}{e^{\beta x} c + e^{-\beta x}}$$

$$p^2 - p' = \beta^2 - 2p' = 2p^2 - \beta^2$$

$$= \beta^2 \left[\left(\frac{e^{\beta x} c - e^{-\beta x}}{e^{\beta x} c + e^{-\beta x}} \right)^2 2 - 1 \right]$$

$$= \beta^2 \frac{2e^{2\beta x} c^2 - 4c + 2e^{-2\beta x} - e^{2\beta x} c - 2c - e^{-2\beta x}}{\left(e^{\beta x} c + e^{-\beta x}\right)^2}$$

$$= \beta^2 \frac{e^{2\beta x} c^2 + 2c + e^{-2\beta x} - 8c}{\left(e^{\beta x} c + e^{-\beta x}\right)^2}$$

$$= \beta^2 - \frac{8c\beta^2}{\left(e^{\beta x} c + e^{-\beta x}\right)^2}$$

So we get this new family of potentials

$$-\frac{d^2}{dx^2} \quad \text{related to} \quad -\frac{d^2}{dx^2} - \frac{8c\beta^2}{(e^{\beta x}c + e^{-\beta x})^2}$$

depends on $c > 0$ which all have the same scattering data, but different norming constants - supposedly.

May 19, 1978:

Suppose $L = -\frac{d^2}{dx^2} + q$ where $q \to 0$ fast as $x \to \pm\infty$. It is not reasonable to suppose q has compact support, but perhaps $q \to 0$ exponentially gives a nice class to work with.

Let $-\beta^2$ be the smallest element of the spectrum of L. We know the spectrum of L consists of $0 \leq \lambda < \infty$ continuous spectrum and a finite set of discrete negative eigenvalues.

May 20, 1978

Given $L = -\frac{d^2}{dx^2} + q$ with q decaying fast as $x \to \pm\infty$ let $-\beta^2$ be $<$ the spectrum $\sigma(L)$. Let $f^{\pm}(x,k)$ be the solutions of $Lu = +k^2 u$ with

$$f^+(x,k) \sim e^{ikx} \qquad x \to +\infty$$

$$f^-(x,k) \sim e^{-ikx} \qquad x \to -\infty$$

These are defined for $\operatorname{Im} k \geq 0$. Let u be a non-zero solution of $Lu = -\beta^2 u$; it is proportional to

$$u = cf^+(x, i\beta) + f^-(x, i\beta).$$

Because $-\beta^2 < \sigma(L)$, one knows that $f^+(x, i\beta) > 0$ for all x and the same for $f^-(x, i\beta)$, so that for $c > 0$ $u > 0$ and

$$u(x) \sim \text{const } e^{\beta x} \qquad x \to +\infty$$

$$\sim \text{const } e^{-\beta x} \qquad x \to -\infty$$

where the constants can be worked out from the scattering as follows. Let

$$A(k)e^{ikx} + B(k)e^{-ikx} \longleftrightarrow e^{ikx}$$

describe the large x behavior. Then one knows $A(k)$ is analytic for $\operatorname{Im} k > 0$, because it gives the leading asymptotic behavior of $f^+(x,k)$:

$$f^+(x,k) \sim A(k)e^{ikx} \qquad \begin{matrix} \operatorname{Im} k > 0 \\ x \to -\infty \end{matrix}$$

Similarly

$$e^{-ikx} \longleftrightarrow -B(-k)e^{ikx} + A(k)e^{-ikx}$$

so

$$f^-(x,k) \sim A(k)e^{-ikx} \qquad x \to +\infty, \ \text{Im } k > 0.$$

Thus

$$u = cf^+(x,i\beta) + f^-(x,i\beta) \sim A(ik)e^{\beta x} \qquad x \to +\infty$$

$$\sim cA(ik)e^{-\beta x} \qquad x \to -\infty.$$

Put

$$p = \frac{u'}{u} \sim \begin{cases} \beta & x \to +\infty \\ -\beta & x \to -\infty \end{cases}$$

Since u satisfies $(L+\beta^2)u = 0$ we have

$$(L+\beta^2) = \left(\frac{d}{dx} + p\right)\left(-\frac{d}{dx} + p\right) \qquad \text{or} \qquad q + \beta^2 = p^2 + p'$$

Now consider the related operator $\tilde{L}$ defined by

$$\tilde{L} + \beta^2 = \left(-\frac{d}{dx} + p\right)\left(\frac{d}{dx} + p\right) \qquad \tilde{q} = p^2 - p' - \beta^2 = q - 2p'$$

A solution of $(\tilde{L} + \beta)v$ is

$$e^{-\int p\,dx} = e^{-\int \frac{u'}{u}dx} = e^{-\ln(u)} = \frac{1}{u}$$

and this is ℓ^2 because u grows exponentially. So $\tilde{L}$ has a bound state with eigenvalues $-\beta^2$. We have

$$\tilde{L}\left(-\frac{d}{dx} + p\right) = \left(-\frac{d}{dx} + p\right)L$$

which allows us to relate the spectrum of L and $\tilde{L}$.

As

$$\left(-\frac{d}{dx} + p\right)f^+(x,k) \sim \left(-\frac{d}{dx} + p\right)e^{ikx} = (-ik + \beta)e^{ikx} \qquad x \to +\infty$$

is an eigenfunction for $\tilde{L}$ with eigenvalues k^2, we have.

$$\tilde{f}^+(x,k) = \frac{1}{\beta - ik} f^+(x,k)$$

Now we can calculate the scattering: As $x \to -\infty$

$$\left(-\frac{d}{dx}+p\right)\frac{f^+(x,k)}{\beta-ik} \sim \frac{1}{\beta-ik}\left(-\frac{d}{dx}+p\right)\left[A(k)\, e^{ikx} + B(k) e^{-ikx}\right]$$

$$= A(k)\frac{(-ik-\beta)}{\beta-ik} e^{ikx} + \frac{(-\beta+ik)}{(\beta-ik)} B(k) e^{-ikx}$$

Thus

$$\boxed{\tilde{A}(k) = \frac{k-i\beta}{k+i\beta} A(k) \qquad \tilde{B}(k) = -B(k)}$$

So

$$\tilde{R}(k) = \frac{\tilde{B}}{\tilde{A}} = -\frac{k+i\beta}{k-i\beta} R(k)$$

$$\tilde{T}(k) = \frac{1}{\tilde{A}} = \frac{k+i\beta}{k-i\beta} T(k)$$

$$\tilde{R}_-(k) = -\frac{\tilde{B}(-k)}{\tilde{A}(k)} = \frac{B(-k)}{\frac{k-i\beta}{k+i\beta} A(k)} = -\frac{k+i\beta}{k-i\beta} R_-(k)$$

So that

$$\tilde{S}(k) = \frac{k+i\beta}{k-i\beta}\begin{pmatrix} -R(k) & T(k) \\ T(k) & -R_-(k) \end{pmatrix}$$

Now we want to reverse the procedure. Start with $\tilde{L} = -\frac{d^2}{dx^2} + \tilde{q}$ having a bound state with eigenvalue $-\beta^2$ at the bottom of the spectrum. Then I know that the corresponding eigenfunction v doesn't vanishes anywhere, so I can put

$$-p = \frac{v'}{v}$$

and I obtain the factorization

$$\tilde{L} + \beta^2 = \left(-\frac{d}{dx} + p\right)\left(\frac{d}{dx} + p\right)$$

Now in fact we know that up to changing v by a scalar

$$* \quad \begin{cases} v \sim e^{-\beta x} & x \to +\infty \\ v \sim \text{const} \cdot e^{\beta x} & x \to -\infty \end{cases}$$

so that again $p \sim \begin{cases} \beta & x \to +\infty \\ -\beta & x \to -\infty \end{cases}$.

Thus if we define L by

$$L + \beta^2 = \left(\frac{d}{dx} + p\right)\left(-\frac{d}{dx} + p\right) = -\frac{d^2}{dx^2} + q + \beta^2$$

then $q = \tilde{q} + 2p' \sim \tilde{q}$ will vanish fast at ∞. It's clear that $\tilde{q}$ is obtained from q and that solution of $Lu = -\beta^2 u$ which has the appropriate c

$$u = cf^+ + f^-$$

such that $v = \frac{1}{u}$ satisfies $(*)$. Somehow this c can be calculated from some sort of norming constant.

The point is to consider

$$\int |v|^2 dx = \int \frac{dx}{(cf^+ + f^-)^2}$$

as a function of c. The derivative is $-2\int \frac{f^+}{(cf^+ + f^-)^3} dx < 0$

and as $c \searrow 0$ it goes toward $\int \frac{dx}{(f^-)^2} = +\infty$,
and as $c \nearrow +\infty$ it approaches 0 by dominated (or monotone) convergence. Thus knowing $c \in (0, \infty)$ is the same as knowing $\|v\|^2$ where $v = \tilde{\underline{f}}(x, i\beta)$.

May 21, 1978

Bound states for the half-line case: Let's treat this using symmetry $x \mapsto -x$. Thus given q on $0 \leq x < \infty$ we can reflect it, i.e. extend it to an even potential on the line.

So consider an even potential $q(x)$ decaying fast as $x \to \infty$ and let $-\beta^2$ be a negative eigenvalue. Since we have the limit point case there can be only one independent eigenfunction which has to be either even or odd, [redacted] and which therefore satisfies $u(0) = 0$ or $u'(0) = 0$. Conversely suppose

$$\left(-\frac{d^2}{dx^2} + q\right) u = \lambda u \qquad 0 \leq x < \infty$$

$$u'(0) = 0$$

$$u(0) = 1 \qquad\qquad u \in L^2(0,\infty)$$

Then taking the solution of the same DE and initial condition over the line, we get an even function which will be square integrable. On the other hand the same DE with initial condition

$$u(0) = 0$$

$$u'(0) = 1$$

will yield an odd eigenfunction on the line.

Let's now consider the Schroedinger equation on the half-line with Dirichlet condition at $x = 0$:

$$u(0) = 0$$

Let $\phi(x,\lambda)$ be the solution of

$$\left(-\frac{d^2}{dx^2} + q\right)\phi = \lambda\phi$$

$$\phi(0,\lambda) = 0$$

$$\phi'(0,\lambda) = 1$$

This is defined for all x and is an odd function of x. Assuming q decays fast as $x \to +\infty$ we have

$$\phi(x, k^2) \sim A(k)e^{ikx} + A(-k)e^{-ikx} \qquad x \to +\infty$$

where $A(k)$ is defined at least for $\operatorname{Im} k \geq 0$ and is analytic for $\operatorname{Im} k > 0$ because e^{-ikx} grows as $x \to +\infty$. The bound states are the zeroes of $A(-k)$ in the UHP. The scattering operator is

$$S(k) = \frac{A(k)}{A(-k)}.$$

Let's see if we can remove bound states. Start the smallest negative eigenvalue $-\beta^2$. The corresponding eigenfunction $\phi(x, -\beta^2)$ vanishes at 0 hence $-\beta^2$ is not the minimum eigenvalue for $L = -\frac{d^2}{dx^2} + q$ on $\mathbb{R}$. Let $-\gamma^2$ be the minimum eigenvalue, and $u = f^+(x, i\gamma)$ the corresponding eigenfunction, and

$$p = \frac{u'}{u} \qquad u \sim e^{-\gamma x} \text{ so } p \sim -\gamma$$

as usual. We've seen u is even and > 0, so p is odd. We have

$$L + \gamma^2 = \left(\frac{d}{dx} + p\right)\left(-\frac{d}{dx} + p\right)$$

and we define $\tilde{L}$ by

$$\tilde{L} + \gamma^2 = \left(-\frac{d}{dx} + p\right)\left(\frac{d}{dx} + p\right). \qquad \tilde{q} + \gamma^2 = p^2 - p' \text{ still even}$$

$\tilde{L}$ has the same spectrum as L but with $-\gamma^2$ removed. So $-\beta^2$ occurs as the lowest eigenvalue for $\tilde{L}$. The corresponding eigenfunction for $\tilde{L}$ is

$$\left(-\frac{d}{dx}+p\right)\left(\phi(x,-\beta^2)\right)$$

and it satisfies the boundary condition

$$u'(0) = 0.$$

In fact we have for all λ

$$\tilde{\phi}(x,\lambda) = \left(\frac{d}{dx}-p\right)\phi(x,\lambda). \qquad \sim A(k)\left(\frac{d}{dx}+\gamma\right)e^{ikx} + .$$

As $p \sim -\gamma$ as $x \to +\infty$ we get

$$\tilde{A}(k) = (ik+\gamma)A(k)$$

so

$$\tilde{S}(k) = \frac{ik+\gamma}{-ik+\gamma}S(k) = \frac{-k+i\gamma}{k+i\gamma}S(k)$$

or it is better to write

$$S(k) = -\frac{k+i\gamma}{k-i\gamma}\tilde{S}(k)$$

to indicate the pole at $k = i\gamma$ belonging to the bound state?? (something wrong with this - evidently poles of $S(k)$ in UHP are also due to poles of $A(k)$ which can occur when q doesn't have compact support.)

~~Review: Given $L = -\frac{d^2}{dx^2}+q$ on $\mathbb{R}$ with q even decaying fast as $x \to \pm\infty$. Suppose we consider the Dirichlet~~

The Dirichlet boundary condition $u(0) = 0$ seems to be special. Consider the other cases:

Given $L = -\frac{d^2}{dx^2} + q$ on $0 \le x < \infty$ where q decays as $x \to +\infty$ and suppose [redacted] given a boundary condition $u'(0) = h u(0)$; let $\phi_\lambda(x)$ satisfy $L\phi_\lambda = \lambda \phi_\lambda$ and $\phi_\lambda(0) = 1$, $\phi_\lambda'(0) = h$. Suppose this problem has bound states and let $-\beta^2$ be the smallest eigenvalue[redacted]. [redacted] The eigenfn. is $\phi_{-\beta^2} = \psi_0$ and we know it doesn't vanish and also that it is asymptotic to [redacted] $e^{-\beta x}$ as $x \to +\infty$ times a scalar. Put

$$p = \frac{\psi_0'}{\psi_0} \sim -\beta \quad \text{as} \quad x \to +\infty$$

so that we get

$$L + \beta^2 = \left(\frac{d}{dx} + p\right)\left(-\frac{d}{dx} + p\right)$$

Then if

$$\tilde{L} + \beta^2 = \left(-\frac{d}{dx} + p\right)\left(\frac{d}{dx} + p\right) \quad \text{we have}$$

$$\tilde{L}\left(-\frac{d}{dx} + p\right) = \left(-\frac{d}{dx} + p\right) L$$

so that $\left(-\frac{d}{dx} + p\right)\phi_\lambda$ is an eigenfunction for $\tilde{L}$ with eigenvalue λ. Also

$$\left(-\frac{d}{dx} + p\right)\phi_\lambda \Big|_{x=0} = -\phi_\lambda'(0) + h\,\phi_\lambda(0) = 0$$

and

$$\frac{d}{dx}\left(-\frac{d}{dx} + p\right)\phi_\lambda \Big|_{x=0} = \underbrace{\left(\frac{d}{dx} + p\right)\left(-\frac{d}{dx} + p\right)}_{L+\beta^2}\phi_\lambda \Big|_{x=0}$$

$$= (\lambda + \beta^2)\,\phi_\lambda(0) = \lambda + \beta^2$$

Therefore suppose I the Dirichlet problem for $\tilde{L}$ and define $\tilde{\phi}_\lambda$ to be the eigenfunction with

$$\tilde{\phi}_\lambda(0)=0 \qquad \tilde{\phi}_\lambda'(0)=1$$

Then I have

$$\left(-\frac{d}{dx}+p\right)\phi_\lambda = (\lambda+\beta^2)\tilde{\phi}_\lambda$$

$$\left(\frac{d}{dx}+p\right)\tilde{\phi}_\lambda = \phi_\lambda$$

The latter follows from: $\left(\frac{d}{dx}+p\right)\tilde{\phi}_\lambda\Big|_{x=0} = 1$

$$\frac{d}{dx}\left(\frac{d}{dx}+p\right)\tilde{\phi}_\lambda\Big|_{x=0} = \left[-\left(-\frac{d}{dx}+p\right)\left(\frac{d}{dx}+p\right)\tilde{\phi}_\lambda + p\left(\frac{d}{dx}+p\right)\tilde{\phi}_\lambda\right]_{x=0}$$

$$= -(\lambda+\beta^2)\tilde{\phi}_\lambda(0)+p(0)\left(\tilde{\phi}_\lambda'(0)+p(0)\tilde{\phi}_\lambda(0)\right) = p(0) = h.$$

From the above formula one sees that if $\phi_\lambda \in L^2$ then so does $\tilde{\phi}_\lambda$ except for $\lambda=-\beta^2$, and that $\tilde{\phi}_\lambda \in L^2 \Rightarrow \phi_\lambda \in L^2$. Thus the spectra of $L, \tilde{L}$ with the given boundary conditions are the same except for $\lambda=-\beta^2$. $\phi_{-\beta^2}$ does $\in L^2$, but $\tilde{\phi}_{-\beta^2}$ doesn't for if it did we could integrate by parts to get

$$0 = \left((\tilde{L}+\beta^2)\tilde{\phi}_{-\beta^2}, \tilde{\phi}_{-\beta^2}\right) = \left\|\left(\frac{d}{dx}+p\right)\tilde{\phi}_{-\beta^2}\right\|^2$$

and any solution of $\left(\frac{d}{dx}+p\right)v=0$ is proportional to

$$\frac{1}{\psi_0} \sim e^{\beta x} \quad \text{as} \quad x\to\infty.$$

So therefore we transform L into $\tilde{L}$ which is a Dirichlet problem, killing the bound state $\lambda = -\beta^2$.

As for the scattering we have

$$\left(\frac{d}{dx} + \underset{-\beta}{p}\right)\left(\tilde{A}(k)e^{ikx} + \tilde{A}(-k)e^{-ikx}\right) \sim A(k)e^{ikx} + A(-k)e^{-ikx}$$

or

$(ik-\beta)\tilde{A}(k) = A(k)$	$-(ik+\beta)\tilde{A}(-k) = A(-k)$

and $A(-k) = 0$ for $k = i\beta$ as it should.

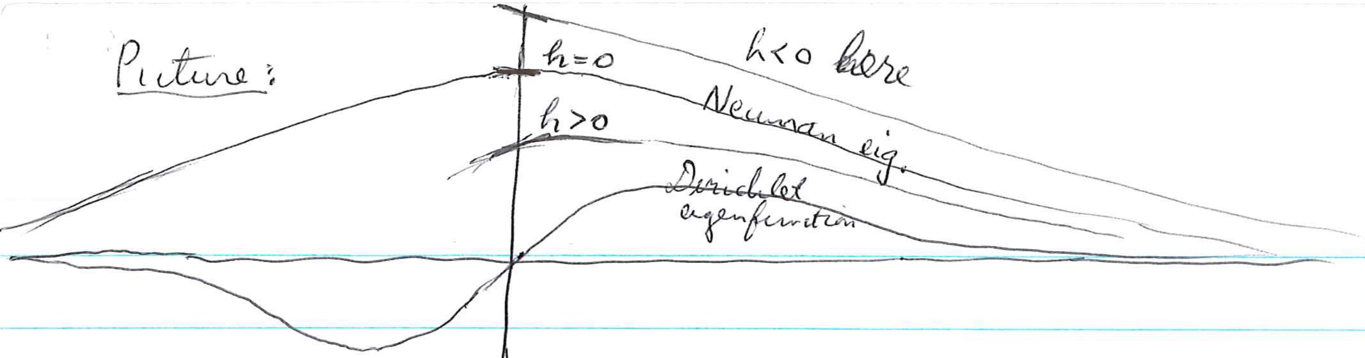
May 22, 1978: Next suppose we start with the Dirichlet boundary condition

$$L = -\frac{\partial^2}{\partial x^2} + q \qquad L\varphi_\lambda = \lambda\varphi_\lambda \qquad \varphi_\lambda(0) = 0,\ \varphi_\lambda'(0) = 1$$

whence if q is even in x, we have that φ_λ is an odd function of x. Suppose this problem has a bound state. Then the operator L on $\mathbb{R}$ has a lower eigenvalue $-\beta^2$ since $\varphi_\lambda(0) = 0$ and since the minimum eigenfunction doesn't vanish. If ψ_0 is the minimum eigenfunction normalized so that $\psi_0 \sim e^{-\beta x}$ as $x \to \infty$, we put as $p = \psi_0'/\psi_0$.

Actually let us choose $-\beta^2 <$ spectrum of L with Dirichlet data and take ψ to be the solution of $L\psi = -\beta^2\psi$ with $\psi \sim e^{-\beta x}$ and put $p = \psi'/\psi$. Put $p(0) = h$. The case $h = 0$ occurs when $-\beta^2$ is the minimum eigenvalue for L on the line with q even.

Picture:



So form $\tilde{L}$ as usual.

$$\left(-\frac{d}{dx}+p\right)\phi_\lambda \Big|_{x=0} = -1$$

$$\frac{d}{dx}\left(-\frac{d}{dx}+p\right)\phi_\lambda\Big|_{x=0} = \left[(L+\beta^2)\phi_\lambda - p\left(-\frac{d}{dx}+p\right)\phi_\lambda\right]_{x=0} = p(0) = h.$$

hence if we define boundary conditions for $\tilde{L}$ to be:

$$\tilde{\phi}_\lambda(0) = 1, \quad \tilde{\phi}_\lambda'(0) = -h$$

then we have

$$\left(-\frac{d}{dx}+p\right)\phi_\lambda = -\tilde{\phi}_\lambda$$

$$\left(\frac{d}{dx}+p\right)\tilde{\phi}_\lambda = -(\lambda+\beta^2)\phi_\lambda$$

We see that bound states for $L, \tilde{L}$ correspond except for $\lambda = -\beta^2$ where it might happen that $\tilde{\phi}_{-\beta^2} \in L^2$ but $\phi_{-\beta^2} \notin L^2$. But

$$\left(\frac{d}{dx}+p\right)\phi_{-\beta^2} = 0 \Rightarrow \phi_{-\beta^2} \text{ prop. to } \frac{1}{\psi} \sim e^{\beta x}$$

so $\phi_{-\beta^2} \notin L^2$. So the spectra for $L, \tilde{L}$ should be the same.

Scattering gives $\quad (-ik-\beta)A(k) = -\tilde{A}(k)$

or

$$\boxed{\begin{aligned}(ik+\beta)A(k) &= \tilde{A}(k)\\ (-ik+\beta)A(-k) &= \tilde{A}(-k)\end{aligned}}\;.$$

So there are no new zeroes for $A(-k)$ in the UHP.