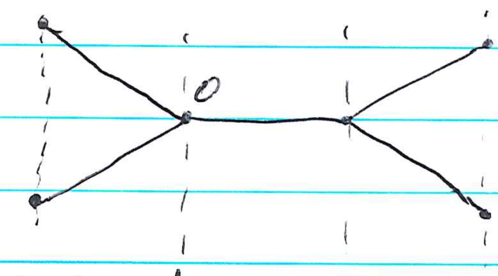
February 19, 1978:

Spectrum of Δ on the modular tree. Lets fix an edge and let K be the compact group of autos fixing the edge. The functions on the vertices which are K-invariant [redacted] form a closed subspace of $C_2^0(X)$ stable under Δ. The operator $\Delta - \lambda$ on this subspace has the form

$$1) \qquad (\Delta-\lambda)f(r) = f(r-1) + 2f(r+1) - (3+\lambda)f(r) \qquad r>0$$

where r denotes distance from the origin.



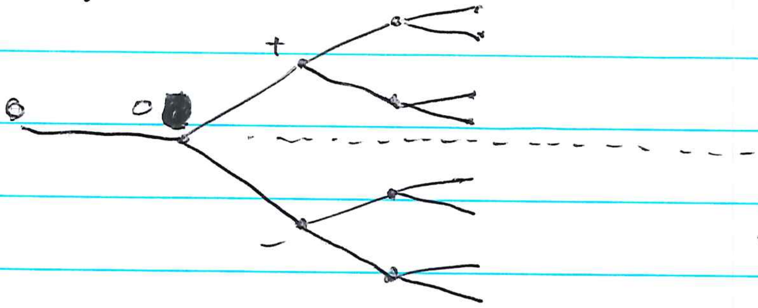
We know that the spectrum of Δ in this subspace is contained in $[-3-2\sqrt{2}, -3+2\sqrt{2}]$, because of our analysis of the difference operator 1).

So let us consider the orthogonal complement; [redacted] this consists of functions whose integral over the vertices with fixed r is zero. Notice that the orthogonal complement is an orthogonal direct sum of two subspaces stable under Δ namely those functions vanishing for $r \le 0$ and for $r \ge 0$. Let W denote [redacted] the subspace of $C_2^0(X)$ consisting of functions u with $u=0$ for $r \le 0$ and such that for all r,

$$\sum_{r(x)=r} u(x) = 0$$

Thus $u(1)=0$ and the two points with $r=2$ have [redacted] u values of opposite sign.

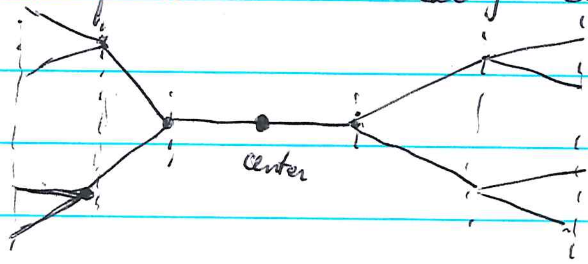
Let's now consider the subgroup of autos of the tree which fix $r \leq 2$ pointwise. Split off from W the invariant functions [redacted] which look like:



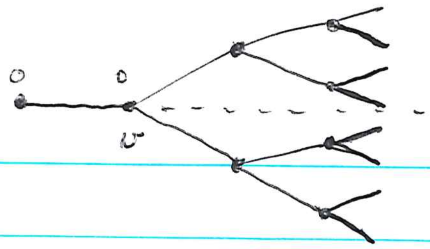
above dotted line the function is vertically constant. Flipping across the line change sign.

On this subspace we know Δ has its spectrum in good interval. The orthogonal complement [redacted] splits into 2 copies, above & below the dotted line, each of which is stable under Δ and isomorphic to W. So the process can be repeated indefinitely and one [redacted] obtains a direct sum decomposition of $C_2^0(X)$ into [redacted] Δ-invariant subspaces each of which is analyzable in terms of a radial difference equation. So it has to follow that the spectrum of Δ is $[-3-2\sqrt{2}, -3+2\sqrt{2}]$.

We get the following orthogonal decomposition of $C_2^0(X)$. First one has functions vertically constant

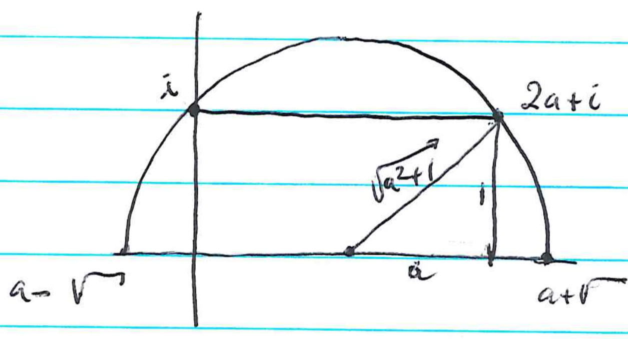


which we can split into those symmetric and anti-symmetric about the center. Then for each vertex [redacted] v we have functions zero except to the right of v, vertically constant above and below the dotted line and which change sign if one reflects across the dotted line.



Each of these pieces is Δ-invariant and [redacted] of multiplicity 1 with respect to Δ.

Consider a Ford and arclength [redacted] along the circle to geodesic length (i.e length of arc versus length of chord). Take Ford circle $y = 1$ and compute the two distances between i and $z = 2a + i$.



Since $ds^2 = \dfrac{dx^2 + dy^2}{y^2}$ the distance along the circle is $2a$. To compute the geodesic distance we use the fractional linear transformation sending

$$i \longmapsto i$$
$$a + \sqrt{a^2+1} \longmapsto \infty$$
$$+a - \sqrt{a^2+1} \longmapsto 0$$

which is

$$w = i\,\frac{z - a + \sqrt{a^2+1}}{-z + a + \sqrt{a^2+1}} \cdot \frac{-i + a + \sqrt{}}{i - a + \sqrt{}}$$

The image of $z = 2a + i$ is

$$w = i\,\frac{a + i + \sqrt{}}{-a - i + \sqrt{}}\;\frac{-i + a + \sqrt{a}}{i - a + \sqrt{}} = i\,\frac{(a + \sqrt{})^2 + 1}{(-a + \sqrt{})^2 + 1}$$

Since $(a + \sqrt{a^2+1})(-a + \sqrt{a^2+1}) = 1$ this is

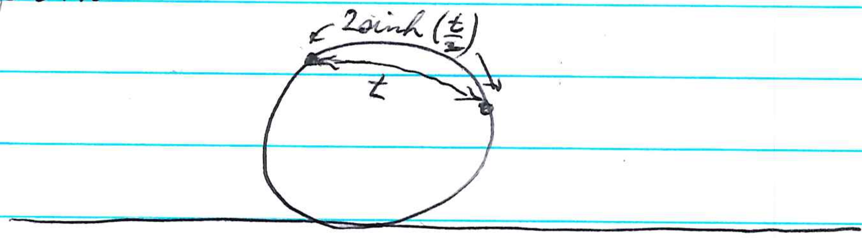
$$w = i\,(a + \sqrt{a^2+1})^2$$ whose non-Euclidean distance

from i is $\quad 2\log(a+\sqrt{a^2+1})$

So if we put $\quad t = 2\log(a+\sqrt{a^2+1}) \qquad e^{t/2} = a+\sqrt{a^2+1}$

or $\qquad a = \sinh\left(\frac{t}{2}\right) \qquad e^{-t/2} = -a+\sqrt{a^2+1}$

Therefore a point on the circle $y=1$ at distance t from i is of distance $2\sinh\left(\frac{t}{2}\right)$ along the circle from i:



February 20, 1978

Let $\Gamma = PSL_2(\mathbb{Z}) \subset G = PSL_2(\mathbb{R})$. Normally one looks at the representation of G on $L^2(\Gamma\backslash G)$. $L^2(\Gamma\backslash G)$ consists of functions on G which are left Γ-invariant. Of especially interest are K-invariant elements of $L^2(\Gamma\backslash G)$, that is, elements of $L^2(\Gamma\backslash G/K)$. However given a character $\chi : \Gamma \longrightarrow \mathbb{C}^*$ we can consider fns. f on G such that $f(\gamma^{-1}g) = \chi(\gamma)f(g)$. Recall that

$$\Gamma = \mathbb{Z}/2 * \mathbb{Z}/3$$

with generators S of order 2 and TS of order 3:

$$z \xmapsto{TS} -\frac{1}{z}+1 \xmapsto{TS} 1-\frac{1}{1-\frac{1}{z}} = 1-\frac{z}{z-1} = \frac{-1}{z-1} = \frac{1}{1-z} \xmapsto{TS} 1-(1-z) = z$$

Consequently there is an interesting character with

$$\chi(S) = -1 \qquad \chi(T) = -1 .$$

Thus we find ourselves looking at functions on the UHP satisfying

$$u(z+1) = -u(z)$$
$$u\left(-\tfrac{1}{z}\right) = -u(z)$$

with the inner product given by integrating over a fundamental domain.

<u>Question:</u> Is the spectrum of Δ in the above situation discrete?

The idea is that up in the cusp we can expand functions in (x,y)-eigenfunctions of the form

$$e^{i\xi x}\, u(y)$$

where to satisfy the boundary conditions we need

$$e^{i\xi} = -1 \qquad \text{or} \qquad \xi \in \pi + 2\pi\mathbb{Z}$$

Then $u(y)$ has to satisfy

$$\left[y^2\left(-\xi^2 + \frac{\partial^2}{\partial y^2}\right) + \frac{1}{4}\right] u = \lambda^2 u$$

and because necessarily $\xi \neq 0$ this leads to a u which either blows up exponentially or decays exponentially.

February 23, 1978

$$G = SL_2(\mathbb{R}), \quad K = SO(2), \quad G/K \xrightarrow{\sim} \mathcal{H}$$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \longmapsto \frac{ai+b}{ci+d}$$

Recall that G-bundles over $\mathcal{H}$ can be identified with K-spaces, in particular, a G-vector bundle over $\mathcal{H}$ can be identified with a representation of K, namely the fibre of the bundle over the origin O of $\mathcal{H}$: $O = i$. Since irreducible representations of K are desribed by characters

$$\rho(\theta) = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \longmapsto e^{im\theta}$$

for $m \in \mathbb{Z}$ any G-bundle over $\mathcal{H}$ is a direct sum of line bundles belonging to these characters.

<u>Tangent bundle</u>: Lets compute the effect of $\rho(\theta)$ on the tangent space to O

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}(i+\varepsilon) = \frac{ai+b+a\varepsilon}{ci+d+c\varepsilon} \cdot \frac{ci+d-c\varepsilon}{ci+d-c\varepsilon}$$

$$= \frac{(ai+b)(ci+d) + [(ai+b)(-c) + a(ci+d)]\varepsilon}{(ci+d)^2}$$

$$= \frac{ai+b}{ci+d} + \frac{1}{(ci+d)^2}\varepsilon$$

Hence

$$\rho(\theta)(i+\varepsilon) = \frac{i\cos\theta + \sin\theta}{-i\sin\theta + \cos\theta} + \frac{1}{(\cos\theta - i\sin\theta)^2}\varepsilon = i + e^{2i\theta}\varepsilon$$

and so the action of K on the tangent space at ε is given by the character $\rho(\theta) \mapsto e^{2i\theta}$.

We are going to be interested in the representations of G obtained by taking sections of a G-vector bundle over

$\mathcal{H}$. The most natural G-bundles such as $\Omega^1 = \Omega^{01} \oplus \Omega^{10}$ give right G-modules. For example a map $g : \mathcal{H} \to \mathcal{H}$ gives $g^* : g^*\Omega \to \Omega$ hence a g^* on $\Gamma(\mathcal{H}, \Omega)$.

Let L be a G-bundle such as Ω where one naturally has $\forall z \in \mathcal{H}$ maps on fibres

$$g^* : L(gz) \longrightarrow L(z)$$

so that the actual G-action is given by $(g^*)^{-1}$. Then the action of K on the fibre $L(0)$ is

$$k \cdot \lambda = (k^*)^{-1} \lambda$$

and we get the isom

$$G \times^K L(0) \longrightarrow L$$

$$(g, \lambda) \longmapsto (g^*)^{-1} \lambda \qquad L_{g0} \xrightarrow{g^*} L_0$$

Now I know that a section s of $G \times^K L(0)$ is representable in the form

$$s(g0) = \text{class of } (g, f(g))$$

where $f : G \longrightarrow L(0)$ satisfies $f(gk) = k^{-1} \cdot f(g) = k^* f(g)$. So I conclude that sections of L can be identified with $f : G \to L(0)$ satisfying

$$f(gk) = k^* f(g)$$

and that the G-action on sections is given by left mult:

$$(g \cdot f)(g_1) = (g^{-1*} f)(g_1) = f(g^{-1} g_1).$$

What does this means for Ω^{10} whose sections are of the form $\varphi(z)\,dz$; z denotes the obvious coordinate on $\mathcal{H}$. Here

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^* \varphi(z)\,dz = \varphi\left(\frac{az+b}{cz+d}\right)\frac{1}{(cz+d)^2}\,dz$$

so the action of K on the fibre over o is

$$\rho(\theta)^* . dz_{(i)} = (-\sin\theta\, i + \cos\theta)^{-2} dz_{(i)} = e^{2i\theta} dz(i),$$

hence Ω^{10} is the G-line bundle belonging to the character $\rho(\theta) \longmapsto e^{-2i\theta}$. A section of Ω^{10} is given by a map from G to $L(o)$ which we can write

$$g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \longmapsto f\begin{pmatrix} a & b \\ c & d \end{pmatrix} dz(i)$$

where f is a function on G with values in $\mathbb{C}$. The problem is to relate this description with the $\varphi(z)dz$ description. Using $G \times^K L(o) \longrightarrow L$, $g, \lambda \mapsto (g^*)^{-1}\lambda$ we see that $g, f(g)dz(i)$ goes to the section

$$s\left(\frac{ai+b}{ci+d}\right) = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}^* f\begin{pmatrix} a & b \\ c & d \end{pmatrix} dz(i)$$

$$= f\begin{pmatrix} a & b \\ c & d \end{pmatrix} \frac{dz}{(-cz+a)^2}\left(\frac{ai+b}{ci+d}\right)$$

this section gets evaluated at

$$= f\begin{pmatrix} a & b \\ c & d \end{pmatrix} \frac{1}{\left(-c\frac{ai+b}{ci+d}+a\right)^2}\, dz\left(\frac{ai+b}{ci+d}\right)$$

$$= f\begin{pmatrix} a & b \\ c & d \end{pmatrix} (ci+d)^2\, dz\left(\frac{ai+b}{ci+d}\right)$$

Thus the formula that a function f on the group with a form $\varphi(z)dz$ is

$$\varphi\left(\frac{ai+b}{ci+d}\right) dz\left(\frac{ai+b}{ci+d}\right) = f\begin{pmatrix} a & b \\ c & d\end{pmatrix}(ci+d)^2\, dz\left(\frac{ai+b}{ci+d}\right)$$

or simply

$$f\begin{pmatrix} a & b \\ c & d\end{pmatrix} = (ci+d)^{-2}\varphi\left(\frac{ai+b}{ci+d}\right)$$

By taking tensor products we can shift to the bundle whose sections are $\varphi(z)dz^{k/2}$ with

$$\begin{pmatrix} a & b \\ c & d\end{pmatrix}^* dz^{k/2} = (cz+d)^{-k} dz^{k/2}$$

(defined for $G = SL_2$ if $k \in \mathbb{Z}$ and for $G = PSL_2$ if $k \in 2\mathbb{Z}$). So it's clear we have the dictionary:

~~Forms $\varphi(z)\,dz^{k/2}$ on $\mathcal{H}$ can be identified with functions f on G by the formula:~~

$$\cancel{f\begin{pmatrix} a & b \\ c & d\end{pmatrix} = (ci+d)^{-k}\varphi\left(\frac{ai+b}{ci+d}\right)}$$

Forms $\varphi(z)\,dz^{k/2}$ on $\mathcal{H}$ can be identified with $\mathbb{C}$-valued functions f on $G = SL_2(\mathbb{R})$ satisfying

$$f(g\rho(\theta)) = e^{ik\theta} f(g)$$

by the formula

$$f\begin{pmatrix} a & b \\ c & d\end{pmatrix} = (ci+d)^{-k}\varphi\left(\frac{ai+b}{ci+d}\right).$$

In other words given the form $\varphi(z)dz^{k/2}$ the corresponding f is

$$f(g) = \cancel{\frac{g^*(\varphi(z)dz^{k/2})}{\varphi(z)dz^{k/2}}} = \varphi(gi)\,\frac{g^*(dz^{k/2})}{dz^{k/2}}(i)$$

So now we see that we can now describe functions on G in terms of forms on $\mathcal{H}$, namely a general function ψ on G can be expanded in Fourier series with respect to the right K-action. Then ψ can be represented as a sum of forms

$$\sum_{k \in \mathbb{Z}} \varphi_k(z)\, dz^{k/2}$$

or precisely

$$\psi\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \sum_k \varphi_k\left(\frac{ai+b}{ci+d}\right)(ci+d)^{-k} \blacksquare$$

The way to remember the correspondence between functions and forms is as follows: [redacted] The form dz^k [redacted] becomes the function

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \longmapsto (ci+d)^{-2k}.$$

and functions on $\mathcal{H}$ like $\varphi(z)$ are pulled-back by $g \mapsto gi$. Thus $\varphi(z)\,dz^k$ becomes the function

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \longmapsto \varphi\left(\frac{ai+b}{ci+d}\right)(ci+d)^{-2k}$$

The idea here seems to be that we have fixed a non-vanishing section $\omega = dz^k$ and then we associate to it the function

$$\frac{g^*\omega}{\omega}(i) = (ci+d)^{-2k}$$

on the group. [redacted] It seems to be better to think of sections of a bundle rather than functions on the group because the latter

seem to depend on the choice of origin. [redacted]
[redacted] Maybe not. The bundle picture has a geometric content, but from the viewpoint of representations, maybe one only wants the modules, not its structure as induced module.

Note that the G-bundles Ω^{01} and Ω^{10} are dual to each other.

Review: I can now identify forms $\varphi(z)dz^k$ on $\mathcal{H}$ with functions f on G satisfying

$$f(g\rho(\theta)) = e^{2ik\theta} f(g)$$

as follows:

$$dz^k \longmapsto \left(\begin{pmatrix} a & b \\ c & d \end{pmatrix} \longmapsto (ci+d)^{-2k}\right) \quad \text{so}$$

$$\varphi\, dz^k \longmapsto \left(\begin{pmatrix} a & b \\ c & d \end{pmatrix} \longmapsto \varphi\left(\frac{ai+b}{ci+d}\right)(ci+d)^{-2k}\right)$$

In the reverse direction given f one sees that $f\begin{pmatrix} a & b \\ c & d \end{pmatrix}(ci+d)^{2k}$ is [redacted] a right [redacted] K-invariant function hence we get a function φ on $\mathcal{H}$:

$$\varphi\left(\frac{ai+b}{ci+d}\right) = f\begin{pmatrix} a & b \\ c & d \end{pmatrix}(ci+d)^{2k}$$

[redacted] Let's now compute the L^2-norm of f:

$$\int_G |f(g)|^2 d\mu = \int_G \left|\varphi\left(\frac{ai+b}{ci+d}\right)\right|^2 |ci+d|^{-4k} d\mu$$

Note that if $z = \dfrac{ai+b}{ci+d}$, then $y = \operatorname{Im} z = \dfrac{1}{|ci+d|^2}$

hence [redacted]

$$\|f\|^2 = \int_G |\varphi(z)|^2 y^{2k}\, d\mu = \int_{\mathcal{H}} |\varphi(z)|^2 y^{2k} \frac{dx\,dy}{y^2}$$

where we use that $d\mu = \frac{dx\,dy}{y^2} \cdot \frac{d\theta}{2\pi}$ is the Haar measure

The above formula shows that if f corresponds to $\varphi(z)dz^k$, then $\|f\|^2$ is $\|\varphi\, dz^k\|^2$ where as usual:

$$\|dz\| = y$$

gives the hermitian metric on T^{10}.

Discrete series representations of $SL_2(\mathbb{R})$: These can be realized as the Hilbert space of square-integrable holomorphic forms $\varphi(z)\,dz^k$ for $k = \frac{m}{2}$, m an integer ≥ 2, according to Serge's book.

Restrict to $PSL_2(\mathbb{R})$ so that k has to be an integer ≥ 1.

Recall that we looked at the representation of $PSL_2(\mathbb{R})$ given by harmonic functions mod constants with the Dirichlet norm. This representation splits into the direct sum of holomorphic functions and anti-holomorphic functions. Now if f is holomorphic its Dirichlet norm is

$$\iint \left\{ \left|\frac{\partial f}{\partial x}\right|^2 + \left|\frac{\partial f}{\partial y}\right|^2 \right\} dx\,dy = 2\int \left|\frac{df}{dz}\right|^2 dx\,dy$$

$$= 2\int \left\| \frac{df}{dz}\, dz \right\|^2 \frac{dx\,dy}{y^2}$$

Hence the representation given by holomorphic functions mod $\mathbb{C}$ in the Dirichlet norm is equivalent to the [redacted] representation using holomorphic differentials. Holomorphic differentials give the discrete series representation with lowest weight [redacted] $= 2$. ($dz^{k/2}$ signifies weight k, in particular weight 1 is arithmetically [redacted] very interesting.)

Question: Consider the space of l^2-cycles on [redacted] the modular tree. This is a representation of the modular group. Does it split because of the orientations [redacted] at each vertex?

If u is a complex-valued harmonic function in an open subset of $\mathbb{C}$, then [redacted] a conjugate harmonic function v is one such that

1) $\begin{cases} u+iv & \text{is holomorphic} \\ u-iv & \text{is anti-holomorphic} \end{cases}$

Thus

$$\frac{\partial}{\partial x}(u+iv) = \frac{1}{i}\frac{\partial}{\partial y}(u+iv)$$

$$\frac{\partial}{\partial x}(u-iv) = i\frac{\partial}{\partial y}(u-iv)$$

so

2) $$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \text{and} \quad -\frac{\partial v}{\partial x} = \frac{\partial u}{\partial y}$$

Conversely 2) $\Rightarrow$ 1). [redacted] These equations determine v up to an additive constant.

Atiyah's ℓ^2-index thm. Let Γ act freely on a manifold $\tilde X$ with $X = \tilde X/\Gamma$ compact and let $D: E \to F$ be an elliptic operator on X with inverse image $\tilde D : \tilde E \to \tilde F$ on $\tilde X$. Define $\dim_\Gamma(\operatorname{Ker}\tilde D)$ as follows. Choose metrics and E and X, and let f_i be an orthonormal basis for $\operatorname{Ker}\tilde D$, so that

$$k(x,y) = \sum_i f_i(x)\bar f_i(y)$$

$\tilde D$ on $\Gamma_{\ell^2}(E)$

is the kernel giving orthogonal projection onto $\operatorname{Ker}\tilde D$. Then [redacted] $\operatorname{tr}(k(x,x))$ is a function on $\tilde X$ invariant under Γ so we can put

$$\dim_\Gamma(\operatorname{Ker}\tilde D) = \int_X \operatorname{tr} k(x,x)$$

Then the ℓ^2-index theorem states that if we put

$$\operatorname{index}_\Gamma(\tilde D) = \dim_\Gamma \operatorname{Ker}\tilde D - \dim_\Gamma \operatorname{Coker}\tilde D$$

then we have

$$\boxed{\operatorname{index}_\Gamma(\tilde D) = \operatorname{index}(D)}$$

Simple application: $\operatorname{index}(D) > 0 \Rightarrow \ell^2\text{-}\operatorname{Ker}\tilde D \neq 0$.

For example let $X =$ [redacted] $\Gamma\backslash\mathcal{H}$ where X is a Riemann surface of genus $g > 1$. Take D to be the $\bar\partial$ operator for the line bundle Ω whose index is

$$h^0(\Omega) - h^1(\Omega) = g-1 > 0$$

It follows that there exist square-integrable holomorphic differentials on $\mathcal{H}$. By generalizing this idea Atiyah and Schmidt construct all the discrete series.

Example: $X = \Gamma \backslash \mathbb{C}$ an elliptic curve. Take D to be $\bar{\partial}$ for a line bundle, say the line bundle given by a simple pole at the origin (of X). Holomorphic sections of $\tilde{E}$ are meromorphic functions on $\mathbb{C}$ with at most simple poles at points of the lattice Γ. Is $\frac{1}{z}$ a square-integrable section? The metric put on the bundle has to allow poles at points of Γ but if we excise small circles around Γ we see that $\frac{1}{z}$ will be ℓ^2 if

$$\int \frac{1}{r^2}\, r\,dr\,d\theta < \infty \qquad \text{NO}$$

But this doesn't work, however

$$\frac{1}{z-\gamma_1} - \frac{1}{z-\gamma_2}$$

for different lattice points γ_1, γ_2 will be in ℓ^2.

February 26, 1978

Analogues of invariant differential operators on the modular tree are what?

What might be the analogue of a G-vector bundle E over G/K? One should want to assign to each vertex a vector space and possibly also something to each edge.

Recall Γ acting on the tree X has strict fundamental domain consisting of an edge with stabilizer $\mathbb{Z}/2\mathbb{Z}$ at one end and $\mathbb{Z}/3\mathbb{Z}$ at the other. Hence the sections of E is a sum of representations induced from the subgroups 1, $\mathbb{Z}/2 = \langle S \rangle$, $\mathbb{Z}/3\mathbb{Z} = \langle TS \rangle$. Any Γ-invariant linear operator from $\Gamma(E)$ to $\Gamma(F)$ is like a map between induced representations, hence has to do with double cosets. However differential operators are local, hence one expects only nearest neighbor relations in the analogue of $D : E \longrightarrow F$.

February 28, 1978:

Clean up your understanding of de Branges theory:

Motivation for definition of Nevanlinna matrix: Consider a Schroedinger equation

$$1) \qquad -u'' + qu = \lambda u$$

on a finite interval $0 \le x \le l$. We can write this as a first order system

$$\frac{d}{dx}\begin{pmatrix} u \\ u' \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ q-\lambda & 0 \end{pmatrix}\begin{pmatrix} u \\ u' \end{pmatrix}$$

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\frac{d}{dx}\begin{pmatrix} u \\ u' \end{pmatrix} = \begin{pmatrix} \lambda - q & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} u \\ u' \end{pmatrix}$$

or

$$\left\{\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\frac{d}{dx} + \begin{pmatrix} q & 0 \\ 0 & 0 \end{pmatrix}\right\}\begin{pmatrix} u \\ u' \end{pmatrix} = \lambda\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} u \\ u' \end{pmatrix}$$

This last DE is a special case of a DE of the form

$$2) \qquad Lu = \left\{P\frac{d}{dx} + Q\right\}u = \lambda R u$$

where now $u = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$ and $P = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ and Q, R are real symmetric matrices with $R \geq 0$.

Let $M(x,\lambda)$ be the matrix whose columns are the solutions of 2) with the values $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ at $x=0$. Thus $M(x,\lambda)\,v$ is the solution of 2) with $u(0)=v$. One knows that

$$\frac{dM}{dx} = P^{-1}(-Q+\lambda R)M$$

and
$$\frac{d}{dx}(\det M) = \operatorname{tr}\left(-P^{-1}Q+\lambda P^{-1}R\right)\cdot \det M .$$

Since Q, R are real symmetric matrices, $P^{-1}Q$, $P^{-1}R$ have trace 0, so we conclude

$$\det(M) = 1 .$$

This is a first property of M. To get another
Recall Green's formula:

$$v^* L u - (Lv)^* u = v^*\left(P\frac{du}{dx} + Qu\right) - \left(P\frac{dv}{dx}+Qv\right)^* u$$

$$= v^* P \frac{du}{dx} + v^* Q u - \frac{dv^*}{dx}P^* u - v^* Q^* u$$

$$= \frac{d}{dx}\left[v^* P u\right]$$

If u is a solution of 2), this implies

$$\frac{d}{dx}\left(\frac{1}{i}u^* P u\right) = \frac{1}{i}\left(u^* L u - (Lu)^* u\right) = \frac{1}{i}\left(u^* \lambda R u - (\lambda R u)^* u\right)$$

$$= 2(\operatorname{Im}\lambda)\; u^* R u .$$

Since $R \geq 0$,
This shows that the quantity

$$\frac{1}{i}(u^*Pu) = \frac{1}{i}\begin{pmatrix}\bar{u}_1 & \bar{u}_2\end{pmatrix}\begin{pmatrix}-u_2 \\ u_1\end{pmatrix} = \frac{1}{i}\left(u_1\bar{u}_2 - \bar{u}_1 u_2\right)$$

$=$ [redacted] $\frac{1}{i}\left(\frac{u_1}{u_2} - \frac{\bar{u}_1}{\bar{u}_2}\right)|u_2|^2$

$$= 2\,\mathrm{Im}\left(\frac{u_1}{u_2}\right)\cdot |u_2|^2$$

is an increasing (resp. decreasing function of x) if $\mathrm{Im}\,\lambda \geq 0$ (resp. ≤ 0). Consequently if u has the property that $\frac{u_1}{u_2}$ [redacted] is on $\mathbb{R}\cup\infty$ at $x=0$ [redacted] then for [redacted] $x \geq 0$ one has $\frac{u_1}{u_2}$ is in the closed upper-half-plane if $\mathrm{Im}\,\lambda \geq 0$.

So we see that if we fix an $x \geq 0$, then $M(x,\lambda)$ has the following properties

$$M(x,\lambda) = \begin{pmatrix} A(\lambda) & B(\lambda) \\ C(\lambda) & D(\lambda)\end{pmatrix}$$

i) its entries are entire functions of λ with real values on the real axis.

ii) $\det M = AD - BC = 1$

iii) For any $t \in \mathbb{R}\cup\infty$ the meromorphic function

$m(\lambda) =$ [redacted] $\dfrac{A(\lambda)t + B(\lambda)}{C(\lambda)t + D(\lambda)}$

has $\mathrm{Im}\, m(\lambda) \geq 0$ for $\mathrm{Im}\,\lambda > 0$

$= 0$ " $= 0$

≤ 0 " < 0

<u>Definition</u>: A Nevanlinna matrix is a 2x2 matrix of entire functions having the above properties.

Condition iii) says that the fractional linear transformation associated to M shrinks the closed UHP for $\operatorname{Im}\lambda \geq 0$ and it shrinks the closed LHP for $\operatorname{Im}\lambda \geq 0$.

Let $\mathcal{N}$ be the set of Nevanlinna matrices. It is a monoid under multiplication containing $SL_2(\mathbb{R})$ as a subgroup of invertible elements.

Let $M \in \mathcal{N}$ be such that $M^{-1} \in \mathcal{N}$. Then $M(\lambda)$ preserves the real axis for all λ, hence for t real

$$m(\lambda) = \frac{A(\lambda)t + B(\lambda)}{C(\lambda)t + D(\lambda)}$$

is [redacted] constant by the open mapping principle. It follows that $M(\lambda)$ is constant. Thus $SL_2(\mathbb{R})$ is exactly the subgroup [redacted] of invertible elements of $\mathcal{N}$.

[redacted] Let $\mathcal{N}_0$ be the submonoid consisting of $M(\lambda)$ with $M(0) = I$. Then clearly

$$\mathcal{N} \xleftarrow{\sim} [redacted]\ SL_2(\mathbb{R}) \times \mathcal{N}_0$$

$$M \longmapsto M(0) \times M(0)^{-1}M$$

so that $\mathcal{N}$ is the semi-direct product of $\mathcal{N}_0$ and $SL_2(\mathbb{R})$ in the obvious sense.

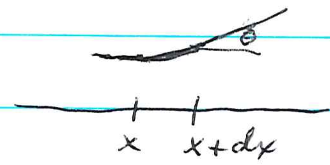
Look at $\mathcal{N}_0$. According to de Branges [redacted] any element [redacted] M of $\mathcal{N}_0$ is joined to the identity by a unique path (unparameterized). One should maybe regard $\mathcal{N}_0$ as a poset by saying $M' \leq M$ if

$$M'' M' = M$$

for some M''. Then according to deB [redacted] the set of $M' \leq M$ is totally-ordered. It's as if N_0 were some kind of continuous tree; or free monoid.

<u>March 2, 1978</u>

Review strings:



$$\sin\theta \approx \frac{\partial u}{\partial x}(x+\Delta x)$$

$$\rho\, dx \cdot \frac{\partial^2 u}{\partial t^2} = \left(T\frac{\partial u}{\partial x}\right)(x+dx) - \left(T\frac{\partial u}{\partial x}\right)(x)$$

$$\rho \frac{\partial^2 u}{\partial t^2} = \frac{\partial}{\partial x}\left(T\frac{\partial u}{\partial x}\right)$$ equation of motion for a string with tension T and density ρ.

Suppose $T=1$ and separate out time $u(x,t) = u(x)e^{i\omega t}$:

$$-\omega^2 \rho u = \frac{d^2 u}{dx^2}$$

We can transform this in various ways

$$\frac{d}{dx}\begin{pmatrix} u \\ \frac{u'}{\omega} \end{pmatrix} = \begin{pmatrix} 0 & \omega \\ -\omega\rho & 0 \end{pmatrix}\begin{pmatrix} u \\ u'/\omega \end{pmatrix}$$

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\frac{d}{dx}\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \omega\begin{pmatrix} \rho & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$$

Introduce time parameter which makes the matrix on the right of determinant 1. $dt = \rho^{1/2}dx$

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\frac{d}{dt}\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \omega\begin{pmatrix} \rho^{1/2} & 0 \\ 0 & \rho^{-1/2} \end{pmatrix}\begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$$

[redacted] Put $\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} \rho^{-1/4} v_1 \\ \rho^{1/4} v_2 \end{pmatrix}$

$$-\frac{d}{dt}\left(\rho^{1/4}v_2\right) = \omega\rho^{1/2}\rho^{-1/4}v_1 \quad \text{or} \quad \rho^{-1/4}\frac{d}{dt}\rho^{1/4}v_2 = -\omega v_1$$

$$\frac{d}{dt}\left(\rho^{-1/4}v_1\right) = \omega\rho^{-1/2}\rho^{1/4}v_1 \qquad \rho^{1/4}\frac{d}{dt}\rho^{-1/4}v_1 = \omega v_2$$

which gives

$$\rho^{-1/4}\frac{d}{dt}\rho^{+1/4}\,\rho^{+1/4}\frac{d}{dt}\rho^{-1/4}\,v_1 = -\omega^2 v_1$$

or

$$\left(\frac{d}{dt}+(\log\rho^{1/4})'\right)\left(\frac{d}{dt}-(\log\rho^{1/4})'\right)v_1 = -\omega^2 v_1$$

Put $\varphi = \rho^{1/4}$. Then the above becomes

$$\left(\frac{d^2}{dt^2}-q\right)v_1 = -\omega^2 v_1$$

where

$$q = \left(\frac{\varphi'}{\varphi}\right)^2 + \frac{d}{dt}\left(\frac{\varphi'}{\varphi}\right) = \frac{\varphi''}{\varphi}$$

I am primarily interested in strings having discrete spectrum. It suffices for this to happen that $q \uparrow \infty$ as $t \to +\infty$. Hence the density ρ has to grow fast enough so that its fourth root is very convex. For example if $\varphi(t) = t^n$, then

$$q = \frac{n(n-1)}{t^2}$$

doesn't work. Nor does $\varphi(t) = e^{at}$ for

$$q = \frac{\varphi''}{\varphi} = a$$

March 4, 1978

Let's consider a Dirac system with δ-function potential.

$$\frac{d}{dx} u = \begin{pmatrix} i\lambda & \bar{p} \\ p & -i\lambda \end{pmatrix} u$$

Suppose $p = c\delta(0)$. I recall that this means that we have the equations

$$\frac{d}{dx} u = \begin{pmatrix} i\lambda & 0 \\ 0 & -i\lambda \end{pmatrix} u$$

on either side of zero glued together by boundary conditions relating their values at $x=0$. These boundary conditions take the form of a frequency-independent transfer matrix:

$$u(0-) = \begin{pmatrix} \alpha & \beta \\ \bar{\beta} & \bar{\alpha} \end{pmatrix} u(0+) \qquad |\alpha|^2 - |\beta|^2 = 1.$$

The transfer matrix is essentially

$$\begin{pmatrix} \alpha & \beta \\ \bar{\beta} & \bar{\alpha} \end{pmatrix}^{-1} = \exp\begin{pmatrix} 0 & \bar{c} \\ c & 0 \end{pmatrix}. \qquad (\Rightarrow \alpha \text{ real } > 0)$$

We wish to set this up using scattering concepts. For $x<0$ we have

$$u(x) = \begin{pmatrix} a_1 e^{i\lambda x} \\ b_1 e^{-i\lambda x} \end{pmatrix}$$

where a_1, b_1 are constants. We think of $a_1 e^{i\lambda x}$ as a wave moving to the right, and $b_1 e^{-i\lambda x}$ as one moving to the left. For $x>0$ we have

$$u(x) = \begin{pmatrix} b_2 e^{i\lambda x} \\ a_2 e^{-i\lambda x} \end{pmatrix}.$$

The interface condition is

$$\begin{pmatrix} a_1 \\ b_1 \end{pmatrix} = \begin{pmatrix} \alpha & \beta \\ \bar{\beta} & \bar{\alpha} \end{pmatrix} \begin{pmatrix} b_2 \\ a_2 \end{pmatrix}$$

and we want to express this in terms of the independent variables a_1, a_2.

$$b_2 = \frac{1}{\alpha}(a_1 - \beta a_2)$$

$$b_1 = \frac{\bar\beta}{\alpha}(a_1 - \beta a_2) + \bar\alpha a_2 = \frac{\bar\beta}{\alpha} a_1 + \left(\bar\alpha - \frac{|\beta|^2}{\alpha}\right) a_2$$

$$\begin{pmatrix} b_1 \\ b_2 \end{pmatrix} = \begin{pmatrix} \frac{\bar\beta}{\alpha} & \frac{1}{\alpha} \\ \frac{1}{\alpha} & -\frac{\beta}{\alpha} \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}$$

(In practice $\alpha > 0$, so the above procedure of replacing transfer by scattering matrices can be visualized as

$$\begin{pmatrix} \frac{1}{\sqrt{1-|h|^2}} & \frac{h}{\sqrt{1-|h|^2}} \\ \frac{\bar h}{\sqrt{1-|h|^2}} & \frac{1}{\sqrt{1-|h|^2}} \end{pmatrix} \longmapsto \begin{pmatrix} \bar h & \sqrt{1-|h|^2} \\ \sqrt{1-|h|^2} & -h \end{pmatrix}$$
).

Time-dependent theory. Here we want to consider the wave equation:

$$\begin{pmatrix} \frac{d}{dx} & -\bar p \\ p & -\frac{d}{dx} \end{pmatrix} \psi = -\frac{\partial \psi}{\partial t}$$

whose single frequency solutions are $\psi = e^{-i\lambda t} u(x)$ with u satisfying the preceding Dirac equation. Thus away from zero one has solutions of the form

$$\psi(x,t) = \begin{pmatrix} f(x-t) \\ g(x+t) \end{pmatrix}$$

and for any t they must agree with the interface conditions at $x=0$. Thus if

$$\psi = \begin{pmatrix} f_1(x-t) \\ g_1(x+t) \end{pmatrix} \qquad x<0$$

$$= \begin{pmatrix} f_2(x-t) \\ g_2(x+t) \end{pmatrix} \qquad x>0$$

then we must have

$$\begin{pmatrix} f_1(-t) \\ g_1(t) \end{pmatrix} = \begin{pmatrix} \alpha & \beta \\ \bar\beta & \text{[redacted]}\ \bar\alpha \end{pmatrix} \begin{pmatrix} f_2(-t) \\ g_2(t) \end{pmatrix}$$

for all t. Maybe the [redacted] notation to use is this

$$\psi(x,t) = \begin{pmatrix} a_1(x-t) \\ b_1(x+t) \end{pmatrix} \qquad x<0$$

$$= \begin{pmatrix} b_2(x-t) \\ a_2(x+t) \end{pmatrix} \qquad x>0$$

so that the interface condition is

$$\begin{pmatrix} a_1(-t) \\ b_1(t) \end{pmatrix} = \begin{pmatrix} \alpha & \beta \\ \bar\beta & \bar\alpha \end{pmatrix} \begin{pmatrix} b_2(-t) \\ a_2(t) \end{pmatrix}$$

hence

$$b_1(t) = \frac{\bar\beta}{\alpha} a_1(-t) + \frac{1}{\alpha} a_2(t)$$

$$b_2(-t) = \frac{1}{\alpha} a_1(-t) - \frac{\beta}{\alpha} a_2(t)$$

so

[redacted]

$$b_1(x) = \frac{\bar\beta}{\alpha} a_1(-x) + \frac{1}{\alpha} a_2(x)$$

$$b_2(x) = \frac{1}{\alpha} a_1(x) - \frac{\beta}{\alpha} a_2(-x)$$

Compute the transfer matrix for an interface with matrix $\begin{pmatrix} \alpha & \beta \\ \bar{\beta} & \bar{\alpha} \end{pmatrix}$ placed at $x = x_0$

Solution $\begin{pmatrix} a_1 e^{i\lambda x} \\ b_1 e^{-i\lambda x} \end{pmatrix}$ $x < x_0$, $\begin{pmatrix} b_2 e^{i\lambda x} \\ a_2 e^{-i\lambda x} \end{pmatrix}$ $x > x_0$

$$\begin{pmatrix} a_1 e^{i\lambda x_0} \\ b_1 e^{-i\lambda x_0} \end{pmatrix} = \begin{pmatrix} \alpha & \beta \\ \bar{\beta} & \bar{\alpha} \end{pmatrix} \begin{pmatrix} b_2 e^{i\lambda x_0} \\ a_2 e^{-i\lambda x_0} \end{pmatrix}$$

or

$$\begin{pmatrix} a_1 \\ b_1 \end{pmatrix} = \begin{pmatrix} e^{-i\lambda x_0} & 0 \\ 0 & e^{i\lambda x_0} \end{pmatrix} \begin{pmatrix} \alpha & \beta \\ \bar{\beta} & \bar{\alpha} \end{pmatrix} \begin{pmatrix} e^{i\lambda x_0} & 0 \\ 0 & e^{-i\lambda x_0} \end{pmatrix} \begin{pmatrix} b_2 \\ a_2 \end{pmatrix}$$

The idea is that the natural description of a solution to the left of the obstacle is obtained by propagating freely to $x=0$ at taking the initial values. Similarly for the right. So the transfer matrix is clearly

$$\begin{pmatrix} \alpha & \beta e^{-2i\lambda x_0} \\ \bar{\beta} e^{2i\lambda x_0} & \bar{\alpha} \end{pmatrix}$$

Next suppose we have interfaces at $x_0 < x_1$. A solution to the right of x_1 is

$$\begin{pmatrix} b_2 e^{i\lambda x} \\ a_2 e^{-i\lambda x} \end{pmatrix}.$$

Its value just to the left of x_1 (i.e. at x_1^- is)

$$\begin{pmatrix} \alpha_1 & \beta_1 \\ \bar{\beta}_1 & \bar{\alpha}_1 \end{pmatrix} \begin{pmatrix} b_2 e^{i\lambda x_1} \\ a_2 e^{-i\lambda x_1} \end{pmatrix}$$

Its value at x_0^- is

$$\begin{pmatrix} e^{+i\lambda(x_0-x_1)} & 0 \\ 0 & e^{-i\lambda(x_0-x_1)} \end{pmatrix}\begin{pmatrix} \alpha_1 & \beta_1 \\ \overline{\beta_1} & \overline{\alpha_1} \end{pmatrix}\begin{pmatrix} b_2 e^{i\lambda x_1} \\ a_2 e^{-i\lambda x_1} \end{pmatrix}$$

and so its value to the left of x_0 is

$$\begin{pmatrix} e^{i\lambda(x-x_0)} & 0 \\ 0 & e^{-i\lambda(x-x_0)} \end{pmatrix}\begin{pmatrix} \alpha_0 & \beta_0 \\ \overline{\beta_0} & \overline{\alpha_0} \end{pmatrix}\begin{pmatrix} e^{i\lambda(x_0-x_1)} & 0 \\ 0 & e^{-i\lambda(x_0-x_1)} \end{pmatrix}\begin{pmatrix} \alpha_1 & \beta_1 \\ \overline{\beta_1} & \overline{\alpha_1} \end{pmatrix}\begin{pmatrix} e^{i\lambda x_1} & 0 \\ 0 & e^{-i\lambda x_1} \end{pmatrix}\begin{pmatrix} b_2 \\ a_2 \end{pmatrix}$$

So therefore it's clear we have

$$\begin{pmatrix} a_1 \\ b_1 \end{pmatrix} = \underbrace{\begin{pmatrix} \alpha_0 & \beta_0 e^{-2i\lambda x_0} \\ \overline{\beta_0} e^{2i\lambda x_0} & \overline{\alpha_0} \end{pmatrix}\begin{pmatrix} \alpha_1 & \beta_1 e^{-2i\lambda x_1} \\ \overline{\beta_1} e^{2i\lambda x_1} & \overline{\alpha_1} \end{pmatrix}}_{\begin{pmatrix} \alpha_0\alpha_1 + \beta_0\overline{\beta_1} e^{2i\lambda(x_1-x_0)} & \alpha_0\beta_1^{-2i\lambda x_1} + \overline{\alpha_1}\beta_0 e^{-2i\lambda x_0} \\ \alpha_1\overline{\beta_0} e^{2i\lambda x_0} + \overline{\alpha_0}\,\overline{\beta_1} e^{2i\lambda x_1} & \overline{\alpha_0}\,\overline{\alpha_1} + \overline{\beta_0}\beta_1 e^{2i\lambda(x_0-x_1)} \end{pmatrix}}\begin{pmatrix} b_2 \\ a_2 \end{pmatrix}$$

Let's take $x_0 = 0$. The reflection coefficient for a wave coming from the left is

$$R = \frac{\overline{B}}{A} = \frac{\alpha_1\overline{\beta_0} + \overline{\alpha_0}\,\overline{\beta_1} e^{2i\lambda x_1}}{\alpha_0\alpha_1 + \beta_0\overline{\beta_1} e^{2i\lambda x_1}} = \frac{\frac{\overline{\beta_0}}{\alpha_0} + \frac{\overline{\alpha_0}}{\alpha_0}\cdot\frac{\overline{\beta_1}}{\alpha_1} e^{2i\lambda x_1}}{1 + \frac{\beta_0}{\alpha_0}\cdot\frac{\overline{\beta_1}}{\alpha_1} e^{2i\lambda x_1}}$$

$$= \frac{R_0 + \frac{\overline{\alpha_0}}{\alpha_0} R_1 e^{2i\lambda x_1}}{1 - \tilde{R}_0 R_1 e^{2i\lambda x_1}} = \frac{R_0 - \det(S_0) R_1 e^{2i\lambda x_1}}{1 - \tilde{R}_0 R_1 e^{2i\lambda x_1}}$$

Here $\begin{pmatrix} R_0 & T_0 \\ T_0 & \tilde{R}_0 \end{pmatrix} = \begin{pmatrix} \frac{\overline{\beta_0}}{\alpha_0} & \frac{1}{\alpha_0} \\ \frac{1}{\alpha_0} & -\frac{\beta_0}{\alpha_0} \end{pmatrix}$ is the scattering matrix S_0 associated to the transfer matrix $\begin{pmatrix} \alpha_0 & \beta_0 \\ \overline{\beta_0} & \overline{\alpha_0} \end{pmatrix}$. Notice that

$$\det S_0 = -\frac{1}{\alpha_0^2} - \frac{|\beta_0|^2}{\alpha_0^2} = \frac{-|\alpha_0|^2}{\alpha_0^2} = -\frac{\overline{\alpha_0}}{\alpha_0}$$

Here is a way to view the formula:

$$R = \frac{\det(S_0)\,(R_1 e^{2i\lambda x_1}) - R_0}{\tilde{R}_0\,(R_1 e^{2i\lambda x_1}) - 1}$$

namely:

$$R = \frac{\frac{\bar{\alpha}_0}{\alpha_0}(R_1 e^{2i\lambda x_1}) + \frac{\bar{\beta}_0}{\alpha_0}}{\frac{\beta_0}{\alpha_0}(R_1 e^{2i\lambda x_1}) + 1} = \begin{pmatrix} \bar{\alpha}_0 & \bar{\beta}_0 \\ \beta_0 & \alpha_0 \end{pmatrix} \begin{pmatrix} e^{i\lambda x_1} & 0 \\ 0 & e^{-i\lambda x_1} \end{pmatrix} (R_1)$$

A way to see this is to recall that if $\begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$ is a frequency λ solution with nothing coming in from the right then

$$R = \frac{u_2}{u_1} \quad \text{at} \quad x = 0.$$

It seems somehow that in the case of a line with attenuation, so all the energy gets reflected, that this reflection coefficient is the impedance of the line.

Somehow it appears that the reflection coefficient is always defined, at least if time elapses before the reflections take place.