June 10, 1977

Idea: Consider a simple harmonic oscillator in 2-dimensions. If we consider states with a fixed angular momentum then these should be described by a radial Schroedinger equation with potential $V(r)$ becoming infinite at both $r=0$ and $r=\infty$.

Schroedinger's equation is

$$-\frac{\hbar^2}{2m}\nabla^2\psi + \left(-E + \frac{1}{2}kr^2\right)\psi = 0$$

Laplacian in polar coordinate r, θ is:

$$\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}$$

Rewrite

$$-\nabla^2\psi + \left(-\underbrace{\frac{2mE}{\hbar^2}}_{\lambda} + \underbrace{\frac{mk}{\hbar^2}}_{a}r^2\right)\psi = 0$$

$$\nabla^2\psi + (\lambda - ar^2)\psi = 0 \qquad a > 0$$

$$\frac{\partial^2\psi}{\partial r^2} + \frac{1}{r}\frac{\partial\psi}{\partial r} + \frac{1}{r^2}\frac{\partial^2\psi}{\partial\theta^2} + (\lambda - ar^2)\psi = 0$$

So we try $\psi = u(r)f(\theta)$ and we find

$$\left[r^2u'' + ru' + r^2(\lambda - ar^2)u\right]f = -uf''$$

so if c is a separation constant we get

$$r^2u'' + ru' + r^2(\lambda - ar^2)u = cu$$

$$f'' = -cf$$

Since we want $f(\theta)$ to be well-defined periodic etc. we have $f(\theta) = e^{in\theta}$ [redacted] $c = n^2$

and so the radial equation for u is

$$(*) \qquad u'' + \frac{1}{r}u' + \left(\lambda - ar^2 - \frac{n^2}{r^2}\right)u = 0$$

Now we know that $e^{-x^2/2}H_m(x)$ is an eigenfunction for $-\frac{d^2}{dx^2} + x^2$ with eigenvalue [redacted] $2m+1$, hence

$$e^{-x^2/2}H_m(x)\, e^{-y^2/2}H_{m'}(y)$$

is an eigenfunction for $-\nabla + r^2$ with the eigenvalue $2m+1+2m'+1$. Moreover these eigenfunctions [redacted] as m, n[redacted] run over integers ≥ 0 give an orthogonal basis in $L^2(\mathbb{R}^2)$. [redacted] Observe that by scaling in r [redacted], say $r \mapsto kr$, one changes a into k^4a, hence we can suppose $a=1$.

Now the rotation group in the plane acts on the space of eigenfunctions of $-\nabla + r^2$ with eigenvalue $2k+2$, which we have seen has dimension $k+1$ as it is spanned by the aforementioned eigenfunctions with $m+m'=k$.

Since, aside from the $e^{-r^2/2}$ factor, there are polys of degree k it would seem reasonable to expect this eigenvalue $2k+2$ space to be the sum of the characters $e^{in\theta}$ for $n=-k,-k+2,\dots,k-2,k$. Thus the D.E. (*) (with $a=1$) has a solution

$$u = e^{-r^2/2}\cdot \text{poly in } r$$

for $\lambda = 2k+2$ and $n=-k,-k+2,\dots,k$.

Better approach: Try the quadratic substitution

$$z = r^2 \qquad \frac{dz}{dr} = 2r$$

$$\frac{du}{dz} = \frac{du}{dr}\cdot\frac{dr}{dz} = \frac{1}{2r}\frac{du}{dr}$$

$$2z\frac{du}{dz} = 2r^2\frac{1}{2r}\frac{du}{dr} = r\frac{du}{dr}$$

So

$$\frac{d^2u}{dr^2}+\frac{1}{r}\frac{du}{dr}+\left(\lambda - r^2-\frac{n^2}{r^2}\right)u=0$$

i.e.

$$\left(r\frac{d}{dr}\right)^2u + (\lambda r^2 - r^4 - n^2)u = 0$$

becomes

$$\left(2z\frac{d}{dz}\right)^2 u + (\lambda z - z^2 - n^2)u = 0$$

or

$$\left(z\frac{d}{dz}\right)^2 u + \left(-\frac{1}{4}z^2+\frac{\lambda}{4}z-\frac{n^2}{4}\right)u=0$$

Now recall

$$z^{\frac12}\left(z\frac{d}{dz}\right)^2 z^{-\frac12}v = \left(z\frac{d}{dz}-\frac12\right)^2 v$$

$$=\left(z^2\frac{d^2}{dz^2}+z\frac{d}{dz}-z\frac{d}{dz}+\frac14\right)v = z^2\frac{d^2v}{dz^2}+\frac14 v$$

So putting $u = z^{-1/2} v$ we get

$$z^2 \frac{d^2 v}{dz^2} + \left(-\frac{1}{4} z^2 + \frac{\lambda}{4} z - \frac{n^2}{4} + \frac{1}{4}\right) v = 0$$

or

$$\boxed{\frac{d^2 v}{dz^2} + \left(-\frac{1}{4} + \frac{\lambda}{4z} - \frac{(n/2)^2 - 1/4}{z^2}\right) v = 0}$$

Whittaker's confluent hypergeometric DE is

$$\frac{d^2 w}{dz^2} + \left(-\frac{1}{4} + \frac{k}{z} - \frac{s^2 - \frac{1}{4}}{z^2}\right) w = 0$$

Go back to the original D.E.

$$\left(r \frac{d}{dr}\right)^2 u + (\lambda r^2 - r^4 - n^2) u = 0$$

and reduce to standard form via the substitution

$$\frac{dx}{dr} = \sqrt{\frac{g}{p}} = \sqrt{\frac{r}{r}} = 1$$

$$u = \beta v \qquad \beta = \frac{1}{\sqrt{gp}} = r^{-1/2}$$

since $r^{1/2} \left(r \frac{d}{dr}\right)^2 r^{-1/2} v = r^2 \frac{d^2 v}{dr^2} + \frac{1}{4} v$ we get

$$r^2 \frac{d^2 v}{dr^2} + \left(\lambda r^2 - r^4 - (n^2 - \tfrac{1}{4})\right) v = 0$$

or

$$\boxed{\frac{d^2 v}{dr^2} + \left(\lambda - r^2 - \frac{(n^2 - \frac{1}{4})}{r^2}\right) v = 0}$$

Hence we are dealing with the potential $V(r) = r^2 + \dfrac{(n^2 - \frac{1}{4})}{r^2}$

The question I have is whether the potential $\frac{1}{r^2}$ is such that there is only one solution up to a constant factor which is square-integrable near 0. The point $r=0$ is a regular singular point of the D.E. with indicial equation

$$\mu^2-\mu-\left(n^2-\tfrac{1}{4}\right)=0$$

$$\left(\mu-\tfrac{1}{2}\right)^2-\tfrac{1}{4}=n^2-\tfrac{1}{4}$$

$$\mu-\tfrac{1}{2}=\pm n$$

$$\boxed{\mu=\tfrac{1}{2}\pm n}$$

So provided $n>\frac{1}{2}$ their will be only one solution vanishing as $r\to 0$, the other grows like $z^{\frac{1}{2}-n}$ so will be square integrable for $-(\frac{1}{2}-n)<\frac{1}{2}$ i.e $n<1$. So once $n\geq 1$ there is only one square integrable solution near 0.

So the problem now is to determine the eigenvalues i.e. those values of λ for which a non-zero solution decaying at 0 and at ∞ can be found.

June 11, 1977:

Finish some aspects of Bessel functions (p. 20).

$$\int_C e^{-u} u^{s} \frac{du}{u} = \left(e^{2\pi i s}-1\right)\Gamma(s) = 2\pi i\, e^{i\pi s}\, \frac{e^{i\pi s}-e^{-i\pi s}}{2i\pi}\,\Gamma(s)$$

$$= 2\pi i e^{-i\pi s}\, \frac{\sin \pi s}{\pi}\,\Gamma(s)$$

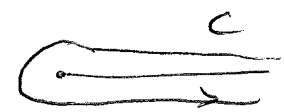
or

$$\boxed{\int_C e^{-u} u^{s}\frac{du}{u} = \frac{2\pi i\, e^{i\pi s}}{\Gamma(1-s)}}$$

Recall $\quad K_s(r) = \int_0^\infty e^{-\frac{r}{2}(t+t^{-1})} t^s \frac{dt}{t} = K_{-s}(r)$

Put

$$f_s(r) = \int_C e^{-\frac{r}{2}(t+t^{-1})} t^{-s}\frac{dt}{t}$$



substitute $\quad \frac{rt}{2} = u \quad$ or $\quad t = \frac{2u}{r} \quad t^{-1} = \frac{r}{2u}$

$$f_s(r) = \left(\frac{r}{2}\right)^s \int_C e^{-u-\frac{r^2}{4u}}\, u^{-s}\frac{du}{u} = \left(\frac{r}{2}\right)^s \int_C e^{-u} \sum \frac{1}{k!}\left(-\frac{r^2}{4}\right)^k u^{-k-s}\frac{du}{u}$$

$$= \left(\frac{r}{2}\right)^s \sum_{k=0}^{\infty} \frac{1}{k!}\left(-\frac{r^2}{4}\right)^k \frac{2\pi i}{\Gamma(1+k+s)}\, e^{i\pi(-k-s)}$$

$$= 2\pi i\, e^{-i\pi s}\left(\frac{r}{2}\right)^s \underbrace{\sum_{k=0}^{\infty} \frac{1}{k!\,\Gamma(1+k+s)}\left(\frac{r}{2}\right)^{2k}}$$

$$i^{-s}\left(\frac{ir}{2}\right)^s \sum_{k=0}^{\infty}\frac{(-1)^k}{k!\,\Gamma(1+k+s)}\left(\frac{ir}{2}\right)^{2k}$$

$$\parallel$$

$$i^{-s} J_s(ir) = I_s(r)$$

$$\boxed{\int_C e^{-\frac{r}{2}(t+t^{-1})}\, t^{-s}\,\frac{dt}{t} = 2\pi i e^{-i\pi s}\left(\frac{r}{2}\right)^{s}\sum_{k=0}^{\infty}\frac{1}{k!\,\Gamma(1+k+s)}\left(\frac{r}{2}\right)^{2k}}$$

$$e^{-\frac{r}{2}(t+t^{-1})} = \sum_{l\geq 0}\frac{1}{l!}\left(-\frac{r}{2}\right)^{l} t^{l}\sum_{k\geq 0}\frac{1}{k!}\left(-\frac{r}{2}\right)^{k} t^{-k}$$

$n = l-k$

$n+k = l$

$$= \sum_{n\in\mathbb{Z}} t^{n}\sum_{k\geq 0}\frac{1}{k!\,(k+n)!}\left(-\frac{r}{2}\right)^{n+2k}$$

$$\boxed{e^{-\frac{r}{2}(t+t^{-1})} = \sum_{n\in\mathbb{Z}} t^{n}(-1)^{n} I_n(r)}$$

$I_n(r) = I_{-n}(r)$ for $n\in\mathbb{Z}$.

$$e^{-r\cos\theta} = \sum_{n\in\mathbb{Z}}(-1)^{n} I_n(r)\, e^{in\theta}$$

$$= I_0(r) + 2\sum_{n=1}^{\infty}(-1)^{n} I_n(r)\cos n\theta$$

So now let's return to the Schroedinger equation for the simple harmonic oscillator in two variables:

$$\frac{\partial^2\psi}{\partial r^2} + \frac{1}{r}\frac{\partial\psi}{\partial r} + \frac{1}{r^2}\frac{\partial\psi}{\partial\theta^2} + (\lambda - r^2)\psi = 0$$

A solution of this for $\lambda = 2m+2$ is $e^{-\frac{x^2}{2}} H_m(x)\, e^{-y^2/2}$

$$\frac{1}{m!}e^{-r^2/2} H_n(x) = e^{-r^2/2}\frac{1}{2\pi i}\int_C e^{-t^2+2xt}\, t^{-m}\frac{dt}{t}$$

$$e^{2xt} = e^{2tr\cos\theta} = \sum_{n\in\mathbb{Z}} I_n(2tr)\, e^{in\theta}$$

So $\psi = \frac{1}{m!} e^{-r^2/2} H_m(x) = \frac{e^{-r^2/2}}{m!} \sum_{n\in\mathbb{Z}} e^{in\theta} \frac{1}{2\pi i} \int_C e^{-t^2} I_n(2tr) t^{-m} \frac{dt}{t}$

Thus it appears that

$$e^{-r^2/2} \int_C e^{-t^2} I_n(2tr)\, t^{-m} \frac{dt}{t}$$

is a solution of the DE

$$\frac{d^2u}{dr^2} + \frac{1}{r}\frac{du}{dr} + \left(\lambda - r^2 - \frac{n^2}{r^2}\right) u = 0$$

where $\lambda = 2m+2$. What this suggests is that we are ~~using~~ going to solve the above DE using contour integrals of the form

$$e^{-r^2/2} \int \phi(t)\, k_n(tr)\, dt$$

where k_n is a solution of the modified ($z = ir$) Bessel DE.

Try $u = e^{-r^2/2} v$

$$e^{r^2/2}\left(\frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr} + \lambda - r^2 - \frac{n^2}{r^2}\right) e^{-r^2/2} v$$

$$= \left(\left(\frac{d}{dr} - r\right)^2 + \frac{1}{r}\left(\frac{d}{dr} - r\right) + \lambda - r^2 - \frac{n^2}{r^2}\right) v$$

$$= \frac{d^2}{dr^2} + \left(-2r + \frac{1}{r}\right)\frac{d}{dr} + \left(\lambda - 2 - \frac{n^2}{r^2}\right) v$$

Now if $v(r) = \int \phi(t)\, k_n(tr)\, dt$, then

$$\left(\frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr} - \frac{n^2}{r^2}\right) v = \int \phi(t)\left[t^2\left[k_n''(tr) + \frac{1}{tr}k_n'(tr) - \frac{n^2}{t^2r^2}k_n(tr)\right]\right] dt$$

$$= \int \phi(t)\, t^2 k_n(tr)\, dt$$

Also

$$-2r\frac{d}{dr}\int \phi(t) k_n(tr)\, dt = -2\int \phi(t)\, t\, r k_n'(tr)\, dt$$

$$= -2\int \phi(t) t \frac{d}{dt} k_n(tr)\, dt = +2\int \frac{d}{dt}(\phi(t)t)\, k_n(tr)\, dt$$

So v will be a solution provided

$$t^2\phi(t) + 2\frac{d}{dt}(t\phi(t)) + (\lambda - 2)\phi(t) = 0$$

$$t^2\phi + 2t\phi' + \lambda\phi = 0$$

$$\frac{\phi'}{\phi} = -\frac{t}{2} - \frac{\lambda}{2t} \qquad \log\phi = -\frac{t^2}{4} - \frac{\lambda}{2}\log t$$

$$\phi(t) = e^{-t^2/4}\, t^{-\lambda/2}$$

Hence provided the contour can be chosen correctly we do get solutions of the form

$$v(r) = \int e^{-t^2/4}\, k_n(rt)\, t^{-\lambda/2}\, dt$$

And I think I should be able to develop these solutions in complete analogy to the Hermite equation.

Look at Polya's elementary proof of the functional equation for Θ. There's a chance Polya didn't know — about the Lee-Yang theorem.

First we want to understand a discrete random walk on the line. Suppose that we have a particle which each second jumps with probability p a distance a to the right and with probability $(1-p)$ jumps a distance b to the left. Assume the particle starts at $x=0$ at $t=0$, the probability of k right jumps and $n-k$ left jumps is

$$\binom{n}{k} p^k (1-p)^{n-k}$$

and the position belonging to this event is

$$ka - (n-k)b.$$

The resulting probability distribution giving the position of the particle after n jumps is

$$d\mu_n(x) = \sum_{k=0}^{n} \binom{n}{k} p^k (1-p)^{n-k} \delta\big(x - ka + (n-k)b\big)$$

The characteristic function of this probability distribution is

$$\sum_{k=0}^{n} \binom{n}{k} p^k (1-p)^{n-k} e^{it(ka-(n-k)b)}$$

$$= \sum_{k=0}^{n} \binom{n}{k} \left(pe^{ita}\right)^k \left((1-p)e^{-itb}\right)^{n-k}$$

$$= \left[pe^{ita} + (1-p)e^{-itb}\right]^n$$

Now let's assume that the [redacted] average motion is zero, i.e. that

$$pa - (1-p)b = 0$$

say in fact that $a = (1-p)c$, $b = pc$. Then

[redacted] $$pe^{ita} + (1-p)e^{-itb} = p\left(1 + ita - \frac{t^2a^2}{2}\right) + (1-p)\left(1 - itb - \frac{t^2b^2}{2}\right) + O(t^3)$$

$$= 1 - \frac{t^2}{2}\left(p(1-p)^2 + (1-p)p^2\right)c^2$$

$$= 1 - \frac{t^2}{2}(p)(1-p)c^2 + O(t^3)$$

Actually I should change t to a neutral variable ξ.

Now I want to pass to a limit as $n \to \infty$, in such a way as to obtain the characteristic function $e^{-t\xi^2}$ of the heat flow distribution. So all we have to do is to arrange that

$$\frac{1}{2}(p)(1-p)c^2 n \longrightarrow t$$

as $n \longrightarrow \infty$.

So take $p = \frac{1}{2}$ and $c^2 = \frac{8t}{n}$ $\quad c = 2\sqrt{\frac{2t}{n}}$

$a = b = \sqrt{\frac{2t}{n}}$.

$$d\mu_n(x) = \sum_{k=0}^{n} \binom{n}{k} 2^{-n}\, \delta\left(x - (2k-n)\sqrt{\frac{2t}{n}}\right)$$

Next let $n \to \infty$. In the interval $x, x+dx$ one has all mass points with

$$x < (2k-n)\sqrt{\frac{2t}{n}} < x + dx$$

so $\quad k - \frac{n}{2} \sim \sqrt{\frac{n}{2t}}\,\frac{x}{2}$ and there are $\frac{1}{2}\sqrt{\frac{n}{2t}}\,dx$ mass pts.

Hence it should be the case that

$$\binom{n}{\frac{n}{2}+\sqrt{\frac{n}{2t}}\,\frac{x}{2}}\, 2^{-n}\,\frac{1}{2}\sqrt{\frac{n}{2t}} \longrightarrow \frac{1}{2\sqrt{\pi t}}\, e^{-\frac{x^2}{4t}}$$

Put $\varepsilon = \sqrt{\frac{n}{2t}}\,\frac{x}{2}$ and use Stirling's formula

$$\frac{n^n\, \cancel{e^{-n}}\, \sqrt{2\pi n}}{\left(\frac{n}{2}+\varepsilon\right)^{n/2+\varepsilon} \cancel{e^{-(\frac{n}{2}+\varepsilon)}} \sqrt{2\pi\left(\frac{n}{2}+\varepsilon\right)}\, \left(\frac{n}{2}-\varepsilon\right)^{(n/2-\varepsilon)} \cancel{e^{-(\frac{n}{2}-\varepsilon)}} \sqrt{2\pi\left(\frac{n}{2}-\varepsilon\right)}}\; 2^{-n}\sqrt{\frac{n}{2t}}\,\frac{1}{2}$$

$$= \frac{n^n}{\left(\frac{n^2}{4}-\varepsilon^2\right)^{n/2} 2^{-n}} \left(\frac{\frac{n}{2}-\varepsilon}{\frac{n}{2}+\varepsilon}\right)^{\varepsilon} \frac{1}{2\sqrt{\pi t}}\; \underset{\substack{\downarrow \\ 1}}{\frac{n}{\sqrt{n^2-4\varepsilon^2}}}$$

$$= \frac{1}{\left(1-\frac{4\varepsilon^2}{n^2}\right)^{n/2}}\; \frac{\left(1-\frac{2\varepsilon}{n}\right)^{\varepsilon}}{\left(1+\frac{2\varepsilon}{n}\right)^{\varepsilon}}\; \frac{1}{2\sqrt{\pi t}} \qquad \frac{2\varepsilon}{n} = \frac{1}{\sqrt{n}}\,\frac{x}{\sqrt{2t}}$$

$$\downarrow$$

$$\frac{1}{e^{-x^2/4t}} \qquad \frac{\left(\left(1-\frac{1}{\sqrt{n}}\frac{x}{\sqrt{2t}}\right)^{\sqrt{n}\,\frac{\sqrt{2t}}{x}}\right)^{\frac{x^2}{2t}\cdot\frac{1}{2}}}{\left(\left(1+\frac{1}{\sqrt{n}}\frac{x}{\sqrt{2t}}\right)^{\sqrt{n}\,\frac{\sqrt{2t}}{x}}\right)^{\frac{x^2}{2t}\,\frac{1}{2}}}\; \frac{1}{2\sqrt{\pi t}}$$

$$\longrightarrow \frac{1}{2\sqrt{\pi t}}\, e^{-\frac{x^2}{4t}}$$

messy!

This argument with Stirling's formula shows that

$$\binom{n}{\frac{n}{2}+\sqrt{\frac{n}{2}}\,u}\, 2^{-n}\sqrt{\frac{n}{2}} \longrightarrow \frac{1}{\sqrt{\pi}}\, e^{-u^2}$$

June 12, 1977.

So next we want to consider a random walk on the cyclic group of order m. This gives a measure

$$\mu_n(x) = \sum_{k=0}^{n} \binom{n}{k} 2^{-n} \begin{Bmatrix} 1 & \text{if } 2k-n \equiv x \ (m) \\ 0 & \text{otherwise} \end{Bmatrix}$$

whose Fourier transform is

$$\int \chi_y \, d\mu_n = \left(\tfrac{1}{2}e^{2\pi i y/m} + \tfrac{1}{2}e^{-2\pi i y/m}\right)^n = \left(\cos(2\pi y/m)\right)^n$$

where $\chi_y(x) = \exp(2\pi i x y/m)$, $y \in \mathbb{Z}/m\mathbb{Z}$. Fourier inversion gives

$$\mu_n(x) = \sum_{k=0}^{n} \binom{n}{k} 2^{-n} \begin{Bmatrix} 1 & \text{if } 2k-n \equiv x \ (m) \\ 0 & \text{otherwise} \end{Bmatrix}$$

$$= \frac{1}{m} \sum_{y=0}^{m-1} e^{-2\pi i x y/m} \left(\cos 2\pi y/m\right)^n$$

Take $x=0$ and let $2k-n = jm$ or $k = \frac{n}{2} + \frac{jm}{2}$ and suppose both n, m even. (Curious: what happens if n is odd and m even). We get

$$\sum_{j=-[\frac{n}{m}]}^{[\frac{n}{m}]} \binom{n}{\frac{n}{2}+j\frac{m}{2}} 2^{-n} m = \sum_{y=0}^{m-1} \left(1 - \tfrac{1}{2}(2\pi y)^2/m^2 + \cdots\right)^n$$

Now let $j\frac{m}{2} = \sqrt{\frac{n}{2}}\, jt$ or $m = \sqrt{\frac{n}{2}}\, 2t$ and take the limit

$$\sum_{j \in \mathbb{Z}} \frac{1}{\sqrt{\pi}} e^{-j^2t^2} 2t = \sum_{y=0}^{\infty} e^{-\frac{1}{2}(2\pi y)^2 \frac{1}{2t^2}} = \sum_{y=0}^{\infty} e^{-\frac{\pi^2 y^2}{t^2}}$$

which is off by a factor of 4, probably because one should take the sum from $y = -\frac{m}{2}+1$ to $\frac{m}{2}$, and because of a stupid error.

The question is whether Polya's proof would suggest approximations to θ by polynomials to which Lee-Yang can be applied. The polynomials obtained would be of the form

$$\sum_n z^n \sum_j \binom{N}{\frac{N}{2}+j\sqrt{\frac{N}{2}}\,q^n} 2^{-N}\sqrt{\frac{N}{2}}\, q^{n/2}$$

which are too complicated.

Analogy:

$$(q^{-s})^{1-g}\zeta_C(s) = \sum_L (q^{-s})^{\deg(L)+1-g}\left(\frac{q^{h^0(L)}-1}{q-1}\right)$$

$$\pi^{-s/2}\Gamma(s/2)\zeta(s) = \int_0^\infty t^{-s}\left[\theta(t^{-1})-1\right]\frac{dt}{t}$$

so we have

$$q^{\deg(L)+1-g} \longleftrightarrow t$$

$$q^{h^0(L)} \longleftrightarrow \theta(t^{-1})$$

$$\int \qquad\qquad \int$$

$$q^{\deg(L)+1-g} \longleftrightarrow t$$

June 14, 1977

Hartman's estimate for $N(\lambda)$:

$$u'' + (\lambda - q)u = 0$$

Put $Q = (\lambda - q)^{1/2}$. We work on the interval $0 \le x \le q^{-1}(\lambda)$ and q is supposed to be increasing and convex.

$$\theta = \arctan\left(\frac{Qu}{u'}\right)$$

$$\theta' = \frac{1}{1+\left(\frac{Qu}{u'}\right)^2}\left(Q'\frac{u}{u'} + Q + Q\frac{uQ^2u}{(u')^2}\right) = \frac{Q'uu'}{u'^2+Q^2u^2} + Q$$

$$\frac{d\theta}{dx} = Q + \frac{Q'}{Q}\sin\theta\cos\theta$$

Now introduce the independent variable

$$y = \int_0^x Q\,dx \qquad \frac{dy}{dx} = Q$$

Thus we get

$$(*) \qquad \boxed{\frac{d\theta}{dy} = 1 + \frac{d}{dy}(\log Q)\sin\theta\cos\theta}$$

$$\frac{d}{dy}(\log Q) = \frac{1}{Q^2}\frac{dQ}{dx} = (\lambda - q)^{-1}\,\tfrac{1}{2}(\lambda-q)^{-1/2}(-q')$$

$$= -\tfrac{1}{2}q'(\lambda - q)^{-3/2}$$

A basic ingredient in Hartman's proof is the fact that $q'(\lambda - q)^{-3/2}$ is increasing.

Hartman's key idea is to write $(*)$ as

$$1 = \frac{d\theta}{dy} - \frac{d}{dy}(\log Q)\sin\theta\cos\theta$$

so

$$\frac{d\theta}{dy} = 1 + \frac{d}{dy}(\log Q)\sin\theta\cos\theta\frac{d\theta}{dy} - \left(\frac{d}{dy}\log Q\right)^2\sin^2\theta\cos^2\theta$$

so

$$\theta(T) = \theta(0) + T + \underbrace{\int_0^T \frac{d}{dy}(\log Q)\,\frac{d}{dy}(\sin^2\theta)\,dy}_{I_1} - \underbrace{\int_0^T \left(\frac{d}{dy}\log Q\right)^2 \sin^2\cos^2\theta\,dy}_{I_2}$$

$$I_1 = \left[\frac{d}{dy}(\log Q)\sin^2\theta\right]_0^T - \underbrace{\int_0^T \frac{d^2}{dy^2}(\log Q)\sin^2\theta\,dy}_{I_3}$$

Put

$$-a = \frac{d}{dy}(\log Q)\Big|_T \qquad -a_0 = \frac{d}{dy}(\log Q)\Big|_0$$

Then because $\frac{d}{dy}(\log Q)$ is decreasing one has

$$0 \le \int_0^T -\frac{d^2}{dy^2}(\log Q)\sin^2\theta\,dy \le \int_0^T -\frac{d^2}{dy^2}(\log Q)\,dy = \text{[redacted]} -a_0 + a$$

so

[redacted] $$|I_1| \le |a| + |a_0| + |a - a_0|$$

$$0 \le I_2 \le \int_0^T \left(\frac{d}{dy}\log Q\right)^2 dy \text{ [redacted]}$$

$$= \int_0^T \frac{dQ}{dy}\left(\frac{1}{Q^2}\frac{dQ}{dy}\right)dy = \int_0^T \frac{dQ}{dy}\left(-\frac{d}{dy}\left(\frac{1}{Q}\right)\right)dy$$

$$= \left[-\frac{1}{Q}\frac{dQ}{dy}\right]_0^T + \int_0^T \frac{1}{Q}\frac{d^2Q}{dy^2}\,dy$$

I have to me more precise:

$$Q = (\lambda - q)^{1/2} \qquad \text{decreasing and } > 0$$

$$-\frac{dQ}{dx} = \frac{1}{2}(\lambda - q)^{-1/2}\frac{dq}{dx} \qquad \text{increasing and } > 0$$

$$-\frac{dQ}{dy} = \frac{1}{2}(\lambda - q)^{-1}\frac{dq}{dx} \qquad \text{increasing and } > 0 .$$

$$\Longrightarrow \quad -\frac{d^2Q}{dy^2} \geq 0 . \qquad \text{So}$$

$$0 \leq I_2 = \int_0^T \left(\frac{d}{dy}\log Q\right)^2 dy = \int_0^T \frac{dQ}{dy}\, Q^{-2}\frac{dQ}{dy}\, dy$$

$$= \left[-\frac{dQ}{dy}\, Q^{-1}\right]_0^T + \int_0^T Q^{-1}\frac{d^2Q}{dy^2}\, dy$$

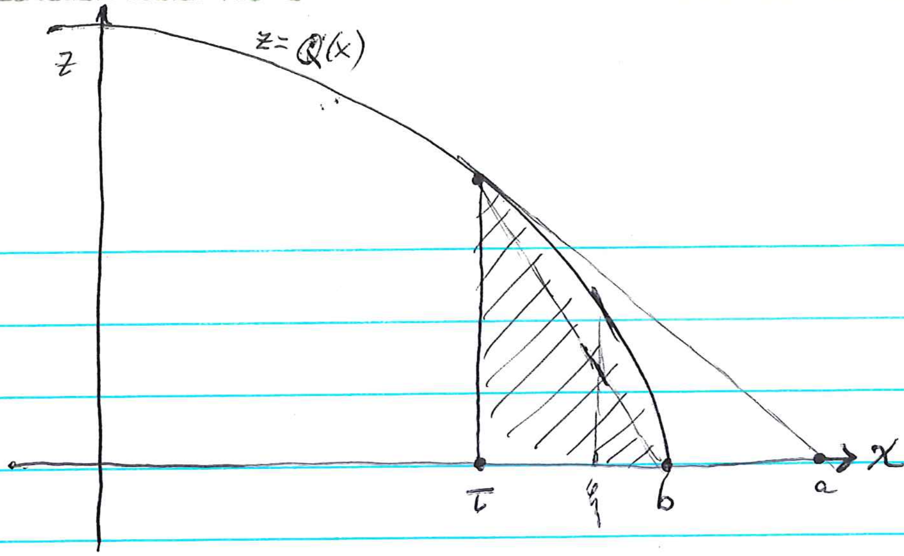
$$\leq a - a_0$$

So we see that $\theta(T) - \theta(0) = T$ + error bounded by $-\frac{1}{Q}\frac{dQ}{dy}$ at 0 and at T.

Hartman chooses T so that $-\frac{1}{Q}\frac{dQ}{dy} = \frac{1}{2}q'(\lambda - q)^{-3/2} = \frac{1}{2}$.

The next point is to estimate the remaining change in θ. First estimate the change in y.

Picture:



$$\int_\tau^b Q(x)\,dx \le \frac{1}{2} Q(\tau)(a-\tau) \qquad Q'(\tau) = \frac{-Q(\tau)}{a-\tau}$$

$$= \frac{1}{2} Q(\tau)\frac{-Q(\tau)}{Q'(\tau)}$$

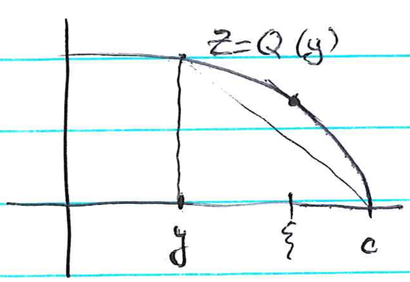
$$= \frac{1}{2}\frac{-(\lambda-g(\tau))}{\frac{1}{2}(\lambda-g(\tau))^{-1/2}(-g'(\tau))} = (\lambda-g(\tau))^{3/2}/g'(\tau)$$

Hartman's estimate is

$$\int_\tau^b Q(x)\,dx \le Q(\tau)(b-\tau) = Q(\tau)\,\overbrace{Q(\tau)\cdot\frac{-1}{Q'(\xi)}}^{\text{Mean Value Thm.}} \qquad \tau<\xi<b$$

$$\le \frac{Q(\tau)^2}{-Q'(\tau)}$$

Now this estimation can be done with y maybe:



$$\frac{Q(y)}{c-y} = \underbrace{-\frac{dQ}{dy}(\xi)}_{\text{increasing}} \ge -\frac{dQ}{dy}(y)$$

$$\therefore \quad c-y \le \left(-\frac{1}{Q}\frac{dQ}{dy}\right)^{-1}$$

Same as Hartman's.

June 15, 1977

Attempt to understand the eigenvalue distribution for the system

$$\frac{du}{dx} = \begin{pmatrix} i\lambda & \bar{p} \\ p & -i\lambda \end{pmatrix} u .$$

If we put $\quad w = \dfrac{u_1}{u_2} \quad$ then

$$w' = \frac{i\lambda u_1 + \bar{p} u_2}{u_2} - \frac{u_1}{u_2}\,\frac{p u_1 - i\lambda u_1}{u_2} = \bar{p} + 2i\lambda w - p w^2$$

so if $\quad w = e^{i\theta} \quad$ then

$$i e^{i\theta}\theta' = \bar{p} + 2i\lambda e^{i\theta} - p e^{2i\theta}$$

$$\theta' = \overline{ipe^{i\theta}} + 2\lambda + ipe^{i\theta}$$

or $$\frac{d\theta}{dx} = 2\left(\lambda + \mathrm{Re}\left(ipe^{i\theta}\right)\right)$$

Better to work with $\theta = 2\phi$. If p real, then we get the equation

$$\boxed{\frac{d\phi}{dx} = \lambda - p\sin(2\phi)}$$

Following Hartman one can rewrite this as follows:

$$1 = \frac{1}{\lambda}\frac{d\phi}{dx} + \frac{p}{\lambda}\sin(2\phi)$$

$$\frac{d\phi}{dx} = \lambda - p\sin(2\phi)\left(\frac{1}{\lambda}\frac{d\phi}{dx} + \frac{p}{\lambda}\sin 2\phi\right)$$

$$= \lambda - \frac{p}{\lambda}\sin(2\phi)\frac{d\phi}{dx} - \frac{p^2}{\lambda^2}\sin^2 2\phi \cdot \frac{d\phi}{dx} - \cdots\cdots$$

or $\quad \lambda = \left(1 - \frac{p}{\lambda}\sin 2\phi\right)^{-1}\frac{d\phi}{dx} = \frac{d\phi}{dx} + \left(\frac{p}{\lambda}\right)\sin(2\phi)\frac{d\phi}{dx} + \cdots\cdots$

This is to be integrated say between $0 \le x \le T$ where $p(T) < \lambda$. Assuming p increasing, one has by the 2nd mean value formula for integrals

$$\int_0^T \left(\frac{p}{\lambda}\right)^n \sin^n(2\phi)\frac{d\phi}{dx} = \left(\frac{p(0)}{\lambda}\right)^n \int_0^{\xi} \sin^n(2\phi)\frac{d\phi}{dx}dx + \left(\frac{p(T)}{\lambda}\right)^n \int_{\xi}^{T} \sin^n(2\phi)\frac{d\phi}{dx}dx$$

where $0 < \xi < T$. Now my idea is that if n is odd the function $\int_0^{\theta} \sin^n(2\phi)\,d\phi$ is oscillatory, hence these integrals might be negligible. What should matter then is the constant non-oscillatory part of $\int_0^{\theta} \sin^{2n}(2\phi)\,d\phi$ for $2n$ even. So we replace $\sin^n(2\phi)$ by its average value.

$$\sin^{2n}(2\phi) = \left(\frac{e^{i2\phi} - e^{-i2\phi}}{2i}\right)^{2n} \mapsto 2^{-2n}(-1)^n(-1)^n\binom{2n}{n}$$

$$2^{-2n}\binom{2n}{n} = \frac{1\cdot 2\cdots\cdots 2n-1\cdot 2n}{n!\,n!\;2^{2n}} = \frac{\frac{1}{2}\,\frac{3}{2}\cdots\, n-\frac{1}{2}}{n!} = (-1)^n\frac{\left(-\frac{1}{2}\right)\cdots\left(-\frac{1}{2}-n+1\right)}{n!}$$

Hence

$$\sum_{n\ge 0}\left(\frac{p}{\lambda}\right)^{2n}\sin^{2n}(2\phi) \longmapsto \sum_{n\ge 0}\left(-\frac{p^2}{\lambda^2}\right)^n\binom{-\frac{1}{2}}{n} = \left(1 - \frac{p^2}{\lambda^2}\right)^{-\frac{1}{2}}$$

This gives the approximate DE for ϕ:

$$\lambda = \frac{d\phi}{dx}\left(1 - \frac{p^2}{\lambda^2}\right)^{-1/2} \quad \text{or} \quad \boxed{\frac{d\phi}{dx} = \left(\lambda^2 - p^2\right)^{1/2}}$$

Unfortunately it seems to be hard to make anything of these heuristics. Compare the equation

$$\frac{d\theta}{dy} = 1 + \left(\frac{Q'}{Q^2}\right)\frac{1}{2}\sin 2\theta \qquad\qquad \frac{Q'}{Q^2} = \frac{1}{Q}\frac{dQ}{dy}$$

with $$\frac{d\phi}{\lambda\cdot dx} = 1 - \frac{p}{\lambda}\sin 2\phi$$

To study the former we used the identity

$$\int\left(\frac{-1}{Q}\frac{dQ}{dy}\right)^2 dy = -\int \frac{dQ}{dy}\frac{dQ^{-1}}{dy}\,dy$$

$$= \left[-\frac{1}{Q}\frac{dQ}{dy}\right] + \int \frac{1}{Q}\frac{d^2Q}{dy^2}\,dy$$

and we checked directly that $-\frac{d^2Q}{dy^2} \geq 0$. To do something similar in the second case we need some sort of estimate for

$$\int_0^{p^{-1}(\lambda)} p^2\,dx.$$

However if $p = e^x$ then $\int_0^{\log\lambda} e^{2x}\,dx = \frac{1}{2}(\lambda^2 - 1)$ which is too big. Confused.

June 16, 1977

Return to

$$\left(\frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr} - \frac{m^2}{r^2} + \lambda - r^2\right) u = 0$$

which arises from separating $(-\nabla^2 + r^2)\psi = \lambda\psi$ in polar coords. The standard form is obtained by putting $u = r^{-1/2} v$

$$r^{1/2}\left(r\frac{d}{dr}\right)^2 r^{-1/2} = \left(r\frac{d}{dr} - \frac{1}{2}\right)^2 = r^2\frac{d}{dr^2} + \cancel{r\frac{d}{dr}}\,\cancel{r\frac{d}{dr}} + \frac{1}{4}$$

$$\left(\frac{d^2}{dr^2} + \lambda - r^2 - \frac{m^2 - \frac{1}{4}}{r^2}\right) v = 0$$

which is the Schroedinger equation on $0 < r < \infty$ with the potential $V(r) = r^2 + \frac{m^2 - \frac{1}{4}}{r^2}$. Note that

$$\left(\frac{d}{dr} + r + \frac{m+\frac{1}{2}}{r}\right)\left(\frac{d}{dr} - r - \frac{m+\frac{1}{2}}{r}\right)$$

$$= \frac{d^2}{dr^2} - r^2 - \cancel{r}\frac{2m+1}{\cancel{r}} - \frac{(m+\frac{1}{2})^2}{r^2} - 1 + \frac{m+\frac{1}{2}}{r^2}$$

$$= \frac{d^2}{dr^2} - (2m+2) - r^2 - \frac{m^2 - \frac{1}{4}}{r^2}$$

Better change $m \mapsto -m$

$$\cancel{\left(\frac{d}{dr} + r + \frac{-m+\frac{1}{2}}{r}\right)\left(\frac{d}{dr} - r - \frac{-m+\frac{1}{2}}{r}\right) = \frac{d^2}{dr^2} + (2m-2) - r^2 - \frac{m^2 - \frac{1}{4}}{r^2}}$$

$$\boxed{\left(\frac{d}{dr} - r + \frac{m+\frac{1}{2}}{r}\right)\left(\frac{d}{dr} + r + \frac{m+\frac{1}{2}}{r}\right) = \frac{d^2}{dr^2} + (2m+2) - r^2 - \frac{m^2 - \frac{1}{4}}{r^2}}$$

$$\left(\frac{d}{dr} + r - \frac{m+\frac{1}{2}}{r}\right) v = 0$$

$$\frac{d}{dr}\log v = -r + \left(m+\tfrac{1}{2}\right)\frac{1}{r}$$

$$v = e^{-r^2/2}\, r^{m+\frac{1}{2}}$$

So we see that

$$u = e^{-r^2/2}\, r^m$$

is an eigenfunction for $-\left(\frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr}\right) + r^2 + \frac{m^2}{r^2}$ with eigenvalue $2m+2$.

$$u = r^{-1/2} v, \quad v = r^{1/2} u$$

$$r^{-1/2}\left(\frac{d}{dr} + r - \frac{m+\frac{1}{2}}{r}\right) r^{1/2} u = \frac{d}{dr} + r - \frac{m+\frac{1}{2}}{r} + \frac{1}{2} r^{-1/2-1/2}$$

$$= \frac{d}{dr} + r - \frac{m}{r}$$

So one has

$$\left(\frac{d}{dr} - r + \frac{m+1}{r}\right)\left(\frac{d}{dr} + r - \frac{m}{r}\right) = \cancel{\frac{d^2}{dr^2} + (2m-2) - r^2}$$

$$= \frac{d^2}{dr^2} + 1 + \frac{m}{r^2} + \frac{1}{r}\frac{d}{dr} - r^2 + m + m + 1 - \frac{m^2+m}{r^2}$$

$$= \frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr} + (2m+2) - r^2 - \frac{m^2}{r^2}$$

The system to study is therefore:

$$\begin{cases} \left(\frac{d}{dr} + r - \frac{m+\frac{1}{2}}{r}\right) u_1 = \lambda u_2 \\ \left(\frac{d}{dr} - r + \frac{m+\frac{1}{2}}{r}\right) u_2 = -\lambda u_1 \end{cases}$$

Thus we are considering the system

$$\frac{d}{dr}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} -p & \lambda \\ -\lambda & p \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$$

with $\quad p(r) = r - \frac{m+\frac{1}{2}}{r}$. $\quad$ Check

$$p^2 - p' = r^2 - (2m+1) + \frac{m^2+m+\frac{1}{4}}{r^2} - 1 - \frac{m+\frac{1}{2}}{r^2}$$

$$= -(2m+2) + r^2 + \frac{m^2-\frac{1}{4}}{r^2}$$

Note that

$$p^2 + p' = -2m + r^2 + \frac{m^2+2m+\frac{3}{4}}{r^2} = -2m + r^2 + \frac{(m+1)^2-\frac{1}{4}}{r^2}$$

which is related to the potential $V_{m+1}(r)$ where $V_m(r) = r^2 + \frac{m^2-\frac{1}{4}}{r^2}$.

Consider Whittaker's DE

$$\frac{d^2W}{dz^2} + \left(-\frac{1}{4} + \frac{k}{z} - \frac{m^2-\frac{1}{4}}{z^2}\right) W = 0$$

If we put $z^{-\frac{1}{2}} W = u$ or $W = z^{\frac{1}{2}} u$ this becomes

$$\left[\left(\frac{d}{dz} + \frac{1}{2z}\right)^2 + \left(-\frac{1}{4} + \frac{k}{z} - \frac{m^2-\frac{1}{4}}{z^2}\right)\right] u = 0$$

$$\left(\frac{d^2}{dz^2}+\frac{1}{z}\frac{d}{dz}+\cancel{\left(\frac{1}{4z^2}-\frac{1}{2z^2}+\frac{1}{4z^2}\right)}-\frac{1}{4}+\frac{k}{z}-\frac{m^2}{z^2}\right)u=0$$

To eliminate the last term put $u=z^m v$

$$\left(\left(\frac{d}{dz}+\frac{m}{z}\right)^2+\frac{1}{z}\left(\frac{d}{dz}+\frac{m}{z}\right)-\frac{1}{4}+\frac{k}{z}-\frac{m^2}{z^2}\right)v=0$$

$$\left(\frac{d^2}{dz^2}+\frac{2m}{z}\frac{d}{dz}+\frac{1}{z}\frac{d}{dz}+\cancel{\frac{m^2}{z^2}}-\cancel{\frac{m}{z^2}}+\cancel{\frac{m}{z^2}}-\frac{1}{4}+\frac{k}{z}-\cancel{\frac{m^2}{z^2}}\right)v=0$$

or $$\left(z\frac{d^2}{dz^2}+(2m+1)\frac{d}{dz}-\frac{z}{4}+k\right)v=0$$

Now use Laplace transform:

$$v=\int e^{tz}\phi(t)\,dt$$

$$-\frac{d}{dt}(t^2\phi)+(2m+1)t\phi+\frac{1}{4}\frac{d}{dt}\phi+k\phi=0$$

$$-t^2\frac{d\phi}{dt}+(2m-1)t\phi+\frac{1}{4}\frac{d\phi}{dt}+k\phi=0$$

$$\left(t^2-\frac{1}{4}\right)\frac{d\phi}{dt}=(2m-1)t\phi+k\phi$$

$$\frac{1}{\phi}\frac{d\phi}{dt}=\frac{(m-\frac{1}{2})2t\phi}{t^2-\frac{1}{4}}+k\left[\frac{1}{t-\frac{1}{2}}-\frac{1}{t+\frac{1}{2}}\right]$$

$$\phi=\left(t^2-\frac{1}{4}\right)^{m-\frac{1}{2}}\left(t-\frac{1}{2}\right)^{k}\left(t+\frac{1}{2}\right)^{-k}$$

$$v=\int e^{tz}\left(t-\frac{1}{2}\right)^{m-\frac{1}{2}+k}\left(t+\frac{1}{2}\right)^{m-\frac{1}{2}-k}dt$$

$$=e^{-\frac{1}{2}z}\int e^{tz}(t-1)^{m-\frac{1}{2}+k}(t)^{m-\frac{1}{2}-k}dt$$

$$=e^{-\frac{1}{2}z}\int e^{-t}\left(-\frac{t}{z}-1\right)^{m-\frac{1}{2}+k}\left(-\frac{t}{z}\right)^{m-\frac{1}{2}-k}\left(-\frac{dt}{z}\right)$$

$$W = z^{m+\frac{1}{2}} V = e^{-\frac{1}{2}z} z^k \int e^{-t}\left(1+\frac{t}{z}\right)^{m-\frac{1}{2}+k} t^{m-\frac{1}{2}-k}\,dt$$

June 17, 1977.

Learn about confluent hypergeometric DE.

$$x\frac{d^2y}{dx^2} + (c-x)\frac{dy}{dx} - ay = 0$$

$x=0$ is a regular singular point. Try series solution $y = x^\mu \sum a_n x^n = \sum a_n x^{n+\mu}$.

$$\sum_n \left(a_n(n+\mu)(n+\mu-1)x^{n+\mu-1} + c a_n (n+\mu) x^{n+\mu-1}\right)$$
$$-\sum\left(a_n(n+\mu)x^{n+\mu} + a a_n x^{n+\mu}\right) = 0$$

$$a_n(n+\mu)(c+\mu+n-1) = a_{n-1}(a+\mu+n-1)$$

For $n=0$ we get the indicial equation

$$\mu(c+\mu-1) = 0$$

with roots $\mu = 0,\ 1-c$. For $\mu=0$ we get the recursion relations

$$a_n = a_{n-1}\frac{(a+n-1)}{n(c+n-1)}$$

yielding the series

$$F(a,c;x) = 1 + \frac{a}{c}\frac{x}{1!} + \frac{a(a+1)}{c(c+1)}\frac{x^2}{2!} + \cdots\cdots$$

which is well-defined for $c \neq 0,-1,-2,\ldots$ and convergent for all x.

For $\mu = 1-c$ we get the ~~[illegible]~~ recursion

$$a_n = a_{n-1}\frac{(a-c+1+n-1)}{(2-c+n-1)n}$$

so the other solution is

$$\boxed{x^{1-c}F(a-c+1, 2-c; x)}$$

provided $c \neq 2, 3, 4, \cdots$; also one wants $c \neq 1$ so that this series is independent of $F(a, c; x)$.

To express these as a contour integral one can use the β-function.

$$\Gamma(x)\Gamma(y) = 2\int_0^\infty e^{-t^2} t^{2x-1}dt \cdot 2\int_0^\infty e^{-u^2}u^{2y-1}du$$

$$= 2\int_0^\infty e^{-r^2} r^{\overbrace{2x-1+2y-1+1}^{}}_{(2x+2y-1)} dr \int_0^{\pi/2} (\cos\theta)^{2x-1}(\sin\theta)^{2y-1}\, 2d\theta$$

$$t = \cos^2\theta \qquad dt = -2\sin\theta\cos\theta\, d\theta$$

$$= \Gamma(x+y)\int_0^1 t^{x-1}(1-t)^{y-1}dt$$

so

$$\boxed{\frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)} = \int_0^1 t^{x-1}(1-t)^{y-1}dt}$$

$$\frac{\Gamma(a)}{\Gamma(c)}F(a,c;x) = \sum_{n\geq 0}\frac{\Gamma(a)a(a+1)\cdots(a+n-1)}{\Gamma(c)c(c+1)\cdots(c+n-1)}\frac{x^n}{n!} = \sum_{n\geq 0}\frac{\Gamma(a+n)}{\Gamma(c+n)}\frac{x^n}{n!}$$

$$\frac{\Gamma(c-a)\Gamma(a)}{\Gamma(c)}F(a,c;x) = \sum_{n\geq 0}\frac{\Gamma(a+n)\Gamma(c-a)}{\Gamma(c+n)}\frac{x^n}{n!}$$

$$= \sum_{n\geq 0}\int_0^1 t^{a+n-1}(1-t)^{c-a-1}\frac{x^n}{n!}dt = \int_0^1 e^{tx}t^{a-1}(1-t)^{c-a-1}dt$$

Thus we get the formula

$$F(a,c;x)=\frac{\Gamma(c)}{\Gamma(a)\Gamma(c-a)}\int_0^1 e^{tx}\,t^{a-1}(1-t)^{c-a-1}\,dt$$

Such a contour integral expression for solutions of the hypergeometric equation could have been obtained by applying the Laplace transformation to the original DE. Putting $t \mapsto 1-t$ in the above gives

$$F(a,c;x)=\frac{\Gamma(c)\,e^{x}}{\Gamma(a)\Gamma(c-a)}\int_0^1 e^{-tx}\,(1-t)^{a-1}\,t^{c-a-1}\,dt$$

The integral is the Laplace transform of the function $(1-t)^{a-1}t^{c-a-1}$ $0\le t\le 1$ and 0 for $1\le t\le\infty$. Now one knows that the asymptotic behavior of the Laplace transform as $x\to+\infty$ depends on the asymptotic expansion of the function as $t\to 0$.
So

$$\int_0^1 e^{-tx}(1-t)^{a-1}t^{c-a-1}\,dt \sim \int_0^\infty e^{-tx}\left(1-(a-1)t+\frac{(a-1)(a-2)}{2!}t^2-\cdots\right)t^{c-a-1}\,dt$$

$$=\frac{\Gamma(c-a)}{x^{c-a}}-\frac{(a-1)\Gamma(c-a+1)}{x^{c-a+1}}+\frac{(a-1)(a-2)}{2!}\frac{\Gamma(c-a+2)}{x^{c-a+2}}-\cdots$$

So as $x\to+\infty$ we have

$$F(a,c;x)\sim\frac{\Gamma(c)}{\Gamma(a)}e^{x}x^{a-c}\left(1-\frac{(a-1)(c-a+1)}{1!}\frac{1}{x}+\frac{(a-1)(a-2)(c-a+1)(c-a+2)}{2!}\frac{1}{x^2}-\cdots\right)$$

which gives an interesting solution when a or $a-c$ is a positive integer.

If in the solution

$$\int_1^\infty e^{(1-t)x}(1-t)^{a-1}t^{c-a-1}\,dt$$

we put $t \mapsto t+1$, then we get the solution

[redacted] $$\int_0^\infty e^{-xt}(1+t)^{c-a-1}t^{a}\frac{dt}{t}$$

asymptotic to $\frac{\Gamma(a)}{x^a}$ as $x \to +\infty$. [redacted] We should think of c as being fixed and of a as being essentially the variable s. Hence put

$$u = \frac{t}{1+t} \qquad u(1+t) = t \qquad u = t(1-u) \qquad t = \frac{u}{1-u}$$

$$1-u = \frac{1}{1+t} \qquad dt = \frac{(1-u)\,du + u\,du}{(1-u)^2} = \frac{du}{(1-u)^2} \qquad 1+t = \frac{1}{1-u}$$

$$\int_0^\infty e^{-xt}(1+t)^{c-a-1}t^{a-1}\,dt = \int_0^1 e^{-x\left(\frac{u}{1-u}\right)}(1-u)^{-c+\cancel{a}+\cancel{1}}\,u^{a-1}(1-u)^{-\cancel{a}+\cancel{1}}(1-u)^{-\cancel{2}}\,du$$

$$= \int_0^1 \frac{e^{-x\left(\frac{u}{1-u}\right)}}{(1-u)^{\text{[redacted]}c}}\,u^{a-1}\,du$$

Lets derive the standard form for the D.E.

$$xy'' + (c-x)y' - ay = 0$$

treating $-a$ as λ. $x^{1/2}\frac{d}{dx} = \frac{d}{dz}$ needed to make the leading coefficient 1.

$$\frac{dz}{dx} = x^{-1/2} \qquad z = 2x^{1/2}$$

If we don't mind changing a by a scalar, we might as well take $z = x^{1/2}$ or $x = z^2$

$$dx = 2z\,dz \qquad \frac{d}{dx} = \frac{1}{2z}\frac{d}{dz}$$

$$\frac{d^2}{dx^2} = \frac{1}{2z}\frac{d}{dz}\cdot\frac{1}{2z}\frac{d}{dz} = \frac{1}{4z^2}\frac{d^2}{dz^2} + \frac{1}{2z}\left(-\frac{1}{2z^2}\right)\frac{d}{dz}$$

$$z^2\left(\frac{1}{4z^2}\frac{d^2}{dz^2} - \frac{1}{4z^3}\frac{d}{dz}\right) + (c-z^2)\frac{1}{2z}\frac{d}{dz} - a$$

$$\frac{1}{4}\frac{d^2}{dz^2} + \left(\frac{-1}{4z} + \frac{c}{2z} - \frac{z}{2}\right)\frac{d}{dz} - a$$

$$\frac{d^2}{dz^2} + \left(\frac{2c-1}{z} - 2z\right)\frac{d}{dz} - 4a$$

To put $u = fv$ where $2f' + \left(\frac{2c-1}{z} - 2z\right)f = 0$

$$\frac{f'}{f} = z - \frac{2c-1}{2z} \qquad \log f = \frac{z^2}{2} + \left(-c+\tfrac{1}{2}\right)\log z$$

$$f = e^{z^2/2} z^{-c+\frac{1}{2}}$$

Thus to replace: $u = e^{z^2/2} z^{-c+\frac{1}{2}} v$

$$e^{-z^2/2} z^{c-\frac{1}{2}} \frac{d}{dz} e^{z^2/2} z^{-c+\frac{1}{2}} v = \left(\frac{d}{dz} + z + \frac{-c+\frac{1}{2}}{z}\right)v$$

$$\left(\frac{d}{dz} + z + \frac{-c+\frac{1}{2}}{z}\right)^2 = \frac{d^2}{dz^2} + 2\left(z + \frac{-c+\frac{1}{2}}{z}\right)\frac{d}{dz}$$
$$+ \left(z^2 + 2\left(-c+\tfrac{1}{2}\right) + \frac{\left(-c+\frac{1}{2}\right)^2}{z^2}\right)$$
$$+ \left(1 - \frac{-c+\frac{1}{2}}{z^2}\right)$$

$$+\left(\frac{2c-1}{z} - 2z\right)\left(\frac{d}{dz} + z + \frac{-c+\frac{1}{2}}{z}\right) = \left(\frac{2c-1}{z} - 2z\right)\frac{d}{dz} - 2\left(z + \frac{-c+\frac{1}{2}}{z}\right)^2$$

$$\frac{d^2}{dz^2} - \left(z^2 - 2c + \cancel{1} + \frac{c^2 - c + \frac{1}{4}}{z^2}\right) + \cancel{1} + \frac{c - \frac{1}{2}}{z^2} - 4a$$

$$\frac{d^2}{dz^2} + 2c - 4a + \frac{-c^2 + 2c - \frac{3}{4}}{z^2} - z^2$$

$$\frac{d^2}{dz^2} + 2c - 4a - \frac{(c-1)^2 - \frac{1}{4}}{z^2} - z^2$$

So we should have $c = m+1$ to make the connection. Thus I conclude that solutions of

$$\left(\frac{d^2}{dz^2} + 2c - 4a - z^2 - \frac{(c-1)^2 - \frac{1}{4}}{z^2}\right) v = 0$$

are of the form $v = e^{-z^2/2} z^{c-\frac{1}{2}} u(z^2)$ where u is a solution of the confluent hypergeometric DE.

$$x u'' + (c-x)u' - a u = 0$$

Now what I am really interested in doing is to find the eigenvalues for the potential $+r^2 + \frac{m^2 - \frac{1}{4}}{r^2}$ on $0 < r < \infty$. I've seen that I should keep $-m + \frac{1}{2} \leq -\frac{1}{2}$ or $m \geq 1$ in order to be in the limit point case as $r \to 0$. Think of $c = m+1$ as being fixed ≥ 2 and a as being the eigenvalue. The solution of the D.E. good at $x = 0$ is

$$F(a, c; x) = \frac{\Gamma(c)}{\Gamma(a)\Gamma(c-a)} \int_0^1 e^{tx} t^{a-1} (1-t)^{c-a-1} dt$$

which we have seen has the asymptotic behavior

$$\frac{\Gamma(c)}{\Gamma(a)} e^x x^{a-c} \quad \text{as} \quad x \to \infty$$

at least when the integral in question makes sense ie. $Re(a) > 0$, $Re(c-a) > 0$.

To define $\int_0^\infty e^{-xt}(1+t)^{c-a-1} t^{a-1}\,dt$ [redacted] beyond

$Re(a) > 0$ [redacted] we consider

$$\frac{1}{2\pi i}\int_C e^{-xt}(1+t)^{c-a-1} t^a \frac{dt}{t}$$

which is an entire function of [redacted] a, [redacted] when $Re(x) > 0$. If $a \in \mathbb{Z}$, then the integrand is single-valued so C can be replaced by a loop around 0. If $a = 1, 2, \ldots$ one gets zero, but not for $a = 0, -1, -2, \ldots$. Thus the good object appears to be

$$\frac{\Gamma(1-a)}{2\pi i}\int_C e^{-xt}(1+t)^{c-a-1} t^a \frac{dt}{t}$$

Now

$$\frac{\Gamma(1-a)}{2\pi i}\left(e^{2\pi i a} - 1\right) = \Gamma(1-a)\frac{\sin \pi a}{\pi} e^{\pi i a} = \frac{e^{\pi i a}}{\Gamma(a)}$$

Hence we have the good solution

$$e^{-\pi i a}\frac{\Gamma(1-a)}{2\pi i}\int_C e^{-xt}(1+t)^{c-a-1} t^a \frac{dt}{t} = \boxed{H(a,c;x) = \frac{1}{\Gamma(a)}\int_0^\infty e^{-xt}(1+t)^{c-a-1} t^a \frac{dt}{t}}$$

which is entire in a and never identically zero. Denote it $H(a,c;x)$. Note that if one uses [redacted] the Taylor expansion for $(1+t)^{c-a-1}$ around zero one gets an asymptotic expansion for H around $x = +\infty$. Thus

$$(1+t)^{c-a-1} = \sum_{n\geq 0} \binom{c-a-1}{n} t^n$$

$$\frac{1}{\Gamma(a)}\int_0^\infty e^{-xt}\,\frac{(c-a-1)\cdots(c-a-1-n+1)}{1\cdots n}\,t^{a+n}\,\frac{dt}{t}$$

$$= \frac{\Gamma(a+n)}{x^{a+n}}\,\frac{1}{\Gamma(a)}\,\frac{(c-a-1)\cdots(c-a-n)}{n!}$$

so

$$\boxed{H(a,c;x) \sim x^{-a}\sum_{n\geq 0}\frac{a(a+1)\cdots(a+n-1)(c-a-1)\cdots(c-a-n)}{n!}\,\frac{1}{x^n}}$$

This sort of asymptotic ~~behavior~~ behavior is valid for all a. In fact if we use the trick of subtracting off [redacted] so many terms of the Taylor series for $(1+t)^{c-a-1}$ we see as in the proof of Watson's lemma that $H(a,c;x)$ is the good solution at $x = +\infty$ for all a. (Note that once multiplied by $e^{-x/2}$ etc. it becomes square integrables.)

Referring to page 54 we see that

$$H(a,c;x) = \frac{1}{\Gamma(a)}\int_0^1 \frac{e^{-x\left(\frac{u}{1-u}\right)}}{(1-u)^c}\,u^{a-1}\,du$$

We can let $x \to 0$ provided $Re(a) > 0$ and $Re(c) < 1$ and we get

$$H(a,c;0) = \frac{1}{\Gamma(a)}\int_0^1 (1-u)^{-c}\,\text{[redacted]}\,u^{a-1}\,du = \frac{\Gamma(\overset{1-c}{\text{[redacted]}})\,\cancel{\Gamma(a)}}{\cancel{\Gamma(a)}\,\Gamma(a+\overset{1-c}{\text{[redacted]}})}$$

$$\frac{dH}{dx}(a,c;0) = -\frac{1}{\Gamma(a)}\int_0^1 (1-u)^{-c-1}\,u^a\,du = -\frac{\Gamma(-c)\,\Gamma(a+1)}{\Gamma(a)\,\Gamma(a+1-c)}$$

This shows that

$$H(a,c;x) = \frac{\Gamma(1-c)}{\Gamma(a+1-c)} F(a,c;x) + (\,?\,)\, x^{1-c} F(a+1-c, 2-c; x)$$

However observe the symmetry: Put $y = x^{1-c}u$ in

$$xy'' + (c-x)y' - ay$$

$$x\left(\frac{d}{dx} + \frac{1-c}{x}\right)^2 + (c-x)\left(\frac{d}{dx} + \frac{1-c}{x}\right) - a$$

$$x\left(\frac{d^2}{dx^2} + 2\,\frac{1-c}{x}\,\frac{d}{dx} + \frac{\cancel{1}-\cancel{2c}+\cancel{c^2}}{x^2} - \frac{\cancel{1}-\cancel{c}}{x^2}\right) + (c-x)\frac{d}{dx} + \frac{\cancel{c}-\cancel{c^2}}{x} - 1 + c - a$$

$$x\frac{d^2}{dx^2} + (2-c-x)\frac{d}{dx} - (a+1-c)$$

Hence $\boxed{H(a,c;x) = x^{1-c} H(a+1-c, 2-c; x)}$ up to a scalar factor which is 1 as $H(a,c;x) \sim x^{-a}$ as $x \to +\infty$. Check:

$$x^{-a} = x^{1-c}\, x^{-(a+1-c)}.$$

So apply this symmetry to the formula at the top of this page

$$H(a,c;x) = x^{1-c} H(a+1-c, 2-c; x)$$

$$= \frac{\Gamma(1-2+c)}{\Gamma(a+1-c+1-2+c)}\, x^{1-c} F(a+1-c, 2-c; x)$$

$$+ (\,?\,)'\, \underline{x^{1-c}\, x^{1-2+c}}\, F(\underset{a}{\underset{\parallel}{a+1-\cancel{c}+1-2+\cancel{c}}}, \underset{c}{\underset{\parallel}{2-2+c}}; x)$$

$$\therefore\ (\,?\,) = \frac{\Gamma(c-1)}{\Gamma(a)}$$

Thus

$$\boxed{H(a,c;x) = \frac{\Gamma(1-c)}{\Gamma(a+1-c)} F(a,c;x) + \frac{\Gamma(c-1)}{\Gamma(a)} x^{1-c} F(a+1-c, 2-c; x)}$$

So supposing that $c \geq 2$ one sees that H is good at $x=0$, i.e. a multiple of F iff $\Gamma(a)=\infty$ i.e. $a = 0, -1, -2, \ldots\ldots$.

Relate modified Bessel's DE and confluent hypergeometric DE.

$$\left[\left(r\frac{d}{dr}\right)^2 - r^2 - n^2\right] u = 0$$

At $r=\infty$ this is roughly $\frac{d^2}{dr^2} - 1$ which has solutions $e^{\pm r}$. At $r=0$ it has solutions like $r^{\pm n}$. On the other hand $xy'' + (c-x)y' - ay = 0$ at $x=\infty$ is like $y''-y'=0$ which has solutions $1, e^x$ and at $x=0$ it has solutions behaving like x^0 and x^{1-c}. So first put $y = e^{-r} w$

$$0 = \left[\left(r\frac{d}{dr} - r\right)^2 - r^2 - n^2\right] w$$

$$= \left[\left(r\frac{d}{dr}\right)^2 - 2r\left(r\frac{d}{dr}\right) + \cancel{r^2} - r - n^2\right] w$$

Next put $w = r^n u$

$$\left(r\frac{d}{dr} + n\right)^2 - 2r\left(r\frac{d}{dr} + n\right) - r - n^2$$

$$\left(r\frac{d}{dr}\right)^2 + 2n\, r\frac{d}{dr} + \cancel{n^2} - 2r^2\frac{d}{dr} - 2nr - r - \cancel{n^2}$$

$$r^2\frac{d^2}{dr^2} + (1+2n-2r)\, r\frac{d}{dr} - (2n+1) r$$

Now put $2r = x$ $\qquad r = \frac{x}{2}$

$$\left[x\frac{d^2}{dx^2} + (2n+1-x)\frac{d}{dx} - \left(n+\frac{1}{2}\right)\right]u = 0$$

Thus a solution of modified Bessel's DE is

$$e^{-r} r^n F\left(n+\tfrac{1}{2}, 2n+1; 2r\right)$$

and this has to be a multiple of $I_n(r)$, which could easily be determined by calculating with the series. Also

$$e^{-r} r^n H\left(n+\tfrac{1}{2}, 2n+1; 2r\right) = e^{-r} r^n \frac{1}{\Gamma\left(n+\frac{1}{2}\right)} \int_0^\infty e^{-2rt} (1+t)^{n-\frac{1}{2}} t^{n-\frac{1}{2}} dt$$

$c - a - 1 = 2n+1-n-\frac{1}{2}-1$
$= n - \frac{1}{2}$ $\qquad$ must be a multiple of $K_n(r)$

$a - 1 = n - \frac{1}{2}$

June 18, 1977

Hermite DE: $\left(\frac{d}{dx}-x\right)\left(\frac{d}{dx}+x\right)u = \left(\frac{d^2}{dx^2}-x^2+1\right)u = -2nu$

If $u = e^{-x^2/2}v$, it becomes

$$\left(\frac{d}{dx}-2x\right)\frac{d}{dx}v = \left(\frac{d^2}{dx^2}-2x\frac{d}{dx}\right)v = 2sv$$

where we put $s=-n$ so the eigenvalues are $s=0,-1,-2,\dots$. By Laplace transform one obtains solutions as contour integrals of the form

$$\int e^{-t^2-2xt}t^s\frac{dt}{t}$$

Consider the contour C. This gives an entire function in x which vanishes for $s=1,2,\dots$ hence to get a non-vanishing for any s solution we have to at least multiply by $\Gamma(1-s)$.

$$e^{-\pi is}\frac{\Gamma(1-s)}{2\pi i}\left(e^{2\pi is}-1\right) = \Gamma(1-s)\frac{\sin\pi s}{\pi} = \frac{1}{\Gamma(s)}.$$

So the good solution is

$$v_s(x) = \frac{1}{\Gamma(s)}\int_0^\infty e^{-t^2-2xt}t^s\frac{dt}{t} = \frac{\Gamma(1-s)}{2\pi ie^{\pi is}}\int_C e^{-t^2-2xt}t^s\frac{dt}{t}$$

Watson's trick gives the asymptotic expansion as $x\longrightarrow+\infty$:

$$\frac{1}{\Gamma(s)}\int_0^\infty\sum\frac{(-t^2)^n}{n!}t^se^{-2xt}\frac{dt}{t} = \sum_{n\geq0}\frac{1}{\Gamma(s)}\frac{(-1)^n}{n!}\frac{\Gamma(s+2n)}{(2x)^{s+2n}}$$

$$V_s(x) \sim \sum_{n\geq 0} \frac{(-1)^n}{n!} \frac{s(s+1)\cdots(s+2n-1)}{(2x)^{s+2n}}$$

which begins with $(2x)^{-s}$.

June 19, 1977.

Recall the L-function for $\mathbb{Z}[i]$ is

$$L(s) = \sum_{n\geq 1} \left(\frac{-1}{n}\right) n^{-s} = \sum_{m\geq 0}{}' (-1)^m (2m+1)^{-s}$$

$$= \frac{1}{\Gamma(s)} \int_0^\infty \sum_{m\geq 0} (-1)^m e^{-(2m+1)t} t^s \frac{dt}{t}$$

$$= \frac{1}{\Gamma(s)} \int_0^\infty \frac{e^{-t}}{1+e^{-2t}} t^s \frac{dt}{t} = \frac{1}{\Gamma(s)} \int_0^\infty \frac{1}{e^t+e^{-t}} t^s \frac{dt}{t}$$

$$= \frac{\Gamma(1-s)}{2\pi i\, e^{i\pi s}} \int_C \frac{1}{e^t+e^{-t}} t^s \frac{dt}{t}$$

Hence if I want to understand Dirichlet L-functions I might as well study the function

$$\sum_{n\geq 1} \frac{e^{2\pi i n y}}{n^s}$$

where y is rational. But note that if $y = \frac{p}{q}$, then this series breaks up into series of the form

$$\sum_{m \geq 0} (r\,[\text{redacted}] + mq)^{-s} \qquad \text{with} \quad 1 < r \leq q$$

which are essentially the same as series of the form

$$\sum_{m \geq 0} \left(\frac{r}{q} + m\right)^{-s}$$

So therefore it is now clear that provided we stick to rational values of x, y we have an essential [redacted] equivalence between series of the form

$$\sum_{n \geq 1} \frac{e^{2\pi i n y}}{n^s} \quad \text{and} \quad \sum_{m \geq 0} (x + m)^{-s}$$

So the problem is now to find the really good gadget on which these series live

June 20, 1977.

The problem is whether one can organize the different meromorphic functions of s of the form $H(x,y,s) = \sum_{n\geq 0}' e^{2\pi i n y}(x+n)^{-s}$ as x, y range over rational numbers $0 < x, y \leq 1$. The hope would be that something simpler, perhaps a one-variable gadget $h(x,s)$ could be found which would play the role of the functions like $K_s(r)$ and $V_s(x)$ found in connection with the Bessel and Hermite DE's.

The simplest functions to start with are arithmetic progression Dirichlet series

$$\sum_{n\geq 0}' (a+nd)^{-s}$$

where $0 < a \leq d$ and $(a,d) = 1$. We can put this in the form

$$d^{-s}\sum_{n\geq 0}' \left(\frac{a}{d}+n\right)^{-s}.$$

Conversely given a Hurwitz ζ-function

$$H(x,s) = \sum_{n\geq 0} (x+n)^{-s}$$

with $x \in \mathbb{Q}$ and $0 < x \leq 1$ it comes from a unique arith. prog. D. series, namely, the one with $x = \frac{a}{d}$ in lowest terms.

The next functions to look at are L-functions

$$L = \sum_{n\geq 1} \chi(n) n^{-s}$$

where $\chi(mn) = \chi(m)\chi(n)$ if m, n are relatively prime. Such an L has an Euler product factorization

$$L = \sum_{p \text{ prime}} \left(\sum_{r\geq 0} \chi(p^r)(p^{-s})^r \right)$$

Now I am interested in those χ such that $\chi(n) = \chi(n')$ if $n \equiv n' \pmod d$ for some d, for these series are the ones which are finite sums of arith. prog. D. series. The case to look at first (the one used by Dirichlet) is where χ is supported on integers relatively prime to d. In this case χ is a character on $(\mathbb{Z}/d\mathbb{Z})^*$ extended by 0 to $\mathbb{Z}_{>0}$. (In effect given m, n relatively prime to d we can find k so that $n + kd \equiv 1 \pmod m$, hence $m, n+kd$ rel. prime, so

$$\chi(mn) = \chi(m(n+kd)) = \chi(m)\chi(n+kd) = \chi(m)\chi(n).)$$

So we want to look at Dirichlet L-functions $L(s,\chi) = \sum_{n\geq 1} \chi(n) n^{-s}$ where χ is a character of the group $(\mathbb{Z}/d\mathbb{Z})^*$ extended to $\mathbb{Z}_{>0}$. It should be the case that the card $(\mathbb{Z}/d\mathbb{Z})^* = \varphi(d)$ (Euler φ function) functions are additively equivalent to the arith. prog. series: $\sum_{n\geq 0} (a+nd)^{-s}$ $0 < a < d$ $(a,d) = 1$. Yes:

$$L(s,\chi) = \sum_a \chi(a) \sum_{n\geq 0} (a+nd)^{-s}$$

and by Fourier inversion

$$\sum_{n\geq 0} (a+nd)^{-s} = \frac{1}{\varphi(d)} \sum_{\chi} \bar{\chi}(a) L(s,\chi)$$

using the orthogonality relations

$$\frac{1}{\varphi(d)} \sum_{\chi} \chi(a)\bar{\chi}(b) = \delta_{ab}$$

Here $\chi \in \mathrm{Hom}((\mathbb{Z}/d\mathbb{Z})^*, S^1)$ and a, b run over integers a prime to d with $0<a<d$.

(1) $$\boxed{\sum_{n\in\mathbb{Z}} e^{-\pi(n+y)^2 t} e^{2\pi i n x} = \frac{e^{-2\pi i x y}}{\sqrt{t}} \sum_{n\in\mathbb{Z}} e^{-\frac{\pi}{t}(n+x)^2} e^{-2\pi i n y}}$$

$$\int_0^\infty e^{-at} t^s \frac{dt}{t} = \frac{\Gamma(s)}{a^s} \qquad \text{if } \operatorname{Re}(a) > 0$$

$$\therefore \int_0^\infty \sum_{n\in\mathbb{Z}} e^{-\pi(n+y)^2 t} e^{2\pi i n x} t^{s/2} \frac{dt}{t} = \pi^{-s/2}\Gamma(s/2) \sum_{n\in\mathbb{Z}} |n+y|^{-s} e^{2\pi i n x}$$

This holds for $y \in \mathbb{R}-\mathbb{Z}$, $x \in \mathbb{R}$, $\operatorname{Re}(s) > 1$.

$$\int_0^\infty \frac{1}{\sqrt{t}} e^{-\frac{\pi}{t}(n+x)^2} t^{s/2} \frac{dt}{t} = \int_0^\infty e^{-\pi(n+x)^2 t} t^{1/2 - s/2} \frac{dt}{t} = \pi^{\frac{s-1}{2}} \Gamma\left(\frac{1-s}{2}\right) |n+x|^{s-1}$$

This holds for $x \in \mathbb{R}-\mathbb{Z}$, $\operatorname{Re}(s) < 1$, but when we sum over n we want $y \in \mathbb{R}$ and $\operatorname{Re}(s) < 0$. Thus from (1) we get

(2) $$\boxed{\pi^{-s/2}\Gamma(s/2) \sum_n |n+y|^{-s} e^{+2\pi i n x} = e^{-2\pi i x y} \pi^{\frac{s-1}{2}} \Gamma\left(\frac{1-s}{2}\right) \sum_n |x+n|^{s-1} e^{-2\pi i n y}}$$

This holds for $x, y \in \mathbb{R}-\mathbb{Z}$ in the sense that one side is the analytic continuation of the other. There is also a sense in which they hold if either x or $y \in \mathbb{Z}$.

Recall $H(x,y,s) = \sum_{n \geq 0} (x+n)^{-s} e^{2\pi i n y}$. Thus for $0 < x < 1$ and y real we have

$$\sum_{n \in \mathbb{Z}} |x+n|^{-s} e^{2\pi i n y} = H(x,y,s) + \sum_{n \geq 0} |x-1-n| e^{2\pi i(-1-n)y}$$

$$= H(x,y,s) + e^{-2\pi i y} H(1-x, y, s)$$

or

$$(3) \qquad \boxed{\sum_{n \in \mathbb{Z}}{}' |x+n|^{-s} e^{2\pi i n y} = H(x,y,s) + e^{-2\pi i y} H(1-x, 1-y, s)}$$

Recall that the L function for $\mathbb{Z}[i]$ is

$$L(s) = \sum_{n \geq 0}{}' (-1)^n (2n+1)^{-s} = 2^{-s} \sum_{n \geq 0} (-1)^n \left(\tfrac{1}{2}+n\right)^{-s}$$

$$(4) \qquad \boxed{L(s) = 2^{-s} H\left(\tfrac{1}{2}, \tfrac{1}{2}, s\right)}$$

However from (3) we have $\sum' \left|\tfrac{1}{2}+n\right|^{-s} (-1)^n = 0$, hence we do not obtain the functional equation for L from (2).

Residue calculation:

$$\Gamma(s) H(x,y,s) = \int_0^\infty \sum_{n \geq 0} e^{-(x+n)t + 2\pi i n y} t^s \frac{dt}{t}$$

$$= \int_0^\infty \frac{e^{-xt}}{1 - e^{-t+2\pi i y}} t^s \frac{dt}{t}$$

So

$$\frac{e^{2\pi i s}-1}{2\pi i}\Gamma(s)H(x,y,s) = \frac{1}{2\pi i}\int_C \frac{e^{-xt}}{1-e^{-t+2\pi i y}}\, t^{s-1}\,dt$$

$-t+2\pi i y = -2\pi i n$

$t = 2\pi i(y+n)$

assume $0<y<1$

$0<x<1$

Change s to 1-s to simplify

$$\frac{e^{-2\pi i s}-1}{2\pi i}\Gamma(1-s)H(x,y,1-s) = \frac{1}{2\pi i}\int_C \frac{e^{-xt}}{1+e^{-t+2\pi i y}}\, t^{-s}\,dt$$

$$= -\sum_{n\in\mathbb{Z}} e^{-x 2\pi i(y+n)}\left(2\pi i(y+n)\right)^{-s}$$

$$= e^{-2\pi i x y}(2\pi)^{-s}\Big\{-\sum_{n\geq 0} e^{-2\pi i n x}\, e^{\frac{i\pi}{2}(-s)}(y+n)^{-s}$$

$$-\sum_{n\geq 0} e^{-2\pi i(-1-n)x}\, e^{\frac{i3\pi}{2}(-s)}|y-1-n|^{-s}$$

$$= e^{-2\pi i x y}(2\pi)^{-s}(-1)\left\{e^{-i\frac{\pi}{2}s}H(y,1-x,s) + e^{-i\frac{3\pi}{2}s}e^{2\pi i x}H(1-y,x,s)\right\}$$

Multiply by $-e^{\pi i s}$ and you get

$$\boxed{\frac{\sin(\pi s)}{\pi}\Gamma(1-s)H(x,y,1-s) = (2\pi)^{-s}e^{-2\pi i x y}\left\{e^{i\frac{\pi}{2}s}H(y,1-x,s) + e^{-i\frac{\pi}{2}s}\underset{e^{2\pi i x}}{\underbrace{}}H(1-y,x,s)\right\}}$$

or even better use

$$\boxed{\Gamma(s)\Gamma(1-s) = \frac{\pi}{\sin \pi s}}$$

to get

$$\boxed{H(x,y,1-s) = (2\pi)^{-s}\Gamma(s)\, e^{-2\pi i xy}\left\{ e^{i\frac{\pi}{2}s} H(y,1-x,s) + e^{-i\frac{\pi}{2}s} \underset{\uparrow\, e^{2\pi i x}}{\bigcirc} H(1-y,x,s)\right\}}$$

Let's put $\quad h(x,y,s) = \pi^{-s/2}\Gamma(s/2)\, H(x,y,s)$. Recall

$$\boxed{\Gamma\left(\frac{s}{2}\right)\Gamma\left(\frac{s+1}{2}\right) = 2^{1-s}\sqrt{\pi}\,\Gamma(s)}$$

$$\frac{\pi^{-s/2}\Gamma(s/2)}{\pi^{-(1+s)/2}\Gamma(1-s/2)}(2\pi)^s\frac{1}{\Gamma(s)} = \sqrt{\pi}\,\frac{\cancel{\Gamma(s/2)}}{\Gamma(1-s/2)}\,2^s\,\frac{2^{1-s}\sqrt{\pi}}{\cancel{\Gamma\left(\frac{s}{2}\right)}\Gamma\left(\frac{s+1}{2}\right)}$$

$$= \cancel{\sqrt{\pi}}\;\pi\, 2\, \frac{\sin\left(\frac{1-s}{2}\pi\right)}{\pi} = 2\cos\left(\frac{s\pi}{2}\right)$$

Hence we get

$$\boxed{h(x,y,1-s) = \frac{e^{-2\pi i xy}}{2\cos\left(\frac{s\pi}{2}\right)}\left\{ e^{i\frac{\pi}{2}s} h(y,1-x,s) + e^{-i\frac{\pi}{2}s} \underset{\uparrow\, e^{2\pi i x}}{\bigcirc} h(1-y,x,s)\right\}}$$

$$\boxed{\text{where } h(x,y,s) = \pi^{-s/2}\Gamma(s/2)\, H(x,y,s)}$$

So for $x = y = \frac{1}{2}$

$$h\left(\tfrac{1}{2},\tfrac{1}{2},1-s\right) = \frac{-i}{2\cos\left(\frac{s\pi}{2}\right)}\left(e^{i\frac{\pi}{2}} - e^{-i\frac{\pi}{2}}\right) h\left(\tfrac{1}{2},\tfrac{1}{2},s\right)$$

$$= \frac{\sin\left(\frac{s\pi}{2}\right)}{\cos\left(\frac{s\pi}{2}\right)}\, h\left(\tfrac{1}{2},\tfrac{1}{2},s\right)$$

whence upon rearranging Γ factors one gets the functional equation for $L(s)$.

June 21, 1977

The following form for the formula at the top of page 70 is due to Lerch:

$$H(x,y,1-s) = \frac{\Gamma(s)}{(2\pi)^s}\Big\{e^{\frac{i\pi s}{2}} e^{-2\pi i x y} H(y,-x,s) + e^{-\frac{i\pi s}{2}} e^{2\pi i x(1-y)} H(1-y,x,s)\Big\}$$

~~Observe that multiplying~~ The point is that

$$e^{2\pi i x y} \sum_{n\in\mathbb{Z}}{}' (x+n)^{-s} e^{2\pi i n y} = \sum_{n\in\mathbb{Z}}{}' (x+n)^{s} e^{2\pi i (x+n) y}$$

is periodic in x, which explains the factors $e^{-2\pi i x y}, e^{2\pi i x(1-y)}$ on the right side.

$$\begin{pmatrix} H(x,y,1-s) \\ H(1-x,1-y,1-s)\end{pmatrix} = \frac{\Gamma(s)}{(2\pi)^s}\begin{pmatrix} e^{\frac{i\pi s}{2}} e^{-2\pi i x y} & e^{-\frac{i\pi s}{2}} e^{2\pi i x(1-y)} \\ e^{-\frac{i\pi s}{2}} e^{-2\pi i (1-x) y} & e^{\frac{i\pi s}{2}} e^{-2\pi i (1-x)(1-y)}\end{pmatrix}\begin{pmatrix} H(y,1-x,s) \\ H(1-y,x,s)\end{pmatrix}$$

This is very complicated and what you should do is work with the eigenvectors of this matrix. We've already found one:

$$\sum_{n\in\mathbb{Z}} |x+n|^{-s} e^{2\pi i n y} = H(x,y,s) + e^{-2\pi i y} H(1-x,1-y,s)$$

(valid for $0<x<1$). The other one according to Weil's book is

$$\sum_{n\in\mathbb{Z}} \operatorname{sgn}(x+n)\,|x+n|^{-S} e^{2\pi i n y} = H(x,y,s) - e^{-2\pi i y} H(1-x,1-y,s)$$

Calculation shows

$$H(x,y,1-s) - e^{2\pi i y} H(1-x, 1-y, 1-s) = \frac{\Gamma(s)}{(2\pi)^s}\left(e^{\frac{i\pi s}{2}} - e^{-\frac{i\pi s}{2}}\right) e^{-2\pi i x y}.$$

$$\left\{ H(y, 1-x, s) - e^{+2\pi i y} H(1-y, x, s) \right\}$$

Digress: Consider a Lee-Yang polynomial

$$P(z_1, \dots, z_n) = \sum_I \prod_{\substack{i \in I \\ j \notin I}} c_{ij} \, z^I$$

where $c_{ij} = \bar{c}_{ji}$ and $0 \leq |c_{ij}| \leq 1$. We put

$$z_i = e^{-h_i s} = \left(e^{+h_i}\right)^{-s} = q_i^{-s}$$

where $h_i > 0$, i.e. $q_i > 1$. Then we get something looking like a Dirichlet series

$$P(s) = \sum_I \prod_{\substack{i \in I \\ j \notin I}} c_{ij} \left(\prod_{i \in I} q_i\right)^{-s}$$

(Another point I recall was that one could make the $c_{ij} \geq 0$ by replacing z_i by $\lambda_i z_i$ with $|\lambda_i| = 1$. In effect from

$$c_{ij} = \bar{c}_{ji}$$

one sees that if we choose $\theta_{ij} = \arg(c_{ij})$ such that $-\theta_{ji} = \theta_{ij}$, then

$$\prod_{\substack{i \in I \\ j \in I'}} c_{ij} = \prod_{\substack{i \in I \\ j \in I'}} |c_{ij}| \, e^{i \sum_{\substack{i \in I \\ j \in I'}} \theta_{ij}}$$

and

$$\sum_{\substack{i \in I \\ j \notin I'}} \theta_{ij} = \sum_{\substack{i \in I \\ j \in I'}} \theta_{ij} + \sum_{\substack{i,j \in I \\ i \neq j}} \theta_{ij} = \sum_{i \in I} \left(\sum_{j \neq i} \theta_{ij} \right),$$

so one takes $\lambda_i = i \prod_{j \neq i} e^{\theta_{ij}}$)

Good form of the functional equations:

Put

$$h^+(x,y,s) = \pi^{-s/2}\Gamma(s/2) \sum_{n\in\mathbb{Z}}{}' |x+n|^{-s} e^{2\pi i n y}$$

$$h^-(x,y,s) = \pi^{-\frac{1+s}{2}}\Gamma\left(\frac{1+s}{2}\right) \sum_{n\in\mathbb{Z}} \operatorname{sgn}(x+n)|x+n|^{-s} e^{2\pi i n y}$$

Then the functional equations are

$$\boxed{\begin{aligned} h^+(x,y,1-s) &= h^+(y,-x,s)\, e^{-2\pi i x y} \\ h^-(x,y,1-s) &= i\, h^-(y,-x,s)\, e^{-2\pi i x y} \end{aligned}}$$

Lets check this by deriving using θ functions.

$$\sum{}' e^{-\pi(x+n)^2 t + 2\pi i n y} = \frac{e^{-2\pi i x y}}{\sqrt{t}} \sum{}' e^{-\pi(y+n)^2/t - 2\pi i n x}$$

Differentiate with respect to x.

$$-2\pi t \sum{}' (x+n) e^{-\pi(x+n)^2 t + 2\pi i n y} = \frac{\partial}{\partial x} \frac{1}{\sqrt{t}} \sum e^{-\pi(y+n)^2/t - 2\pi i (y+n) x}$$

$$= \frac{1}{\sqrt{t}} \sum e^{-\pi(y+n)^2/t - 2\pi i (y+n) x} \left(-2\pi i (y+n)\right)$$

or

$$\boxed{\sum (x+n) e^{-\pi(x+n)^2 t + 2\pi i n y} = i\, t^{-3/2} e^{-2\pi i x y} \sum (y+n) e^{-\pi(y+n)^2/t - 2\pi i n x}}$$

$$\int_0^\infty \sum (x+n) e^{-\pi(x+n)^2 t + 2\pi i n y}\, t^{\frac{1+s}{2}} \frac{dt}{t}$$

$$= \pi^{-\frac{1+s}{2}} \Gamma\left(\frac{1+s}{2}\right) \sum{}' (x+n)\left(|x+n|^2\right)^{-\frac{1+s}{2}} e^{2\pi i n y}$$

$$= \pi^{-\frac{1+s}{2}} \Gamma\left(\frac{1+s}{2}\right) \sum{}' \operatorname{sign}(x+n)\, |x+n|^{-s} e^{2\pi i n y}$$

$$= h^-(x, y, s)$$

$$i \int_0^\infty \sum (y+n) e^{-\pi(y+n)^2/t - 2\pi i n x}\, t^{-\frac{3}{2}+\frac{1+s}{2}} \frac{dt}{t}\, e^{-2\pi i x y}$$

$$= i \int_0^\infty \sum{}' (y+n) e^{-\pi(y+n)^2 t - 2\pi i n x}\, t^{1-\frac{s}{2}} \frac{dt}{t}\, e^{-2\pi i x y} \qquad 1-\frac{s}{2} = \frac{1+(1-s)}{2}$$

$$= i\, h^-(y, -x, 1-s)\, e^{-2\pi i x y}$$

Now it is necessary to understand what these formulas mean when x, y are rationals.

First look at the modulus 2 case, that is, series of the form

$$\sum_{m \geq 1}{}' a_n n^{-s}$$

where $n \equiv n' \bmod 2 \Rightarrow a_n = a_{n'}$. Then these series can be expressed in terms of

$$\sum_{n\geq 1} (2n)^{-s} = 2^{-s} \zeta(s) = 2^{-s} H(1, 0, s)$$

$$\sum_{n\geq 0} (2n+1)^{-s} = (1-2^{-s}) \zeta(s) = 2^{-s} H\left(\tfrac{1}{2}, 0, s\right)$$

What is the modulus for the series $H(x,y,s)$? It should have something to do with the subgroup of $\mathbb{Q}/\mathbb{Z}$ generated by x, y.

We should first of all consider only series $H(1,y,s)$ and $H(x,0,s)$ first.

Take the case $p=3$. Then we have

$$H(x,0,s) \qquad x = \tfrac{1}{3}, \tfrac{2}{3}, 1$$

where $H(1,0,s) = \zeta(s)$ has occurred before. We also have

$$H(1,y,s) \qquad y = 0, \tfrac{1}{3}, \tfrac{2}{3}$$

where [redacted] $y=0$ gives $\zeta(s)$. Now the interesting functions perhaps are the two Dirichlet L-series:

$$L(s,\chi_0) = \sum_{\substack{n \not\equiv 0\ (3) \\ n \geq 1}} n^{-s} = (1-3^{-s})\,\zeta(s)$$

$$L(s,\chi) = \cancel{\sum_{n \equiv 0 (3)} \omega^n n^{-s} \qquad \omega = e^{2\pi i/3}}$$

$= \sum_{\substack{n \equiv 1\ (3) \\ n \geq 1}} n^{-s} + \sum_{\substack{n \equiv 2\ (3) \\ n \geq 1}}$ [redacted] $(-)n^{s}$

Milne formula [4]

$$\pi N(\lambda) \sim \int_0^{\phi(\lambda)} (\lambda - q)^{1/2}\, dt \text{ as } \lambda \to \infty, \text{ where } q\big(\phi(\lambda)\big) \equiv \lambda, \tag{3}$$

holds if q has a continuous derivative satisfying $q' t^3 \to \infty$, as $t \to \infty$ (for example, if $q' \geqslant \text{const.} > 0$ for large t). In some of his theorems on the convergence of the expansions of arbitrary functions into series of eigenfunctions, belonging to (2) and a homogeneous boundary condition, Titchmarsh needs the sharper formula

$$\pi N(\lambda) = \int_0^{\phi(\lambda)} (\lambda - q)^{1/2}\, dt + O(1), \text{ as } \lambda \to \infty; \tag{4}$$

[5], Chap. IX. He proves (4) under the assumption that q has a continuous second derivative satisfying $0 \leqslant q'' \leqslant q'^{\gamma}$ for large t, where $1 < \gamma < \frac{4}{3}$; [5], Chap. VII. The object of this note is to prove that

(*) *Formula* (4) *holds whenever* q *is a continuous, increasing, convex function.*

Let $q'(t)$ denote, for all t, the limit $q'(t+0)$. With the same convention for $\theta'(t)$, the formulae

$$\theta = \operatorname{arc\,tan}\big((\lambda - q)^{1/2} y/y'\big) \text{ and } 0 \leqslant \theta(0) < \pi \tag{5}$$

determine for large λ a continuous function on $0 \leqslant t < \phi(\lambda)$ such that

$$\theta' = (\lambda - q)^{1/2} - \tfrac{1}{4} q' (\lambda - q)^{-1} \sin 2\theta, \tag{6}$$

by virtue of (2). If the last term is multiplied by

$$1 = \theta' (\lambda - q)^{-1/2} + \tfrac{1}{4} q' (\lambda - q)^{-3/2} \sin 2\theta,$$

an integration shows that

$$\theta(t) = \theta(0) + \int_0^t (\lambda - q)^{1/2}\, dt - \tfrac{1}{4} I_1 - \tfrac{1}{16} I_2, \tag{7}$$

where $0 < t < \phi(\lambda)$,

$$I_1 = I_1(t) = \int_0^t q' (\lambda - q)^{-3/2} \theta' \sin 2\theta\, dt \tag{8}$$

and

$$I_2 = I_2(t) = \int_0^t q'^2 (\lambda - q)^{-5/2} \sin^2 2\theta\, dt. \tag{9}$$

The product $q'(\lambda - q)^{-3/2}$ is increasing on $0 < t < \phi(\lambda)$, since the first factor is non-decreasing and positive and the second factor is increasing. In fact, $q'(\lambda - q)^{-3/2}$ increases, from a value which is less than 1, if λ is sufficiently large, to ∞, as t increases from 0 to $\phi(\lambda)$. Hence, for every fixed large λ, there exists a unique number T satisfying $0 < T < \phi(\lambda)$ and

$$q'(T)\big(\lambda - q(T)\big)^{-3/2} = 1, \tag{10}$$

where $T = T(\lambda)$.

The monotony of $q'(\lambda-q)^{-3/2}$ and the second mean value theorem of integral calculus imply the existence of a number $s = s(t)$ with the properties that $0 < s < t$ and that the absolute value of (8) does not exceed $q'(t)\big(\lambda-q(t)\big)^{-3/2}$ times

$$\left|\int_0^s \theta' \sin 2\theta \, dt\right| + \left|\int_s^t \theta' \sin 2\theta \, dt\right| \leqslant 2.$$

Hence, by (10) and the choice $t = T$,

$$|I_1(T)| \leqslant 2. \tag{11}$$

In order to majorize $I_2(T)$, let the sin in (9) be replaced by 1. An integration by parts shows that $I_2(t)$ does not exceed $\frac{2}{3}$ times

$$\Big[q'(\lambda-q)^{-3/2}\Big]_0^t - \int_0^t (\lambda-q)^{-3/2}\, dq'.$$

Since the assumption that q is convex means that $dq' \geqslant 0$, it follows that $I_2(t)$ is majorized by $\frac{2}{3}$ times the first term of the last formula line. Consequently, by (10),

$$0 \leqslant I_2(T) \leqslant \tfrac{2}{3}. \tag{12}$$

Let $N(\lambda) = N_1 + N_2$, where N_1 is the number of zeros of $y(t)$ on $0 \leqslant t < T$ and N_2 is the number of zeros on $T \leqslant t < \infty$. It is clear from (5) and (6) that θ increases through an integral multiple of π at (and only at) the zeros of $y(t)$. Hence $|\pi N_1 - \theta(T)| \leqslant 2\pi$, and so (7), (11) and (12) show that

$$\pi N_1 = \int_0^T (\lambda-q)^{1/2} dt + O(1) \text{ as } \lambda \to \infty. \tag{13}$$

Up to a possible additive correction of 1, N_2 is the number of zeros of $y(t)$ on the interval $T \leqslant t \leqslant \phi(\lambda)$. Since, on this interval, $\lambda - q(t) \leqslant \lambda - q(T)$, it follows from the Sturm comparison theorem that

$$\pi N_2 \leqslant \big(\lambda - q(T)\big)^{1/2}\big(\phi(\lambda) - T\big) + O(1).$$

As $\phi(\lambda)$ is concave and $\phi'(\lambda)$ exists, in the sense

and is non-increasing, it follows that $\phi(\lambda) - T = \phi(\lambda) - \phi\big(q(T)\big)$ does [not] exceed $\lambda - q(T)$ times $1/q'(T)$. Hence, $\pi N_2 \leqslant \big(\lambda - q(T)\big)^{3/2}/q'(T) + O(1)$. Thus, by (10),

$$\pi N_2 = O(1).$$

The argument leading to (14) also supplies the second of the inequalities

$$\int_T^{\phi(\lambda)} (\lambda-q)^{1/2} dt \leqslant \big(\lambda - q(T)\big)^{1/2}\big(\phi(\lambda) - T\big) \leqslant 1. \tag{14}$$

In view of $N(\lambda) = N_1 + N_2$, the formula (4) follows from (13), (14), (15). This proves (*).

2. The method used in proving (13) allows some results of [3] to [be] completed:

(**) *Let $q(t)$ possess a continuous n-th order derivative on $0 \leqslant t < \infty$, where $n \geqslant 0$, and let $q = q^{(0)}, q', \ldots, q^{(n)}$ be bounded on $0 \leqslant t < \infty$. [Let] $N(T, \lambda)$ denote the number of zeros of a solution $y = y(t) = y(t, \lambda) \not\equiv 0$ o[f] ... in $0 < t \leqslant T$. Then for large λ and T, $\pi N(T, \lambda)$ can be written as the [sum] of $\lambda^{1/2} T + O(1) + O(\lambda^{-\frac{1}{2}(n+3)} T)$ and*

$$\sum_{k=1}^{n+2} \lambda^{-\frac{1}{2}k} \int_0^T P_k(q, q', \ldots, q^{(k-3)})\, dt + O(\lambda^{-\frac{1}{2}(n+1)}) \int_0^T w_n(t, \pi/\lambda^{1/2})\, dt,$$

where $\quad w_n(t, \delta) = \text{l.u.b.}\, |q^{(n)}(s) - q^{(n)}(t)|$ *for* $|s - t| \leqslant \delta$

and $\quad P_1(x_0),\ P_2(x_0),\ P_3(x_0),\ P_4(x_0, x_1),\ \ldots,\ P_{n+2}(x_0, x_1, \ldots, x_{n-1})$

are polynomials (independent of q and T).

The proofs of the theorems (I), (I′), (I″) in [3] depend on the c[ases] $n = 0, 1, 2$ of (**). Analogous consequences of the other cases of [(**)] for the spectral theory of boundary value problems associated with ... can be immediately drawn from the general theorem in [1].

The proof of (**) will only be indicated. In terms of y, define a [con]tinuous $\theta = \theta(t, \lambda)$ by

$$\theta = \operatorname{arc\,tan}(\lambda^{1/2} y/y') \text{ and } 0 \leqslant \theta(0) < \pi.$$

Then, by (2), $\quad \theta' = \lambda^{1/2} - \frac{1}{2}\lambda^{-1/2} q + \frac{1}{2}\lambda^{-1/2} q \cos 2\theta,$

or $\quad \theta(T) = \theta(0) + \lambda^{1/2} T - \frac{1}{2}\lambda^{-1/2} \int_0^T q(t)\, dt + \frac{1}{2}\lambda^{-1/2} \int_0^T q \cos 2\theta dt.$

Insert $1 = \theta' \lambda^{-1/2} + \frac{1}{2}\lambda^{-1} q - \frac{1}{2}\lambda^{-1} q \cos 2\theta$ into the last integral and elimi[nate]