August 21, 1976

Suppose $J^*(P)$ is the subgroup of P gen. by abelian subgroups of maximal rank $r = r_p(P)$. If $x \in \Omega_1 Z J^*(P)$, and if $A \subset P$ is elem. abelian of rank r, one has $A \subset J^*(P)$, so $[x, A] = 1$, and $\langle x \rangle A$ is abelian. Since $x^p = 1$, one has $x \in \Omega_1(A) = A$.

$$ZJ^*(P) = \{ x \in P \mid [x, M] = 1 \text{ all abelian } M,\ r_p(M) = r_p(P) \}.$$

$$\subset \bigcap \{ M \mid M \text{ max'l abel. } r_p(M) = r \}$$

M max. abel of rank r, $x \in Z(J^*P) \Rightarrow \langle x \rangle M$ abelian of rank $r \Rightarrow x \in M$. If T abelian of rank r, then $T \subset M$ M max. ab. of rank r, so $x \in M \Rightarrow [x, T] = 1$. $\therefore$

$$Z J^*(P) = \bigcap \{ M \mid M \text{ max. abelian of rank } r \}.$$

But $x \in \Omega_1 Z J^*(P)$, A elem. ab. of rank r $\Rightarrow \langle x \rangle A$ elem ab. $\Rightarrow x \in A$. Conversely suppose $x \in A$ all elem. ab. A of rank $r \Rightarrow x \in M$ all abelian M of rank $r \Rightarrow x \in \Omega_1 Z J^*(P)$. Thus

$$\Omega_1 Z J^*(P) = \bigcap \{ A \mid A \text{ elem. abelian rank } r \}.$$

Suppose P now a Sylow p-subgroup of G and that $ZJ^*(P) \triangleleft G$. Then if A is elem. abelian of rank r in G, we have $A^x \subset P$ so $A^x \supset \Omega_1 Z J^*(P)$, so $A \supset \Omega_1 Z J^*(P)$, as $\Omega_1 Z J^*(P) \triangleleft G$. Thus assuming $ZJ^*(P) \triangleleft G$ we see that

$$\Omega_1 ZJ^*(P) = \bigcap \{A \mid A \text{ elem. ab. in } G \ \operatorname{rank}(A) = r\}$$

Consider next a solvable group G with $O_p(G) = 1$ and $r_p(G) = r$. Put $H = O_{p'}(G)$, $\bar{G} = G/H$. Assuming G is of Glauberman type, I know that

$$\Omega ZJ^*(\bar{P}) = \bigcap \{B \subset \bar{G} \mid B \text{ elem. ab rank } r\}$$

is a non-trivial normal elem. ab. subgroup of $\bar{G}$. Let K be the inverse image of this group in G, so that $\bar{K} = \Omega_1 ZJ^*(\bar{P})$. Let $\mathcal{J}$ denote the poset of those B in $\mathcal{A}_p(\bar{G})$ such that $\bar{K} \subset B$. Now we cover $X = \mathcal{A}_p(G)$ by the sets

$$X_B = \mathcal{A}_p(\pi^{-1}B).$$

as B ranges over $\mathcal{J}$. Given $A \in \mathcal{A}_p(G)$ one has $A \in X_B \iff A \subset \pi^{-1}B \iff \pi A \subset B \iff \pi A \cdot \bar{K} \subset B$.

So it is clear this family doesn't cover X but only the "regular" part i.e. that part consisting of elem. ab subgroups contained in a elem. ab. subgp of rank r.

Assume instead that

$$\bigcap \{ B \in \mathcal{A}_p(\bar{G}) \mid B \text{ max. in } \mathcal{A}_p(\bar{G}) \} = K/H$$

is non-trivial. For example if $\bar{G}$ is a p-group, then $\bar{K}$ contains $\Omega_1 Z(\bar{G})$. Then in this case given any $\bar{A} \in \mathcal{A}_p(\bar{G})$ one has that [redacted] $\bar{A} \subset$ some maximal B, hence $\bar{A}\bar{K}$ is the least elem. ab. subgroup of $\bar{G}$[redacted] containing $\bar{A}$ and $\bar{K}$. Hence $\forall A \in \mathcal{A}_p(G) = X$, $\{B \mid A \in X^B\} = \{B \mid \bar{A}\bar{K} \subset B\}$ is contractible. This should mean that we have a map of some sort

$$X \xrightarrow{f} J$$

with fibres $f^{-1}(B) = X_B$, and hence a spectral sequence

$$E^2_{st} = H_s(J,\ B \mapsto \underset{\substack{\parallel \\ \mathcal{A}_p(\pi^{-1}B)}}{\tilde{H}_t(X_B)}) \Longrightarrow \tilde{H}_{s+t}(X)$$

(can put $\sim$ in because J is contractible). Now I assume known that X_B is spherical of dim rank$(B)-1$. Let $k = \text{rank}(\bar{K})$. Then

$$E^2_{st} = 0 \qquad t < k-1.$$

and for $t = k-1$, $B \mapsto \tilde{H}_t(X_B)$ is supported on the unique minimal element $B = \bar{K}$ of J. Thus it seems that I get

$$\tilde{H}_{k-1}(\mathcal{A}_p(G)) = \tilde{H}_{k-1}(\mathcal{A}_p(K))$$

which is non-trivial.

Something is wrong for this would allow X to be disconnected if $\bar{K}$ has rank 1.

August 22, 1976.

Suppose G solvable and $O_p(G) = 1$. Put $H = O_{p'}(G)$, and $\bar{G} = G/H$. Let $\pi : G \to \bar{G}$ denote the projection. Consider the family $X_B = \mathcal{A}_p(\pi^{-1}B) \subset X = \mathcal{A}_p(G)$ as B ranges over $J = \mathcal{A}_p(\bar{G})$. Thus we look at the map $f : X \to J$ given by $f(A) = \pi A$ and $X_B = \{A \in X \mid fA \leq B\} = f/B$. Remark that $\forall A$, $\{B \mid A \subset X_B\}$ has the least element $f(A)$. So now form

$$\bigcup_{|B_0 < \cdots < B_k| \in |J|} |B_0 < \cdots < B_k| \times |X_{B_0}|$$

which maps to $|X|$ with contractible fibres. Better I want the spectral sequence of the map f:

$$E^2_{st} = H_s(J, B \mapsto H_t(f/B)) \Rightarrow H_{s+t}(X)$$

Since $J = \mathcal{A}_p(\bar{G})$ is contractible, one should have a spectral

sequence
$$E^2_{st} = H_s\big(J, B \mapsto \tilde{H}_t(f(B))\big) \Longrightarrow \tilde{H}_{s+t}(X)$$
$$\tilde{H}_t(f(B)) = \tilde{H}_t(\mathcal{a}_p(\pi^{-1}B)) \qquad \tilde{H}_{s+t}(X) = \tilde{H}_{s+t}(\mathcal{a}_p(G))$$

But now use the fact that $\mathcal{a}_p(\pi^{-1}B)$ is a bouquet of rank$(B)-1$ spheres so that $B \mapsto \tilde{H}_t(\mathcal{a}_p(\pi^{-1}B))$ is concentrated on rank$(B) = t+1$.

<u>Lemma</u>: Let L be an abelian group and $L_{j_0} : J \to \mathcal{Ab}$ the functor sending j_0 to L and all other j in J to zero. Then
$$H_i(J, L_{j_0}) = H_i(\{\geq j_0\}, \{> j_0\}; L)$$

<u>Proof.</u> Suppose $S \subset J$ closed under generalization and let $L_S : J \to \mathcal{Ab}$ be zero outside of S and L on S. Then
$$H_i(J, L_S) = H_i(S, L)$$

as one sees from the spectral sequence of $i : S \to J$: i/j is empty if $j \notin S$ and has final elt j if $j \in S$. Now use exact sequence
$$0 \to L_{\{>j_0\}} \to L_{\{\geq j_0\}} \to L_{j_0} \to 0$$
$$\to H_i(\{>j_0\}, L) \to H_i(\{\geq j_0\}, L) \to H_i(J, L_{j_0}) \to \cdots$$

So

$$E^2_{st} = H_s\left(J,\ B \mapsto \tilde{H}_t(f/B)\right)$$

$$= \bigoplus_{r(j)=t+1} H_s\left(\{\geq j\}, \{>j\}\right) \otimes \tilde{H}_t(f^{-1}j)$$

Note: If we replace j by B, then $\{>j\}$ becomes $\mathcal{A}_p(\bar{G})_{>B} = \mathcal{A}_p(C_{\bar{G}}(B))_{>B}$. Suppose this conically contractible: $A \leq AB_0 \geq B_0$ some $B_0 > B$; then $B_0 \subset$ all max. A in $\mathcal{A}_p(C_{\bar{G}}(\bar{B}))$; ~~conversely this~~ conversely if $B < B_0 \subset$ all max. A, then given A one can put A in A' maximal, whence $A' \supset B_0$ so AB_0 is elementary abelian and $\mathcal{A}_p(C_{\bar{G}}(B))_{>B}$ is conically contractible. This shows that if $\Omega_1 Z(\bar{G}) \not\subset B$ then $\mathcal{A}_p(G)_{>B}$ is contractible.

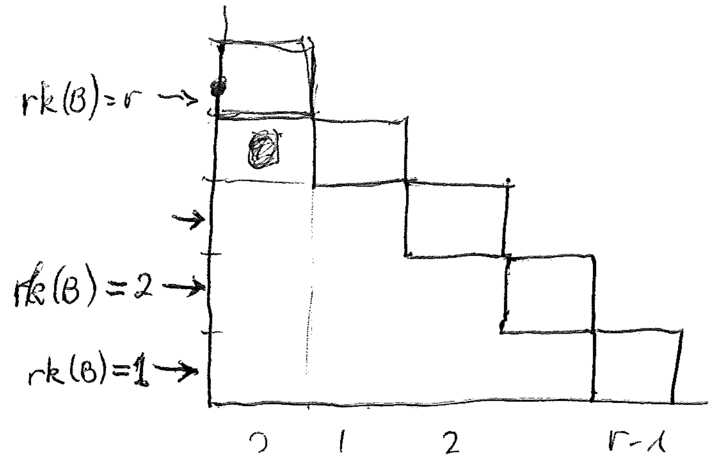
If B is a max elem. ab. group, $\mathcal{A}_p(\bar{G})_{>B}$ is empty. So look at $\tilde{H}_{r-1}(X)$. This involves the terms

$$E^2_{0,r-1} = \bigoplus_{r(B)=r} \tilde{H}_{r-1}\left(\mathcal{A}_p(\pi^{-1}B)\right)$$

$$E^2_{1,r-2} = \bigoplus_{r(B)=r-1} \tilde{H}_0\left(\mathcal{A}_p(\bar{G})_{>B}\right) \otimes \tilde{H}_{r-2}\left(\mathcal{A}_p(\pi^{-1}B)\right)$$

etc.

Picture:



Note that $|\mathcal{A}_p(\bar{G})_{>B}|$ has dimension $r - rk(B) - 1$ so that

$$H_s\left(\{\geq B\}, \{>B\}\right) \neq 0 \implies \begin{array}{l} s \leq r - rk(B) \\ \quad r - t - 1 \\ \text{or } \ s + t \leq r - 1 \end{array}$$

Seems that therefore $E^2_{0,r-1}$, $E^2_{1,r-2}$ appear intact in E^∞. Also $E^2_{0,t} \neq 0$

means that there are maximal elementary abelian groups of rank $t+1$.

<u>Question</u>: If all maximal elem. ab. p-groups have the same rank r, does it follow that $\mathcal{A}_p(G)$ is spherical of dim. $r-1$?

It seems from Glauberman's thm. that if all max. elem. ab. p-groups of G have rank r, then their intersection is $\Omega_1 Z(J^*(P))$ which is non-trivial, so we get some sort of connectivity for $|\mathcal{A}_p(G)|$.

Here is a good class of groups G: those such that for every A in $\mathcal{A}_p(G)$ and $A = 1$, the complex $\mathcal{A}_p(G)_{>A} = \mathcal{A}_p(C_G(A))_{>A}$ is spherical of dimension $r_p(G) - r_p(A) - 1$.

Example: 1) Solvable groups with abelian Syl$_p$-groups. Note that if $G = H \rtimes Q$ and $A \subset Q$, then

$$C_G(A) = H^A \rtimes Q$$

if Q is abelian, so that if Q is elementary abelian

$$\mathcal{A}_p(C_G(A))_{>A} = \mathcal{A}_p(H^A \rtimes Q/A)$$

is spherical of dim $r_p(Q/A) - 1 = r_p(G) - r_p(A) - 1$.

2) If $\bar{G}$ is in the class, $\bar{G} = G/H$ where H is a solvable p'-group, then G is also in the class.

3) If $G_1 \times G_2 \supset A$, let $A_i = \mathrm{Im}\{A \to G_i\}$. Then

$$C_{G_1 \times G_2}(A) = C_{G_1}(A_1) \times C_{G_2}(A_2)$$

is contractible unless $A = A_1 \times A_2$ in which case $C_{G_1 \times G_2}(A_1 \times A_2)_{> A_1 \times A_2}$ should deform into the join of the $C_{G_i}(A_i)$. $\therefore$ The class is closed under products.

4). Take $G = P$ of the form

$$1 \longrightarrow \mathbb{Z}/p \longrightarrow P \longrightarrow V \longrightarrow 1$$

central extension. Suppose p odd, whence I can replace

P by $\Omega_1 P$ which is of exponent p without changing elem. ab. groups. In this case one has that $\mathcal{A}_p(G)_{>A}$ is contractible unless A contains $\mathbb{Z}/p$ in which case A can be identified with an isotropic subspace for the commutator pairing $\Lambda^2 V \longrightarrow \mathbb{Z}/p$. Supposing this non-degenerate one has $\mathcal{A}_p(G)_{>\mathbb{Z}/p}$ = the complex of non-trivial isotropic subspaces which is the building for a symplectic group. Rest is clear.

5) $r_p(G) = 1$, forces this property.

$\mathcal{A}_p^*(G)$ = subset of $\mathcal{A}_p(G)$ consisting of A such that $\exists$ a $B \in \mathcal{A}_p(G)$ with $A \leq B$, $r(B) = r_p(G)$.

Suppose G solvable, and let $H \triangleleft G$ be such that G/H is cyclic of prime order. If $(G:H) \not\equiv 0 \pmod p$ then $\mathcal{A}_p^*(H) = \mathcal{A}_p^*(G)$. Similarly this holds if $\Omega_1 G \subset H$ so consider the case when $\exists\, x$ of order p in $G \ni x \notin H$.

Let S be the set of cyclic order p subgroups C of G complementary to H. If we fix a generator of G/H, then S can be viewed as the set of order p elements $x \ni xH$ is this generator.

$C \in \mathcal{A}_p^*(G)$ means $\exists\, A \in \mathcal{A}_p(G)$ of rank $r = r_p(G)$ with $C \subset A$, whence $A \cong H \cap A \oplus C$. As $H \cap A \subset H^C$ one has $r_p(H^C) = r-1$. Let $S^* = S \cap \mathcal{A}_p^*(G)$. The elements of S^* are vertices of $|\mathcal{A}_p^*(G)|$ and the link of C

$$\text{Link of } C \text{ in } |\mathcal{A}_p^*(G)| = |\{A \in \mathcal{A}_p^*(G) \mid A > C\}|$$

$$\cong |\mathcal{A}_p^*(H^C)|$$

$A \in \mathcal{A}_p(G)$, $A > C \Rightarrow A = (A \cap H) \times C \subset H^C \times C = C_G(C)$.

$A \in \mathcal{A}_p(G)$, $A > C \Leftrightarrow A \cap H \in \mathcal{A}_p(H^C)$. same holds with $*$. ~~On the old~~

On the other hand upon removing the vertices $C \in S^*$ from $\mathcal{A}_p^*(G)$ the rest contracts into $\mathcal{A}_p^*(H)$ by the map $A \mapsto A \cap H$. Be careful. The problem occurs if $r_p(H) = r_p(G)$. We have to be sure that if A is of rank r and $G = AH$, then $A \cap H$ can be extended inside H to something of rank r. ?

~~Suppose G has a normal p-complement~~ ~~$\mathcal{A}_p(G)$ elementary abelian. Then in the preceding argument~~

What the above argument shows is that $\mathcal{A}_p(G)$ is obtained from $\mathcal{A}_p(H)$ by attaching the cone on $\mathcal{A}_p(H^C)$ for each C in S. Suppose that $G = O_{p'}(G) \cdot A$ where A is elementary abelian of rank r. Then $H = O_{p'}(G)(A \cap H)$.

If A is chosen so as to contain C, then

$$H^C = O_{p'}(G)^C (A \cap H)$$

so $\mathcal{A}_p(H^C)$ is a bouquet of $(r-2)$-spheres by induction. Unfortunately $\mathcal{A}_p(H)$ is also a bouquet of $(r-2)$-spheres.

Review

$$1 \longrightarrow H \longrightarrow G \xrightarrow{\varphi} \mathbb{F}_p \longrightarrow 1$$

S = set of complements to H. $\simeq \{x \in G \mid x^p = 1, \varphi(x) = 1\}$.
Assume S non-empty. If $C \in S$, then

$$C_G(C) = C_H(C) \times C$$

so any $A \geq C$ is of the form $(A \cap H) \times C$, hence the link of C in $\mathcal{A}_p(G)$ can be identified with $\mathcal{A}_p(H^C)$. So

$$\mathcal{A}_p(G) = \bigcup_{C \in S} \mathcal{A}_p(H) \cup_{\mathcal{A}_p(H^C)} pt$$

If you wanted to use this to understand a group G with abelian Sylow p-groups you want to show that $\mathcal{A}_p(H) \hookrightarrow \mathcal{A}_p(G)$ is null homotopic. No simple way to see this it seems.

August 28, 1976

Assume $X = \mathcal{a}_p(G)$ is spherical of dim $r-1$, $r = r_p(G)$, and let $J = \tilde{H}_{r-1}(X)$, so that $J_{(p)}$ is a projective $\mathbb{Z}_{(p)}[G]$-module. One has an exact sequence

$$\cdots \longrightarrow H_G^*(pt, \mathbb{F}_p) \to H_G^*(X, \mathbb{F}_p) \longrightarrow H_G^{*-r+1}(pt; \check{J} \otimes \mathbb{F}_p) \longrightarrow \cdots$$

$$\| $$

$$\begin{cases} 0 & * \neq r-1 \\ (\check{J} \otimes \mathbb{F}_p)^G & * = r-1 \end{cases}$$

$$(\check{J} \otimes \mathbb{F}_p)^G = \mathrm{Hom}_G(J, \mathbb{F}_p) = \tilde{H}^{r-1}(X, \mathbb{F}_p)^G$$

Now it might be possible to calculate J_G by characters, and show that it is zero. If so, one has

$$H^*(G, \mathbb{F}_p) = H_G^*(X, \mathbb{F}_p)$$

so we have possibilities of induction.

character of J can be calculated via Lefschetz:

$$\chi_{H^*(X)}(g) = \chi(X^g)$$

Thus

$$\pm \chi_J(g) = \chi(X^g) - 1$$

and

$$X^g = \mathcal{a}_p(G)^g = \text{complex of elem. ab. subgroups normalized by } g$$

K. Brown's argument shows $\chi(X^g) - 1 \equiv 0 \mod |C_G(g)|_p$. The good case is where for each p'-element one has

$$\chi(X^g) - 1 = \pm |C_G(g)|_p$$

because then Feit shows $[J]$ generates the ideal of projectives in $R_{\overline{\mathbb{F}}_p}(G)$.

Question is whether $(J\otimes \mathbb{F}_p)_G = 0$.

$$0 \longrightarrow J_{(p)} \xrightarrow{\;p\;} J_{(p)} \longrightarrow (J\otimes\mathbb{F}_p) \longrightarrow 0$$

$$0 \to (J_{(p)})_G \xrightarrow{\;p\;} (J_{(p)})_G \longrightarrow (J\otimes \mathbb{F}_p)_G \longrightarrow 0$$

By Nakyama $\quad (J\otimes\mathbb{F}_p)_G \neq 0 \iff (J_{(p)})_G \neq 0.$

Point: Work directly with $J_{(p)}$ which is projective of $\mathbb{Z}_{(p)}[G]$, hence

$$J_{(p)\,G} \xrightarrow{\;\sim\;} J_{(p)}^{\;G}$$

as $\check{H}^0(G, J_{(p)}) = 0$. Now $J_{(p)}^{\;G} \neq 0 \iff J^G\otimes\mathbb{Q} \neq 0$, so we can see this using the character.

$$(J_G)\otimes\mathbb{Q} \cong H_{n-1}(X/G)\otimes\mathbb{Q}.$$

So it seems that $J_G = 0 \iff \chi(X/G) = 1$.

September 2, 1976.

H normal p'-group in G, $\bar{G} = G/H$. Then we have $\pi : \mathcal{A}_p(G) \longrightarrow \mathcal{A}_p(\bar{G})$ and $\pi/B = \mathcal{A}_p(\pi^{-1}B)$, $(\pi A \le B \iff A \le \pi^{-1}B)$. So we get a spectral sequence

$$E^2_{st} = H_s(\mathcal{A}_p(\bar{G}), B \mapsto H_t(\mathcal{A}_p(\pi^{-1}B))) \Longrightarrow H_{s+t}(\mathcal{A}_p(G)).$$

Now I know $\mathcal{A}_p(\pi^{-1}B) \sim$ bouquet of $(r(B)-1)$-spheres.

Construct this spectral sequence by hand. Consider the poset consisting of pairs (A, B) $A \in \mathcal{A}_p(\pi^{-1}B)$, $B \in \mathcal{A}_p(\bar{G})$. The ordering is $(A,B) \le (A', B') \iff A' \le A$ and $B \le B'$. Thus a simplex is a chain

$$A_k \le \cdots \le A_0 \qquad B_0 \le \cdots \le B_k \qquad A_0 \subseteq \pi^{-1}B_0.$$

Thus we are inside of $|\mathcal{A}_p(G)| \times |\mathcal{A}_p(G')|$. We are covering $X = |\mathcal{A}_p(G)|$ with $|\mathcal{A}_p(\pi^{-1}B)| = X_B$, $B \in \mathcal{A}_p(G') = J$. So we form

$$X_J = \bigcup_{B_0 \le \cdots \le B_k} X_{B_0} \times |B_0 \le \cdots \le B_k| \subset X \times |J|.$$

The map $X_J \longrightarrow X$ is a homotopy equivalence because $\{B \mid A \subset \pi^{-1}B\} = \{B \mid \pi A \le B\}$ has the least elt. πA.

So now we want to filter $|J|$ by decreasing rank. If $r_p(G') = r$, then we start with

$$F_0 \mathcal{A}_p(G') = \{B \mid r(B) = r\}$$

$$F_s \mathcal{A}_p(G') = \{B \mid r(B) \ge r - s\}$$

So $\quad F_0 X_J = \coprod_{r(B)=r} X_B$

Somehow this is the wrong approach because $F_0 X_J$ is not a bouquet of $(r-1)$-spheres; somehow one has worked in $F_0 J$.

Better approach. Use homotopy theory to conclude that $\mathcal{A}_p(G)$ is spherical. Use the spectral sequence

$$E^2_{st} = H_s(\mathcal{A}_p(\bar{G}), B \mapsto H_t(\mathcal{A}_p(\pi^{-1}B))) \Longrightarrow H_{s+t}(\mathcal{A}_p(G)).$$

to show $\mathcal{A}_p(\bar{G})$ has trivial reduced homology in degrees $< r-1$. Then show directly $\mathcal{A}_p(G)$ is simply-connected if $r \geq 3$.

Suppose $r \geq 3$ and let $E : \mathcal{A}_p(G) \to \text{Sets}$ be a morphism-inverting functor. I propose to show $E \simeq f^*\bar{E}$ for some $\bar{E} : \mathcal{A}_p(\bar{G}) \to \text{Sets}$. If $B \in \mathcal{A}_p(\bar{G})$ has rank ≥ 3, I know $\mathcal{A}_p(\pi^{-1}B)$ is simply-connected hence I can define $\bar{E}(B) = \varinjlim_{A \in \mathcal{A}_p(\pi^{-1}B)} E(A)$ and I know that $E(A) \xrightarrow{\sim} \bar{E}(B)$ for any $A \in \mathcal{A}(\pi^{-1}B)$. Suppose B has rank 2. Then because $\exists\, \tilde{B}$ of rank 3 containing B, I know that $\mathcal{A}_p(\pi^{-1}B)$ contracts to a point in $\mathcal{A}_p(\pi^{-1}\tilde{B})$, so E restricted to $\mathcal{A}_p(\pi^{-1}B)$ is trivial and so I can define $\bar{E}(B)$ so that $E(A) \xrightarrow{\sim} \bar{E}(B)$ for $A \in \mathcal{A}(\pi^{-1}B)$. Now you have to check that if B is of rank 1, then $\mathcal{A}_p(\bar{G})_{>B}$ is connected, for then you can define $\bar{E}(B) = \varprojlim_{\tilde{B} > B} \bar{E}(\tilde{B})$, and you know that $\bar{E}$ on $\mathcal{A}_p(\bar{G})_{>B}$ is constant because you can pick an A over B.

Spectral sequence. Assuming skyscraper lemma we know that for $t \geq 1$

$$E^2_{st} = H_s\left(\mathcal{A}_p(\bar{G}),\ B \mapsto H_t(\mathcal{A}_p(\pi^{-1}B))\right) = \bigoplus_{r(B)-1=t} \tilde{H}_{s-1}(\mathcal{A}_p(\bar{G})_{>B}) \otimes \tilde{H}_t(\mathcal{A}_p(\pi^{-1}B))$$

By assumption $\mathcal{A}_p(\bar{G})_{>B}$ has zero homology outside of degree $r-r(B)-1$, so $E^2_{st}=0$ outside of $s-1=r-r(B)-1$ i.e. outside of $s+t = r-r(B)+t = r-1$. For $t=0$ one uses the exact sequence

$$0 \longrightarrow \tilde{H}_0(\mathcal{A}_p(\pi^{-1}B)) \longrightarrow H_0(\mathcal{A}_p(\pi^{-1}B)) \longrightarrow \mathbb{Z} \longrightarrow 0$$

supported on $\operatorname{rank}(B)=1$.

So

$$\longrightarrow \bigoplus_{r(B)=1} \tilde{H}_{s-1}(\mathcal{A}_p(\bar{G})_{>B}) \otimes \tilde{H}_0(\mathcal{A}_p(\pi^{-1}B)) \longrightarrow E^2_{s0} \longrightarrow H_s(\mathcal{A}_p(\bar{G})) \longrightarrow \cdots$$

contributes only for $s=r-1$

so E^2_{s0} is zero outside of $s=r-1$. Thus we see that $\tilde{H}_n(\mathcal{A}_p(G))$ is zero outside $n=r-1$, and that it admits a filtration with quotients:

$$E^2_{r-1,0} = \bigoplus_{r(B)=1} \tilde{H}_{r-2}(\mathcal{A}_p(\bar{G})_{>B}) \otimes \tilde{H}_0(\mathcal{A}_p(\pi^{-1}B))$$

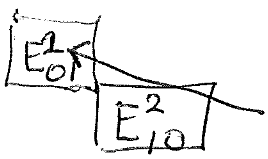
$$E^2_{r-l,l-1} = \bigoplus_{r(B)=l} \tilde{H}_{r-l-1}(\mathcal{A}_p(\bar{G})_{>B}) \otimes \tilde{H}_{l-1}(\mathcal{A}_p(\pi^{-1}B))$$

for $l = 1, \ldots, r = r(G)$.

and the last quotient is $\tilde{H}_{r-1}(\mathcal{A}_p(\bar{G}))$.

Let's review why $r \geq 3 \Rightarrow \mathcal{A}_p(G)$ simply-conn. Idea is to see if this works more generally. I want to take $\bar{G} = G/O_{p'}(G)$, $H = O_{p'}(G)$. So I need to know that each B of rank 2 in $\mathcal{A}_p(\bar{G})$ is not maximal, and that for each B of rank 1, $\mathcal{A}_p(\bar{G})_{>B}$ is connected.

Actually as $\mathcal{A}_p(\bar{G})$ is contractible, the degree 1 line of the spectral sequence is:



so that $H_1(\mathcal{A}_p(G)) \twoheadrightarrow E^2_{10} \longrightarrow 0$

$$E^2_{10} = \bigoplus_{r(B)=1} \tilde{H}_0(\mathcal{A}_p(\bar{G})_{>B}) \otimes \tilde{H}_0(\mathcal{A}_p(\pi^{-1}B))$$

Question: ① Does $r(\bar{G}) \geq 3 \Rightarrow$ no maximal $B \in \mathcal{A}_p(\bar{G})$ of rank 2? ② Does $r(\bar{G}) \geq 3 \Rightarrow \mathcal{A}_p(\bar{G})_{>B}$ conn. for all $r(B) = 1$?

①. Can suppose $\bar{G}$ is a p-groups P.

Both are false for $P = \mathbb{Z}/p\mathbb{Z} \wr (\mathbb{Z}/p\mathbb{Z})$ which is the Sylow p-subgp of Σ_{p^2}.

Sept. 9, 1976

$\mathcal{A}_p(G) \neq \phi$, and $O_p(G) = 1$

Suppose G solvable. I have seen that if G contains a maximal elem. abelian subgp of rank r, then $\tilde{H}_{r-1}(\mathcal{A}_p(G)) \neq 0$. Thus $\mathcal{A}_p(G)$ is not simply-connected if there is a maximal elem. abelian group of rank 2.

Recall that $\pi_0 \mathcal{A}_p(G) \neq pt \iff r_p(G) = 1$. So assume $r_p(G) > 1$, whence $\mathcal{A}_p(G)$ is connected. Note that then by the above no A of rank 1 is maximal in $\mathcal{A}_p(G)$; this can also be seen directly. (If $|A| = p$ in a p-group P of rank ≥ 2, then either $A \subset \Omega_1 Z(P) \subset$ any max. elt of $\mathcal{A}_p(P)$, or $A \not\subset \Omega_1 Z(P)$ whence $A < A\Omega_1 Z(P)$, so A is not maximal in $\mathcal{A}_p(P)$)

Suppose then that $A \in \mathcal{A}_p(G)_{max} \Rightarrow r_p(A) \geq 3$. I want to show then that $\mathcal{A}_p(G)$ is 1-conn. I have seen that it is enough to show that in $\bar{G} = G/O_{p'}(G)$ we have no maximal elements of rank 2 in $\mathcal{A}_p(\bar{G})$, which is clear, and that $\operatorname{rank}(B) = 1 \Rightarrow \mathcal{A}_p(\bar{G})_{>B}$ is connected.

First consider the case where $B = \Omega_1 Z(P)$ has rank 1, P is a p-group of rank ≥ 3 having no max elementary ab. subgroups of rank 2. Let A_0 be a max. normal elem. abelian subgroup of P. Assuming p-odd, it follows that $A_0 > B$, because A_0 is a max. elt of $\mathcal{A}(P)$ hence $\operatorname{rank}(A_0) \geq 3$. Let $R \in \mathcal{A}_p(P)_{>B}$. Choose a max. $Q \supset R$ in $\mathcal{A}_p(P)$. Make Q act on A_0. Let $A_1 \subset A_0$, $A_1 \triangleleft P$, $\operatorname{rank} A_1 = 2$. Then $Q/Q^{A_1} \hookrightarrow \operatorname{Aut}(A_1)$, so $\operatorname{rank} A_1 = 2 \Rightarrow rk(Q/Q^{A_1}) \leq 1 \Rightarrow$

$rk(Q^{A_1}) \geq rk(Q) - 1 \geq 2$. So we have

$$R \subset Q \supset Q^{A_1} \subset Q^{A_1}\cdot A_1 \supset A_1$$

which shows $\mathcal{A}_p(P)_{>B}$ is connected. So have proved:

<u>Lemma:</u> Let [redacted] p be odd, let P be a p-group with [redacted] $B = \Omega_1 Z(P)$ [redacted] cyclic. Then $\mathcal{A}_p(P)_{>B}$ is connected provided all max. elements of $\mathcal{A}_p(P)$ have rank ≥ 3.

Now suppose $B = \Omega_1 Z(\bar{G})$ [redacted] and P is a Sylow p-subgroup of $\bar{G}$ as in the lemma. ?

Suppose we set up the spectral sequence for $\mathcal{A}_p(G)_{>Z}$ where $Z \subset \Omega_1 Z(G)$. [redacted] This should be similar.

$$f : \mathcal{A}_p(G)_{>Z} \longrightarrow \mathcal{A}_p(\bar{G})_{>\bar{Z}}$$

$\pi^{-1}\bar{Z} = H \times Z$. $\bar{Z} < \bar{A} \subset \bar{G}$ $\pi^{-1}\bar{Z} = H \times Z < \pi^{-1}\bar{A} = H \rtimes A$ and clearly as $Z \triangleleft G$, we must have $Z < A$. Thus [redacted] we have [redacted]

$$f/B = \{ A \in \mathcal{A}_p(G)_{>Z} \mid \bar{A} \subset B \}$$

$$= \mathcal{A}_p(\pi^{-1}B)_{>Z} = \mathcal{A}_p(\overset{\pi^{-1}B}{\cancel{H \times B}}/Z)$$

It follows we get the same sort of spectral sequence. Look at

$$E^2_{00} = H_0(\mathcal{A}_p(\bar{G})_{>\bar{Z}},\ B \mapsto H_0(\mathcal{A}_p(\overset{\pi^{-1}B}{\cancel{H \times B}}/Z)))$$

Since $r(B) > r(Z)$ for $B \in \mathcal{A}_p(\bar{G})_{>\bar{Z}}$, one has

$a_p(\pi^{-1}B/Z) \neq \phi$, hence

$$0 \to \tilde{H}_0(a_p(\pi^{-1}B/Z)) \to H_0(a_p(\pi^{-1}B/Z)) \to \mathbb{Z} \to 0$$

$\tilde{H}_0(a_p(\pi^{-1}B/Z))$: zero for $r(B) \geq r(Z)+2$.

hence

$$H_0(a_p(\bar{G})_{>\bar{Z}}, B \mapsto \tilde{H}_0(a_p(\pi^{-1}B/Z))) \to E^2_{00} \to H_0(a_p(\bar{G})_{>Z}) \to 0$$

The first group ought to be zero if $a_p(\bar{G})_{>B} \neq \phi$ for each $B \supset \bar{Z}$ with $r(B) = 1 + r(\bar{Z})$. But it's the second group we have no control over once $O_p(\bar{G}) = 1$.

September 6, 1976

Go back to the spectral sequence for $f: a(G) \to a(\bar{G})$ in the case where $G = H \rtimes A$, $\bar{G} = G/K = H/K \rtimes A$.

$$E^2_{st} = \bigoplus_{rg(B)=\cancel{t}+1} H_s\left(a(\bar{G})_{\geq B}, a(\bar{G})_{>B}\right) \otimes \tilde{H}_{\cancel{t}}\left(a(\pi^{-1}B)\right).$$

In the case $G = HA$, fix $B \in a(\bar{G})$ and suppose A chosen so that $B \subset A$. Then

$$a(\bar{G})_{>B} = a(C_{\bar{G}}(B))_{>B} = a(\bar{H}^B \rtimes A/B)$$

is a bouquet of $r(A) - r(B) - 1$ spheres, which is non-trivial if A/B acts faithfully on $\bar{H}^B$. In the case $\bar{H}$ = a simple non-trivial $\mathbb{Z}[A]$-module, this means exactly that $B = A_0$ or $B = A$.

Recall that $H_{r-1}(a(G))$ had two kinds of elements in it:

$$\tilde{H}_{r-1}(a(G)) \cong \tilde{H}_{r-1}(a(KA))^{|\bar{H}|} \oplus \tilde{H}_{r-2}(a(KA_0))^{|\bar{H}|-1}.$$

I can clearly see the first kind of elements. Since A/A_0 acts freely on $\bar{H}$, there are $|\bar{H}|$ elements B/A_0 in $a(\bar{H} \rtimes A/A_0)$. Thus there are $|\bar{H}|$ different maximal elements in $H/K \rtimes A$. Specifically you choose a coset in $\bar{H}$; this gives you a $B \subset \bar{H} \rtimes A$, hence a map

$$\tilde{H}_{r-1}\left(a(\pi^{-1}B)\right) \longrightarrow \tilde{H}_{r-1}(a(G))$$

which is one of the direct summands of interest.

Next one has [redacted] elements arising from $B = A_0$. Somehow there should be a way of seeing the join of $\mathcal{a}(KA_0)$ and $\mathcal{a}(\bar{H} \rtimes A/A_0)$ inside $\mathcal{a}(G)$.

<u>September 7, 1976</u>

$G = H \rtimes A$, suppose H is abelian, [redacted] say H is an $\mathbb{F}_\ell$-vector space, on which A acts faithfully. [redacted] Then

$$H/H^A = \bigoplus_{B \in \mathbb{P}(A^\vee)} H^{B}/H^A.$$

where B runs over $\mathbb{P}(A)^\vee$. For each $B_1, \dots, B_r \in \mathbb{P}(A^\vee)$ [redacted] such that

$$A \longrightarrow \prod_{i=1}^{r} A/B_i$$

we can consider the subgroup $\sum_{i=1}^{r} H^{B_i}$ and we know that

$$\mathcal{a}\Big(\sum_i H^{B_i} \rtimes A\Big) = \mathcal{a}\Big(\bigoplus H^{B_i}/H^A \rtimes A\Big)$$
$$\cong \mathop{\ast}_i \mathcal{a}\big(H^{B_i}/H^A \rtimes A/B_i\big)$$

is a bouquet of $(r-1)$-spheres. It is more or less clear that we can generates $\tilde{H}_{r-1}(\mathcal{a}_r(G))$ using these subgroups.

Go back to the situation $G = H \rtimes A$, $\bar{H} = H/K$ where A/A_0 acts freely on $\bar{H} - 1$. Then to get all the "spheres" in $a(HA)$ I used the covering $a(KA^h)$ as h runs over $\bar{H}$. In addition to the spheres obtained from some $a(KA^h)$, I used spheres obtained from the intersection:

$$
\begin{tikzcd}
 & a(KA) \arrow[dr, phantom, sloped, "\subset"] & \\
a(KA_0) \arrow[ur, phantom, sloped, "\subset"] \arrow[dr, phantom, sloped, "\subset"] & & a(HA) \\
 & a(KA^h) \arrow[ur, phantom, sloped, "\subset"] &
\end{tikzcd}
$$

To realize such spheres I have to start with an $(r-2)$-sphere in $a(KA_0)$ and then use a contraction of it in $a(KA)$ and $a(KA^h)$. Note $A \cap A^h = A_0$ can be assumed since without changing KA^h, I can replace h by kh and so suppose $kh \in H^{A_0}$. So

September 10, 1976. Alperin's thm.

Let P be a fixed Sylow p-subgroup of G. A subgroup H of P will be called "nice" if
i) $N_p(H) \in Syl_p(N_G(H))$ ii) $\pi_0(S_p(N_G(H)/H))$ ~~connected~~ $\neq pt$.
If $Q, R \in Syl_p(G)$ we write $R \sim Q$ if $\exists$ sequence H_i, y_i $i=1,\dots,m$ with H_i nice, $y_i \in N_G(H_i)$ such that

$$Q = R^{y_1 \cdots y_m}$$

$$(P \cap R)^{y_1 \cdots y_{i-1}} \subset H_i \qquad i=1,\dots,m$$

We want to prove $R \sim P$ for all $R \in Syl_p(G)$.

Lemma 1. $S \sim R,\ R \sim Q \Rightarrow S \sim Q$.

Only have to check that if $S \sim R$ via $\{H_i, y_i,\ 1 \le i \le m\}$ then $(P \cap S)^{y_1 \cdots y_m} \subset P \cap R$. But $(P \cap S)^{y_1 \cdots y_m} = ((P \cap S)^{y_1 \cdots y_{m-1}})^{y_m}$ $\subset (H_m)^{y_m} = H_m \subset P$ and $(P \cap S)^{y_1 \cdots y_m} \subset S^{y_1 \cdots y_m} = R$, so clear.

Lemma 2. Assume $P \cap R \subset P \cap Q$ and $Q \sim P$ via $\{H_i, y_i,\ i=1,\dots,m\}$. Then $R \sim R^{y_1 \cdots y_m}$ via the same sequence.

Have to check $(P \cap R)^{y_1 \cdots y_{i-1}} \subset H_i$, which is clear since $P \cap R \subset P \cap Q$ and $(P \cap Q)^{y_1 \cdots y_{i-1}} \subset H_i$

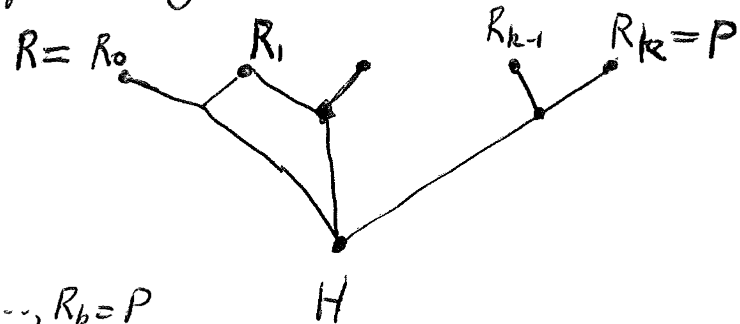
~~Lemma 3. Assume $Q \sim P$ for all $Q \in Syl_p(G)$ such that $|P \cap Q| \ge p^b$. Then $R \sim P$ for all $R \in Syl_p(G)$~~

~~Lemma 3. Let H be a subgroup of P and let $R \in Syl_p(G)$. Assume R, P are in the same component of $S_p(G)_{>H}$~~

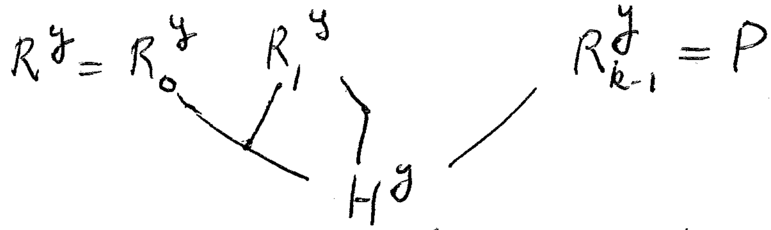
Lemma 3. Let $H \subset P \cap R$ with $R \in Syl_p(G)$. Assume P, R are in the same component of $S_p(G)_{>H}$

and that $Q \sim P$ for all $Q \in Syl_p(G) \ni |P \cap Q| > |H|$. Then $R \sim P$.

<u>Proof</u>. By assumption we have a chain of Sylow



$R = R_0, R_1, \dots, R_k = P$ groups containing H such that $R_{i-1} \cap R_i > H$. Prove the lemma by induction on k. Clear if $k = 1$. So $R_{k-1} \sim P$ via $\{H_i, y_i\}$. By lemma 2, $R \sim R^y$ if $y = y_1 \cdots y_m$. So we get



to which we can apply the inductive version of the lemma to get $R^y \sim P$. Then lemma 1 gives $R \sim P$.

Proof that $R \sim P$ for all $R \in Syl_p(G)$: Using decreasing induction on $|P \cap R|$, we can suppose $Q \sim P$ for all $Q \in Syl_p(G) \ni |P \cap Q| > |P \cap R|$. ~~Apply the~~ Put $H = P \cap R$. ~~[illegible]~~ Let $Q \in Syl_p(G)$ be such that $N_Q(H) = Q \cap N_G(H)$ is a Syl_p subgp of $N_G(H)$ containing $N_P(H)$. Since $N_P(H) > H$ $(H = P \Rightarrow R = P)$, one has $Q \sim P$ via some sequence (H_i, y_i) whence $R \sim R^y$ by lemma 2 where $y = y_1 \cdots y_m$. Also $|H| = |P \cap R| = |(P \cap R)^y| \leq |H_m \cap R^y| \leq |P \cap R^y|$, so either $|P \cap R^y| > |H|$ whence $R^y \sim P$ and we are done by induction, or else $|P \cap R^y| = |H|$ and $P \cap R^y = H^y$

so that $P = Q^y$ is a S_p-subgp of $N_G(H^y)$. Thus we can assume that R is such that $N_p(H) \in Syl_p(N_G(H))$. According to Lemma 3 if R, P are in the same component of $\mathcal{S}_p(G)_{>H}$ we have $R \sim P$. So we have to consider the case where R, and P are in different components, whence because $\mathcal{S}_p(G)_{>H} \sim \mathcal{S}_p(N_G(H)/H)$, we see that H is "nice". In this case $\exists\, y \in N_G(H) \ni$ R^y and P are in the same component of $\mathcal{S}_p(G)_{>H}$. We have $R \sim R^y$ and $R^y \sim P$, so done.

Application: Let N be the stabilizer of the component of $\mathcal{S}_p(G)$ containing P. I claim N is generated by $\{N_G(H)\}$ as H ranges over the non-identity nice subgroups of P. In effect let M be the subgroup generated by these normalizers, so that $M \subset N$ and $N_G(P) \subset M$. To show that if $Q \in Syl_p(G)$ and $Q \cap M > 1$, then $Q \subset M$. (If this holds then $M \cap M^x \supset H \in \mathcal{S}_p(G)$ $\Rightarrow H \subset R^x$, $R \in Syl_p(G)$ $R \subset M$, whence $R^x \cap M > 1 \Rightarrow R^x \subset M$ $\Rightarrow \exists\, m \ni xm \in N_G(R) \subset M \Rightarrow x \in M$.) Can suppose $Q \cap M \subset P$, whence $Q \cap P > 1$. By Alperin, $Q^y = P$ where $y = y_1 \cdots y_m$, $y_i \in N_G(H_i)$, H_i nice in P. But $N_G(H_i) \subset M$, so $y \in M$, so $Q \in M$. QED.

Sept 18, 1976

Thompson normal p-complement thm. If $C_G(Z(G_p))$ and $N_G(J(G_p))$ have normal p-complements, so does G.

Main reductions lead to the following situation: $O_{p'}(G)=1$, $H=O_p(G)>1$, G/H has a normal p-complement K/H and for any subgroup $G_p\subset M<G$, M has a normal p-complement. Note $G=KG_p$

So G is p-solvable with $O_{p'}(G)=1$, so we know $C_G(H)\subset H$, $C_G(K/H)\subset K$.

Next use Sylow theory to construct $1<Q/H\subset K/H$ normalized by G_p, Q/H elem ab. Then QG_p has no normal p-complement, as $Q\not\subset C_G(H)$, so $QG_p=G$, and $K=Q$. Can suppose K/H irreducible under G_p.

If $J(G_p)\subset H$, then $J(G_p)=J(H)$ char H so $J(G_p)\triangleleft G$ contradicting $N_G(J(G_p))$ has a normal p-complement. Thus $J(G_p)\not\subset H$, so we get $A\subset G_p$ abelian of maximal rank $A\not\subset H$. Let K_0/H be the non-trivial A submodule part of K/H; it exists for $A\not\subset H=C_G(K/H)$.

Put $P_0=HA$, $G_0=K_0A$. Then G_0 has no normal p-complement as K_0 doesn't. $P_0<N_G(J(P_0))\subset N_{G_0}(HJ(P_0))=N_{G_0}(P_0)$ $=P_0$ (because AH/H faithfully rep. in K_0/H. Also $Z(P_0)\supset Z(H)\supseteq Z(G_p)$ so $C_{G_0}(Z(P_0))\subset C_G(Z(G_p))$ has a normal p-comp. Thus $G_0=G$,

$J_n(X) = \langle A \mid A \text{ abelian} \subset X \text{ with } m(A) \geq d(X) - n \rangle$

$J(X) = J_0(X) \subseteq J_1(X) \subseteq \cdots \subseteq J_{d(X)-1}(X) = \cdots = X$

<u>Thm 1</u>. G p-solvable, $G_p \in Syl_p(G)$, $O_{p'}(G) = 1$. Assume one of:
a) $p \geq 5$
b) $p = 3$ and $SL(2,3)$ not involved in G
c) $p = 2$ " $SL(2,2)$ " " " ".

Let $H = \bigcap_g C_G(Z(G_p))^g$. Then

$$G = H \cdot N_G(J(G_p))$$

and if $p > 2$ $G = H \cdot N_G(J_1(G_p))$. In particular

$$G = C_G(Z(G_p)) \cdot N_G(J(G_p)).$$

<u>Proof</u>. Put $W_1 = Z(G_p)^G = \langle Z(G_p)^g \mid g \in G \rangle$.
$W = \Omega_1(W_1)$.

Then $H = C_G(W_1)$ and $H = O_p(G \bmod H)$.

(first clear. 2nd $O_p(G \bmod H) = \bigcap_{P \in Syl_p(G)} PH$
so $HG_p \subset C_G(Z(G_p)) \Rightarrow H \subset \bigcap_{P \in Syl_p(G)} PH \subset H$.)

It suffices to show $J_1(G_p) \subset H$ if $p \geq 5$ and $J(G_p) \subset H$ if $p \leq 3$. (clear by Frattini argument: If $J(G_p) \subset H$, then $J(G_p) \subset H \cap G_p$ so $J(G_p) = J(H \cap G_p)$, so given $g \in G$ $J(G_p)^g = J((H \cap G_p)^g) = J((H \cap G_p)^h) = J(G_p)^h$ for some $h \in H$, etc.)

Let G be a minimum counterexample. Let A be abelian, $\subset G_p$, $A \not\subset H$ and $m(A) \geq d(G_p) - \delta$ ($\delta = 0$ if $p \leq 3$, $\delta = 1$ if $p \geq 5$). Put $K = O_{p'}(G \bmod H)$, $L = KA$. Since $G_p \cap L \in Syl_p(L)$ the theorem is false for L ($A \subset J(G_p \cap L)$). Have to see that in L, L is not the product of $N_L(J(G_p \cap L))$ and $\bigcap_l C_L(Z(G_p \cap L))^l$?

Maybe one wants to prove $J(G_p) \subset C_G(Z(G_p))^g$ for any $g \in G$, and this is false in L.

As $A \subset J(G_p \cap L)$

can we get $C_L(Z(G_p \cap L)) \subset H$?

$$H = C_G(\langle Z(G_p)^g \rangle)$$

$$G_p \cap L = (H \cap G_p)A$$

$$Z(G_p \cap L) \supset Z(G_p)^g$$

$$Z(G_p) \subset O_p(G) \subset H$$

To prove $J(G_p)$ centralizes $Z(G_p)^g$ for all g

$A \subset J(G_p \cap L)$ centralizes $Z(G_p \cap L)^l$ for all l

Is it true that $Z(G_p) \subset H$? If so $Z(G_p) \subset Z(G_p \cap L)$

so perhaps it works.)

Thus $L = KA = G$. Minimality of G forces $A/A \cap H$ to be cyclic and K/H to be a special q-group. On the other hand, since $m(A) \geq d(G_p) - \delta$ it follows that $|W : W \cap A| \leq p^{1+\delta}$. If $p \geq 5$ use Thm B of Hall-Higman to get a contradiction. If $p \leq 3$, compute as in Thompson's J. Alg. paper.

Sept 19, 1976. Thompson normal p-complement situation.

The critical minimal case looks as follows. G p-solvable, $O_{p'}(G)=1$, $H=O_p(G)$, $K=O_{p,p'}(G)$, G/K cyclic of order p, $G = KA$ A abelian p-group of maximal rank, K/H an elem. ab. q-group. Why doesn't this work? Sylow groups of G are conjugates of HA and the conjugating element can be taken in K. $Z(HA) = C_{Z(H)}(A)$. $W = C_{\Omega_1 Z(H)}(A)^K$ = smallest K-inv. subspace of $\Omega_1 Z(H)$ containing the A-fixpts. Because A has maximal rank $C_W(A) = W\cap A$ is of codim ≥ 1 in W because $A/A\cap H$ cyclic. Hence you contradict Thm B, for a generator of $A/A\cap H$ can't have quadratic minimum polynomial.

Sept 20, 1976. G p-solvable, $O_{p'}(G)=1$. Thompson's claims to show that if A is abelian of max. rank then A centralizes $Z(G_p)^G$. Note that $Z(G_p)^G \subset Z(O_p(G))$ (because $Z(G_p) \subset C_G(O_p(G)) = Z(O_p(G))$), hence $Z(G_p)^G \subset G_p$, and so $Z(G_p)^G$ is the weak closure in G_p of $Z(G_p)$.

(Basic fact: If $X \subset O_p(G)$, then X^G = weak closure in G_p of X.)

Thompson's result says A abelian $\subset G_p$ of max rank $\Rightarrow$ A centralizes $Z(G_p)^G$. So $J(G_p)$ centralizes $Z(G_p)^G$. But if $x \in C_G(J(G_p))$ and A is max. ab. of max. rank then x centralizes $A \subset J(G_p)$, so $x \in A \subset J(G_p)$. Thus

$C_{G_p}(J(G_p)) = Z(J(G_p))$. In fact

$$C_{G_p}(J(G_p)) = Z(J(G_p)) = \bigcap_{\substack{A \text{ max. ab} \subset G_p \\ \text{rank}(A) = \text{rank}(G_p)}} A$$

So we know this contains $Z(G_p)^G$ by Thompson. So we get:

G p-solv + $O_{p'}(G) = 1$ (p odd)

Thompson's thm. $\Rightarrow$ $Z(J(G_p))$ contains a non-trivial normal p-subgp of G, namely $Z(G_p)^G$.

If we want to prove $Z(J(G_p)) \triangleleft G$ we can therefore replace G by $G_0 = C_G(Z(G_p)^G)$. Point is that if $Z(J(G_p)) \triangleleft G_0$ then it is a characteristic subgp. of G_0 hence $\triangleleft G$. So I can suppose $Z(G_p) = Z(G)$.

~~I still don't understand Thompson's reduction~~

G p-solvable, $O_{p'}(G) = 1$. To show that if A is an abelian p-subgrp of maximal rank, then A centralizes $B = Z(G_p)^G$. Introduce $H = C_G(B) = \bigcap C_G(Z(G_p))^g$.

Then

$$H \subset \bigcap_g H G_p^g \subset \bigcap_g C_G(Z(G_p))^g = H$$

so $H = O_p(G \bmod H)$. Let $K = O_{p'}(G \bmod H)$ and look at $L = KA$, $L/H = K/H \rtimes A/A \cap H$ acts faithfully on B. Claim $C_B(A/A\cap H)$ not invariant under K/H, because otherwise $C_B(A/A\cap H)$ would be a repn of L/H whose kernel contains $A/A\cap H$ hence all of L/H as $O_p(L/H)=1$. So K/H would acts trivially on $C_B(A/A\cap H)$. The splitting

$$B = C_B(K) \oplus [K, B]$$

is A-invariant because $K \lhd G$. But then as

$$C_B(A/A\cap H) \subseteq C_B(K)$$

we have $C_{[K,B]}(A/A\cap H) = 0$, hence $[K,B]=0$ so K acts trivially on B which is non-sense.

Note $HA \in Syl(L)$ and

$$Z(HA)^L \supseteq C_{Z(H)}(A)^K \supset C_B(A)^K > C_B(A)$$

Consequently A acts non-trivially on $Z(HA)^L$. Thus using induction on $|G|$ we have $G = KA$.

~~In the preceding we could have chosen K more efficiently. At the moment $G = KA$ and $B = C_{Z(H)}(A)^K$ So let $B = Z(G_p)^G$, $H = C_G(B)$ as before, but this time choose K/H to be a q-group of G/H stabilized and non-trivial under A. Again if K leaves $C_B(A)$ invariant, $C_B(A)$ is a $L = KA$ module on which A acts trivially!~~

To make the above work, I needed K to be normalized by A and for A to act non-trivially on K/H. Check this. If $C_B(A)$ is K-invariant, then it is an $L = KA$ module on which A acts trivially, hence on which the normal subgroup N gen. by A in $L = KA$ and by H acts trivially. So $N \cap K > H$ as A acts non-trivially on K/H. But $B = C_B(A \cap K) \oplus [N \cap K, B]$ is an A-invariant ~~splitting~~ and so $C_B(A) \subset C_B(N \cap K) \Rightarrow C_{[N \cap K, B]}(A) = 0 \Rightarrow [N \cap K, B] = 0$ $\Rightarrow N \cap K$ acts trivially on B, a contradiction.

So I can suppose K/H is a ~~[illegible]~~ q-subgp, ~~[illegible]~~ even a special q-group. In this case $A/A \cap H$ has to be cyclic. Now if $W = \Omega_1 B$ you have $r(A) \geq r(P) - 1$ $\Rightarrow |W/W \cap A| \leq p^2$ (~~[illegible]~~ A assumed maximal abelian of rank $\geq r(P) - \delta$ ($\delta = 1$ if $p \geq 5$ which I assume) hence $C_W(A) = W \cap A$. Have $r(A \cap H) \geq r(A) - 1$ and

$$r(W \cdot A \cap H) = r(W) - r(W \cap A) + r(A \cap H) \leq r(P)$$

$$r(W) - r(W \cap A) \leq r(A) - r(A \cap H) \leq r(P) - r(A) + 1$$
$$\leq 2.$$

But now this implies that the p-length of W as a $A/A \cap H$ module is ≤ 3.

$$\underbrace{W > \cdot > A \cap W > 0}$$

whereas Hall - Higman thm. B. $\Rightarrow$ length $\geq p$.

More on Thompson's thm:

Put $J_n(X) = \langle A \mid A \text{ abelian} \subset X \text{ and } r(A) \geq r(X) - n \rangle$
here X is a p-group. One has

$$J(X) = J_0(X) \subseteq J_1(X) \subseteq \cdots \subseteq J_{r(X)-1} = \cdots = X$$

and these are characteristic subgps.

<u>Thm</u> (Thompson Pac J. Math 16 (1966) 371-2). G p-solvable $O_{p'}(G) = 1$. Assume either a) $p \geq 5$ b) $SL_2(\mathbb{F}_3)$ not involved in G, c) $SL_2(\mathbb{F}_2)$ not involved in G. Put $H = \bigcap_{g \in G} C_G(Z(G_p))^g$. Then

$$G = H . N_G(J(G_p))$$

and $G = H \cdot N_G(J_1(G_p))$ if $p \geq 5$. In particular

$$G = C_G(Z(G_p)) \cdot N_G(J(G_p))$$

In what follows we will work with just $J(G_p)$.

Better formulation: The hypotheses G p-solv. $O_{p'}(G) = 1$ imply $Z(G_p) \subset C_G(O_p(G)) \subset Z O_p(G) \subset G_p$. Thus

$$H = C_G(Z(G_p)^G)$$

where $Z(G_p)^G = \langle Z(G_p)^g \mid g \in G \rangle$ is the smallest G-normal subgroup containing $Z(G_p)$. It suffices to show that $J(G_p) \subset H$ for then $J(G_p) = J(G_p \cap H)$ so $J(G_p)^g = J(H \cap G_p)^g = J((H \cap G_p)^g) = J((H \cap G_p)^h)$ $= J(G_p)^h$ for some $h \in H$ etc. Thus what we have to show is that any $A \subset G_p$ of maximal rank centralizes $B = Z(G_p)^G$. Assume A doesn't.

$$H \subset O_p(G \bmod H) = \bigcap_g HG_p^g \subset \bigcap C_G(Z(G_p))^g = H.$$

Thus as G is p-solvable $K_0 = O_{p'}(G \bmod H) > H$.
$C_{G/H}(K_0/H) \subset K_0/H \Longrightarrow C_A(K_0/H) = A \cap H < A$. So A acts non-trivially on K_0/H, and one knows $\exists$ a special subgroup K/H of K_0/H, invariant under A such that A acts non-trivially on K/H.

Put $L = KA$. $HA \in Syl_p(L)$ and $Z(HA) \subset C_G(H) = Z(H)$, so $Z(HA) = C_{ZH}(A) \supset C_B(A)$. Claim that $C_B(A)$ is not K-invariant. Assuming this we have

$$Z(HA)^L \supset C_B(A)^K > C_B(A)$$

So $C_B(A)^K$ is subgroup of $Z(HA)^L$ not centralized by A. (More precisely: $Z(HA)^L = C_{ZH}(A)^K$ contains $C_B(A)^K$ which is not centralized by A for if it were we would have $C_B(A) \subset C_B(A)^K \subseteq C_B(A)$ hence $C_B(A)$ would be K-invariant). To establish the claim suppose $C_B(A)^K = C_B(A)$; then $C_B(A)$ would be a L/N module, $N = (HA)^L$. But because A acts non-trivially on K, $N \cap K > H$, hence as $N \cap K/H$ is a p'-group

$$B = C_B(N \cap K) \oplus [N \cap K, B]$$

whence since A normalizes $N \cap K, B$ and N centralizes $C_B(A)$.

~~$H^0(A,B) = H^0(A, C_B(N \cap K)) \oplus H^0(A, [N \cap K, B])$~~

$\le H^0(N,B)$

$$C_B(N) = C_B(A) = C_{C_B(N\cap K)}(A) \oplus C_{[N\cap K,\, B]}(A)$$

$$\cap$$

$$C_B(N\cap K)$$

Conclude $C_{[N\cap K, B]}(A) = 0$

so $[N\cap K, B] = 0$, so $N\cap K$ centralizes B which contradicts $N \cap K > H$.

So one sees that L contradicts the theorem. Specifically we have $B \subset Z(L_p)^L$, $B \triangleleft L$, and $\exists A$ of max. rank not centralizing B. If we use induction we can claim $L = G$, hence K/H special and $A/A\cap H$ cyclic. But I would like to see the counterexample explicitly.

So far we have $L = KA$ acting on $B \subset Z(H)$. Now replace B by $B_1 = C_B(A)^K$. Then $O_p(L) = \bigcap_{k\in K} HA^k$ acts trivially on B_1, so B_1 is a module for

$$L/O_p(L) = K/H \rtimes A/C_A(K/H)$$

(Note that $L/H = K/H \rtimes A/A\cap H$, hence $O_p(L/H) = C_{A/A\cap H}(K/H)$ so $O_p(L) = H\cdot C_A(K/H)$.) Now $\bar{L} = L/O_p(L)$ is a p-solvable group with $O_p(\bar{L}) = 1$ acting on B_1 so we can use Thm. B to conclude something about no quadratic min. polys. In particular as $A \cap O_p(L)$ is cyclic, one has $B/C_{B_1}(A)$ at most cyclic and you get a contradiction.

October 4, 1976:

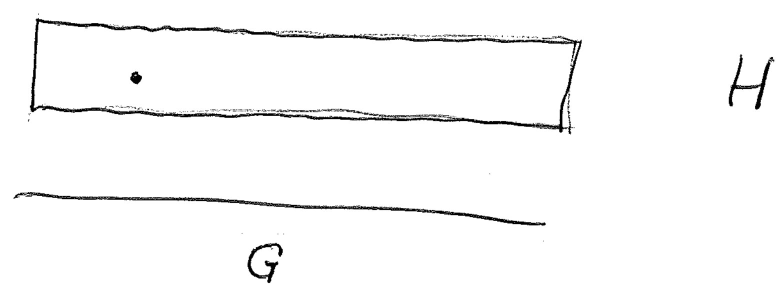
Problem: I want a direct proof of the Frob. thm. on normal p-complement which doesn't use induction. Suppose H is a subgroup of G and I want to construct a homomorphism of retraction: $G \to H$. One method would be to consider G as a subgroup of the autos of G as a right H-set. Then I want to trivialize this H-torsor, that is, find a partition of π into sets σ which are cross-sections for $G \to G/H$ such that π is invariant under H right mult and G-left multiplication. Then H will act simply-transitively on π, so we get a homom. $G \to H$.

Idea: Can we construct the graph in $G \times H$ of the desired homomorphism. Think of Lie theory in differential geometry. I want to construct an isomorphism between two Lie groups G, H with isomorphic Lie algebras. So what one does is to form $G \times H$ and define a distribution on $G \times H$. Actually one finds the Lie subgroup K corresponding to the graph of the isom. $\mathfrak{g} \to \mathfrak{h}$. Then one gets $G \leftarrow K \rightarrow H$ covering maps, hence isos. if G, H are simply-conn.

To construct a subgp H of G corresp. to a subalg. $\mathfrak{h}$ of $\mathfrak{g}$ one constructs the foliation of G into cosets gH. Thus one translates $\mathfrak{h}$ over G to obtain a left-invariant distribution on G, and the fact that $\mathfrak{h}$ is a subalgebra implies this distribution is integrable.

Critical example: $|G| = 2m$ $\quad m$ odd. $\quad H = \mathbb{Z}/2\mathbb{Z}$.
I want to construct a non-trivial homom. $G \longrightarrow H$.

So we are seeking a partition of $G \times H$. It might be appropriate to look for nearby members of the same partition block. Thus look for the equivalence relation generated by something.



Maybe we can find an interesting graph inside of $G \times H$. Idea: Go back to the idea of sets ${}_xH_y$

Possibility: Whatever geometric gadget is to appear it has to consist of subsets of $G \times H$ and it must be $G \times H$-invariant. For example given a subset σ of $G \times H$ can one define what it means for σ to be flat? First criterion is that it be the graph of a map from a <u>subset</u> of G into H.

Let $\mathcal{F}$ be a hereditary family of subsets of G such that $\sigma \in \mathcal{F} \Longrightarrow g\sigma \in \mathcal{F}$ and $\sigma_1, \sigma_2 \in \mathcal{F}$, $\sigma_1 \cap \sigma_2 \neq \emptyset \Longrightarrow \sigma_1 \cup \sigma_2 \in \mathcal{F}$. Then the maximal elements of $\mathcal{F}$ are disjoint and are the cosets gH, where H = the maximal element containing $\mathbf{1}$.

October 6, 1976

~~Suppose a Sylow p subgroup of G is cyclic of order p~~

Suppose $|G| = 2m$, m odd, $H \subset G$ cyclic of order 2. The problem is to produce the normal complement K to H as the solution of some kind of extremal problem. I've seen that maybe I ought to think of the foliation of G defined by K i.e. all subsets σ of G such that [redacted] $\sigma \subset Kx$ for some x. Can I locate such subsets σ directly? First criterion is that $\sigma \cap \sigma h = \phi$.

Another possibility would be to consider a function f on G such that $f(gh) = -f(g)$ which is non-zero. This is somehow the same as [redacted] linear combinations of [redacted] sets σ such that $\sigma \cap \sigma h = \phi$.

Now can you spot the extremal problem? Observe that if I find a one-dimensional space of f satisfying $f(gh) = -f(g)$ invariant under [redacted] G, then I get $\theta : G \to H$ a retraction.

The idea is to look at way of partitioning all the different cosets xH. The number of [redacted] $\sigma \subset G$ transversal to the H-right orbits is 2^m, so the number of partitions is 2^{m-1}. [redacted] Exactly one of these is fixed under G. What can one say about the wrong partitions?

Suppose S [redacted] is a subset of a group G. Then we can look at it stabilizer $R = \{g \in G \mid gS = S\}$ which is the largest subgroup such that S is a union of cosets Rg. Then $|R|$ divides $|S|$, so

$$|S| \cdot \frac{|G|}{|R|} = \frac{|S|}{|R|} \cdot |G| > |G|$$

unless $|R| = |S|$ in which case S is a right coset for R.

Suppose S is transversal to right H-action: $S \cap Sh = \phi$. Then $Rs_1 \cap Rs_2 h = \phi$ for each $s_1, s_2 \in S$.

S subset of G. Assume $\forall g \in G$ either $S \cap gS = \phi$ or $S = gS$. Then S is a ~~left~~ coset of a subgroup.

October 16, 1976

Let V be a vector space of finite dimensional over a field F on which an algebra R over F acts. Claim the poset $T(V)^R$ of proper subspaces of V invariant under R is spherical. Better: Suppose V is a finite length R-module. Then the poset $T(V,R)$ of proper submodules of V is spherical.

Case 1. V is not semi-simple. Then if V_0 is the socle of V we have $0 < V_0 < V$ and $W \cap V_0 > 0$ for any $W \in T(V,R)$. Thus $T(V,R)$ is conically contractible.

Case 2. V is semi-simple. Write $V = \oplus V_\chi$, decomposition into isotypical semi-simple modules. A submodule W of V can be identified with a family of submodules $W_\chi \subset V_\chi$. Let $L(V,R)$ denote the poset of layers in the poset of submodules. Then

$$L(V,R)_{<(0,V)} = \{(W_0, W_1) \mid W_1 < V\} \cup \{(W_0, W_1) \mid W_0 > 0\}$$

is the union of two contractible sets with intersection homotopy equivalent to $T(V,R)$. Thus $L(V,R)_{<(0,V)} \sim$ suspension of $T(V,R)$. Clearly

$$L(V,R) = \prod L(V_\chi, R)$$

But now use

Lemma: Let $J_1, \ldots, J_s$ be posets and $x_i \in J_i$. Then

$$(J_1 \times \cdots \times J_s)_{<(x_1, \ldots, x_s)} \sim \mathop{*}_{i=1}^{s} (J_i)_{>x_i}$$

The lemma implies

$$\mathrm{Susp}\ T(V,R) \sim \underset{\substack{\text{[redacted]}\\ x}}{\ast} \mathrm{Susp}\ T(V_x, R).$$

On the other hand [redacted] $T(V_x, R)$ is known to be spherical, because [redacted] $T(V_x, R) \cong T(E, D)$, where E is a vector space over a skew-field D.

This shows $\mathrm{Susp}\ T(V,R)$ is [redacted] ~ bouquet of spheres, but I still want to [redacted] see that $T(V,R)$ itself is ~ bouquet of spheres. Here's how. Let me suppose $V = V_1 \oplus V_2$. Put $J(V)$ for the poset of all subspaces of V. Then

$$Z_{12} = \left(J(V_1)_{<V_1} \times J(V_2)_{<V_2} \right)_{>(0,0)} = \left\{ (W_1, W_2) \;\middle|\; \substack{0 < (W_1, W_2) \\ W_i < V_i} \right\}$$

by the lemma is homotopic to the join

$$T(V_1) * T(V_2).$$

Similarly

$$Z_1 = \left(J(V_1) \times J(V_2)_{<V_2} \right)_{>(0,0)} \simeq J(V_1)_{>0} * T(V_2) \sim pt$$

$$Z_2 = \left(J(V_1)_{<V_2} \times J(V_2) \right)_{>(0,0)} \sim pt.$$

However Z_i are closed in $\{ (W_1, W_2) \mid 0 < W_1 \oplus W_2 < V \} = T(V,R)$ and $Z_1 \cup Z_2 = T(V,R)$, $Z_1 \cap Z_2 = Z_{12}$. Thus we see that

$$T(V,R) \sim \Sigma\, T(V_1) * T(V_2).$$

The general lemma should be

$$\{ (x_1, \ldots, x_s) \in J_1 \times \cdots \times J_s \mid (a_1, \ldots, a_s) < (x_1, \ldots, x_s) < (b_1, \ldots, b_s) \}$$

$$\sim \Sigma^{s-1} \mathop{\ast}_{i=1}^{s} \{ x_i \in J_i \mid a_i < x_i < b_i \}.$$

You can prove this by induction using:

$$\{ (x_1, x_2) \in J_1 \times J_2 \mid (a_1, a_2) < (x_1, x_2) < (b_1, b_2) \}$$
$$= \{ x_1 \mid a_1 < x_1 < b_1 \} * \Sigma^{1} * \{ x_2 \mid a_2 < x_2 < b_2 \}$$

already established.

Now suppose p [redacted] is a prime number dividing $q-1$. I want to show that the complex of elementary abelian p-subgroups of $GL_n(\mathbb{F}_q)$ containing $\Delta\mu_p$ is spherical of dimension $n-2$. [redacted]

Put $G = GL_n(\mathbb{F}_q)$, $Z = \Delta\mu_p$, so that the complex in question is $\mathcal{Q}_p(G)_{>Z}$. I want to form a complex J with maps:

$$\mathcal{Q}_p(G)_{>Z} \xleftarrow{f_1} J \xrightarrow{f_2} T(G)$$

such that f_i/y is spherical of dim $d(y)$.

Candidate for J is pairs (A, W) where [redacted] W is a subspace invariant under A, or maybe it should be a flag invariant under A. Think over covering $T(G)$ by the $T(G)^A$ as A runs over $\mathcal{A}_p(G)_{>Z}$. The fibre [redacted] $\{A \mid$ [redacted] $x \in T(G)^A\} = \{A \mid A \in G_x\}$ $\cong \mathcal{A}_p(G_x)_{>Z}$. G_x has a normal [redacted] p-subgroup modulo which it is a product of smaller linear groups, so one should win by induction that $\mathcal{A}_p(G_x)_{>Z}$ is [redacted] spherical of the right dim.

So suppose then we look at $T(G)^A$ for fixed A. This is spherical of dim $n-2$.

[redacted]

<u>October 19, 1976:</u> [redacted] Goal: to show $GL_n(\mathbb{F}_q)$ has the property γ for all primes p not dividing q.

From now on p-torus means elem. ab. p-subgroup. Let B be a p-torus in $GL_n(\mathbb{F}_q) = G$. We decompose $V = \mathbb{F}_q^n$ into disjoint repns.

$$V = \bigoplus V_H \oplus V^B$$

where V_H is the direct sum of those non-trivial irreducible subrepns. of V which are reps. of B/H, H a hyperplane in B.

Choose an isom $B/H \xrightarrow{\sim} \mu_p$ = pth roots of 1 in a $\overline{\mathbb{F}}_q$. Then V_H becomes a vector space over $F(\mu_p)$, $F = \mathbb{F}_q$. Now recall

$$a_p(G)_{>B} = a_p(C_G(B))_{>B}$$

and

$$\text{①} \qquad C_G(B) = \prod_H \operatorname{Aut}_{F(\mu_p)}(V_H) \times \operatorname{Aut}_F(V^B)$$

If this splitting is non-trivial, then by induction on n, we know that each of the factors has property γ, so $C_G(B)$ does also by

<u>Prop.</u> If G_1, G_2 satisfy γ, so does $G_1 \times G_2$.

<u>Proof.</u> Let B be a p-torus in $G_1 \times G_2$ with projections B_i in G_i. If $B < B_1 \times B_2$, then $a_p(G_1 \times G_2)_{>B}$ is conically contractible. (For if $A > B$ then A centralizes B, hence A centralizes B_1 and B_2 $\left((b_1, b_2)^{(a_1,a_2)} = (b_1, b_2) \Longrightarrow b_1^{a_1} = b_1 \Longrightarrow (b_1, e)^{(a_1,a_2)} = (b_1, e)\right)$, so A centralizes $B_1 \times B_2$). Thus we can suppose $B = B_1 \times B_2$.

If $A \in a_p(G_1 \times G_2)_{>B}$, then $A \leq A_1 \times A_2$ where $A_i = pr_i(A)$. So we def $a_p(G_1 \times G_2)_{>B}$ into the subset of A with $A = A_1 \times A_2$ and either $A_1 > B_1$ or $A_2 > B_2$. This poset is the join up to homotopy equivalence of $a_p(G_1)_{>B_1}$ and $a_p(G_2)_{>B_2}$. QED.

Suppose ① is trivial i.e. either $B=1$ and $V=V^B$ or $B=\mu_p\subset F^*$, so that $V_H=V$. Thus by induction one has to consider 2 cases

2) p does not divide $(q-1)$.

1) p divides $(q-1)$ and $B=\mu_p\cdot I$ in $GL_n(\mathbb{F}_q)$.

Consider case 1). Put $Z=\mu_p\cdot I$ in $GL_n(\mathbb{F}_p)$. We consider the poset of pairs (A,φ) where $A\in \mathcal{A}_p(G)_{>Z}$ and φ is a flag (simplex in $T(V)$) invariant under A. Call this set J so that we have functors

$$\mathcal{A}_p(G)_{>Z} \xleftarrow{p_1} J \xrightarrow{p_2} T(V)$$

p_1, p_2 are fibred. $p_2^{-1}(\varphi)=\mathcal{A}_p(G_\varphi)_{>Z}$. Now G_φ is a proper parabolic group of G, hence $G_\varphi/\mathrm{rad}\,G_\varphi$ is a product of smaller general linear groups, so it has property (γ). But $\mathrm{rad}(G_\varphi)$ is a p'-group solvable & normal in G_φ. So G_φ has property γ with the same rank as G. Thus all fibres of p_2 have the h-type of bouquets of $(n-2)$-spheres. $p_1^{-1}(A)=T(V)^A=$ poset of A-invariant proper subspaces of V. I saw this $\sim$ bouquet of $(n-2)$-spheres.

~~Next look at the fibre $p_1^{-1}(A)$~~

Case 2. We will have to consider the subcomplex of $T(V)$ ~~~~ made up of proper subspaces of dimension $\equiv 0 \pmod r$.

Assertion: Remove from $T(V)$ all subspaces of dimensions $a_1, \ldots, a_s$, then the new complex is spherical, at least as far as homology is concerned.

Lemma: Let J be a poset, S a set of unrelated elements of S. Assume J is homol.-spherical of dim d, that the links $L_s = \{x \in J \mid x < s \text{ or } x > s\}$

$$= J_{<s} * J_{>s}$$

are spherical of dim $d-1$, and that $J-S$ has dimension $d-1$. Then $J-S$ is spherical of dim. $d-1$.

Proof: One has a cofibration

$$\underset{\text{dim } d-1}{J-S} \longrightarrow \underset{\underset{\vee S^d}{\wr}}{J} \longrightarrow \bigvee_{s \in S} (\Sigma L_s)$$

So now look at the homology sequence

$$0 \longrightarrow \underset{\underset{0}{\shortparallel}}{\tilde{H}_d(J-S)} \longrightarrow \tilde{H}_d(J) \longrightarrow \tilde{H}_d(\vee \Sigma L_s) \longrightarrow \tilde{H}_{d-1}(J-S) \to 0$$

Apply this successively to $T(V)$ removing the subspaces of dims $a_1, a_2, \ldots$ ~~to~~ get $T_{a_1, a_2, \ldots, a_r}(V)$.

Put $J = T_{a_1, \ldots, a_{r-1}}(V)$, S = subspaces of dim a_r. But if W is of dim a_r, then $J_{<W} = T_{a_1, \ldots, a_{r-1}}(W)$ is sph. by induction.

Suppose now p doesn't divide $q-1$ and let $r=[F(\mu_p):F]$. This time I compare $\mathcal{A}_p(G)$, $G=GL_n(F)$ with the poset of proper subspaces W of $V=F^n$ such that $\dim(W)\equiv 0 \pmod{r}$; call this poset $T_r(V)$. Again we will look at the poset J consisting of pairs (A,φ) with $A\in\mathcal{A}_p(G)$ and $\varphi\in \mathrm{Simp}\, T_r(V)$ such that φ is A-invariant.

$$\mathcal{A}_p(G) \xleftarrow{\ p_1\ } J \xrightarrow{\ p_2\ } \mathrm{Simp}(T_r(V))$$

Again $p_2^{-1}(\varphi)=\mathcal{A}_p(G_\varphi)$. Suppose φ is given by

$$0<W_1<\cdots<W_s<V$$

Then $G_\varphi/\mathrm{rad}(\varphi) = \mathrm{Aut}(W_1)\times \mathrm{Aut}(W_2/W_1)\times\cdots\times \mathrm{Aut}(V/W_s)$
By assumption $\dim(W_i/W_{i-1})\equiv 0 \pmod{r}$ $i=1,\ldots,s$ and by induction $\mathrm{Aut}(W_i/W_{i-1})$ and $\mathrm{Aut}(V/W_s)$ satisfy γ. Thus $\mathcal{A}_p(G_\varphi)$ is spherical, so one only has to check that its dimension is $[\frac{n}{r}]-1$. But the dimension p-rank of G_φ is

$$\frac{\dim W_1}{r}+\frac{\dim W_2/W_1}{r}+\cdots+\left[\frac{\dim V/W_s}{r}\right]=\left[\frac{n}{r}\right].$$

so it works.

$$p_1^{-1}(A)=\mathrm{Simp}\ T_r(V)^A \sim T_r(V)^A.$$

Now decompose: $V=\bigoplus_B V_B\oplus V^A$ where B runs over hyperplanes in A. An invariant subspace W of V is the same as a family of invariant subspaces in each summand.

Lemma: Let a p-torus A act on V. Then $T_r(V)^A$ is spherical of dim $\left[\frac{n}{r}\right]-2$.

Proof. Go by induction on n. OK if A acts trivially on V. Otherwise we have $V = V_1 \oplus V_2$ where V_1, V_2 are disjoint subrepns and V_1 is a direct of 2 scin non-trivial irred. repns of A ~~doesn't contain the trivial repn~~. Then a subspace $W \subset V$ is A-invariant $\iff W = W_1 \oplus W_2$ with W_i invariant in V_i. Because $V_1{}^A = 0$, we know $\dim W_1 \equiv 0 \pmod r$. Hence

$$T_r(V)^A = \{W \in \mathrm{Sub}_r(V)^A / 0 < W < V\}$$

$$\cong \{(W_1, W_2) \in \mathrm{Sub}_r(V_1)^A \times \mathrm{Sub}_r(V_2)^A \mid (0,0) < (W_1, W_2) < (V_1, V_2)\}$$

$$\approx T_r(V_1)^A * S^1 * T_r(V_2)^A$$

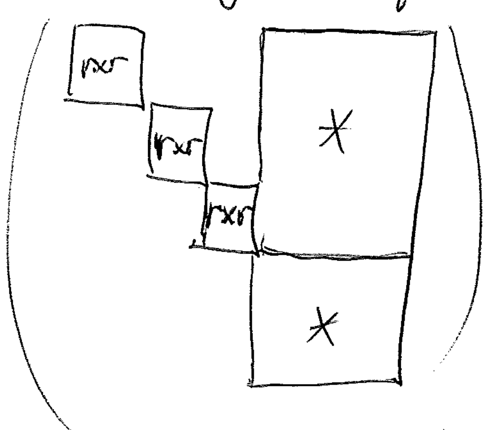
Now this reduces us to the case of $V_1 = V$ i.e. V is a direct sum of $F(\mu_p)$ with A acting thru $\chi : A \twoheadrightarrow \mu_p$. But then $T_r(V_1)^A = T(V_1$ as an $F(\mu_p)$ vector space$)$.

October 20, 1976

There is a problem with the proof when $p \nmid q-1$. The problem is that when $n = rm$, then $\mathcal{a}_p(GL_n)$ has dimension $m-1$ while $T_r(V)$ is spherical of dim. $m-2$.

Try $m=2$. $T_r(V)$ is zero-dimensional whereas we are trying to get $\mathcal{a}_p(GL_n)$ to be a connected graph.

Idea: suppose instead of $T_r(V)$ we consider the simplicial complex $K_r(V)$ of r-dimensional subspaces and independent subsets. I know for an infinite field this complex is spherical of dimension $m-1$, $m = [\frac{n}{r}]$. To each simplex σ in $K_r(V)$ we associate the subgroup of $G = Aut(V)$ preserving σ element-wise. Call this subgroup G_σ. It is a group of the form:



and so by induction $\{A \mid A \subset G_\sigma\} = \mathcal{a}_p(G_\sigma)$ will be spherical of dimension $m-1$. On the other side we have to examine $\{\sigma \mid A \subset G_\sigma\}$. This will be the simplicial complex of all r-diml. subspaces fixed by A.

<u>Lemma</u>: Let a p-torus act on V. Then the subcomplex of $K_r(V)$ consisting of simplices whose vertices are fixed by A is spherical of dimension $\left[\frac{n}{r}\right]-1$.

<u>Proof.</u> I am assuming this known when A acts trivially on V. Otherwise we have $V = V_1 \oplus V_2$ where these are disjoint repns. and V_1 is a direct sum of isom. non-trivial irred. repns. Then the subcomplex $K_r(V)_A$ I am looking at is the join

$$K_r(V_1, A) * K_r(V_2, A)$$

because vertices invariant under A fall in either V_1 or V_2. So I win by induction unless $V_1 = V$. In this case $V \simeq F(\mu_p)^m$ with A acting thru a character $\chi : A \twoheadrightarrow \mu_p$, so $K_r(V, A)$ = $K_1(V$ reg. as a vector space over $F(\mu_p))$, which I know is spherical of dimension $m-1$. QED.

Still I need to prove.

<u>Question</u>: $K_r(V)$ = simplicial complex consisting of subspaces of dimension r of V in which simplices are subsets $\{W_1, \dots, W_s\}$ such that $W_1 \oplus \cdots \oplus W_s \hookrightarrow V$. Is $K_r(V)$ spherical?

~~Suppose $n = \dim(V) = m \cdot r + e$ where $0 \le e < r$~~
~~Look at $F_{r+1}, 2r+1, \dots, mr+1$ (V) and to each W in this I associate $K_r(W)$~~

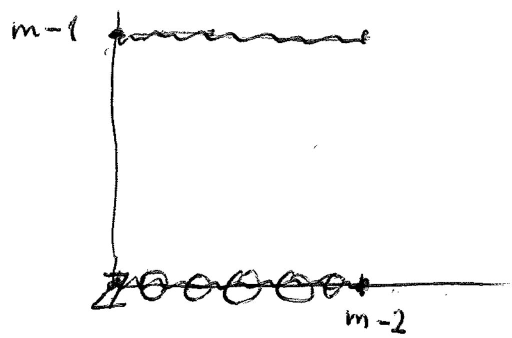
October 22, 1976

Here is a way to fill above gap. $G = GL_n(F)$, ~~where~~ $p \neq \mathrm{char}(F)$, and $r = [F(\mu_p):F] > 1$. Then I want to prove $G \in (\mathcal{O})$ by induction on n and I use the diagram

$$\mathcal{A}(G) \xleftarrow{\ p_1\ } \underset{\overset{\parallel}{J}}{\{(A,\varphi) \mid A \subset G_\varphi\}} \xrightarrow{\ p_2\ } T_r(V)$$

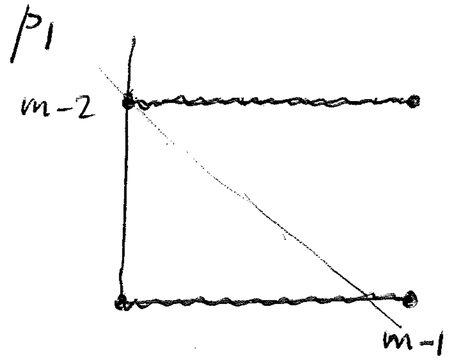
If $n = mr + e$ with $0 \leq e < r$ and $e > 0$, then $T_r(V)$ is spherical of dim. $m-1$. The same is true for $p_2^{-1}(\varphi) = \mathcal{A}(G_\varphi)$ by induction and for $p_1^{-1}(A) = T_r(V)^A$, so we can conclude that $\mathcal{A}(G)$ is spherical. The problem arises when $e = 0$ because then $T_r(V)$ and $T_r(V)^A$ are spherical of dim. $m-2$.

But in this case the spectral sequence for p_2



$\tilde{H}_i(J) = 0 \quad i < m-2$

shows that $H_{m-2}(J) \xrightarrow{\sim} H_{m-2}(T_r(V))$, and the spectral sequence for p_1



gives us that $0 = H_i(J) \xrightarrow{\sim} H_i(\mathcal{A}(G))$ for $i < m-2$ and an exact sequence

$$\begin{tikzcd}
E^2_{m-1,0} \arrow[r] \arrow[d, equal] & E^2_{0,m-2} \arrow[r, "\alpha"] \arrow[d, equal] & H_{m-2}(J) \arrow[r] \arrow[d, equal] & E^2_{m-2,0} \arrow[r] \arrow[d, equal] & 0 \\
H_{m-1}(a(G)) & H_0(a(G), A \mapsto H_{m-2}(T_r(V)^A)) & H_{m-2}(T_r(V)) & H_{m-2}(a(G)) &
\end{tikzcd}$$

Now it seems reasonable that the map α is induced by the inclusion $T_r(V)^A \longrightarrow T_r(V)$. So the question is whether the maps $H_{m-2}(T_r(V)^A) \longrightarrow H_{m-2}(T_r(V))$ generate as A runs over $a(G)$. Possibly true if we restrict just to the case A maximal, in which case $T_r(V)^A$ is an $(m-2)$-sphere

Conjecture: Given $0 < a_1 < \cdots < a_r < n$, for each splitting $V = V_1 \oplus \cdots \oplus V_{r+1}$ with $\dim(V_i) = a_i - a_{i-1}$ I get a map of an $(r-1)$-sphere into $J_{a_1, \ldots, a_r}(V)$ namely

not defined

$$\{\sigma \mid \sigma \subset \{1, \ldots, r+1\}, \sigma \neq \phi\} \longrightarrow J_{a_1, \ldots, a_r}(V)$$

$$\sigma \longmapsto \bigoplus_{j \in \sigma} V_j .$$

Is it true that one obtains generators for $J_{a_1, \ldots, a_r}(V)$ in this way?

Recall the sequence

October 23, 1976

Prop: Let α be a subset of $\{1, 2, \dots, n-1\}$ and let $J_\alpha(V)$ denote the poset of subspaces W of V with $\dim(W) \in \alpha$.

a) $J_\alpha(V)$ is spherical

b) Generators for the top homology of $J_\alpha(V)$ can be obtained as follows. Let β be the complement of α. If σ is a top dimensional simplex in $J_\beta(V)$, then one has an inclusion

$$(*) \qquad \text{Link of } \sigma \text{ in } T(V) \hookrightarrow J_\alpha(V)$$

and

$$\cong \quad \mathop{\ast}_{i=1}^{r} T(W_i/W_{i-1})$$

if $\sigma : 0 < W_1 < \cdots < W_{r-1} < V$. Claim that the different maps on top homology induced by $(*)$ generate the top homology of $J_\beta(V)$, as σ ranges over the top simplices of $J_\beta(V)$.

Proof by induction on $|\beta|$. If $|\beta| = 0$, then $J_\alpha(V) = T(V)$ is spherical and b) is clear. So consider adding an element b to β. Say $\beta' = \beta \cup \{b\}$, so $\alpha' = \alpha - \{b\}$. Put S = all subspaces of dim b, $J = J_\alpha(V)$, $J - S = J_{\alpha'}(V)$. Then for $W \in S$

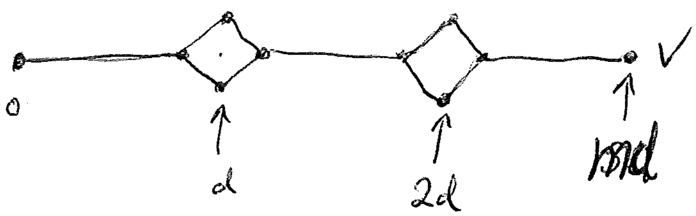
$$J_{<W} = J_{\alpha \cap \{1, \dots, b-1\}}(W)$$

$$J_{>W} = J_{\alpha \cap \{b+1, \dots, n\}}(V)_{>W}$$

are spherical by induction. Etc.

Now from Tits' theorem we know that $T(V)$ has generators given by the spheres associated to the apartments.

How generators of $T_d(V)$ look dim $V = n = md$.



Thus for each $i = 1, \ldots, m-1$ we given subspaces W_i, W_i' ~~such~~ of dim im such that $W_i \cap W_i'$ is of dim $im - 1$ and $W_i + W_i' \subset W_{i+1} \cap W_{i+1}'$. Then the subcomplex of $T_d(V)$ with the vertices W_i, W_i' is clearly the join

$$\begin{Bmatrix} W_1 \\ W_1' \end{Bmatrix} * \begin{Bmatrix} W_2 \\ W_2' \end{Bmatrix} * \cdots * \begin{Bmatrix} W_{(m-1)d} \\ W_{(m-1)d}' \end{Bmatrix}$$

which is a sphere of dimension $(m-2)$.

~~Let U be a common complement to W_1 and W_1'.~~