Notes on Buildings, symm. spaces, etc.

Third part: Reductive complex group associated to a compact Lie grp; real reductive group assoc. to (K,σ).

Let K be a compact Lie group (not necessarily connected). Suppose K acts on a Hilbert space H; one assumes the map $K \longrightarrow$ unitary gp of H is continuous for the strong topology on the latter, i.e. $\forall v \in H$, $g \mapsto g\cdot v$ is continuous from K to H. (This last notion is forced if one wants the translation action of K on $L^2(K)$ to be considered.)

Let E be the operator $E = x(?,y)$ with $x, y \in H$. Then denoting action of g by T_g, one has $T_g E T_g^{-1} = T_g x(T_g^{-1}?, y) = T_g x(?, T_g y)$. Clearly $g \mapsto T_g E T_g^{-1}$ is continuous from K to End(H) equipped with the uniform topology and

$$\|T_g E T_g^{-1}\| = \|x\|\,\|y\|, \text{ so}$$

$$\left\| \int_{g\in K} T_g E T_g^{-1} dg \right\| \leq \int_{g \in K} \|T_g E T_g^{-1}\| \, dg = \|x\|\cdot\|y\|.$$

Now I suppose H is separable, choose an orthonormal basis $y_1, \dots$, and put $A = \sum \lambda_i y_i (?, y_i)$, where $\lambda_1 > \lambda_2 > \dots > 0$ decrease rapidly so that A is of trace class. By the previous estimates

$$B = \int_g T_g A T_g^{-1}\, dg = \sum_i \lambda_i \int_g T_g y_i (?, T_g y_i)\, dg$$

is a trace class self-adjoint operator, strictly >0. Moreover it evidently commutes with the action of K. The eigenspaces of B are finite-dimensional and their sum is dense in H; these are invariant under the K-action. Thus we obtain the following special case of the Peter-Weyl thm.

<u>Prop. 1</u>: The union of the finite dimensional subspaces invariant under K is dense in H.

Take $H = L^2(K)$. We have two actions of K on H given by left and right translation:

$$(L_g f)(x) = f(g^{-1}x) \qquad (R_g f)(x) = f(xg).$$

These commute, so we have an action of $K \times K$ on $L^2(K)$ given by

$$\left(T_{(g_1, g_2)} f\right)(x) = f(g_1^{-1} x g_2).$$

Let V be a finite-diml subspace of H invariant under the operator R_g, $g \in K$. Let $f_1, \dots, f_n$ be a basis for V whence we get $\rho : K \to GL_n\mathbb{C}$ a continuous hom. such that

$$T_g f_i = \sum_j \rho_{ji}(g) f_j \qquad \text{or}$$

$$(1) \qquad f_i(xg) = \sum_j f_j(x)\,\rho_{ji}(g)$$

One knows ρ is an analytic homomorphism, hence taking $x = 1$ we see f_i is analytic. Moreover V is contained in the subspace W spanned by the ρ_{ij}, which is invariant under both L_g and R_g since

$$\rho_{ij}(xy) = \sum_l \rho_{il}(x)\,\rho_{lj}(y).$$

Thus one sees that in H, the union of the finite dimensional subspaces invariant for the $\{R_g\}$ is the same as for the $\{T_{g_1,g_2}\}$, hence for the $\{L_g\}$. This union we denote by $A(K)$; its elements are called representative functions.

Given a repn. V of K we get a map

$$(2) \qquad \begin{aligned} V^* \otimes V &\longrightarrow A(K) \\ \lambda \otimes v &\longmapsto (g \mapsto \lambda(gv)) \end{aligned}$$

which is equivariant for the action of $K \times K$:

$$\begin{aligned} g_1\lambda \otimes g_2 v \longmapsto \big(g \mapsto (g_1\lambda)(g g_2 v) &= \lambda(g_1^{-1} g g_2 v)\big) \\ &= T_{(g_1,g_2)}\big(g \mapsto \lambda(gv)\big). \end{aligned}$$

The map $V \to A(K)$ $\quad v \mapsto (g \mapsto \lambda(gv))$, where λ is fixed, is equivariant for R_g-operators. V embeds in $A(K)$ iff λ is a cyclic vector for V^* (the ~~gd~~ span V^*). Thus a finite diml. subspace $V \subset A(K)$ invariant under $\{R_g\}$ can be identified with a representation V equipped with a cyclic vector in V^*.

~~Prop. 2: $\mathrm{Hom}_K(V, A(K)) \simeq V^*$ for any (finite diml.) representation V of K.~~

~~Proof: Given $h: V \to A(K)$ we define $\lambda_h(v) = h(v)(1)$.~~

Adjoint to (2) is the map

$$(3) \qquad V^* \longrightarrow \mathrm{Hom}_K(V, A(K))$$
$$\lambda \longmapsto (v \mapsto (g \mapsto \lambda(gv)))$$

which is a K-module map for the action on the Hom induced by $\{L_g\}$. Given $h: V \to A(K)$ we can define $\lambda_h \in V^*$ by $\lambda_h(v) = h(v)(1)$. Then one sees easily that $h \mapsto \lambda_h$ is inverse to the above map (3). So

<u>Prop. 2</u>: $V^* \xrightarrow{\sim} \mathrm{Hom}_K(V, A(K))$.

It follows that each irreducible repn. of K occurs in $A(K)$ with multiplicity equal to its dimension; the same is true of $L^2(K)$. Let V_i, $i \in I$, be representatives for the different iso. classes of irreducible representations of K. By Schur's lemma

$$\mathrm{Hom}_K(V, V) \simeq \mathbb{C}$$

if V is irreducible. Since $L^2(K)$, hence $A(K)$, is an orthogonal sum of irreducibles we get an orthogonal decomposition

$$A(K) \simeq \bigoplus_i \mathrm{Hom}_K(V_i, A(K)) \otimes V_i$$

so we get:

<u>Prop. 3</u>: $$A(K) \xleftarrow{\ \simeq\ } \bigoplus_i V_i^* \otimes V_i$$ (orthogonal direct sum for inner product in L^2)

$$(g \mapsto \lambda(gv)) \longleftarrow\!\shortmid \quad \lambda \otimes v$$

where V_i, $i \in I$, are representatives for the different iso. classes of irreducible reps. of K.

<u>Remark</u>: This isomorphism is compatible with $K \times K$ action, hence an irreducible rep. W of $K \times K$ ($W \simeq V_1 \boxtimes V_2$ where V_1, V_2 are irred. over K) occurs in $A(K)$ at most once, and it occurs iff W has a fixed vector under ΔK (in which case $V_1 \simeq V_2^*$). (This generalizes to the symmetric space case.)

~~in A(K) iff λ is a cyclic vector for V^*. Thus a finite-dimensional subspace of A(K) invariant under $\{R_g\}$ can be identified with a representation V of K equipped with a cyclic vector in V^*.~~

A(K) is an algebra over $\mathbb{C}$ with product given by multiplication of functions. (Since product of functions commutes with the T_{g_1,g_2}-action, the product of invariant subspaces is invariant). In addition it is a bialgebra with coproduct induced by the multiplication in K (see (1)) and antipode by $g \mapsto g^{-1}$. Therefore A(K) defines an affine algebraic group G over $\mathbb{C}$ by

$$G = \mathrm{Hom}_{\mathbb{C}\text{-alg.}}(A(K), \mathbb{C}),$$

and we have a homomorphism $K \longrightarrow G$. I can think of G as being a complex Lie group such that every f in A(K) extends uniquely to a holomorphic function on G whose translates form a finite-dimensional space.

<u>Prop. 4</u>: A compact Lie gp. K has a faithful finite-dimensional unitary representation, i.e. $\exists\ K \hookrightarrow U_m$. (Consequently $K \hookrightarrow G$.)

<u>Proof</u>: For each finite-dimensional subspace V of A(K) invariant under the right repn., consider its

kernel (those elements of K acting trivially). As V increases the kernel decreases. Since the closed subgroup of K satisfy the d.c.c. there is a representation $V < A(K)$ whose kernel N acts trivially on $A(K)$. But $A(K)$ is dense in $L^2(K)$, ~~so~~ we get a contradiction if $N \neq 1$. QED.

bad notation as Θ = Cartan invol.

Suppose we have an embedding $\Theta : K \hookrightarrow U_m$. Since $\overline{\Theta(k)} = (\Theta(k)^t)^{-1}$ = a poly in the $\Theta_{ij}(k)$ and $(\det \Theta(k))^{-1}$, it follows that the subalgebra A' of $A(K)$ generated by the functions Θ_{ij} and $(\det \Theta)^{-1}$ is closed under conjugation and separates points (as Θ is injective). By the Weierstrass thm. A' is dense in $L^2(K)$. ~~This forces $A' = A(K)$.~~

If $A' < A(K)$, then because A' is stable

~~Corollary to Prop. 1: Let W be the union of the finite dimensional invariant subspaces of H, and let W' be a subspace of W invariant under K. If $W' = H$, then $W' = W$.~~

under $K \times K$, there would exist a subspace $V_i^* \otimes V_i$ as in Prop. 3 orthogonal to A'. This contradicts A' being dense in $L^2(K)$. Thus we have

<u>Prop. 5:</u> If $\Theta : K \hookrightarrow U_m$ is an embedding, then the functions Θ_{ij}, $1 \le i, j \le m$, and $(\det \Theta)^{-1}$ generate the algebra $A(K)$. In particular $\Theta^* : A(U_m) \twoheadrightarrow A(K)$.

Let us consider the homomorphism
$$\mathbb{C}[X_{ij}, (\det X)^{-1}] \longrightarrow A(U_m)$$
sending X_{ij} to the function which assigns to a matrix its (i,j)-th entry; here the X_{ij}, $1 \le i,j \le m$ are indeterminates. According to Prop. 5 this homomorphism is surjective. Suppose $f(X) = (\det X)^{-r} p(X)$ is in the kernel, where p is a polynomial in the X_{ij}. Consider the function $f \circ \exp : \mathfrak{gl}_m \to GL_m \to \mathbb{C}$, where f is interpreted in the obvious way as a holomorphic function on GL_m. The function $f \circ \exp$ is holom. on $\mathfrak{gl}_m$ and by assumption it vanishes on $\mathfrak{u}_m$ which is the fixed subspace for the conjugation $\theta(X) = -\bar{X}^t$. Thus I have a holom. fn. on $\mathbb{C}^k$ vanishing on $\mathbb{R}^k$ essentially, so $f \circ \exp = 0$. Thus $f = 0$ as a function on GL_m, and this implies that $f = 0$ as an element of $\mathbb{C}[X_{ij}, (\det X)^{-1}]$. So:

<u>Prop. 6</u>: We have an isomorphism
$$\mathbb{C}[X_{ij}, (\det X)^{-1}] \xrightarrow{\sim} A(U_m).$$

Consequently $GL_m = GL_m\mathbb{C}$ is the complex algebraic group associated to U_m.

~~Suppose $\phi : K \to U_m$ is an imbedding. Then by Prop. 5, $\phi^* : A(U_m) \to A(K)$ has~~

From Prop. 5 we get generators for the alg. $A(K)$. We now wish to understand the relations between these generators.

Suppose K is a closed subgp of the compact Lie group H. Let $i: K \to H$ be the inclusion. Then $i^*: A(H) \longrightarrow A(K)$ is surjective, because if we embed $H \hookrightarrow U_m$, we have $A(U_m) \twoheadrightarrow A(K)$ by Prop. 5. Let $\mathfrak{a} = \operatorname{Ker} i^*$. Because $A(H)$ is noetherian, the ideal $\mathfrak{a}$ is finitely generated, hence we can find a finite-dimensional subspace V stable under the right translation action such that $W = V \cap \mathfrak{a}$ generates $\mathfrak{a}$ as an ideal.

It is clear that W is stable for R_k, $k \in K$. Conversely suppose $h \in H$ is such that $R_h W = W$. As R_h is a ring automorphism $R_h \mathfrak{a} = \mathfrak{a}$, i.e. any $f \in A(H)$ vanishing on K also vanishes at h. ~~Consider~~ ~~$L^2(K\backslash H) = \{f \in L^2(H) \mid L_k f = f \text{ all } k \in K\}$. By Prop. 1 $A(K\backslash H) = \{f \in A(H) \mid L_k f = f \text{ all } k \in K\}$ is dense in $L^2(K\backslash H)$~~ ~~As usual we identify $L^2(K\backslash H)$ with the subspace of $L^2(H)$ consisting of f such that $L_k f = f$ for all k in K. By prop. 1, $A(K\backslash H) = A(H) \cap L^2(K\backslash H)$ is dense in $L^2(K\backslash H)$. If $h \notin K$, then choose a~~

If $h \notin K$, choose a continuous function on H vanishing on K and equal to 1 on hK and approximate it uniformly by an f in $A(H)$ (this is possible, see below:). If the approximation is close, then $f'=\int_K R_k f$ will be in $A(K)$ and it will have different values on the cosets K, hK. Then $f'-f'(1)$ will vanish on K and be $\neq 0$ at h. Thus we have proved

Lemma 1:
$$K = \{h \in H \mid R_h W = W\}.$$

In the course of the proof we used the following ingredient in the proof of Prop. 5.

Prop. 5': Any continuous function on K can be uniformly approximated by representative functions.

In effect, if $\phi: K \hookrightarrow U_m$, then the subalg. $A' \subset A(K)$ generated by ϕ_{ij} and $(\det\phi)^{-1}$ separates points and is closed under conjugation, so this follows from the Weierstrass thm.
In fact one gets C^n approximation if one wants.

Returning to $K \subset H$, $W \subset V$, let $v_1, \dots, v_n$ be an orthonormal basis for V such that $v_1, \dots, v_p$ span W. Let $\rho: H \to U_n$ be the homo. given by R_h action on V:

(1) $$v_j(xh) = \sum_j v_i(x)\rho_{ij}(h).$$

Let τ be the diagonal matrix with entries $(\underbrace{1,\dots,1}_{p}, \underbrace{-1,\dots,-1}_{n-p})$ and put $\rho'(h) = \tau\rho(h)\tau^{-1}$. Then $\rho'(h) = \rho(h)$ iff $\rho(h)$ commutes with τ, i.e. R_h preserves W. Thus $\rho'(h) = \rho(h)$

iff $h \in K$ by Lemma 1.

Let X_{ij} denote the function in $A(U_n)$ sending a matrix to its (i,j)-th entry. Then $\rho^*(X_{ij}) = \rho_{ij}$ and

$$\rho'^*(X_{ij}) = \begin{cases} \rho_{ij} & i,j \leq p \text{ or } > p \\ -\rho_{ij} & i \leq p < j \text{ or } j \leq p < i. \end{cases}$$

Thus $\mathrm{Im}(\rho^* - \rho'^*)$ contains ρ_{ij} for $j \leq p < i$. From (1) we have

$$v_j(h) = \sum_{i > p} v_i(e) \rho_{ij}(h)$$

since $v_i \in \mathfrak{a}$ for $i \leq p$. Thus $\mathrm{Im}(\rho^* - \rho'^*)$ contains $v_1, \ldots, v_p$, and as it is obviously contained in $\mathfrak{a}$ (because $\rho = \rho'$ on K), it generates the ideal $\mathfrak{a}$. Thus we have established:

<u>Lemma 2</u>: If K is a closed subgroup of H, then there exist homomorphisms $H \overset{\rho}{\underset{\rho'}{\rightrightarrows}} U_n$ such that $K = \{h \mid \rho(h) = \rho'(h)\}$ and such that $\mathfrak{a} = \mathrm{Ker}\,\{A(H) \to A(K)\}$ is generated by $\mathrm{Im}(\rho^* - \rho'^*)$.

For completeness we prove:

<u>Lemma 3</u>: ρ, ρ' can be chosen such that $\mathfrak{a} = \mathrm{Im}(\rho^* - \rho'^*)$.

<u>Proof</u>: Start with ρ, ρ' as in lemma 2, choose an embedding $\varepsilon : H \times U_n \hookrightarrow U_{n_1}$, and let ρ_1

be the composite homomorphism

$$H \xrightarrow{\rho_0=(id,\rho)} H\times U_n \overset{\varepsilon}{\hookrightarrow} U_{n_1}$$

and define ρ_0', ρ_1' similarly. Because ρ_0, ε are embeddings, ρ_1^* is surjective. Using $pr_1 : H\times U_n \to H$, one sees that for any $x \in A(H)$, there exists $y \in A(U_{n_1})$ with $\rho_1^*(y) = \rho_1'^*(y) = x$. Now given $z \in A(U_{n_1})$, take $x = \rho_1'(z)$; then

$$\rho_1^*(z) - \rho_1'^*(z) = \rho_1^*(z-y) \quad \text{where}$$

$$\rho_1'^*(z-y) = 0.$$

Thus $\mathrm{Im}(\rho_1^* - \rho_1'^*) \subset \rho_1^*(\mathrm{Ker}\,\rho_1^*)$, whence these two are equal, since the reverse inclusion is obvious. But ρ_1^* being surjective, it carries the ideal $\mathrm{Ker}\,\rho_1^*$ to an ideal of $A(H)$. Thus $\mathrm{Im}(\rho_1^*-\rho_1'^*)$ is an ideal in $A(H)$ contained in $\mathfrak{a}$. Finally

$$\mathrm{Im}(\rho_1^*-\rho_1'^*) = \mathrm{Im}(\rho_0^*-\rho_0'^*) \supset \mathrm{Im}(\rho_0^*-\rho_0'^*)pr_2^* = \mathrm{Im}(\rho^*-\rho'^*)$$

so this ideal must be all of $\mathfrak{a}$. QED.

Combining the preceding lemmas with Props. 4 + 5 we get:

<u>Prop. 7</u>: Given a compact Lie group K, one can find homomorphisms

$$(1) \qquad K \xrightarrow{i} U_m \overset{\rho}{\underset{\rho'}{\rightrightarrows}} U_n$$

such that

i) i is an isomorphism of K with $\{h \in U_m \mid \rho(h) = \rho'(h)\}$.

ii) The sequence

$$(2) \qquad 0 \leftarrow A(K) \xleftarrow{i^*} A(U_m) \xleftarrow{\rho^* - \rho'^*} A(U_n)$$

is exact.

Let $G = \mathrm{Hom}_{\mathbb{C}\text{-alg}}(A(K), \mathbb{C})$ be the alg. gp. over $\mathbb{C}$ associated to K. Combining (2) with Prop. 6, it is clear that we have a diagram

$$(3) \qquad G \xrightarrow{i} GL_m \overset{\rho}{\underset{\rho'}{\rightrightarrows}} GL_n$$

which is exact in the sense that i is an isomorphism of G with the equalizer of ρ, ρ' as an algebraic subgrp. of GL_m.

It is clear that the complex conjugate of a representative function on K is again a representative function. Given $g \in G$ the composite

$$A(K) \xrightarrow{-} A(K) \xrightarrow{g} \mathbb{C} \xrightarrow{-} \mathbb{C}$$

is a $\mathbb{C}$-algebra homomorphism, hence we get an element of G which we denote θg. Thus if we write $f(g)$ for the image of $f \in A(K)$ under the homo. g we have

$$f(\theta g) = \overline{f(g)}.$$

The product on G is defined as follows: Let $\Delta : A(K) \longrightarrow A(K) \otimes A(K)$ be defined by $\Delta f = \sum f_i' \otimes f_i''$ iff $f(xy) = \sum f_i'(x) f_i''(y)$ for all $x, y \in K$. Then for $g_1, g_2 \in G$ the product is defined by the formula

$$f(g_1 g_2) = \sum f_i'(g_1) f_i''(g_2).$$

It follows that

$$f(\theta(g_1 g_2)) = \overline{f(g_1 g_2)} = \sum \overline{f_i'(g_1)}\; \overline{f_i''(g_2)}$$

$$= \sum f_i'(\theta g_1) f_i''(\theta g_2)$$

$$= f(\theta g_1, \theta g_2)$$

hence θ is an endomorphism of G. It's clear θ is of order 2, i.e. an involution of G. θ is an automorphism of the underlying real algebraic group to G. It is called the Cartan involution of G.

~~Complex conjugation on $A(U_n)$ carries X_{ij} into the function $\overline{X_{ij}}$ which is a polynomial in the X_{ij} and $(\det X)^{-1}$ which can be computed from the formula~~

$$\cancel{\overline{X} = (X^t)^{-1} = (\det X)^{-1} \operatorname{Cof}(X^t).}$$

~~Now we've identified GL_n with homs. $g : A(U_n) \to \mathbb{C}$ by associating to g the matrix $X(g)$. Hence~~

~~$X(\theta g) = \overline{X}(g) = (X(g)^t)^{-1}$ and [redacted] so we see the Cartan involution on GL_n is $A \mapsto (A^t)^{-1}$~~

Let's compute θ for GL_n. If A is a unitary matrix, then $A^{-1} = \bar{A}^t$ so $\bar{A} = (A^t)^{-1} = (\det A)^{-1} \operatorname{cof}(A^t)$. Thus complex conjugation on $A(U_n) = \mathbb{C}[X_{ij}, (\det X)^{-1}]$ sends $\lambda \mapsto \bar{\lambda}$ for $\lambda \in \mathbb{C}$ and $X_{ij} \mapsto \bar{X}_{ij}$, where $\bar{X} = (X^t)^{-1} = (\det X)^{-1} \operatorname{cof}(X^t)$. Let $A \in GL_n$, and let g be the corresponding homomorphism $g : A(U_n) \longrightarrow \mathbb{C}$, so that $X(g) = A$. Then $X(\theta g) = \bar{X}(g) = (X(g)^t)^{-1} = (X(g)^*)^{-1}$. Therefore the Cartan involution [redacted] on GL_n is

$$(4) \qquad \theta(A) = (A^*)^{-1}$$

and so the fixpt. grp. for θ is just U_n.

Returning to (3), [redacted] taking fixpts. for θ, and comparing with (1) we get:

Prop. 8: If θ is the Cartan involution on G, then $K =$ [redacted] G^θ.

Let $A(K;\mathbb{R})$ be the fixed ring of conjugation on $A(K)$, i.e. the algebra of real-valued representation functions. Then

$$A(K;\mathbb{R}) \otimes_{\mathbb{R}} \mathbb{C} = A(K).$$

$A(K;\mathbb{R})$ is a bigebra over $\mathbb{R}$, hence it yields an affine group scheme over $\mathbb{R}$ which I denote temporarily by $\underset{\sim}{G}$. One has $\underset{\sim}{G}(\mathbb{C}) = G$; θ is induced by conjugation on $\mathbb{C}$. Prop. 8 says $\underset{\sim}{G}(\mathbb{R}) = K$. In classical language G is defined over $\mathbb{R}$ and K is the group of its $\mathbb{R}$-valued points.

For any algebra R over $\mathbb{R}$ we have

$$
\begin{aligned}
\underset{\sim}{G}(R) &= \mathrm{Hom}_{\mathbb{R}\text{-alg}}(A(K;\mathbb{R}), R) \\
&= \{ g \in \mathrm{Hom}_{\mathbb{C}\text{-alg.}}(A(K), R\otimes\mathbb{C}) \mid \overline{f}(g) = f(g) \} \\
&= \underset{\sim}{G}(R\otimes\mathbb{C})^{\theta}
\end{aligned}
$$

where θ is the effect of the conjugation on $R\otimes\mathbb{C}$. From now on we write $G(R)$ instead of $\underset{\sim}{G}(R)$ as long as this causes no confusion.

Returning to the situation of Prop. 7, if R is a $\mathbb{R}$-algebra it is clear from ii) that the diagram

$$
(5) \qquad G(R) \longrightarrow GL_m(R) \rightrightarrows GL_n(R)
$$

is exact.

Let $\mathfrak{g}$ be the Lie algebra of G, and recall the formula:

$$
\mathfrak{g} = \mathrm{Hom}_{\text{aug } \mathbb{C}\text{-algs}}(A(K), \mathbb{C}[\varepsilon]/(\varepsilon^2))
$$

where $A(K)$ is augmented by evaluation at 1. Comparing (5) for $R = \mathbb{C}[\varepsilon]/(\varepsilon^2)$ and $R = \mathbb{C}$, one gets an exact diagram

(6) $$\mathfrak{g} \longrightarrow \mathfrak{gl}_m \rightrightarrows \mathfrak{gl}_n .$$

If $\mathfrak{k}$ is the Lie algebra of K we have an exact diagram

(7) $$\mathfrak{k} \longrightarrow \mathfrak{u}_m \rightrightarrows \mathfrak{u}_n$$

because it is clear from (1) that a 1-parameter subgroup $\mathbb{R} \to K$ is the same as a 1-parameter subgroup in U_m equalized by ρ, ρ'. Comparing (6) with the complexification of (7) we get

<u>Prop. 9</u>: $\mathfrak{k} \otimes \mathbb{C} = \mathfrak{g}, \qquad \mathfrak{k} = \mathfrak{g}^{\theta}$.

G being an alg. group over $\mathbb{C}$ is in particular a complex Lie group so it has an exp. map

$$\exp : \mathfrak{g} \longrightarrow G \qquad X \longmapsto e^{X} .$$

We consider the map

(8) $$K \times \mathfrak{k} \longrightarrow G \qquad (k, X) \longmapsto k e^{iX} .$$

In the case $K = U_m$, this map is bijective. In effect every invertible matrix A can be uniquely factored $A = UP$, with U unitary and P positive definite hermitian, and P can be uniquely expressed $P = e^{H}$ with H hermitian. Furthermore (8) is a diffeomorphism, because it is a C^{∞} map whose differential one can show is everywhere non-singular.

Now upon taking the submanifolds on both sides of (8) defined by $\rho = \rho'$ we deduce:

<u>Prop. 10:</u> [redacted] One has a diffeomorphism $K \times \mathfrak{k} \xrightarrow{\sim} G$ given by $(k, X) \longmapsto k e^{iX}$. Consequently K and G have the same homotopy type.

~~[redacted] Next suppose we have an involution σ on K. σ extends to an involution $\tilde{\sigma}$ of G as algebraic group over $\mathbb{C}$ and one has $\theta\tilde{\sigma} =$ [redacted] $\tilde{\sigma}\theta$. [redacted] $\theta\tilde{\sigma}$ is an involution of G.~~

Let σ be an involution on K. By functorality it induces an involution $\tilde{\sigma}$ of G as an alg. group over $\mathbb{C}$. One has $\theta\tilde{\sigma} = \tilde{\sigma}\theta$, hence $\theta\tilde{\sigma}$ is an involution of G as an algebraic group over $\mathbb{R}$, which reverses the complex structure. We [redacted] define σ on G to be $\theta\tilde{\sigma}$; it is the unique anti-holomorphic involution of G agreeing with σ on K. [redacted]

σ can be described in algebra terms as follows. The [redacted] homom. $\sigma^*: A(K) \to A(K)$ induces $\tilde{\sigma}$ on [redacted] $G = \mathrm{Hom}_{\mathbb{C}\text{-alg}}(A(K), \mathbb{C})$. Given $g \in G$, then σg is the composition

$$A(K) \xrightarrow{\sigma^*} A(K) \xrightarrow{\ -\ } A(K) \xrightarrow{g} \mathbb{C} \xrightarrow{\ -\ } \mathbb{C}$$

i.e.

$$f(\sigma g) = \overline{(\overline{f\sigma})(g)} \left(= f(\theta\tilde{\sigma} g)\right).$$

Now σ on G can be interpreted as descent data for G relative to $\mathbb{R} \subset \mathbb{C}$, i.e. $f \mapsto \overline{\sigma^*(f)}$ is a conjugation on $A(K)$ whose invariants form a bigebra over $\mathbb{R}$. Thus G^σ is a real algebraic group with complexification G.

On $\mathfrak{g} = \mathfrak{k} \otimes \mathbb{C}$, $\tilde{\sigma}$ is the complexification of σ and θ is the conjugation which is the identity on $\mathfrak{k}$. We have

$$\begin{array}{ccccccccc} \mathfrak{g} & = & \mathfrak{k}^+ & \oplus & \mathfrak{k}^- & \oplus & i\mathfrak{k}^+ & \oplus & i\mathfrak{k}^- \\ \theta & : & +1 & & +1 & & -1 & & -1 \\ \tilde{\sigma} & : & +1 & & -1 & & +1 & & -1 \\ \sigma & : & +1 & & +1 & & -1 & & +1 \end{array}$$

hence $\mathfrak{g}^\sigma = \mathfrak{k}^+ \oplus i\mathfrak{k}^-$ is the Lie algebra of G^σ. We put $\mathfrak{p} = i\mathfrak{k}^-$. Note that multiplication by i gives an isom. $\mathfrak{p} \simeq \mathfrak{k}^-$ commuting with the adjoint action of K^σ and $\mathfrak{k}^+$.

Taking σ invariants in Prop. 10 we get:

<u>Prop. 10′</u>: (Cartan decomposition). One has a diffeom.

$$K^\sigma \times \mathfrak{p} \longrightarrow G^\sigma \qquad (k, Y) \longmapsto ke^Y.$$

Hence K^σ and G^σ have the same homotopy type.

Because $\mathfrak{p} \simeq \mathfrak{k}^-$ as K^σ-modules, maximal abelian subspaces of $\mathfrak{p}$ are the same as maximal abelian subspaces of $\mathfrak{k}^-$. In fact in virtue of the Cartan decomposition, our previous analysis of the K^σ orbit structure on $\mathfrak{k}^-$

amounts to an analysis of the K^σ-orbit structure of the space $G^\sigma/K^\sigma \simeq \mathfrak{p}$.

Examples:

1) $K = U_n$, $G = GL_n\mathbb{C}$, σ = complex conjugation on U_n. The extension σ to G is again complex conjugation, as it is anti-holomorphic. $\tilde{\sigma} = \Theta\sigma$ and $\Theta A = (A^*)^{-1}$, so $\tilde{\sigma}A = (A^t)^{-1}$. We have $K^\sigma = O_n$ and $G^\sigma = GL_n\mathbb{R}$. $\mathfrak{k}^- = i$ (real symm.), $\mathfrak{k}^+$ = real skew-symm. $\mathfrak{p}$ = real symm. matrices. G^σ/K^σ = pos. def. real matrices.

2) $K = U_{2n}$, $\sigma x = J(\bar{x})J^{-1}$, where $J = \left(\begin{array}{c|c} 0 & I_n \\ \hline -I_n & 0 \end{array}\right)$ and $\bar{\ }$ is complex conjugation. Here $K^\sigma = Sp_n$. σ on G is again $\sigma x = J\bar{x}J^{-1}$ and $\sigma x = x$ means that x is an auto. of $\mathbb{C}^{2n} \simeq \mathbb{H}^n$ via $j v = J\bar{v}$. Thus $G^\sigma = GL_n(\mathbb{H})$.

3) $K = U_{p+q}$, σ = conjugation by $S = \left(\begin{array}{c|c} -I_p & \\ \hline & I_q \end{array}\right)$, whence $K^\sigma = U_p \times U_q$. $\tilde{\sigma}$ = conjugation by S on GL_n. $\sigma A = S(\Theta A)S^{-1} = S(A^*)^{-1}S^{-1}$. $\sigma A = A \iff S = ASA^*$. Thus G^σ is the subgroup preserving the hermitian form

$$-|z_1|^2 - \cdots - |z_p|^2 + |z_{p+1}|^2 + \cdots + |z_{p+q}|^2.$$

$A \in \mathfrak{g}^\sigma \iff -SA^*S^{-1} = A$. As $\Theta A = -A^*$ on $\mathfrak{gl}_n$, this means $\mathfrak{p} = \{A \mid A = A^*,\ SAS^{-1} = -A\} = \left\{ \left(\begin{array}{c|c} 0 & B \\ \hline B^* & 0 \end{array}\right) \,\middle|\, B \in M_{pq}(\mathbb{C}) \right\}$.

General examples:

4) Modifications of (K,σ): suppose σ extended to G. Make G act on itself by $g * x = g x \sigma(g)^{-1}$. Take a $y \in \tilde{R}_\sigma = \{ y \in K \mid y \cdot \sigma y \in$ center $K \}$. If τ is the modified involution $\tau(x) = y\,\sigma(x)\,y^{-1}$, then τ on G is given by the same formula. It is probably true that G^τ / K^τ is the G^σ-orbit ~~orbit~~ of y.

5) Normal form - G^σ is the Chevalley group over $\mathbb{R}$ with complexification G. In this case K/K^σ has the same rank as K, i.e. $\exists$ maximal torus of K reversed by σ. Ex. 1) is of this type

6) Suppose H is the complex group assoc. to a compact group U, let $K = U \times U$ with $\sigma(x,y) = (y,x)$. Then $G = H \times H$ and $\tilde{\sigma} =$ interchange, hence for G $\sigma(x,y) = \theta(y,x) = (\theta y, \theta x)$. So $\sigma(x,y) = (x,y) \iff y = \theta x$. Thus $G^\sigma \simeq H$ embedded as $\Gamma_\sigma \subset H \times H$. In this case the non-compact symm. space G^σ/K^σ is H/U and the dual compact symm. space is $K/K^\sigma = U\times U/\Delta U \simeq U$ with conjugation action.