December 5, 1975

Problem: To filter the K-theory of coherent sheaves on a smooth variety X in a different way. The idea is that filtration p should be made up of sheaves of codimension p with homotopies coming from sheaves of cod $p-1$ etc.

First step. To compute $K_0(F_p \mathcal{M}(X))$ which should be a quotient of $K_0(\mathcal{M}^p(X))$. Thus I seek

$$K_0(\mathcal{M}^p)/\text{relns.}$$

which should modulo torsion embed into $F_p K_0(X) = \operatorname{Im}\{K_0\mathcal{M}^p \longrightarrow K_0\mathcal{M}\}$. I want an exact sequence

$$\begin{tikzcd}
 & & 0 \\
K_0\mathcal{M}^{p+1}/\text{relns} \arrow[r] & K_0\mathcal{M}^p/\text{relns.} \arrow[r] & A^p(X) \arrow[r] \arrow[u] & 0 \\
K_0\mathcal{M}^{p+1} \arrow[r] \arrow[u] & K_0\mathcal{M}^p \arrow[r] \arrow[u] & K_0(\mathcal{M}^p/\mathcal{M}^{p+1}) \arrow[r] \arrow[u] & 0 \\
K_1(\mathcal{M}^p/\mathcal{M}^{p+1}) \arrow[r, "0"] \arrow[u, "\partial"] & K_1(\mathcal{M}^{p-1}/\mathcal{M}^p) \arrow[r, phantom, "="] \arrow[u, "\partial"] & K_1(\mathcal{M}^{p-1}/\mathcal{M}^p) \arrow[u, "\partial"]
\end{tikzcd}$$

So it is clear that what I want is

$$K_0(\mathcal{F}_p) = \operatorname{Coker}\{K_1(\mathcal{M}^{p-1}/\mathcal{M}^p) \longrightarrow K_0(\mathcal{M}^p)\}$$

Let me now try to understand the map

$$\partial : K_1(\mathcal{M}^{p-1}/\mathcal{M}^{p}) \longrightarrow K_0(\mathcal{M}^{p})$$
$$\wr\!\!\parallel$$
$$\bigoplus_{x \in X^{p-1}} k(x)^*$$

Let $x \in X^{p-1}$ and let $f \in k(x)^*$. Then $\overline{\{x\}} = Y$ is an irreducible subvariety of codim $p-1$ and f is a rational map of Y to $\mathbb{P}^1$. So we can form the graph of f in $Y \times \mathbb{P}^1$ and close it to obtain a subvariety $\tilde{Y} \subset Y \times \mathbb{P}^1$ mapped birationally to Y.

$\tilde{Y}$

$\mathbb{P}^1$

Y

Then perhaps $\partial(f)$ is $\tilde{Y}_0 - \tilde{Y}_\infty$.

Another version: $f \in k(Y)^*$ is a rational function on Y, hence it determines a Cartier divisor (f).

So what appears to be the case is this. Given $f \in k(Y)^*$, one can find an ideal I in $\mathcal{O}_Y$ such that

$$I \begin{smallmatrix} \subset \mathcal{O}_Y \subset \\ \subset f\mathcal{O}_Y \subset \end{smallmatrix} k(Y)^*$$

and then $[\mathcal{O}_Y/I] - [f\mathcal{O}_Y/I] \in K_0(\mathcal{M}^p)$ is the thing one wants to look at.

Another version:

$$\begin{tikzcd} I \arrow[r, hook] \arrow[dr, hook, bend right] & \mathcal{O}_Y \\ & f\mathcal{O}_Y \arrow[u, "s"'] \end{tikzcd}$$

hence one gets a diagram in $\mathcal{M}^{p-1}$

$$\begin{tikzcd} I \arrow[r, hook, shift left, "\alpha"] \arrow[r, hook, shift right, "\beta"'] & \mathcal{O}_Y \end{tikzcd}$$

such that the cokernel of α and β are in $\mathcal{M}^p$.
I guess that $\tilde{Y}$ = closure of graph f is what one obtains by taking $I = \mathcal{O}_Y \cap f\mathcal{O}_Y$. (roughly)

Review regular sheaves on $\mathbb{P}^1_A$.

<u>Prop</u>: TFAE for a quasi-coherent sheaf F on $\mathbb{P}^1_A$.

i) $H^1(F(-1)) = 0$ (i.e. F regular)

ii) $\exists$ presentation $0 \to \mathcal{O}(-1) \otimes L_1 \to \mathcal{O} \otimes L_0 \to F \to 0$

iii) F generated by $H^0(F)$.

ii) $\Rightarrow$ iii) obvious

iii) $\Rightarrow$ i). $0 \to K \to \mathcal{O} \otimes H^0(F) \to F \to 0$

$$0 \simeq H^1(\mathcal{O}(-1) \otimes H^0(F)) \to H^1(F(-1)) \to H^2(F(-1)) = 0$$

i) $\Rightarrow$ ii) From $0 \to F(n-2) \to F(n-1)^2 \to F(n) \to 0$ and the fact that $H^2 = 0$ shows $H^1(F(n)) = 0$ all $n \geq -1$.

$\therefore$

$$H^0(F(n-1))^2 \twoheadrightarrow H^0(F(n)) \to \underset{\substack{\| \\ 0 \text{ if } n \geq 1.}}{H^1(F(n-2))}$$

so $A[T_0,T_1] \otimes H^0(F) \twoheadrightarrow \bigoplus_{n \geq 0} H^0(F(n))$, so F is generated by $H^0(F)$. This gives us

$$0 \to K \to \mathcal{O} \otimes H^0(F) \to F \to 0$$

with $H^0(K) = H^1(K) = 0$, and $H^1(K(-1)) = H^0(F(-1))$. Now if $H^0(F(-1)) = 0$, then K would be regular, hence generated by $H^0(K) = 0$, so $K = 0$. This shows $H^1(F(-1)) = H^0(F(-1)) = 0 \iff F = \mathcal{O} \otimes H^0(F)$. Applying this in the general case to K, we get $K(+1) = \mathcal{O} \otimes H^0(K(1))$ or $K = \mathcal{O}(-1) \otimes L_1$ for some L_1 proving ii).

Next suppose given $L_1 \overset{\alpha}{\underset{\beta}{\rightrightarrows}} L_0$ a diagram of A-modules. Associate the map

$$A[t] \otimes L_1 \xrightarrow{\alpha - t\beta} A[t] \otimes L_0$$

to this diagram. Claim β injective $\Rightarrow$ $\alpha - t\beta$ injective. For if

$$(\alpha - t\beta)(x_0 + t x_1 + \cdots + t^m x_m) = 0$$

$$\|$$

$$\alpha(x_0) + t[\alpha(x_1) - \beta(x_0)] + \cdots + t^m[\alpha(x_m) - \beta(x_{m-1})] - t^{m+1}\beta(x_m) = 0$

so $\beta(x_m) = 0 \Rightarrow x_m = 0$ etc. Thus β injective implies we have an exact sequence

$$0 \longrightarrow A[t]\otimes L_1 \xrightarrow{\alpha - t\beta} A[t]\otimes L_0 \longrightarrow M \longrightarrow 0$$

where $M = \mathrm{Coker}(\alpha - t\beta)$. Furthermore $\mathrm{Tor}_1^{A[t]}(A[t]/(t=0), M) = \mathrm{Ker}\,\alpha$.

We've seen that a regular quasi-coherent sheaf F on $\mathbb{P}^1_A$ can be identified with a diagram of A-modules

$$L_1 \underset{\beta}{\overset{\alpha}{\rightrightarrows}} L_0$$

such that $\mathcal{O}(-1)\otimes L_1 \xrightarrow{T_0\alpha - T_1\beta} \mathcal{O}\otimes L_0$

is injective. For the homotopy problem which involves pulling back to A via $t=0$ and $t=\infty$ we want F to be flat with respect to these pull-backs, i.e. we want α, β to be injective. Thus

<u>Prop.</u> Regular sheaves on $\mathbb{P}^1_A$ having no associated primes in the divisors $t=0$, $t=\infty$ can be identified with diagrams of A-modules

$$L_1 \underset{\beta}{\overset{\alpha}{\rightrightarrows}} L_0$$

such that α, β are both injective.

December 6, 1975

I have seen that the image of $K_1(\mathcal{M}^{p-1}/\mathcal{M}^p) \xrightarrow{\partial} K_0(\mathcal{M}^p)$ is generated by $[L_0/\alpha L_1] - [L_0/\beta L_1]$, where

$$L_1 \underset{\beta}{\overset{\alpha}{\rightrightarrows}} L_0$$

are sheaves in $\mathcal{M}^{p-1}$ such that $L_0/\alpha L_1$, $L_0/\beta L_1$ are in $\mathcal{M}^p$. The above diagram is the same thing as a regular sheaf F on $\mathbb{P}^1 \times X$ flat with respect to the divisors $0 \times X$ and $\infty \times X$. Question: Is it possible to make F flat over $\mathbb{P}^1$ without changing the element $[L_0/\alpha L_1] - [L_0/\beta L_1] = [F_0] - [F_\infty]$ in $K_0(\mathcal{M}^p)$?

Flat over $\mathbb{P}^1$ means that F as a $\mathcal{O}_{\mathbb{P}^1}$-module is torsion-free; since F is flat over $0, \infty \in \mathbb{P}^1$, this means that F as a $k[t]$-module is torsion-free. Let F' be the torsion-submodule, so we have an exact sequence on $\mathbb{P}^1 \times X$

$$0 \longrightarrow F' \longrightarrow F \longrightarrow F'' \longrightarrow 0,$$

where F' is killed by some monic polynomial in t (if X is affine say). Hence $H^1(F(n)) = 0$ for all n, so F'' is regular. Also F'' is torsion-free over $k[t]$, hence it is flat over $\mathbb{P}^1$. Since the support of F' doesn't intersect $0 \times X$, $\infty \times X$ we have $F'_\infty = F_\infty$

and $F_0' = F_0$.

Conclude: We can delete from F all submodules which are finite over X without affecting $[F_0]-[F_\infty]$. So we can suppose $F \in \mathcal{M}(\mathbb{P}^1 \times X)$ has no non-zero submodules finite over X. (When $k=\bar{k}$ and X is proper, this is the same as F being flat over $\mathbb{P}^1$).

Next we should understand transitivity.

Suppose we have $f, g \in k(Y)^*$ Y irred of codimension d. Then to define $\partial(f)$ we used

$$\mathcal{O}_Y \cap f\mathcal{O}_Y \underset{f^{-1}\cdot\mathrm{in}}{\overset{\mathrm{in}}{\rightrightarrows}} \mathcal{O}_Y = L_0$$

Thus $F_0 = L_0/\alpha L_1$ is to be the "zeros" of f i.e. $\mathcal{O}_Y/\mathcal{O}_Y \cap f\mathcal{O}_Y$, and $L_0/\beta L_1 = F_\infty$ is to be the "∞'s" of f: $\mathcal{O}_Y/f^{-1}(\mathcal{O}_Y \cap f\mathcal{O}_Y) = \mathcal{O}_Y/f^{-1}\mathcal{O}_Y \cap \mathcal{O}_Y$. Similarly to define $\partial(g)$ we use

$$\mathcal{O}_Y \cap g\mathcal{O}_Y \underset{\cdot g^{-1}}{\overset{\hookrightarrow}{\rightrightarrows}} \mathcal{O}_Y$$

Now we could have shrunk L_1 so for transitivity we shall want to use:

$$\begin{tikzcd} \mathcal{O}_Y \cap f\mathcal{O}_Y \cap gf\mathcal{O}_Y \arrow[r, hook, shift left=3] \arrow[r, hook, "f^{-1}"] \arrow[r, hook, shift right=3, "g^{-1}f^{-1}"'] & \mathcal{O}_Y \end{tikzcd}$$

Thus the problem becomes to interpret a diagram

$$\begin{tikzcd} L_1 \arrow[r, hook, shift left=3, "\alpha"] \arrow[r, hook, "\beta"] \arrow[r, hook, shift right=3, "\gamma"'] & L_0 \end{tikzcd}$$

in terms of <u>projective lines.</u>

<u>Question</u>: Consider all coherent sheaves on X, better, consider $\mathrm{Iso}(\mathrm{Modf}(X))$. Call two sheaves F_1 and F_2 equivalent if there exists a coh sheaf F on $\mathbb{P}^1 \times X$ and two points $a, b \in \mathbb{P}^1$ such that

$$\iota_a^* F \cong F_1 \qquad \iota_b^* F \cong F_2$$

and such that F is flat with respect to these pull-backs. Is this an equivalence relation?

I've seen that I can suppose F flat with respect to $\mathbb{P}^1$ and that it is regular.

December 8, 1975

Consider how transitivity works for cycles. Call two cycles $\alpha, \beta \in C(X)$ equivalent if $\exists\ \lambda \in C(\mathbb{P}^1 \times X)$ such that

$$i_0^*(\lambda) = \alpha$$

$$i_\infty^*(\lambda) = \beta$$

(it is understood that these pull-backs are OK). If in addition

$$i_0^*(\mu) = \beta$$

$$i_\infty^*(\mu) = \gamma$$

then we have

$$i_0^*\left(\lambda + \mu - p^*(\beta)\right) = \alpha + \beta - \beta = \alpha$$

$$i_\infty^*\left(\lambda + \mu - p^*(\beta)\right) = \beta + \gamma - \beta = \gamma$$

which gives us transitivity.

Next symmetry: $\quad i_0^*(\lambda) = \alpha, \ i_\infty^*(\lambda) = \beta$

$$i_0^*\left(p^*(\alpha+\beta) - \lambda\right) = \alpha + \beta - \alpha = \beta$$

$$i_\infty^*\left(p^*(\alpha+\beta) - \lambda\right) = \alpha + \beta - \beta = \alpha.$$

December 10, 1975

Original idea: Call two sheaves F, G on X lin. equivalent iff there exists a sheaf H on $\mathbb{P}^1 \times X$ such that $H_0 \cong F$, $H_\infty \cong G$.

The trouble with this definition is that lin. equivalence is not transitive.

Here is what one does with cycles. If λ, μ are cycles on $\mathbb{P}^1 \times X$ such that $\lambda_\infty = \mu_0$, then

$$(\lambda+\mu)_0 - (\lambda+\mu)_\infty = \lambda_0 + \mu_0 - \lambda_\infty - \mu_\infty$$
$$= \lambda_0 - \mu_\infty .$$

In other words although λ_0 and μ_∞ are not joined by a family, one has that $\lambda_0 + \nu$ and $\mu_\infty + \nu$ are joined, where $\nu = \lambda_\infty = \mu_0$.

0-cycles on X. $A_0(X)$ is the quotient of $C_0(X)$ by linear equivalences. $C_0(X)$ is the free abelian group associated to the free abelian monoid $\coprod_n S^n X$. $\tilde{A}_0(X)$ is the subgroup of cycles of degree 0. Any element of $\tilde{A}_0(X)$ is of the form $\alpha - \beta$ where $\alpha, \beta \in S^n X$ for some n. $\alpha - \beta = 0$ in $\tilde{A}_0(X)$ iff $\exists\, \lambda \in C_1(\mathbb{P}^1 \times X) \ni \lambda_0 - \lambda_\infty = \alpha - \beta$

in $C_0(X)$. Let $\lambda = \lambda^+ - \lambda^-$

$$\lambda_0^+ + \lambda_\infty^- - \lambda_0^- - \lambda_\infty^+ = \alpha - \beta$$

Put $\mu = \lambda^+ +$ reverse of λ^- for $t \to t^{-1}$ on $\mathbb{P}^1$, then one get a positive cycle μ on $\mathbb{P}^1 \times X$ with

$$\mu_0 - \mu_\infty = \alpha - \beta$$

or $$\mu_0 + \beta = \mu_\infty + \alpha$$

Therefore we have that $\mu_0 + \alpha \sim \mu_\infty + \alpha = \mu_0 + \beta$.
This shows that

$$\alpha, \beta \in S^n X \Rightarrow \left(\alpha = \beta \text{ in } A_0(X) \Leftrightarrow \nu + \alpha \sim \nu + \beta \text{ for some } \nu \in S^m X \right).$$

Idea: Take the additive category $\mathcal{M}$ of coherent sheaves on X and form the pair category $\mathcal{M}^{-1}\mathcal{M} = \langle \mathcal{M} \backslash \mathcal{M} \times \mathcal{M} \rangle$. Then introduce the linear equivalence relations: Because exact sequences split up to linear equivalence, one should get the correct K_0.

Try the same thing with finite sheaves over $\mathbb{G}_m \times X$

Consider $\mathbb{P}^1$ a bit more closely. It is

$$\mathrm{Proj}\ k[T_0, T_1]$$

and the embedding $0 \hookrightarrow \mathbb{P}^1 \hookleftarrow \infty$ are respectively induced by the quotient maps

$$k[T_0, T_1] \underset{\left\{\begin{smallmatrix} T_0 \mapsto 0 \\ T_1 \mapsto T \end{smallmatrix}\right.}{\overset{\left\{\begin{smallmatrix} T_0 \mapsto T \\ T_1 \mapsto 0 \end{smallmatrix}\right.}{\rightrightarrows}} k[T]$$

We can get a cosimplicial scheme without degeneracies in this fashion.

Here's the basic problem: to describe what should be called "loops", that is, linear equivalences of 0 with itself. You want to give a "cycle" over $\mathbb{P}^1 \times X$ with trivializations over $0 \times \mathbb{P}^1$ and $\infty \times \mathbb{P}^1$.

So let $\gamma \in C(\mathbb{P}^1 \times X)$ and suppose $\gamma_0 = 0$, $\gamma_\infty = 0$. If I write $\gamma = \alpha - \beta$, then $\alpha_0 = \beta_0$ and $\alpha_\infty = \beta_\infty$. So I have a path from α_0 to $\alpha_\infty = \beta_\infty$ and then back to $\beta_0 = \alpha_0$

Now what I really ought to examine carefully is the stuff on generalized Jacobians, where one looks at cycles disjoint from a finite set modulo special kinds of linear equivalence (linear equivalence with respect to a conductor $\mathfrak{m}$).

X a complete non-singular curve say $\mathbb{P}^1$, S is a finite set of closed points of X, $\mathfrak{m}$ a positive divisor with support S. Then

$$J_{\mathfrak{m}} = \tilde{C}_0(X-S)/\{(f) \mid f \in k(X)^* / f \equiv 1 \ (\mathfrak{m})\}$$

This means that f is a rational function on X regular at points P of S such that $v_P(f-1) \geq$ coefficients of P in $\mathfrak{m}$.

I want to take $X = \mathbb{P}^1$, $S = \{0, \infty\}$. Recall that any divisor of degree 0 is the divisor of a function f unique up to a scalar. Thus if I specify $D \in \tilde{C}_0(X-S)$, and fix $f_0 \ni (f_0) = D$, then translation by f_0 identifies $\{D + (f) \mid f \equiv 1 \ (\mathfrak{m})\}$ with $\{f \equiv f_0 \ (\mathfrak{m})\}$.

$$\begin{tikzcd}
 & & k^* \arrow[d] & & \\
0 \arrow[r] & \{f \mid f \equiv 1 \ (\mathfrak{m})\} \arrow[r] \arrow[dr] & \{f \mid f \text{ S-unit}\} \arrow[r] \arrow[d] & \Gamma(\mathcal{O}/\mathfrak{m}\mathcal{O})^* \arrow[r] & 0 \\
 & & \tilde{C}_0(X-S) \arrow[d] & & \\
 & & 0 & &
\end{tikzcd}$$

Thus

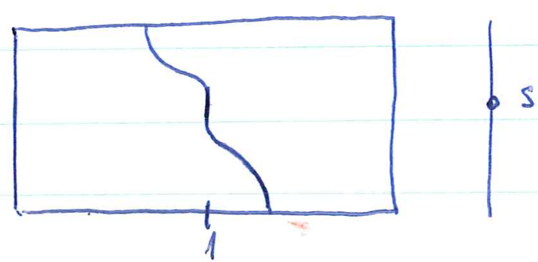
$$J_m \cong \Gamma(\mathcal{O}/m\mathcal{O})^*/k^*$$

Ignoring p-torsion this means I have $k^* \times k^*/k^*$ taking m to be of the first order at $0, \infty$.

The question now is whether all this can be adapted to give a different sort of K_1. Specifically one wants to start with sheaves on $\mathbb{G}_m \times X$ finite over X. These are the same as sheaves on X with an automorphism. Next I want to define a suitable notion of linear equivalence.

Suppose C is a complete n.s. curve and S is a finite set of points in C. Let F be a sheaf over $\mathbb{P}^1 \times C$ of codim. 1 such that F_0 and F_∞ have support in $C-S$. What does it mean to say that F is the identity near S?

Example: Suppose F is the graph of a rational function $f: C \longrightarrow \mathbb{P}^1$. Then first of all we want $f(S) = 1$.



So this translates to

$$\mathrm{Supp}(F) \cap \mathbb{P}^1 \times S \subset 1 \times \mathbb{P}^1$$

So the condition is something like $t-1$ is zero on the restriction of F to $\mathbb{P}^1 \times S$. Recall that S is a given subscheme of C, i.e. $\mathfrak{m}$ is an ideal in $\mathcal{O}_C$. The condition is that $t-1$ kills $F/\mathfrak{m}F$.

If F is regular on $\mathbb{P}^1 \times C$, then it is given by a diagram

$$L_1 \overset{\alpha}{\underset{\beta}{\rightrightarrows}} L_0$$

where L_i are coherent sheaves on C. Then assuming F flat over the points of S, $F/\mathfrak{m}F$ is given by the diagram

$$L_1/\mathfrak{m}L_1 \overset{\breve{\alpha}}{\underset{\breve{\beta}}{\rightrightarrows}} L_0/\mathfrak{m}L_0$$

and the condition that $t-1$ kills $F/\mathfrak{m}F$ means that $\bar{\alpha} = \bar{\beta}$ are isomorphisms.

In fact since $F_0 = L_0/\alpha L_1$ has support off S we know $\mathfrak{m}L_0 + \alpha L_1 = L_0$ so $\bar{\alpha}$ is onto. Also $\bar{\beta}$.

Let us make the following definition. To define K groups of X modulo a closed subscheme Z defined by an ideal m, we consider the Grothendieck group of coherent sheaves on X with support in $X-Z$, and we divide out by elements $[F_0]-[F_\infty]$ obtained from a diagram

$$L_1 \overset{\alpha}{\underset{\beta}{\rightrightarrows}} L_0 \qquad \text{of sheaves on } X$$

such that

$$F_0 = L_0/\alpha L_1$$
$$F_1 = L_0/\beta L_1$$

have support off Z and such that the two isoms.

$$\bar{\alpha}, \bar{\beta} \;:\; L_1/m_{L_1} \overset{\sim}{\underset{\sim}{\rightrightarrows}} L_0/mL_0$$

are equal. Does this give a good $K(X,Z)$?

If I tried applying this to $\mathbb{G}_m^n = X - Z$, $X = (\mathbb{P}_1)^n$ then I would get what?

December 13, 1975

Question: Let S be a curve on a surface X (say a divisor with normal crossings) such that $X-S$ is affine say. (Example $X = \mathbb{P}^1 \times \mathbb{P}^1$, $X - S = \mathbb{G}_m \times \mathbb{G}_m$). Consider all pairs (C, f) where C is an irreducible curve on X such that $C \not\subset S$, and where $f \in k(C)^*$ is such that $f = 1$ on the subscheme $C \cap S$. What is the meaning of the group

$$C_0(X-S) \Big/ \text{divisors of } (C,f)$$

and how can it be computed?

Problem: Let Z be a closed subscheme of X. The problem is to obtain a good theory of sheaves on $X-Z$ with support closed in X, leading to groups $K_*(X, Z)$.

It might be the case that I can construct such a K-theory naively using sheaves on $X-Z$ with support closed in X, which ~~might not~~ would agree with the good groups high up. Like connected k-theory.

December 14, 1975

Try to derive the Brown-Gersten spectral sequence:
$$E_2^{pq} = H^p(X, \mathcal{K}'_{-q}) \Longrightarrow K'_{-p-q}(X)$$

My feeling is that if $\mathcal{U}$ is a finite covering of X, then there should exist a descent spectral sequence
$$E_2^{pq} = \check{H}^p(\mathcal{U}, U \mapsto K'_{-q}(U)) \Longrightarrow K'_{-p-q}(X).$$

For example if $X = U_1 \cup U_2$, then one has a MV sequence
$$\cdots \longrightarrow K'_0(X) \longrightarrow K'_0(U_1) \times K'_0(U_2) \longrightarrow K'_0(U_1 \cap U_2) \longrightarrow 0$$

Murthy example: $K_1 \neq K_1^{Bass}$. Let A be the local ring of the ordinary double point  and let $\bar{A}$ be the normalization  of A. Then $\bar{A}$ is semi-local with two maximal ideals.

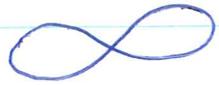
$$\begin{tikzcd}
K_1k \times K_1k \arrow[r] \arrow[d, "+"] & K_1'\bar{A} \arrow[r] \arrow[d] & K_1F \arrow[r] \arrow[d, equal] & \mathbb{Z} \oplus \mathbb{Z} \arrow[r] \arrow[d, "+"] & K_0'\bar{A} \arrow[r] \arrow[d] & K_0F \arrow[r] \arrow[d, equal] & 0 \\
K_1k \arrow[r] & K_1'A \arrow[r] & K_1F \arrow[r] & \mathbb{Z} \arrow[r] & K_0'A \arrow[r] & K_0F \arrow[r] & 0
\end{tikzcd}$$

Now $\bar{A}$ being regular semi-local one has $K_q'\bar{A} = K_q\bar{A}$, hence $K_1'\bar{A} = \bar{A}^*$, $K_0'\bar{A} = \mathbb{Z}$. Diagram chasing shows $K_0'\bar{A} \twoheadrightarrow K_0'A$, so we get

$$\begin{tikzcd}
0 \arrow[r] & \bar{A}^* \arrow[r] \arrow[d, phantom, "\cap"] & F^* \arrow[r] \arrow[d, equal] & \mathbb{Z} \oplus \mathbb{Z} \arrow[r] \arrow[d, "+"] & 0 \\
0 \arrow[r] & K_1'A \arrow[r] & F^* \arrow[r] & \mathbb{Z} \arrow[r] & 0
\end{tikzcd}$$

Now the point is that if M is a f.t. A-module with an autom θ, then the determinant of θ on $M \otimes_A F$ $= \Lambda^r\theta$ on $(\Lambda^r M) \otimes_A F$ is an element of $\bar{A}$ because it preserves the "lattice" $\mathrm{Im}\{\Lambda^r M \to \Lambda^r M \otimes_A F\}$. Therefore $\bar{A}^* = $ Image of $\{K_1^{Bass}(\mathrm{Mod}_f A) \to K_1'A\}$.

Let X be the projective line pinched together



and $\bar{X} = \mathbb{P}^1$ its normalization.

$$
\begin{tikzcd}
 & & & & \mathbb{Z}^2 \arrow[d, equal] & \mathbb{Z} \arrow[d, equal] & \\
(K_1 k)^2 \arrow[r] \arrow[d, "+"] & K_1 \bar{X} \arrow[r] \arrow[d] & K_1(\mathbb{G}_m) \arrow[r] \arrow[d, equal] & \mathbb{Z}^2 \arrow[r] \arrow[d, "+"] & K_0 \bar{X} \arrow[r] \arrow[d] & K_0 \mathbb{G}_m \arrow[r] \arrow[d, equal] & 0 \\
(K_1 k) \arrow[r, hook] & K_1' X \arrow[r] & K_1'(\mathbb{G}_m) \arrow[r] & \mathbb{Z} \arrow[r, hook] \arrow[d, equal] & K_0' X \arrow[r] \arrow[dl, "\delta_X"] & K_0 \mathbb{G}_m \arrow[r] & 0 \\
 & & & \mathbb{Z} & & &
\end{tikzcd}
$$

$$
\begin{tikzcd}
0 \arrow[r] & K_q k \arrow[r] \arrow[dr, "\sim"'] & K_q' X \arrow[r] \arrow[d, "\delta_X"] & K_q \mathbb{G}_m \arrow[r] & 0 \\
 & & K_q k & &
\end{tikzcd}
$$

Thus one gets

$$
\begin{tikzcd}
0 \arrow[r] & K_1 k \arrow[r] \arrow[d, equal] & K_1 \bar{X} \arrow[r] \arrow[d] & K_1 \mathbb{G}_m \arrow[r] \arrow[d, equal] & \mathbb{Z} \arrow[r] & 0 \\
0 \arrow[r] & K_1 k \arrow[r] & K_1' X \arrow[r] & K_1 \mathbb{G}_m \arrow[r] & 0 &
\end{tikzcd}
$$

so $\boxed{0 \longrightarrow K_1 \bar{X} \longrightarrow K_1' X \longrightarrow \mathbb{Z} \longrightarrow 0}$

More generally $\boxed{0 \longrightarrow K_q \bar{X} \longrightarrow K_q' X \longrightarrow K_{q-1} k \longrightarrow 0}$

If M is a coh. sheaf on X with an auto θ, then since $K_1 \mathbb{G}_m = (k[T,T^{-1}])^* = k^* \times \mathbb{Z}$, one sees as before that the image of $cl(M,\theta) \in K_1'X$ in $K_1\mathbb{G}_m$ viewed as a section of $\mathcal{O}^*$ over $\mathbb{G}_m$ is integral over $\mathcal{O}_X$, hence is in $\Gamma(\bar{X}, \mathcal{O}^*) = k^*$. Thus again $K_1^{Bass}(\mathrm{Modf}\, X) \neq K_1'X$.

Significance of these examples. I wanted to represent elements of $K_1'X$ by coh. sheaves on $\mathbb{P}_1 \times X$ with support in $\mathbb{G}_m \times X$. But if $Z \subset \mathbb{G}_m \times X$ is closed in $\mathbb{P}_1 \times X$, then Z is affine and proper hence finite over X. Therefore a coherent sheaf on $\mathbb{P}_1 \times X$ with support in $\mathbb{G}_m \times X$ is the same thing as a coherent sheaf M on X with an automorphism θ. Hence only elements in $K_1^{Bass}(\mathrm{Modf}\, X)$ can be represented in the desired way.

Example: Let A be a discrete valuation ring with residual field k and suppose A is a k algebra, e.g. $A = k[[x]]$. Let $B = k + m^N \subset A$. (B is a pinched curve). Then A, B have a common fraction field F.

$$\begin{tikzcd}
K_q k \arrow[r, "0"] \arrow[d, equal] & K_q A \arrow[r] \arrow[d] & K_q F \arrow[r, "\partial"] \arrow[d, equal] & K_{q-1} k \arrow[r, "0"] \arrow[d, equal] & {} \\
K_q k \arrow[r] & K_q' B \arrow[r] & K_q' F \arrow[r, "\partial"] & K_{q-1} k \arrow[r] & {}
\end{tikzcd}$$

Therefore we see that $K_q A \xrightarrow{\sim} K'_q B$.

December 17, 1975.

Brown-Gersten spectral sequence.

Let X be a noetherian scheme and let $\mathcal{U}$ be an open covering of X. I want to construct a spectral sequence

$$E_2^{pq} = \check{H}^p(\mathcal{U}, U \mapsto K'_q(U)) \Longrightarrow K'_{-p-q}(X)$$

(descent spectral sequence for the covering $\mathcal{U} \to X$).

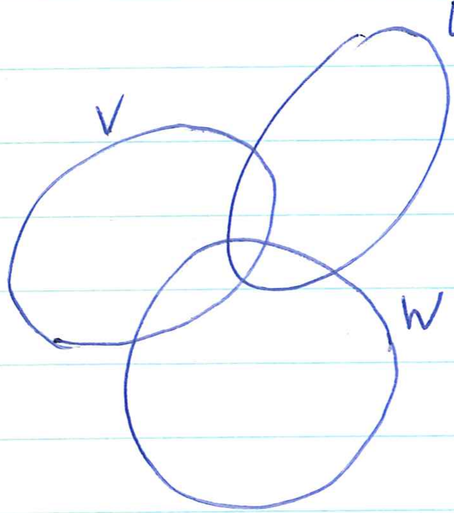
Example 1: $X = U \cup V$ where U, V are open sets. Let h^q denote a coh. functor say: $h^q(U) = H^q(U, F)$ for some complex F of sheaves on X. Then

$$\begin{array}{ccccccccc}
\longrightarrow & h^q(X,U) & \longrightarrow & h^q(X) & \longrightarrow & h^q(U) & \longrightarrow & h^{q+1}(X,U) & \longrightarrow \\
& \downarrow \wr & & \downarrow & & \downarrow & & \downarrow \wr & \\
\longrightarrow & h^q(V, U\cap V) & \longrightarrow & h^q(V) & \longrightarrow & h^q(U \cap V) & \longrightarrow & h^{q+1}(V, U\cap V) &
\end{array}$$

The vertical isoms. come from excision. So we get a Mayer-Vietoris sequence which yields:

$$0 \longrightarrow \check{H}^1(\mathfrak{U}, \underline{h}^{q-1}) \longrightarrow h^q(X) \longrightarrow \check{H}^0(\mathfrak{U}, \underline{h}^q) \longrightarrow 0 .$$

We see in this proof that we need $h^*(U,V)$ relative groups for $V \subset U$.



So take in this case the filtration

$$\underset{X^0}{\overset{\shortparallel}{X}} \supset \underset{X^1}{\overset{\shortparallel}{(U\cap V)\cup(U\cap W)\cup(V\cap W)}} \supset \underset{X^2}{U\cap V\cap W}$$

Then
$$\begin{aligned} h^q(X,X^1) &= h^q(U, U\cap(V\cup W)) \\ &\oplus h^q(V, V\cap(U\cup W)) \\ &\oplus h^q(W, W\cap(U\cup V)) \end{aligned}$$

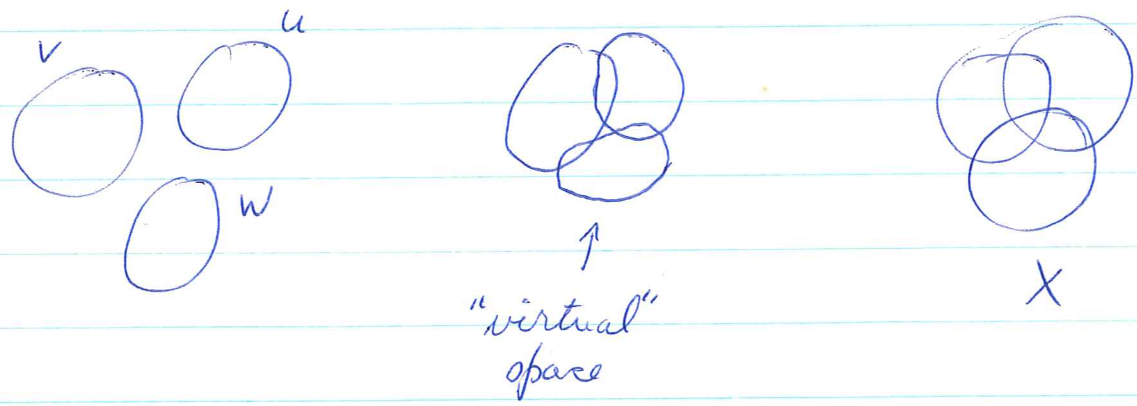
$$\begin{aligned} h^q(X^1,X^2) = h^q(U\cap V, U\cap V\cap W) &\oplus h^q(U\cap W, U\cap V\cap W) \\ &\oplus h^q(V\cap W, U\cap V\cap W) \end{aligned}$$

So one gets a spectral sequence with the wrong E_1-term. But the E_2 might be OKAY.

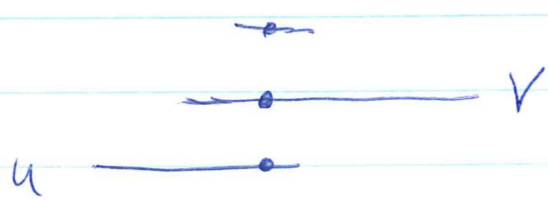
$$\bigoplus_{i<j<k} \mathbb{Z}_{U_i \cap U_j \cap U_k} \longrightarrow \bigoplus_{i<j} \mathbb{Z}_{U_i \cap U_j} \longrightarrow \bigoplus_{i} \mathbb{Z}_{U_i} \longrightarrow \mathbb{Z}_X \longrightarrow 0$$

So what does Segal's machinery do? It essentially gives us the natural filtration of the above exact sequence.

The filtration arises from the following spaces:



There is no way to glue together



without effectively getting X. Therefore I need something which will give me the K-theory of this virtual space.

$$m(U \cap V) \times m(U \cap W) \times m(V \cap W) \leftleftarrows m(U) \times m(V) \times m(W)$$

This will work, except I don't know what it is.
So the problem seems to be this: Given a functor

$$J \longrightarrow \begin{matrix}\text{Permutative} \\ \text{Cats.}\end{matrix}$$

form the associated K-theory consisting of chains on J with coefficients in this functor. Also cochains on J.

December 23, 1975

Brown-Gersten (continued).

I still want to understand the case of a covering $X = U \cup V \cup W$ by three open sets. I want to carefully set up the B-G spectral sequence. I should look for new style K-theories which might serve for further development.

I have a partially ordered set J of open sets in X. For each $U \in J$ I have a space $Q(M(U))$ varying contravariantly in U. But even better, $M(U)$ is like a monoid, so I get a contravariant functor on J to monoid categories.

If you have a functor $J \xrightarrow{F} Ab$ you can form chains and cochains on J with coefficients in F. Cochains form a cosimp. abelian gp.

$$\prod_{U} F(U) \rightrightarrows \prod_{U \subset V} F(U) \;\substack{\longrightarrow\\ \longrightarrow\\ \longrightarrow}\; \prod_{U \subset V \subset W} F(U)$$

whose coh. groups are $H^*(J, F)$. Chains form a simp. abelian gp.

$$\coprod_{U} F(U) \leftleftarrows \coprod_{U \subset V} F(V) \;\substack{\longleftarrow\\ \longleftarrow\\ \longleftarrow}\; \coprod_{U \subset V \subset W} F(W)$$

So I have the space of cochains with values in the system $\mathcal{F}: U \longmapsto Q(M(U))$. Two problems:

a) $M(X) \longrightarrow C(\mathcal{U}, \mathcal{F})$ heq.

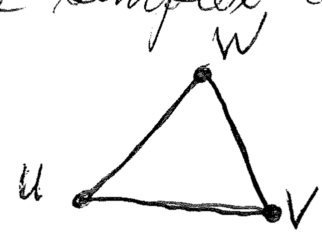
b) spectral sequence $H^p(\mathcal{U}, K'_{-q}) \Longrightarrow K'_{-p-q}(X)$.

a) should follow from Mayer-Vietoris
b) should follow from the Postnikov system of the functor $U \mapsto Q(M(U))$ and the fact that the cochain functor preserves fibrations.

Go back to three sets and see what you need to set up the spectral sequence

$$\begin{array}{ccc} & X & \\ U & V & W \\ U\cap V & U\cap W & V\cap W \\ & U\cap V\cap W & \end{array}$$

Can use the 2-simplex as a model



[redacted] Use the skeletal filtration of [redacted] the simplex

$$\begin{tikzcd} h^{p+q-1}(X_p) \arrow[d] & & \\ h^{p+q-1}(X_{p-1}) \arrow[r, "\delta"] & h^{p+q}(X_p/X_{p-1}) \arrow[r] & h^{p+q}(X_p) \arrow[d] \\ & & h^{p+q}(X_{p-1}) \end{tikzcd}$$

$$E_1^{pq} = h^{p+q}(X_p/X_{p-1}) \xrightarrow{d_1} h^{p+q+1}(X_{p+1}/X_p)$$

Now these skeleta are [redacted] to be rigged so that

[redacted] $X_{-1} = \phi$

$X_0 = U \amalg V \amalg W$

$$X_1 = \left\{ \begin{matrix} U\cap V \\ \amalg \\ U\cap W \\ \amalg \\ V\cap W \end{matrix} \rightrightarrows \begin{matrix} U \\ \amalg \\ V \\ \amalg \\ W \end{matrix} \right\}$$

$$X_2 = \left\{ U\cap V\cap W \; \substack{\to\\\to\\\to} \; \begin{matrix} U\cap V \\ + \\ U\cap W \\ + \\ V\cap W \end{matrix} \rightrightarrows \begin{matrix} U \\ + \\ V \\ + \\ W \end{matrix} \right\}$$

December 24, 1975

Question: Can the Bass relative group $K(\Phi)$ be made into a higher K-theory?

Here $\Phi : \mathcal{A} \to \mathcal{B}$ is an exact functor. The relative group is formed out of triples $A \xrightarrow{u} A'$ such that $\Phi(u)$ is an isomorphism. The relations of exactness and composition are introduced, also triviality of $A \xrightarrow{id} A$. The obvious simplicial gadget to look at is

$$\mathrel{\substack{\longrightarrow\\ \longrightarrow\\ \longrightarrow\\ \longrightarrow}} \{A \to A' \to A''\} \mathrel{\substack{\longrightarrow\\ \longrightarrow\\ \longrightarrow}} \{A \xrightarrow{u} A'\} \rightrightarrows \{A\}$$

Vertically, one takes the straight K-theory of diagrams in $\mathcal{A}$. In the typical localization situation, we can identify $K_*\{A \xrightarrow{u} A'\}$ with $K_*(\mathcal{A}) \oplus K_*(\mathcal{T})$, where $\mathcal{T}$ = torsion modules

This construction is reasonable when in some sense stable isoms. in the $\mathcal{B}$-category can be lifted to morphisms in the $\mathcal{A}$-category.

Possible K_1 would be

$$\mathrel{\substack{\longrightarrow\\ \longrightarrow\\ \longrightarrow\\ \longrightarrow}} \{A \xrightarrow{\theta_1} A \xrightarrow{\theta_2} A\} \mathrel{\substack{\longrightarrow\\ \longrightarrow\\ \longrightarrow}} \{A \xrightarrow{\theta} A\} \rightrightarrows \{A\}$$

modulo the obvious factor of $K_*(A)$.

This involves looking at proper support elts in

$$\mathbb{G}_m \times \mathbb{G}_m \times \mathbb{G}_m \mathrel{\substack{\textstyle\rightarrow\\[-0.6ex]\textstyle\rightarrow\\[-0.6ex]\textstyle\rightarrow\\[-0.6ex]\textstyle\rightarrow}} \mathbb{G}_m \times \mathbb{G}_m \mathrel{\substack{\textstyle\rightarrow\\[-0.6ex]\textstyle\rightarrow\\[-0.6ex]\textstyle\rightarrow}} \mathbb{G}_m \rightrightarrows pt$$

Check if it is correct conjecturally. We know

$$K_{!q}(\mathbb{G}_m) = K_q(A) \oplus K_{q+1}(A)$$

$$K_{!q}(\mathbb{G}_m^2) = K_q(A) \oplus K_{q+1}(A)^2 \oplus K_{q+2}(A)$$

Offhand it looks like the normalization of

$$p \longmapsto K_{!q}(\mathbb{G}_m^p)$$

is $K_{q+p}(A)$ in degree p. ~~So~~ the E^1 term:

$$\begin{array}{l} \longrightarrow K_2 A \longrightarrow K_2 A \longrightarrow K_1 A \\ \longrightarrow K_2 A \longrightarrow K_1 A \longrightarrow K_0 A \end{array}$$

with differentials probably zero

This shows me that the simplicial object at the top of this page (toral homotopy) will not be a model for $K_!$. ~~However~~

Toral homotopy groups. (an old approach of Fox).

Let X be a pointed connected space. Consider

$$[\underbrace{S^1\times\cdots\times S^1}_{r}, \Omega X]$$

(basepoint-preserving maps). No

Let B be a loop space. We have

$$(S^1)^r \twoheadrightarrow S^r$$

hence a map $\pi_r B \longrightarrow [(S^1)^r, B]$. The point is I believe that this map is injective always. For example, from the cofibration

$$S^1 \vee S^1 \longrightarrow S^1\times S^1 \longrightarrow S^2$$

one gets

$$\begin{tikzcd} 0 \arrow[r] & \pi_2 B \arrow[r] & {[S^1\times S^1, B]} \arrow[r] & \pi_1 B \oplus \pi_1 B \arrow[r] \arrow[l, bend right, dashed] & 0 \end{tikzcd}$$

because the dotted arrow comes from the two projections $S^1\times S^1 \longrightarrow S^1$.

Basic formula making this work is

$$S^1\wedge(S^1\times Y) \sim (S^1\wedge Y)\vee(S^2\wedge Y)$$

For

$$[S^1 \wedge (S^1 \times Y), X] = [S^1 \times Y, \Omega X]$$

$$\begin{tikzcd}[S^1 \wedge Y, \Omega X] \arrow[r] & [S^1 \times Y, \Omega X] \arrow[r] & {[Y, \Omega X]} \arrow[l, bend right, dashed]\end{tikzcd}$$

hence one has a canonical isomorphism

$$[S^1 \wedge Y, \Omega X] \times [Y, \Omega X] \xrightarrow{\sim} [S^1 \times Y, \Omega X]$$

etc.

It should be true that

$$[\underbrace{S^1 \times \cdots \times S^1}_{r}, B] = \bigoplus_{q=0}^{r} \Lambda^q(\mathbb{Z}^r)^\vee \otimes \pi_q B$$

as $GL_r(\mathbb{Z})$ modules.

$(*)$ $\pi_p \left\{ p \longrightarrow \bigoplus_q \Lambda^q(\mathbb{Z}^p)^\vee \otimes \pi_q B \right\}$

Now I know that $p \longmapsto \mathbb{Z}^p$ is $K(\mathbb{Z}, 1) = \Sigma(\mathbb{Z})$ and that

$$\pi_* \Lambda^q(\Sigma\mathbb{Z})^\vee$$

is $\mathbb{Z}$ concentrated in degree q. Thus $(*)$ above $\simeq \pi_q B$.

Consider the simplicial space

$$S^1 \times S^1 \;\substack{\longrightarrow\\ \longrightarrow\\ \longrightarrow}\; S^1 \;\substack{\longrightarrow\\ \longrightarrow}\; pt$$

whose realization is $BS^1 = \mathbb{C}P^\infty$. Let h^* be a generalized coh. theory.

$$E_1^{pq} = h^q((S^1)^p) \Longrightarrow h^{p+q}(\mathbb{C}P^\infty)$$

$$\parallel$$

$$\bigoplus_{t \geq 0} h^{q-t}(pt) \otimes \Lambda^t(\mathbb{Z}^p)^\vee$$

$$E_2^{pq} = \bigoplus_{t\geq 0} h^{q-t}(pt) \otimes \begin{cases} 0 & p \neq t \\ \mathbb{Z} & p = t \end{cases}$$

$$= h^{q-p}(pt)$$

If h connected, then E_2-term looks like:

$$\begin{array}{c|cccc}
q & & & & \\
 & & & & h^0 \\
 & & & h^0 & h^{-1} \\
 & & h^0 & h^{-1} & h^{-2} \\
\hline
 & h^0 & h^{-1} & h^{-2} & \quad p \\
 & h^{-1} & h^{-2} & & \\
 & h^{-2} & & & \\
\end{array}$$

$h^0(\mathbb{C}P^\infty) = h^0 \oplus h^{-2} \oplus h^{-4} \oplus \cdots$

$h^1(\mathbb{C}P^\infty) = h^{-1} \oplus h^{-3} \oplus \cdots$ so it does work.

Bass K-groups:

Look at the simplicial gadget $\mathrm{Nerv}(\mathbb{G}_m)$

$$\mathbb{G}_m \times \mathbb{G}_m \;\substack{\longrightarrow\\ \longrightarrow\\ \longrightarrow}\; \mathbb{G}_m \;\substack{\longrightarrow\\ \longrightarrow}\; pt$$

$$K_{!q}(\mathbb{G}_m^n, A) \text{ "=" } \bigoplus_{t \geq 0} \Lambda^t(\mathbb{Z}^n) \otimes K_{q+t}(A)$$

you get groups $\pi_p\left(p \mapsto \bigoplus \Lambda^t(\mathbb{Z}^p) \otimes K_{q+t}(A)\right) = K_{q+p}(A)$.

Thus I form a simplicial abelian group which in degree d is the Grothendieck group of objects with d commuting automorphisms.

If F is an algebraically closed field then

$$K_{!0}(\mathbb{G}_{m,F}^n) = \mathbb{Z}[(F^\circ)^n]$$

so

$$K_q^{Bass}(F) = H_q(F^\circ, \mathbb{Z}).$$

Now $\qquad F^\circ \cong \overset{\mu}{\mathbb{Q}/\mathbb{Z}} \otimes V \qquad V$ a $\mathbb{Q}$-module

$$H_*(F^\circ; \mathbb{Z}) = H_*(\overset{\mu}{\mathbb{Q}/\mathbb{Z}}, \mathbb{Z}) \otimes \Lambda^* V$$

$$H_*(\mu, \mathbb{Z}) = \mathbb{Z}, \mu, 0, \mu^{(2)}, 0, \mu^{(3)}, \ldots$$

Hence the Bass K_q is right on the torsion at least.

Notice: This calculation shows that the Bass groups are not the same as mine. For one has examples where the relation $\{f, 1-f\}$ is non-trivial in K_2.

Topological K-theory. Consider the K-theory of bundles over X with commuting automorphisms, better:

$$K^0(S^1 \times S^1 \times X) \overset{\longrightarrow}{\underset{\longrightarrow}{\longrightarrow}} K^0(S^1 \times X) \rightrightarrows K^0(X)$$

Still this remains confusing.

Let C be a complete curve over F alg. cl.

Then

$$K_{!0}\left(\mathbb{G}^n_{m\,C}\right) = K_0(C) \otimes \mathbb{Z}[(F^\circ)^n]$$

so

$$K_q^{Bass}(C) = H_q(F^\circ, K_0(C))$$

$$= H_q(F^\circ) \otimes K_0(C) \oplus \mathrm{Tor}_1(H_{q-1}(F^\circ)_{tors}, \mathrm{Pic}^\circ C).$$

which is close to what I want

Can one compute Bass K-groups for a finite field F?

Let $X \xrightarrow{f} Y$ be a finite Galois covering with group π. Relate zero cycles on X and Y.

$$\begin{array}{cc} K_0(\mathcal{T}_X) & K_0(\mathcal{T}_Y) \\ \| & \| \\ C_0(X) & C_0(Y) \\ \| & \\ \bigoplus_{x \in X_{cl}} \mathbb{Z} & \bigoplus_{y \in Y_{cl}} \mathbb{Z} \end{array}$$

Thus

$$\mathcal{O}_x/m_x \longleftarrow \mathcal{O}_y/m_y \qquad\qquad y = f(x)$$

and so

$$f_*(1\cdot x) = [k(x) : k(y)]\cdot y$$

$$f^*(1\cdot y) = \sum_{x \in f^{-1}(y)} 1\cdot x$$

Thus $$f^* : C_0(Y) \xrightarrow{\sim} C_0(X)^\pi$$

So the **Bass K-groups** of F should be the homology groups of the complex

$$\Rrightarrow \mathbb{Z}[F^{\cdot 2}]^\pi \rightrightarrows \mathbb{Z}[F^{\cdot}]^\pi \rightrightarrows \mathbb{Z}$$

where $\pi = \mathrm{Gal}(\bar{F}/F)$.

Now I know for $F = \mathbb{F}_q$ that the complex

$$L : \quad \mathbb{Z}[\bar{F}^{\cdot 2}] \;\substack{\longrightarrow\\ \longrightarrow\\ \longrightarrow}\; \mathbb{Z}[\bar{F}^{\cdot}] \rightrightarrows \mathbb{Z}$$

has homology groups $K_i(\bar{F})$, so I get a spectral sequence

$$E_2^{pq} = H^p(\pi, K_{-q}(\bar{F})) \Longrightarrow H^{p+q}(\pi, L)$$

To show the other spectral sequence

$$E_1^{pq} = H^q(\pi, \mathbb{Z}[\bar{F}^{\cdot p}]) \Longrightarrow H^{p+q}(\pi, L)$$

degenerates I would need to know that the modules $\mathbb{Z}[\bar{F}^{\cdot p}]$ are cohomologically trivial. Now $\bar{F}^{\cdot p}$ breaks up into π-orbits each having a stabilizer isomorphic to $\pi = \hat{\mathbb{Z}}$. So

$$H^q(\pi, \mathbb{Z}[\bar{F}^{\cdot p}]) \simeq \mathbb{Z}[\bar{F}^{\cdot p}]^{\pi} \otimes \underbrace{H^q(\pi, \mathbb{Z})}_{\begin{cases} \mathbb{Z} \\ 0 \\ \mathbb{Q}/\mathbb{Z} \end{cases}}$$

I don't understand the H^2 terms. They seem to introduce a discrepancy between $\mathrm{Bass}(K_i)$ and K_i.

$$H^2(\pi, \mathbb{Z}[F^{\cdot}P]) \cong H^1(\pi, \mathbb{Q}/\mathbb{Z}[F^{\cdot}P]) \qquad \text{canon}$$
$$\cong (\mathbb{Q}/\mathbb{Z}[F^{\cdot}P])_{\pi} \qquad \text{canon}$$
$$= \mathbb{Q}/\mathbb{Z}[F^{\cdot}P/\pi]$$

Philosophy. I have this homotopy process of converting bundles into higher K-groups. I also see how useful "bundles decomposed relative to T" are in topological K.

Basic idea: We must think of a 0-cycle on T as being generalized by a coherent sheaf with finite support, i.e. a vector space V of finite dimension over the ground field k decomposed relative to T. In top. K-theory I know how to make a K-theory out of such gadgets. I search for an analogue in algebraic K-theory.

Simplest example. Take $T = \mathbb{G}_m$. Then the objects to make the K-theory out of are vector spaces + autos. (V, θ).

The basic monoid is

$$M = \coprod_n PGL_n \times^{GL_n} (GL_n) \qquad \text{inner auto action.}$$

(This is the space to consider in the topological case)

Basic problem: How to make a K-theory out of zero cycles.

If X is a variety then $S^n X = X^n / \Sigma_n$ is a variety whose points are 0-cycles ≥ 0 of degree n on X. The Dold-Thom theorem tells me that

$H_q X = \pi_q$ of the group-completion of $\coprod_{n \geq 0} S^n X$

So we have two functors which don't make sense algebraically: group-completion and π_q.

In the topological context the group-completion problem is sometimes solved by choosing a basepoint to define an inductive system

$$\lim_{n \to \infty} S^n X$$

which works when X is connected.

X, complete curve over k alg. closed. For n large $S^n X$ is a fibre bundle over the Jacobian with projective spaces for fibres.

Let F be a **torsion** coh sheaf over $\mathbb{P}^1$. $V = \Gamma(F)$ $V' = \Gamma(F \otimes \mathcal{O}(-1))$. Let

$$V' \overset{\alpha}{\underset{\beta}{\rightrightarrows}} V$$

be induced by T_0 and T_1 respectively. Then $\alpha - t\beta$ is an isomorphism for some $t \in k$. Thus we can identify V' with $\{(\alpha v', \beta v') \in V \times V\}$ and α, β with the projections. So to F we have assoc. a subspace V' of $V \times V$ of the same dimension as V. Moreover V' determines F as a quotient of $\mathcal{O}_{\mathbb{P}^1} \otimes V$.

Next suppose given $V' \subset V \times V$ of the same dimension as V. Does V' come from a torsion sheaf on $\mathbb{P}^1$? Thus given two maps

$$V' \overset{\alpha}{\underset{\beta}{\rightrightarrows}} V$$

such that $\operatorname{Ker} \alpha \cap \operatorname{Ker} \beta = 0$, does it follow that $\exists\, t \in k$ such that $\operatorname{Ker}(\alpha - t\beta) = 0$?

So form the

$$\begin{tikzcd}
0 \arrow[r] & R \arrow[r] & \mathcal{O}(-1)\otimes V' \arrow[rr, "T_0\alpha - T_1\beta"] \arrow[dr, two heads] & & \mathcal{O}\otimes V \arrow[r] & F \arrow[r] & 0 \\
& & & I \arrow[ur, hook] & & &
\end{tikzcd}$$

$$0 \longrightarrow H^0(I) \longrightarrow V \longrightarrow H^0(F) \longrightarrow H^1(I) \longrightarrow 0 \to H^1(F) \to 0$$

$$H^1(\mathcal{O}(-1)\otimes V') \twoheadrightarrow H^1(I)$$
$$\overset{\shortparallel}{0}$$

$\therefore\ H^1(I) = 0$ similarly $H^1(F(-1)) = 0$ so F is regular. So we get

$$\begin{tikzcd}
0 \arrow[r] & I \arrow[r] \arrow[d] & \mathcal{O}\otimes V \arrow[r] \arrow[d] & F \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & \mathcal{O}(-1)\otimes T \arrow[r] & \mathcal{O}\otimes H^0(F) \arrow[r] & F \arrow[r] & 0
\end{tikzcd}$$

So it's clear that $\mathrm{Im}(\alpha - t\beta) \subset \mathrm{Ker}\{V \to H^0(F)\}$ for all $t \in F$. / NO

Consider then two lines L, L' in V' and a hyperplane H in V. $\alpha : V'/L \xrightarrow{\sim} H$ $\beta : V'/L' \xrightarrow{\sim} H$. If $L \neq L'$, then $\mathrm{Ker}\,\alpha = \mathrm{Ker}\,\beta = L \cap L' = 0$ but $\mathrm{Im}(\alpha - t\beta) \subset H$ so, $\alpha - t\beta$ is always singular.

Dec. 28, 1975

I want to determine the space Z of torsion sheaves F on $\mathbb{P}^1$ with $\Gamma(F) = V$ ~~[illegible]~~ Actually Z is to be the space of quotients $F \twoheadleftarrow \mathcal{O}\otimes V$ such that $V \xrightarrow{\sim} \Gamma(F)$. Recall that the space Q of quotients of $\mathcal{O}\otimes V$ with Hilbert poly n is compact, and on Q we have a canonical vector bundle E of dimension n whose fibre at F is $\Gamma(F)$. Thus on Q we have a canonical morphism

$$\varphi : \mathcal{O}_Q \otimes V \longrightarrow E$$

of vector bundles of rank n. Z is the open set of Q where this morphism φ is an isomorphism.

Given $F \in Z$, we have a canonical resolution

$$0 \longrightarrow \mathcal{O}(-1)\otimes \tilde{T}_1(F) \longrightarrow \mathcal{O}\otimes \tilde{T}_0(F) \longrightarrow F \longrightarrow 0$$

where $\tilde{T}_0(F) = \Gamma(F)$

$$\begin{array}{ccccccc} 0 \longrightarrow & \tilde{T}_1(F) & \longrightarrow & \tilde{T}_0(F)^2 & \longrightarrow & \Gamma(F(1)) & \longrightarrow 0 \\ & \parallel & & \parallel & & & \\ & V' & \longrightarrow & V\times V & & & \end{array}$$

$$(v_1, v_2) \longmapsto T_0 v_1 + T_1 v_2$$

$$v' \longmapsto (\alpha(v'), -\beta(v'))$$

$$\mathcal{O}(-1)\otimes V' \longrightarrow \mathcal{O}\otimes V$$

$$1\otimes v' \longmapsto T_0\otimes\alpha(v') - T_1\otimes\beta(v')$$

Therefore we have associated to each F in Z a subspace $\operatorname{Ker}\{V^2 \to \Gamma(F(1))\}$ of V^2 of the same dimension as V, which determines F. Thus we get an embedding

$$Z \hookrightarrow \mathrm{Grass}_n(V\times V)$$

Now we want to determine the image of this map. So start with $V' \subset V\times V$ with projections $\alpha, \beta : V' \rightrightarrows V$, and form

$$0 \longrightarrow R \longrightarrow \mathcal{O}(-1)\otimes V' \xrightarrow{T_0\alpha - T_1\beta} \mathcal{O}\otimes V \longrightarrow F \longrightarrow 0$$
$$\mathcal{O}(-1)\otimes V' \twoheadrightarrow I \hookrightarrow \mathcal{O}\otimes V$$

Since $0 = H^1(\mathcal{O}(-1)\otimes V) \twoheadrightarrow H^1(F(-1))$, F is regular, so we have a canonical sequence

$$0 \longrightarrow \mathcal{O}(-1)\otimes \tilde{\Gamma}_1(F) \longrightarrow \mathcal{O}\otimes\Gamma(F) \longrightarrow F \longrightarrow 0$$

Also $H^1(I) \twoheadleftarrow H^1(\mathcal{O}(-1)\otimes V') = 0$, so $V \twoheadrightarrow H^0(F)$.

~~So we get a diagram, but~~

$$\cancel{\operatorname{Ker}\{V \to H^0(F)\} \leftarrow H^0(I)}$$

$$0 \to H^0(I) \longrightarrow V \longrightarrow \Gamma(F) \longrightarrow 0$$

$$\begin{tikzcd}
 & 0 \arrow[d] & & \\
 & \mathcal{O}\otimes H^0(I) \arrow[d] & 0 \arrow[d] & \\
\mathcal{O}(-1)\otimes V' \arrow[r] \arrow[d] & \mathcal{O}\otimes V \arrow[r] \arrow[d] & F \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & \mathcal{O}(-1)\otimes \tilde{T}_1(F) \arrow[r] \arrow[d] & \mathcal{O}\otimes \Gamma(F) \arrow[r] \arrow[d] & F \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

~~So now specialize~~ $T_0, T_1 \mapsto 1, \lambda$

$$\begin{tikzcd}
 & V' \arrow[r, "\alpha-\lambda\beta"] \arrow[d] & V \arrow[r] \arrow[d] & F(\lambda) \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & \tilde{T}_1(F) \arrow[r] & \Gamma(F) \arrow[r] \arrow[d] & F(\lambda) \arrow[r] & 0 \\
 & & 0 & &
\end{tikzcd}$$

λ generic

~~which shows that~~ $\mathrm{Im}\{(\alpha-\lambda\beta) : V' \to V\}$ ~~is contained in~~ K

So the only result I get from this calculation is that the image of Z_1 consists of those $V' \subset V \times V$ such that $\alpha - t\beta$ is generically injective

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Good problem: What is the topological K theory arising from torsion sheaves over $\mathbb{P}^1_{\mathbb{C}}$?

December 30, 1975

The basic problem remains to find the 'good' definition of 'algebraic' K-groups for torsion sheaves on a curve C over k.

I have some idea of how to define 'topological' K-groups for torsion sheaves on C when $k = \mathbb{C}$.

$\mathcal{T}$ = torsion sheaves on C.
We have an exact functor

$$\mathcal{T} \longrightarrow \mathrm{Modf}(k)$$

$$F \longmapsto \Gamma(C, F).$$

The fibres of this functor have natural structures as alg. varieties. If $V \in \mathrm{Modf}(k)$, we put

$$D(V, C) = \left\{ \begin{array}{l} \text{set of torsion sheaves } F \text{ with} \\ \Gamma(C, F) = V \end{array} \right.$$

This can be identified with the open subscheme of the scheme of quotients: $\mathcal{O}_C \otimes V \twoheadrightarrow F$ such that F is torsion and $V \xrightarrow{\sim} \Gamma(F)$.

Is $D(V, C)$ non-singular? There is a map

$$D(V, C) \longrightarrow \mathrm{Sym}^n C \qquad n = \dim V.$$

which associates to each quotient $O_C \otimes V \twoheadrightarrow F$, the quotient $O_C \otimes \Lambda^n V \twoheadrightarrow \Lambda^n F$ which determines a positive divisor of degree n. This divisor is the support of the torsion sheaf counted with multiplicity.

To show $D(V, C)$ is non-singular, it suffices to show the associated complex analytic space is non-singular. Better: Given a point $\eta \in \mathrm{Sym}^n C$ one can find an etale map $U \to A'$ where U is an open nbd. of the points appearing in η. For an etale map $C \to C'$ it is clear that

$$\begin{array}{ccc} D(V,C) & \longrightarrow & \mathrm{Sym}^n C \\ \downarrow & & \downarrow \scriptstyle{\text{etale}} \\ D(V,C') & \longrightarrow & \mathrm{Sym}^n C' \end{array}$$

is cartesian. But $D(V, A') = \mathrm{End}(V)$ is non-singular.

We also have filtered torsion sheaves to worry about

$$D(0 \subset V_1 \subset V_2 \subset \cdots \subset V_p, C) \longrightarrow \mathrm{Sym}^{a_1} C \times \cdots \times \mathrm{Sym}^{a_p} C.$$

$$a_i = \dim(V_i / V_{i-1}).$$

$D(V_1 \subset V_2, C)$ consists of quotients:

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{O}_C \otimes V_1 \arrow[r] \arrow[d] & \mathcal{O}_C \otimes V_2 \arrow[r] \arrow[d] & \mathcal{O}_C \otimes V_2/V_1 \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & F_1 \arrow[r] & F_2 \arrow[r] & F_2/F_1 \arrow[r] & 0
\end{tikzcd}$$

bottom row consists of torsion sheaves, vertical arrows are Γ isos. Is the obvious map

$$D(V_1 \subset V_2, C) \longrightarrow D(V_1, C) \times D(V_2/V_1, C)$$

a homotopy equivalence?

Consider the exact sequence

$$0 \longrightarrow V_1 \xrightarrow{i} V_2 \xrightarrow{p} V'' \longrightarrow 0$$

Form over $\mathbb{A}^1$ the family

$$0 \longrightarrow W(t) \longrightarrow V_2 \oplus V'' \xrightarrow[p - t \cdot id_{V''}]{} V'' \longrightarrow 0$$

Then
$$W(0) = V_1 \oplus V''$$
$$W(t) \xleftarrow{\sim} V_2 \qquad t \neq 0.$$
$$(tx, p(x)) \longleftarrow\!\shortmid x$$

If $s : V'' \longrightarrow V_2$ is a section of p, then

$$V'' \longrightarrow V_2 \oplus V''$$
$$y \qquad\quad s(y) \oplus 0$$

is a section of [redacted] $p - t \cdot id_{V''}$. If $x \in V_2$

$$x = sp(x) + \underset{\substack{\cap \\ V_1}}{x - sp(x)} \longmapsto \big(tsp(x) + x - sp(x) \underset{\substack{\cap \\ V_2 \oplus V''}}{,} p(x)\big)$$

Thus

$$\begin{array}{ccc} x & \longmapsto & (tsp(x) + x - sp(x), p(x)) \\ V_2 & \longrightarrow & W(t) \end{array}$$

is an isomorphism for all t. Therefore given the exact sequence and quotients

$$\begin{tikzcd} 0 \arrow[r] & \mathcal{O}_C \otimes V' \arrow[r] \arrow[d, "f"] & \mathcal{O}_C \otimes V' \oplus \mathcal{O}_C \otimes V'' \arrow[r] \arrow[d, "f"] & \mathcal{O}_C \otimes V'' \arrow[r] \arrow[d, "f"] & 0 \\ 0 \arrow[r] & F' \arrow[r] & F \arrow[r] & F'' \arrow[r] & 0 \end{tikzcd}$$

I form the family

$$F_t = \mathrm{Ker}\left\{ F \oplus F'' \xrightarrow{p - tid_{F''}} F'' \right\}$$

which is a good quotient of $\mathcal{O}_C \otimes W(t)$ where

$$W(t) = \mathrm{Ker}\left\{ V \oplus V'' \xrightarrow{p\,tid} V'' \right\}$$

$$\begin{tikzcd} 0 \arrow[r] & V' \arrow[r, "z \mapsto (z,0)"] \arrow[d, equal] & W(t) \arrow[r, "pr_2"] & V'' \arrow[r] \arrow[d, equal] & 0 \\ 0 \arrow[r] & V' \arrow[r] & V' \oplus V'' \arrow[r] \arrow[u, "z \oplus y \,\mapsto\, (z+ty, y)"'] & V'' \arrow[r] & 0 \end{tikzcd}$$

Therefore I do see that

$$D(V_1 \subset V_2, C) \longrightarrow D(V_1, C) \times D(V_2/V_1, C)$$

is a homotopy equivalence.

Suppose I consider the fibre of $D(V, C)$ over the divisor $n \cdot P$ of $\mathrm{Sym}^n C$, where $n = \dim V$. If F is in this fibre, then

$$V \longrightarrow \mathcal{O}_P \otimes V \twoheadrightarrow F$$

so I get an action of $\mathcal{O}_P$ on V, such that m_P acts **nilpotently**. Thus if I choose a gen.^{z} for m_P, I get a nilpotent endomorphism ν on V which determines F via

$$0 \longrightarrow \mathcal{O}_P \otimes_k V \xrightarrow{z \otimes 1 - 1 \otimes \nu} \mathcal{O}_P \otimes_k V \longrightarrow F \longrightarrow 0$$

Thus the fibre is the space of nilpotent endomorphisms on V.

From the above one sees that $D(V, C)$ can be identified with those "lattices" in $\mathcal{O}_C \otimes V$ which are complementary to $1 \otimes V$.

Because exact sequences split ~~s~~ I know that the bundles I am after are just vector bundles equipped with an $\mathcal{O}_C$-action, i.e. decomposed with respect to C. Now in this category we have ~~[illegible]~~ no problem with commutativity. So if I ~~[illegible]~~ work modulo a basepoint of C, then the representing space should be

$$\lim_{n\to\infty} D(k^n, C)$$

So back to the problem of the vector bundles over $\mathbb{P}^1$. Bundles of rank n topologically were

$$BU_n^{S^2}$$

which fibres

$$\begin{array}{c} \Omega^2 BU_n \longrightarrow BU_n^{S^2} \longrightarrow BU_n \\ \parallel \\ \Omega U_n \end{array}$$

Thus we can think of ΩU_n as being made of vectors bundles ~~of~~ rank n on $\mathbb{P}^1$ trivialized at 0. Such a bundle is trivial on A^1 so the key point always is ~~[illegible]~~ what happens at ∞. So we take a bundle on $\mathbb{P}^1$

trivialized over $A^1 = \mathbb{P}^1 - \infty$ and then consider the possible extensions to an "algebraic" bundle over $\mathbb{P}^1$. Remarkable fact is then I get all lattices [redacted] for O_∞ in $k(z)^n$. [redacted]

$M = k[z]^n$ is a given bundle on the affine line. You seek extensions [redacted] at ∞. The extensions $\approx O^n$ correspond to unimodular subspaces of M.

Fix O^n [redacted] on $\mathbb{P}^1$ and consider all lattices. [redacted]

Question: What is the homotopy type of the space of all vector bundles inside [redacted] $k(C)^n$? (adele viewpoint)

[redacted]

Example: Consider the unit disk D in $\mathbb{C}$. [redacted] Scattering theory allows me to identify lattices in A^n A = holomorphic functions on D with rational maps $S^1 \to$ [redacted] U_n. Thus I may [redacted] be able to describe the "space" of rank n bundles in $k(C)^n$ and to identify its homotopy type at least stably.

Discuss topology first. Let E be a rank n vector bundle over the complete curve C. Choose a basis for the generic fibre E_η, whence we get

$$\begin{tikzcd}[column sep=small, row sep=small] & E \arrow[dr, phantom, "\subset"] & \\ E' \arrow[ur, phantom, "\subset"] \arrow[dr, phantom, "\subset"] & & E_\eta \\ & \mathcal{O}^n \arrow[ur, phantom, "\subset"] & \end{tikzcd}$$

with cokernels E/E' and $\mathcal{O}^n/E'$ finite and disjoint support. I propose to determine a neighborhood of E in the space of all rank n bundles in E_η?

Point maybe is that the set of $E' \subset E$ with $\mathrm{length}(E/E') = n$ is compact. Thus I can consider all E such that

$$\mathrm{length}(\mathcal{O}^n/E\cap\mathcal{O}^n), \quad \mathrm{length}(E/E\cap\mathcal{O}^n) \leq N$$

and this <u>should be a compact space</u>.

<u>December 31, 1975</u>

Consider the set of all fractionary ideals in $\mathbb{C}(z)$. This is a free abelian group generated by points in $\mathbb{P}^1 = S^2$. Presumably it gets a natural topology. Note that if X is a simplicial set and A is a top. abelian group, then

$$\mathrm{Hom}_{\text{top ab}}\left(|\mathbb{Z}[X]|, A\right) = \mathrm{Hom}_{s.ab.}\left(\mathbb{Z}[X], \mathrm{Sing}\, A\right)$$
$$= \mathrm{Hom}_{s.sets}\left(X, \mathrm{Sing}\, A\right)$$
$$= \mathrm{Hom}_{spaces}\left(|X|, A\right).$$

Therefore the set of fractionary ideals in $\mathbb{C}(z)$ should have a natural inductive limit topology. The homotopy groups of this space should be the homology groups of $\mathbb{P}^1$. The same conclusion should hold for any curves.

Now I want to consider the space of rank n subbundles in $\mathbb{C}(z)^n$. First consider those E inside of $\mathbb{C}(z)^n$ which coincide with $\mathcal{O}^n$ in a nbd of $z = \infty$. This space can be filtered according to the size of this nbd. Thus I might as well assume the nbd is the disk $D = \{z \mid |z| \geq 1\}$. To each such E I have a scattering matrix. It should be true that this space has as deformation retract the E which agree with $\mathcal{O}^n$ off D, a space which we have seen has the homotopy type ΩU_n.

So next I need to see what effect the

[redacted] stalk of E at ∞ has. The result is a stratification of the space in question with strata indexed by $\mathbb{Z}$ each having the homotopy type of [redacted] $\Omega U_n \times (\Omega U_n)_0$.

[redacted] Specifically let us fix the stratum Z_k where [redacted] E_∞ has index k with respect to $\mathcal{O}_\infty^n$. As E_∞ varies one gets [redacted] a space of the homotopy type of $(\Omega U_n)_0$. (the 0 means I take the connected component).

It appears there is some similarity between rank n [redacted] subbundles in $\mathbb{C}(z)^n$ and configurations in [redacted] an n-manifold. So what I need is a space [redacted] where things are allowed to disappear or appear along the boundary. [redacted] Specifically, [redacted] I could take the space of rational maps $S^1 \to U_n$ modulo the subgroup of those having no [redacted] singularities at 0.

Consider subbundles with support in an annulus. What is the homotopy type? See what happens for $n=1$ i.e. divisors. In this case one wants to work with the [redacted] closed boundary.

Make some guesses. Try for a Segal & McDuff approach should be

This says that the space of subbundles of $k(z)^n$ with "support in U is homotopy equivalent to the space of maps $U \cup \infty \longrightarrow BU_n$.

Try $n=1$. Then $BU_1 = K(\mathbb{Z}, 2)$ so π_* of the space of maps in question is

$$\pi_0\Big(\mathrm{Map}_{pt}(U \cup \infty, BU_1)\Big) = \tilde{H}^2(U \cup \infty, \mathbb{Z})$$

$$\pi_i\Big(\mathrm{Map}_{pt}(U \cup \infty, BU_1)\Big) = \tilde{H}^{2-i}(U \cup \infty, \mathbb{Z})$$

$$= H_c^{2-i}(U, \mathbb{Z}) \cong H_i(U, \mathbb{Z})$$

because U is a 2-manifold with orientation. So it works.

Thus it should be possible to make a series of guesses.

1) $U = D$ disk. Then $U \cup \infty = S^2$ so

$$\mathrm{Map}_{pt}(S^2, BU_n) = \Omega U_n$$

which I know to be true (Garland–Raghunathan)

2) U annulus. Then $U \cup \infty$ is S^2 with 2 points collapsed

$$S^0 \longrightarrow S^2 \longrightarrow U \cup \infty$$

so $\quad BU_n \xrightarrow{U \cup \infty} \Omega U_n \xrightarrow{0} BU_n$

so

$$\mathrm{Maps}(U \cup \infty, BU_n) \equiv U_n \times \Omega U_n$$

3) $U = \mathbb{P}^1$ so

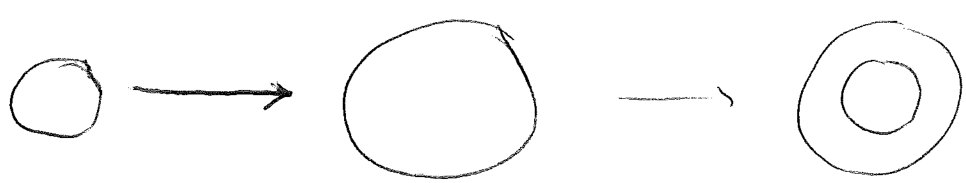
$$\mathrm{Maps}(U \cup \infty; BU_n) = BU_n \times \Omega U_n$$

Check that

$$\begin{tikzcd}
BU_n \times \Omega U_n \arrow[r] \arrow[d] & \Omega U_n \arrow[d, "(0, id)"] \\
\Omega U_n \arrow[r, "(0, id)"'] & U_n \times \Omega U_n
\end{tikzcd}$$

is h. cartesian.

Justification of sorts for 2):

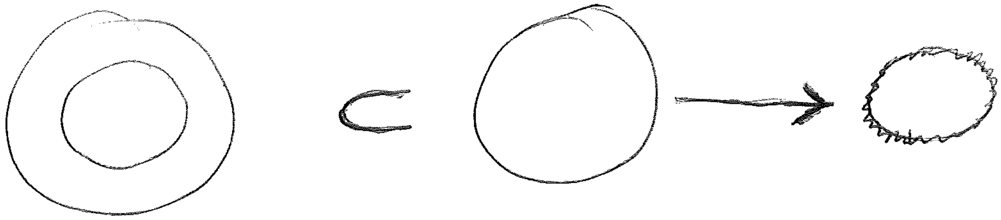


$\Omega U_n \qquad \Omega U_n \qquad ?$

<u>no</u> because points are free to move inside

Here ΩU_n acts

Possible method of interpretation



wavy edges mean no triviality for bundles

$$U_n \times \Omega U_n \longrightarrow \Omega U_n \xrightarrow{\partial} BU_n$$

What about torsion sheaves?

Note: your model for $\mathbb{P}^1$ stably is

$$BU \times \Omega U = BU \times \mathbb{Z} \times BU$$

so as you expected for torsion sheaves one of the $\mathbb{Z}$'s has gone from $K_0(\mathbb{P}^1)$.

Question: I have seen that not all elements of K_1 can be represented by bundles + autos, i.e. torsion sheaves on $\mathbb{G}_m$. However, it appears that there is something more general than a torsion sheaf on $\mathbb{G}_m$, namely, a "subbundle" of $k(t)^n$. Can one make a definition of K_1 based on these more general gadgets.

General base A, and you take a "subbundle" E of $A(t) \otimes_A M$ over $\mathbb{P}^1_A$ such that $E = \mathcal{O}^n$ over the 0 and the ∞ sections. What is $A(t)$?