<u>October 17, 1975</u> Grayson's discovery

I want to review some old ideas first. Let A be a ring, I an ideal, ~~[illegible]~~ ~~$\mathcal{E}$ denote the exact [illegible] category~~ and assume that $I \in \mathcal{P}_A$. Then we have

$$\mathcal{P}_{A/I} \subset \mathcal{P}^1_A.$$

Let $\mathcal{E}$ denote the exact category consisting of exact sequences

$$0 \longrightarrow L_1 \longrightarrow L_0 \longrightarrow M \longrightarrow 0$$

where $M \in \mathcal{P}_{A/I}$ and $L_i \in \mathcal{P}'_A$. We see that objects of $\mathcal{E}$ admit a canonical filtration

$$\begin{array}{c} 0 \to L_1 \to L_1 \longrightarrow 0 \longrightarrow 0 \\ \downarrow \\ 0 \longrightarrow L_1 \to L_0 \longrightarrow M \longrightarrow 0 \\ \downarrow \\ 0 \longrightarrow 0 \longrightarrow M \longrightarrow M \longrightarrow 0 \end{array}$$

where the sub and quotient objects are in $\mathcal{E}$. Thus by exactness thm.

$$K_*(\mathcal{E}) = K_*(\mathcal{P}^1_A) \oplus K_*(\mathcal{P}_{A/I}) = K_*(A) \oplus K_*(A/I)$$

$$[0 \to L_1 \to L_0 \to M \to 0] \longmapsto [L_1] + [M]$$

Note that the map from $K_*(\mathcal{E})$ to $K_*(A)$ induced by $[0\to L_1\to L_0\to M\to 0]\longmapsto[L_0]$ is the difference $[L_1]$ $-\, i_*[M]$, $i_*: K_*(A/I)\longrightarrow K_*(A)$ denoting the transfer. Thus to prove $i_*=0$, we have to show $[L_0]=[L_1]$ on the K_* level.

Next let $\mathcal{E}_1$ denote the full subcat. of $\mathcal{E}$ consisting of the exact sequences with $L_0\in P_A$ (hence $L_1\in P_A$). $\mathcal{E}_1$ is closed under extensions, and the hypotheses of the resolution thm. hold:

$E'\rightarrowtail E\twoheadrightarrow E''$ exact $\quad E''\in\mathcal{E},\ E\in\mathcal{E}_1 \Longrightarrow E'\in\mathcal{E}_1$. ✓

$\forall E\in\mathcal{E}\quad\exists\quad \bar{E}\twoheadrightarrow E$ with $\bar{E}\in\mathcal{E}_1$. ✓ So

$$K_*(\mathcal{E}_1) = K_*(\mathcal{E})$$

But I claim that in $\mathcal{E}_1$ every exact sequence splits. In effect, suppose we have

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & P_1' \arrow[r] \arrow[d] & P_0' \arrow[r] \arrow[d] & M' \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & P_1 \arrow[r] \arrow[d] & P_0 \arrow[r] \arrow[d] & M \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & P_1'' \arrow[r] \arrow[d] & P_0'' \arrow[r] \arrow[d] & M'' \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

The M-sequence splits as $M'' \in \mathcal{P}_{A/I}$. Choose a section $s : M'' \longrightarrow M$. We want to lift the map $P_0'' \to M'' \xrightarrow{s} M$ to a map $P_0'' \to P_0$ which is a section of the epim. going the other way. Applying $\mathrm{Hom}(P_0'', ?)$ to the diagram, we are reduced to showing that elements $x \in P_0''$, $y \in M$ with the same image in M'' can be simultaneously lifted to P_0, i.e. that

$$P_0 \to P_0'' \oplus M \longrightarrow M''$$

is exact. This is well-known. So once we have compatible sections $P_0'' \to P_0$ and $M_0'' \longrightarrow M$, the bottom row is a direct factor of the middle row. QED.

So it is now clear that $\mathcal{E}_1$ is $\sim$ the category of finitely generated projective modules of the ring of endos. of the sequence

$$0 \longrightarrow \begin{matrix} I \\ \oplus \\ A \end{matrix} \longrightarrow \begin{matrix} A \\ \oplus \\ A \end{matrix} \longrightarrow A/I \longrightarrow 0$$

which is a generator. So I find this ring is the ring of matrices of the form

$$\begin{pmatrix} A & I \\ A & A \end{pmatrix} = \left\{ \alpha : \begin{matrix} A \\ \oplus \\ A \end{matrix} \to \begin{matrix} A \\ \oplus \\ A \end{matrix} \;\middle|\; \alpha\begin{pmatrix} I \\ \oplus \\ A \end{pmatrix} \subset \begin{pmatrix} I \\ \oplus \\ A \end{pmatrix} \right\}$$

Thus we obtain:

<u>Theorem</u>: $K_*\left(\begin{bmatrix} A & I \\ A & A \end{bmatrix}\right) = K_*(A) \oplus K_*(A/I)$
if I is <u>an ideal in A</u> such that $I \in \mathcal{P}_A$.

Let's make explicit the arrows: ~~They~~

$$K_*(A/I) \longleftarrow K_*\begin{bmatrix} A & I \\ A & A \end{bmatrix} \xrightarrow{\theta} K_*(A)$$

which give the isomorphism of the theorem. The left arrow is induced by the homomorphism

$$\begin{bmatrix} A & I \\ A & A \end{bmatrix} \longrightarrow A/I$$

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \longmapsto a \bmod I$$

The right arrow θ is induced from the inclusion

$$\begin{bmatrix} A & I \\ A & A \end{bmatrix} \subset \mathrm{End}\begin{pmatrix} I \\ \oplus \\ A \end{pmatrix}$$

and the fact $I \oplus A \in \mathcal{P}_A$. To be more specific, suppose I is principal: $I = A\pi$. Then $I \oplus A \simeq A^2$

$$\begin{bmatrix} \pi^{-1} & 0 \\ 0 & 1 \end{bmatrix} : \begin{matrix} I \\ \oplus \\ A \end{matrix} \longrightarrow \begin{matrix} A \\ \oplus \\ A \end{matrix}$$

so the map θ is induced by the homomorphism

$$(*)\qquad \begin{bmatrix} a & b \\ c & d \end{bmatrix} \longmapsto \begin{bmatrix} \pi^{-1} & \\ & 1 \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} \pi & \\ & 1 \end{bmatrix} = \begin{bmatrix} a & \pi^{-1}b \\ \pi c & d \end{bmatrix}$$

$$\begin{bmatrix} A & I \\ A & A \end{bmatrix} \longrightarrow \begin{bmatrix} A & A \\ A & A \end{bmatrix}$$

together with the natural isom. $K_*\left(\begin{bmatrix} A & A \\ A & A \end{bmatrix}\right) = K_*(A)$.

The map induced by the inclusion $\begin{bmatrix} A & I \\ A & A \end{bmatrix} \subset \begin{bmatrix} A & A \\ A & A \end{bmatrix}$ corresponds to the functor $\mathcal{E}_1' \to \mathcal{P}_A$ given by taking the ~~total~~ object of an exact sequence. In view of the remark at the top of page 2 we have

<u>Assertion</u>: The transfer $K_*(A/I) \longrightarrow K_*(A)$ is zero iff the two maps

$$K_*\left(\begin{bmatrix} A & I \\ A & A \end{bmatrix}\right) \rightrightarrows K_*\left(\begin{bmatrix} A & A \\ A & A \end{bmatrix}\right) = K_*(A)$$

induced by $(*)$ and the inclusion coincide.

Before going on, let me change notation, and replace the ring $\begin{bmatrix} A & I \\ A & A \end{bmatrix}$ by the isomorphic ring $\begin{bmatrix} A & A \\ I & A \end{bmatrix}$, the isomorphism being given by conjugation by $\begin{bmatrix} \pi & \\ & 1 \end{bmatrix}$. Now we have an isomorphism:

$$K_*\left(\begin{bmatrix} A & A \\ I & A \end{bmatrix}\right) \xrightarrow{\sim} K_*(A/I) \oplus \overset{K_*(A)}{\overset{\wr}{K_*\left(\begin{bmatrix} A & A \\ A & A \end{bmatrix}\right)}}$$

induced by $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \mapsto a \bmod I$, and the inclusion.

The theorem on page 4 can be expressed as saying that the homo~~m~~

$$GL_n\begin{bmatrix} A & A \\ I & A \end{bmatrix} \longrightarrow GL_n(A/I) \times GL_n\begin{bmatrix} A & A \\ A & A \end{bmatrix}$$

induces isoms. of homology as $n \to \infty$. Using the "block matrix" isom. $GL_n\begin{bmatrix} A & A \\ A & A \end{bmatrix} \simeq GL_{2n}A$, the group $GL_n\begin{bmatrix} A & A \\ I & A \end{bmatrix}$ can be identified with

$$\begin{bmatrix} M_nA & M_nA \\ M_nI & M_nA \end{bmatrix}^* .$$

$*$ denotes invertible elements

Thus the theorem implies that at least after $+$ing one has a fibration

$$\begin{bmatrix} 1 & 0 \\ 0 & M_nA \end{bmatrix}^* \longrightarrow \begin{bmatrix} M_nA & M_nA \\ M_nI & M_nA \end{bmatrix}^* \longrightarrow GL_n(A/I)$$

in the limit as $n \to \infty$. This suggests that the inclusions

$$\begin{bmatrix} 1 & \\ & GL_n A \end{bmatrix} \subset \begin{bmatrix} 1+M_n I & M_n A \\ M_n I & M_n A \end{bmatrix}^* \subset GL_{2n} A$$

induces isos. on homology as $n \to \infty$.

Grayson proves a stronger result, namely that for any r the inclusions

$$\begin{bmatrix} 1_r & \\ & GL_n A \end{bmatrix} \subset \begin{bmatrix} 1+M_r I & M_{r\times n} A \\ M_{n\times r} I & M_n A \end{bmatrix}^* \subset GL_{r+n} A$$

induce isos. on homology as $n \to \infty$. His result includes mine (which is the case $I=0$).

Proof goes like this: Fix $M \in \mathcal{P}_A^1$ and let $\mathcal{E}_M$ be the groupoid consisting of maps $P \twoheadrightarrow M$ with $P \in \mathcal{P}_A$ and all isoms. over M. $\mathcal{E}_0 = \mathrm{Iso}(\mathcal{P}_A)$ acts on $\mathcal{E}_M$ by $Q * (P \twoheadrightarrow M) = (Q \oplus P \xrightarrow{pr} P \twoheadrightarrow M)$. According to my results one has a quasi-fibn.

$$\mathcal{E}_0^{-1}\mathcal{E}_0 \xrightarrow{i} \mathcal{E}_0^{-1}\mathcal{E}_M \longrightarrow \mathcal{E}_0 \backslash \mathcal{E}_M$$

where i is induced by letting $\mathcal{E}_0$ act on a fixed basept. $P_0 \twoheadrightarrow M$ of $\mathcal{E}_M$.

Thm. $\mathcal{E}_0^{-1}\mathcal{E}_0 \longrightarrow \mathcal{E}_0^{-1}\mathcal{E}_M$ is a heq.; equivalently $\mathcal{E}_0 | \mathcal{E}_M$ is contractible.

To prove this one equips $\mathcal{E}_0 \backslash \mathcal{E}_M$ with the product induced by the fibre product

$$(P \twoheadrightarrow M) \cdot (P' \twoheadrightarrow M) = (P \times_M P' \twoheadrightarrow M).$$

A map in $\mathcal{E}_0 | \mathcal{E}_M$ from $P \twoheadrightarrow M$ to $P' \twoheadrightarrow M$ can be identified with a pair of arrows (i,p) over M.

$$\begin{tikzcd} P \arrow[rr, shift left, "i"] \arrow[dr] & & P' \arrow[ll, shift left, "p"] \arrow[dl] \\ & M & \end{tikzcd}$$

such that $pi = id_P$. From the pair

$$P \underset{\Delta}{\overset{pr_1}{\leftrightarrows}} P \times_M P$$

one then gets that the product in $\mathcal{E}_0 \backslash \mathcal{E}_M$ is homotopy idempotent: $\xi^2 \sim \xi$.

$\mathcal{E}_0 \backslash \mathcal{E}_M$ is connected, for if $P \twoheadrightarrow M$ and $P_0 \twoheadrightarrow M$ are two objects, then one has

$$\begin{tikzcd} P \arrow[r, shift left, dashed, "i_1"'] & P \times_M P_0 \arrow[l, shift left, "pr_1"'] \arrow[r, shift left, "pr_2"] & P_0 \arrow[l, shift left, dashed, "i_2"] \end{tikzcd}$$

where i_1, i_2 exist because P, P_0 are projective. If the product on $\mathcal{E}_0 | \mathcal{E}_M$ had an identity, $\mathcal{E}_0 \backslash \mathcal{E}_M$ would be an H-space which is connected, hence it would have a homotopy inverse, whence $\xi^2 \sim \xi$ would $\Longrightarrow$ $\xi \sim pt$.

Unfortunately $\mathcal{E}_0 \backslash \mathcal{E}_M$ has no identity to be seen, so one has to proceed differently.

Proof that $\tilde{H}_*(\mathcal{E}_0 \backslash \mathcal{E}_M) = 0$. Let ε be the obvious generator of $H_0(\mathcal{E}_0 \backslash \mathcal{E}_M)$. The product on $\mathcal{E}_0 \backslash \mathcal{E}_M$ is commutative and associative, hence $H_*(\mathcal{E}_0 \backslash \mathcal{E}_M)$ is a ring, associative & anti-commutative. Since $X \xrightarrow{\Delta} X \times X \xrightarrow{\mu} X$ is homotopic to id_X $(X = \mathcal{E}_0 \backslash \mathcal{E}_M)$, for any element $\alpha \in \tilde{H}_*(X)$ which is primitive $(\Delta_* \alpha = \varepsilon \otimes \alpha + \alpha \otimes \varepsilon)$, we have

$$\alpha = \mu_* \Delta_*(\alpha) = \mu_*(\varepsilon \otimes \alpha + \alpha \otimes \varepsilon)$$

$$= \varepsilon \cdot \alpha + \alpha \cdot \varepsilon = 2\varepsilon \cdot \alpha = \varepsilon \cdot (2\alpha).$$

Thus multiplication by ε on $\mathrm{Prim}(\tilde{H}_*(X))$ is invertible; as $\varepsilon^2 = \varepsilon$ it is also idempotent, thus $\varepsilon \cdot \alpha = \alpha \implies \alpha = 2\varepsilon \cdot \alpha = 2\alpha \implies \alpha = 0$. Thus $\mathrm{Prim}(\tilde{H}_*(X)) = 0 \implies \tilde{H}_*(X) = 0$.

Now Grayson only has to prove that $\pi_1(X) = 0$. I note the above argument that $\mathrm{Prim}(\tilde{H}_*(X)) = 0$ would imply that $\pi_*(X) = 0$ directly, except for basepoint trouble. When one tries to show $\varepsilon \cdot \alpha = \alpha \cdot \varepsilon$ on the level of π_1, it turns out one has to know that the two maps

$$P_0 \rightrightarrows P_0 \times_M P_0$$

given by (pr_1, Δ) and (pr_2, Δ) give a homotopically trivial loop at P_0. So without going into all details, I'll show that the interchange auto. of $P_0 \times_M P_0$ represents a trivial loop. First look at

$$P_0 \times_M P_0 \times_M P_0 \times_M P_0 \underset{\Delta}{\overset{pr_4}{\rightleftarrows}} P_0 .$$

As interchanging the first two factors commutes with Δ and pr_4 it follows, that any transposition of factors of $(P_0/M)^4$ gives a trivial loop, hence any permutations of factors gives a trivial loop. Now use

$$(P_0 \times_M P_0) \underset{\Delta}{\overset{pr_1}{\leftrightarrows}} (P_0 \times_M P_0) \times_M (P_0 \times_M P_0)$$

to conclude that $P_0 \times_M P_0$ represents a trivial loop.

October 18, 1975. On MacPherson's construction

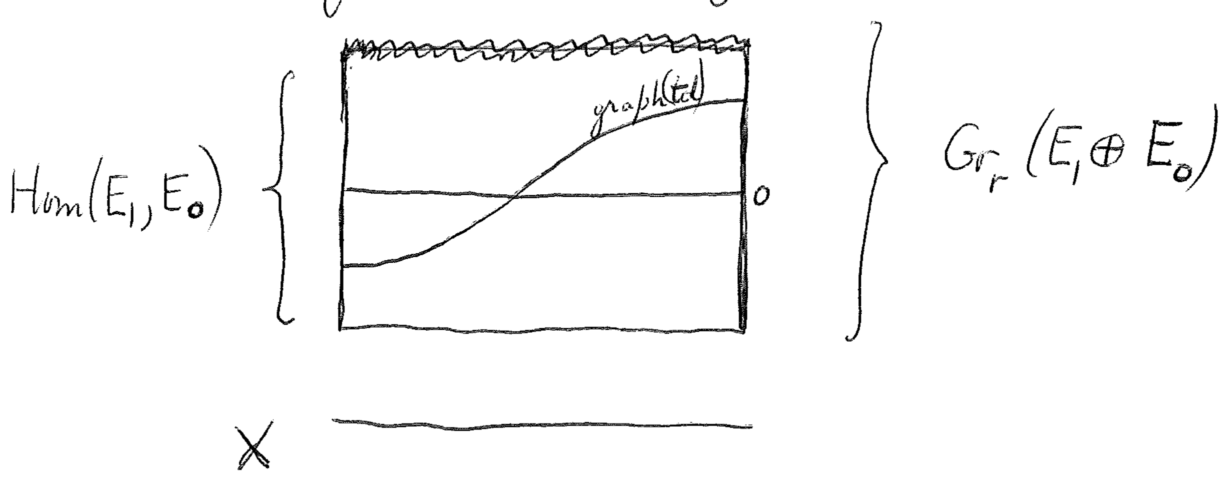
Let X be a non-singular variety, let $d: E_1 \to E_0$ be a map of vector bundles over X. If $r =$ rank of E_1 the graph of d gives us a section of the bundle $Gr_r(E_1 \oplus E_0)$ over X. One can consider the section

$$X \times \mathbb{C} \subset Gr_r(E_1 \oplus E_0) \times \mathbb{C}$$

$$(x, t) \longmapsto ((\text{graph } td), t)$$

and form the closure W of the image of $X \times \mathbb{C}$ inside $Gr_r(E_1 \oplus E_0) \times \mathbb{P}^1$. I want to understand what W looks like.

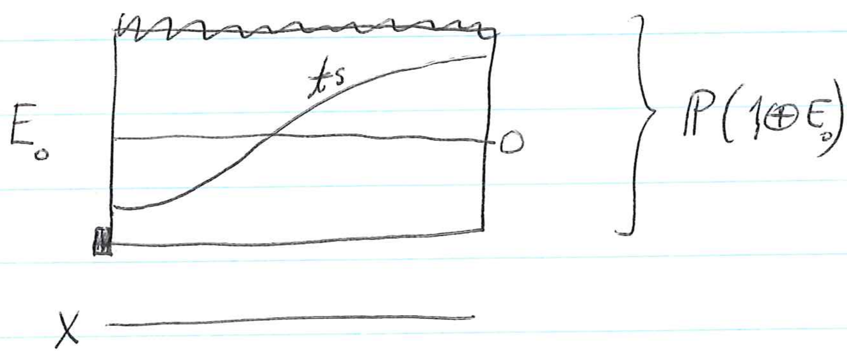
First of all we ought to draw the picture



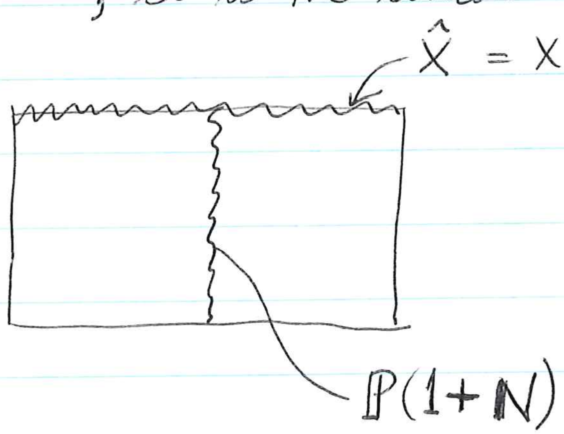
and think of $\mathrm{Hom}(E_1, E_0)$ as being the open Schubert cell in the Grassmannian. As $t \to \infty$, over points of X where $d(x) \neq 0$ the graph will push off .

to the complement of the fat cell. [redacted]

Suppose $\dim E_{+1} = r = 1$, so E_1 is the trivial line bundle so that d is just a section s of E_0. Then I am looking at



Suppose $Y = \{x \mid s(x) = 0\}$ is smooth, and further that $Y = s^{-1}0$ as schemes. It's more or less clear what W is at $t = \infty$; it is the union



(To blow up $Y \subset X$ you find Y as the 0-sub-scheme of a section $s \in \Gamma(X, E_0)$, then you have a section of $\mathbb{P}(E_0/X)$ over $X - Y$ whose closure is $\hat{X}$.)

where N = normal bundle to $Y \subset X$.

Suppose d is everywhere injective, hence it defines a section of $Gr_r(E_0) \subset Gr_r(E_1 \oplus E_0)$. It is clear that W_∞ is just this section.

<u>Digression:</u> Let E be a vector bundle over X non-singular affines. I have been trying to describe the good families of sections of E. If T is a parameter variety, then I want to understand nice $s \in \Gamma(X_T, E_T)$, $X_T = T \times X$, $E_T = pr_2^*(E)$. Idea: What we really would like is to have each s_t transversal to the zero section, i.e. if $s(t,x) = 0$, then $ds_t : T_X(x) \twoheadrightarrow E(x)$. This is not always possible, however it might be possible that ds as a map from $T_{X_T/T}$ to E_T be generic.

Review MacPherson theorem background.

Let $f: X \to Y$ be a proper smooth map with Y smooth.

$$0 \to T_f \to T_X \to f^* T_Y \to 0$$

$$\Rightarrow \quad c(T_X) = c(T_f)\, f^* c(T_Y)$$

$$\Rightarrow \quad f_* c(T_X) = \left(f_* c(T_f)\right) \cdot c(T_Y)$$

Now if f has relative dimension d, than f_* kills $H^i(X)$ for $i<d$, so $f_* c(T_f) = f_* c_d(T_f) \in H^0(Y, \mathbb{Z})$. If Y is connected, this function gives just $\chi(f^{-1}(y))$. Thus

we have

$$f_* c(T_X) = \chi(f^{-1}(y)) \cdot c(T_y). \qquad y \in Y$$

Perhaps I can understand MacPherson's paper if I consider the following special situation. Let $f: X \longrightarrow Y$ be a proper morphism of smooth varieties such that $\chi(f^{-1}(y)) = n$ for all y. To show $f_* c(X) = n \cdot c(Y)$.

So in his paper I can take $N = Y$. Then we put

$$Z_\lambda = \text{Image of } \left\{ X \xrightarrow{x \mapsto \text{graph } \lambda df(x)} Gr_d(T_X \oplus f^* T_Y) \right\}$$

for each $\lambda \in \mathbb{C}$, and let Z_∞ be the limit of the Z_λ as $\lambda \to \infty$. Z_∞ is a cycle $= \sum m_i V_i'$

$$Z = \bigcup_{\substack{\lambda \in \mathbb{R} \cup \infty \\ 0 \le \lambda \le \infty}} Z_\lambda . \qquad V_i = Im\{V_i' \longrightarrow Y\}$$

~~Somehow what he saw is this.~~ What he is doing is to calculate $f_* c(T_X)$ in some way, in fact he computes it in terms of Mather Chern classes of subvarieties of the image variety Y.

Start his calculation $\quad f_* \, c(T_X) = f_* \, s_0^* \, c(\xi)$

$$= f_* \pi_* s_{0*} s_0^* c(\xi) = f_* \pi_* \left(s_{0*} 1 \cdot c(\xi)\right) = f_* \pi_* \left(\sum m_i \mu_{i*} 1_{P_i} \cdot c(\xi)\right)$$

$$= \sum m_i \nu_{i*} \rho_{i*}\Big(\overbrace{\rho_i^* c(T_{V_i})\, c\big(\xi/\rho_i^* T_{V_i}\big)}^{c(\mu_i^* \xi)}\Big)$$

$$= \sum m_i \nu_{i*} \, c(T_{V_i}) \underbrace{\rho_{i*} \, c\big(\xi/\rho_i^* T_{V_i}\big)}_{\text{an integer } p_i} = \sum m_i p_i \, \nu_{i*} \, c(T_{V_i}).$$

It seems to me that a key point occurs when you can talk about the tangent bundle to the fibres.

<u>Question:</u> Let $u : E_1 \to E_0$ be a vector bundle map over X. MacPherson constructs a cycle Z_∞ in $Gr_r(E_1 \oplus E_0)$. What is the significance of this cycle?

First of all since Z_∞ is homologous to Z_0 which is the image of a section of $\pi : Gr_r(E_1 \oplus E_0) \longrightarrow X$, we know that

$$\int_X \alpha = \int_{Z_\infty} \pi^* \alpha$$

and more generally for any $f : X \to Y$, $\quad f_*(\alpha) = f_* \pi_* (Z_\infty \cdot \pi^* \alpha)$

I ought to be able to see what the class of Z_∞ is in terms of the cohomology of of $Gr_r(E_1 \oplus E_0)$.

Suppose I want $ch(E_0) - ch(E_1) \in K(X)$. Then on $Gr_r(E_1 \oplus E_0)$ I have the canonical subbundle ξ which pulls back via s_1 to E_1. So

$$
\begin{aligned}
ch(E_1) &= s_0^* \, ch(\xi) \\
&= \pi_*(s_0)_* s_0^* \, ch(\xi) = \pi_*\left[s_{0*}1 \cdot ch(\xi)\right] \\
&= \pi_*\left[Z_\infty \cdot ch(\xi)\right].
\end{aligned}
$$

The same formula holds for all the char. classes:

$$\varphi(E_1) = \pi_*\left[Z_\infty \cdot \varphi(\xi)\right]$$

In the case where $u : E_1 \to E_0$ is an isomorphism off a subvariety Y of X, the cycle Z_∞ contains a component X' mapped birationally to X, in fact, off Y, $Z_\infty = X'$. For if u is an isomorphism then the limit of graph(λu) is the subspace E_0. In fact X' = the image of the section of π given by $E_0 \subset E_1 \oplus E_0$. Thus

$$\pi_*\left[(Z_\infty - X') \cdot \varphi(\xi)\right] = \varphi(E_1) - \varphi(E_0).$$

Oct. 19, 1975

Question: Can MacPherson's idea be used to define the Chern classes of a vector bundle?

Idea: Given a vector bundle E over X of rank r I can choose a non-zero map $\mathcal{O}^r \longrightarrow E$ and use Mac's construction. Better, choose a sequence of sections $s_1, \ldots, s_r$ as in Serre's theorem.

October 20, 1975.

Let E be a vector bundle over a manifold X (maybe it would be better to think of X as a non-singular affine variety over the field $\mathbb{C}$). I understand what a generic section s of E is. It is a section transversal to the zero section.

Suppose s_1 is a fixed generic section of E and $Y = s_1^{-1} 0$. I want to describe what I should mean by a section s_2 which is generic with respect to the choice of s_1.

First of all I want s_2 to induce a generic section of E over Y and a generic section of E/s_1 over $X-Y$. Let Z_1 be the subset where s_2, s_1 are lin. dep. Then $Z_1 \cap (X-Y)$ [redacted] is a submanifold, and s_2 is a section of the line bundle generated by s_1 [redacted] over this submanifold. So I probably also want s_2 to be transversal as a section of this line bundle.

Critical thing to examine first is the following. Consider a point $x \in Y$ where s_2 vanishes. Let [redacted] Z_2 be the subspace where s_1 [redacted] and s_2 vanish. We know

<u>October 22, 1975</u>

Let E be a vector bundle over a non-singular affine variety X over $\mathbb{C}$. If V is a space of sections generating E, the transversality theorem shows that any generic element s of V is transv. to the zero section:

$$\begin{tikzcd}
s^{-1}(0) \arrow[r] \arrow[d] & E' \arrow[r] \arrow[d] & X \arrow[d, "0"] \\
X \arrow[r] \arrow[d] & X \times V \arrow[r, "ev"] \arrow[d] & E \\
pt \arrow[r, "s"] & V &
\end{tikzcd}$$

$\tilde{s}$ is transversal to zero $\Longleftrightarrow$ s is a regular point for the map $E' \longrightarrow V$.

So suppose now that we choose a section s_1 of E tranversal to zero; let $Y = s_1^{-1}(0)$. We next wish to choose s_2 in generic fashion. Clearly we want s_2 restricted to Y to be transversal to zero. Let $Z = \mathrm{Zero}(s_1, s_2)$. ds_1 induces an isom $T_X|_Y / T_Y \xrightarrow{\sim} E|_Y$ and ds_2 induces an isom of $T_Y|_Z / T_Z \xrightarrow{\sim} E|_Z$. Thus

$$T_X|_Z / T_Z \xrightarrow{\sim} \mathrm{Hom}(\mathcal{O}^2, E)|_Z .$$

So we know a tubular nbd of Z in X can be identified with the bundle $\mathrm{Hom}(\mathcal{O}^2, E)|_Z$ in such a way that Z becomes the zero section and Y the subbundle where s_1 vanishes (evident meaning).

To understand the normal structure of Z in X we can suppose $Z = pt$ and $X = \mathrm{Hom}(F^2, V)$ where V is a vector space over F.

So the behavior of s_2 near $Y = \mathrm{Zero}(s_1)$ is clear. What happens around a point where $s_1 \neq 0$. Then s_1 defines a line subbundle $\langle s_1 \rangle \subset E$ over $X - Y$, and we can ask that s_2 satisfies i) the section $\bar{s}_2$ of $E/\langle s_1 \rangle$ induced by s_2 is transversal to zero, and where $\bar{s}_2$ vanishes it is transversal to zero as a section of $\langle s_1 \rangle$ ii) s_2 is transversal to zero as a section of E. Clearly i) $\Rightarrow$ ii).

<u>Lemma</u>: Let $0 \to E' \to E \xrightarrow{\beta} E'' \to 0$ be an exact sequence of vector bundles, let $s \in \Gamma(E)$ and assume $\beta(s) \in \Gamma(E'')$ is transversal to zero. TFAE

i) s is transv. to zero.

ii) If $W = \mathrm{Zero}\,\beta(s)$, then the section of E' over W induced by s is transv. to zero.

<u>Proof</u>:

$$\begin{tikzcd}
s^{-1}(0) \arrow[r] \arrow[d] & W \arrow[r] \arrow[d, "\bar{s}"] & X \arrow[d, "s"] \\
X \arrow[r, "0"] & E' \arrow[r, hook] \arrow[d] & E \arrow[d, "\beta"] \\
& X \arrow[r, hook, "0"'] & E''
\end{tikzcd}$$

Because $\beta(s)$ is transversal to zero, the square $\begin{smallmatrix} W & X \\ X & E'' \end{smallmatrix}$ is tr. cart; as $\begin{smallmatrix} E' & E \\ X & E'' \end{smallmatrix}$ is also it follows that $\begin{smallmatrix} W & X \\ E' & E \end{smallmatrix}$ is too. But then I know that $\bar{s}$ is transversal to $0 : X \to E''$ iff s is trans. to $0 : X \to E$. QED.

At the moment it is clear that a generic choice of s_1, s_2 consists in choosing s_1 trans. to 0, then choose s_2 traversal to zero i) as a section of E over $\text{Zero}(s_1)$, ii) as a section of $E/\langle s_1 \rangle$ over $X - \text{Zero}(s_1)$, iii) as a section of E over $X - \text{Zero}(s_1)$.

<u>Question</u>: Is a generic choice of s_1 then s_2 the same as a generic map $\mathcal{O}^2 \longrightarrow E$? NO

Clear at points of $\text{Zero}(s_1, s_2)$, and points where s_1, s_2 are ind. Suppose we consider a point x where s_1, s_2 become dependent. <u>Case 1</u>: $s_1(x) = 0$, $s_2(x) \neq 0$. OKAY because $ds_1(x) : T_X(x) \longrightarrow E(x)$ is onto, hence

$$T_X(x) \longrightarrow \mathrm{Hom}\Big(\overset{Fs_1(x)}{\overset{\|}{\mathrm{Ker}\,u(x)}},\ \overset{E/Fs_2(x)}{\overset{\|}{\mathrm{Cok}\,u(x)}}\Big)$$ is onto.

Case 2: $s_1(x) \neq 0$, $s_2(x) = 0$. Then $ds_2 : T_X(x) \longrightarrow E(x)$ is onto, in particular onto modulo s_1.

So I see that a generic choice of s_1, then s_2 is not the same as a generic map $\mathcal{O}^2 \xrightarrow{u} E$; $u(e_i) = s_i$. In effect at points of rank 1, say where $s_1(x) = 0$ $s_2(x) \neq 0$, the generic map condition says that

$$ds_1 : T_X(x) \longrightarrow \mathrm{Hom}(\ , E/\langle s_2\rangle)$$

is onto. (Recall we take a section of $\mathcal{O}^2$ say e_1 $\ni$ $e_1(x) \in \mathrm{Ker}(u(x))$ and apply tangent vectors to $u(e_1) = s_1$.)

It is clear that a generic choice of (s_1, s_2) is the same as a generic map $\mathcal{O}^2 \xrightarrow{u} E$ such that s_1, s_2 are transversal to zero.

Question: Is it possible to arrange that all sections $\lambda s_1 + s_2$ be transversal to zero, or does one encounter a singularity?

The point is that once the section $\bar{s}_2 = \overline{s_2 + \lambda s_1}$ $\in \Gamma(X - \mathrm{Zero}(s_1), E/\langle s_1\rangle)$ is transversal to zero one knows by the lemma that $s_2 + \lambda s_1$ is trans. iff

$s_2+\lambda s_1$ restricted to W [redacted] is transversal. [redacted] [redacted] W is the set where $s_1 \neq 0$ [redacted] and [redacted] s_1, s_2 are dependent hence $\exists!\ f : W \to \mathbb{C}$ $\ni$ $s_2 = f s_1$. So $s_2 + \lambda s_1 = (f+\lambda) s_1$. This section over W is transversal to zero iff $f+\lambda : W \to \mathbb{C}$ has simple zeroes which means that $-\lambda$ is not a critical value of f. So certainly one can't perturb a given s_2 to the good situation.

Example: Take E to be $\mathcal{O}(1)$; [redacted] a generic pair (s_1, s_2) is essentially a generic pencil of hyperplane sections (Lefchetz pencil). The singularities which occur have been well-studied.

[redacted]

Problem: Describe generic subspaces of $\Gamma(E)$ of a given dimension.

Suppose rank$(E) = 1$. Let V be a space of sections such that $\mathcal{O}_X \otimes V \longrightarrow J_1(E)$ is onto. Let $K = \mathrm{Ker}\{\mathcal{O}_X \otimes V \longrightarrow J_1(E)\} = \{(x, s) \mid x \in X, s \in V, j_1(s)(x) = 0\}$ and because E is a line bundle $j_1(s)(x) = 0$ means that s is not transversal to zero at x. The bad sections are in the image of $p_2 : K \longrightarrow V$. One has $\dim(K) = \dim X + \mathrm{rank}(K) = \dim X + \dim V - \mathrm{rank}\ J_1(E)$

$= \dim X + \dim V - (\dim X + 1) = \dim V - 1.$

One might try defining a subspace W of V to be generic if the inclusion $W \subset V$ is transversal to $p_2 : K \longrightarrow V$.

Let E be a vector bundle of rank r, and let V be a space of sections of E which generate $J_1(E)$. A section s of E is transversal to zero provided for each x such that $s(x) = 0$, one has $ds(x) \in (E \otimes T^*)(x) = \mathrm{Hom}(T(x), E(x))$ is surjective. Let $Y \subset E \otimes T^*$ be the bundle of non-surjective maps. If $\dim(X)+1 \geq \mathrm{rank}(E)$, we can resolve Y by

$$\tilde{Y} = \{(u, H) \mid u \in \mathrm{Hom}(T, E),\ H \in \check{\mathbb{P}}(E),\ \mathrm{Im}\, u \subset H\}$$

which is a bundle over $\check{\mathbb{P}}E$.

$$\begin{aligned} \dim \tilde{Y} &= \dim \check{\mathbb{P}}E + (\dim X)(\mathrm{rank}\, E - 1) \\ &= \dim X + \mathrm{rank}(E) - 1 + (\dim X)(\mathrm{rank}\, E - 1) \\ &= \dim X \cdot \mathrm{rank}\, E + \mathrm{rank}(E) - 1 \end{aligned}$$

$\dim J_1(E) = \dim X + \dim X\, \mathrm{rank}\, E + \mathrm{rank}\, E$

Thus the map $\tilde{Y} \longrightarrow J_1(E)$ has relative codimension $\dim X + 1$. It follows that if we form:

$$\begin{tikzcd} K \arrow[r] \arrow[d] & \check{Y} \arrow[d] \\ X \times V \arrow[r] \arrow[d] & J_1(E) \\ V & \end{tikzcd}$$

then $\dim K = \dim X + \dim V - (\dim X + 1) = \dim V - 1$, and so any generic section will be transversal to O.

If $\dim X < \operatorname{rank}(E)$, transversal is the same as empty intersection, so we look at

$$\begin{tikzcd} E' \arrow[r] \arrow[d] & X \arrow[d] \\ X \times V \arrow[r] \arrow[d] & E \\ V & \end{tikzcd}$$

$\dim E' = \dim X + \dim V - r = \dim V - (r - \dim X)$.

October 24, 1975.

Let me continue to try to understand a generic pair (s_1, s_2) in $\Gamma(X, E)$. So far I have the concept of the map $O^2 \to E$ being generic in the sense that the section of $\operatorname{Hom}(O^2, E)$ is transversal to the natural stratification. And I have the idea of the map $X \times \mathbb{C}^2 \longrightarrow J_1(E)$ being transversal to $\tilde{Y}$. How are these two concepts related?

[redacted] Thom's philosophy: [redacted] Generic elements are structurally stable. What this means is that after you succeed in putting enough conditions to define generic you can then prove a conjugacy theorem.

Suppose G acts on a space J. Over X I have a principal G-bundle P and I form the associated fibre bundle $P \times^G J = E$. I then want a model for the space $\Gamma(E)$ made up somehow of generic elements. [redacted] This looks too hard.

Basic approach is this: One defines generic in terms of some natural stratification of $J_k(E)$. (Say E is a vector bundle). [redacted] Describe a natural stratification. First thing to try is to use the orbits of $\mathrm{Aut}(E)$ on $J_k(E)$.

Start with a vector bundle E over X and a space [redacted] V of sections generating $J_k(E)$. I want to select inside of $J_k(E)$ a stratification. So the idea is to write down a finite number of submanifolds in $J_N(E)$ to define generic. (Or an infinite number of conditions of increasing codimension).

If X is a curve, the orbit structure of $\underline{Aut}(E)$ on $J_k(E)$ can be analyzed.

Orbit structure: Orbits of $GL_r(A)$ on A^r where A is a d.v.r. One orbit for each integer $k \geq 0$, depending on the order of vanishing of the section.

The general problem looks too hard.

Let J be a finite poset; I think of J as the poset of simplices in a finite simplicial complex. Let F be a flask sheaf on J with values in f.d. k-modules where k is a field. [redacted]
Put
$$E_x = \mathrm{Ker}\,\{F(U_x) \twoheadrightarrow F(\partial U_x)\}$$
$$= \mathrm{Ker}\,\{F(\text{[redacted]} \geq x)\ \text{[redacted]} \twoheadrightarrow F(> x)\}$$

(sections [redacted] near x having support $\overline{\{x\}}$.) Then by splitting exact sequences one gets an isomorphism
$$F \cong \text{[redacted]} \prod_{x \in J} (i_x)_*(E_x).$$

i.e.
$$F(U) \cong \prod_{x \in U} E_x$$

with the evident restriction maps.

Geometric picture is this. In the direct sum theory, we [redacted] look at a sky-scraper sheaf[redacted] on $|J|$ and collapse all the points in the same stratum. This gives rise to an F as above.

Next I want to define the concept of one sheaf specializing to another. So I start with the direct sum case: Given $\bigoplus_x E_x$, to specialize it to $\bigoplus_y E'_y$ we split each E_x up: $E_x = \bigoplus_{y \leq x} E''_{y,x}$

and give isomorphisms

$$E'_y \cong \bigoplus_{x \geq y} E''_{y,x} .$$

This means that I have a sort of decomposed gadget $\bigoplus_{y \leq x} E''_{y,x}$ on $\mathrm{Ar}\, J$ which defines a specialization starting with

$$\bigoplus_x E_x = \bigoplus_x \Big(\bigoplus_{y \leq x} E''_{y,x} \Big) \quad : \quad \prod_x (i_x)_* \Big(\bigoplus_{y \leq x} E''_{y,x} \Big)$$

$$\Gamma\Big(U, \prod_x (i_x)_* \Big(\bigoplus_{y \leq x} E''_{y,x} \Big)\Big) = \prod_{\substack{y \leq x \\ x \in U}} E''_{y,x}$$

$$= \Gamma\Big(t^{-1}(U), \prod_{y \leq x} (i_{y \leq x})_* E''_{yx} \Big) .$$

and ending with $\quad \prod_y (i_y)_* \Big(\prod_{x \geq y} E''_{y,x} \Big) = s_* \Big(\prod_{y \leq x} (i_{y \leq x})_* E''_{yx} \Big)$

Thus it appears that a specialization map from F to F' is given by an F'' on $(\mathrm{Ar}\, J)$ together with isos. $t_*(F'') \simeq F$, $s_*(F'') \simeq F'$.

So I can now write down a simplicial groupoid of chains on J with coefficients in $\mathcal{A}$.

October 25, 1975

Flask sheaf F on J may be identified with a module $M = \Gamma(J, F)$ equipped with a filtration

$$M_Z = \Gamma_Z(J, F) \subset \Gamma(J, F)$$

indexed by the closed subsets Z of J such that the Mayer-Vietoris property holds:

$$0 \longrightarrow M_{Z_1 \cap Z_2} \longrightarrow M_{Z_1} \oplus M_{Z_2} \longrightarrow M_{Z_1 \cup Z_2} \longrightarrow 0$$

(Can also say $Z \longmapsto M_Z$ is a lattice homomorphism). Note that the filtration is determined by the submodules $M_{\overline{\{x\}}}$ ~~suppose I have a specialization~~ as $x \in J$. If I want, then, a sheaf is a module M together with a functor $\{x\} \longmapsto M_{\overline{\{x\}}}$ from J to submodules such that when extended to all closed sets it satifies the Mayer-Vietoris property.

Now consider a specialization from F to F' given by an F'' on $Ar(J)$. We have an identification of M, M', and M''. So M'' is a filtration of M indexed by points of $Ar(J)$. Thus we give $M''_{\overline{\{(x,y)\}}} \subset M$ for each $x \leq y$ such that

$$M_y = M''_{\{(a,b) \mid b \leq y\}}$$

$$M'_x = M''_{\{(a,b) \mid a \leq x\}}$$

and we want the flaskness condition to be satisfied. Clearly this means

$$M_x = M''_{\{(a,b) \mid b \leq x\}} \subset M''_{\{(a,b) \mid a \leq x\}} = M'_x$$

Note that

$$M_y \cap M'_x = M''_{\left\{(a,b) \mid \begin{smallmatrix} a \leq x \\ b \leq y \end{smallmatrix}\right\}} = M''_{\overline{\{x,y\}}}$$

when $x \leq y$.

Converse question: Given two filtrations M_x and M'_x of $M = M'$ satisfying MV such that $M_x \subset M'_x$ for all x. Put $M''_{(x,y)} = M'_x \cap M_y$ for $x \leq y$. Does it follow always that M'' is flask?

<u>Example</u>: Recall that a $\mathbb{C}$-vector space V decomposed relative to $I = [0,1]$ is the same as a self-adjoint operator A on V_n such that $0 \leq A \leq 1$. Let $0 < t_1 < \cdots < t_p < 1$ be the eigenvalues of A not 0, or 1, and let

$$V = W_0 \oplus \cdots \oplus W_{p+1}$$

be the eigenspace decompositions where $A = 0$ on W_0, $A = 1$ on W_{p+1}, and $A = t_i$ on W_i. Then we get a flag

$$0 \leq V_0 < V_1 < \cdots < V_p \leq V$$

which together with $t_1 < \dots < t_p$ determines A. In this manner I can identify self-adjoint A, $0 \leq A \leq 1$ with the geometric realization of the poset of subspaces of V.

Another way of doing this is to give an increasing filtration V_t of V $0 \leq t \leq 1$ continuous from above $V_t = V_{t+0} = \lim\limits_{\varepsilon \searrow 0} V_{t+\varepsilon}$.

[redacted]

October 26, 1975

Important Point (maybe): Any functor $F: J \longrightarrow$ modules defines [redacted] an element of the K-theory to be associated to J.

October 27, 1975.

The problem as I see it is to describe [redacted] efficiently the K-theory of chains on J with coefficients in $\mathcal{M}$. [redacted] As a first approximation it is the exact category of functors from J to $\mathcal{M}$. But then I have to work in the specialization theory.

[redacted] $\exists$ basic discrepancy

between alg. + top. K-theory concerning automorphisms.

Bundle over S^1 is the same as a pair consisting of a bundle E and and auto. θ unique up to homotopy. Obvious [redacted] category model for S^1:

$$\bullet \rightrightarrows \bullet$$

or if you want a poset
$$\begin{tikzcd} \bullet \arrow[r] \arrow[dr] & \bullet \\ \bullet \arrow[r] \arrow[ur] & \bullet \end{tikzcd}$$

Category of diagrams $V_1 \overset{\alpha}{\underset{\beta}{\rightrightarrows}} V_0$ where the V_i are [redacted] in $Mod_f(F)$ is clearly of finite homological dimension. Projectives

$$0 \overset{0}{\underset{0}{\rightrightarrows}} F \qquad \text{represents} \qquad [V_1 \rightrightarrows V_0] \longmapsto V_0$$

$$F \overset{in_1}{\underset{in_2}{\rightrightarrows}} F \oplus F \qquad \text{———} \qquad [\qquad] \longmapsto V_1.$$

Projectives are those $\ni$ $V_0 \oplus V_0 \xrightarrow{\alpha+\beta} V_1$ injective.

October 29, 1975

Let V be a vector space over a field. I want describe carefully a poset Y attached to V which should be a generalized building. Here's how elements of Y should be. Start with a decomposition of V

$$V = V_1 \oplus \cdots \oplus V_k .$$

(This is the same as a "standard" reductive subgroup of $\mathrm{Aut}\, V$.) Next we give a partial ordering on the set $\{1, \ldots, k\}$. Then we associate to each closed subset Z for the partial ordering a subspace $\bigoplus_{j \in Z} V_j$. We get a filtration of V indexed by closed subsets of $\{1, .., k\}$ for the partial ordering. It is this filtration that we are to regard as an element of Y.

Let me start with a distributive lattice $\mathcal{L}$ of subspaces of V. Let J be the poset of join irreducibles in $\mathcal{L}$. (Recall that if $\mathcal{L}$ is a finite distributive lattice, then every element of $\mathcal{L}$ has a unique decomposition into irreducibles). For each $j \in J$, let us choose a subspace V_j complementary to $F_{\{<j\}}V$ in $F_{\{\leq j\}}V$. Then $V = \bigoplus_{j \in J} V_j$.

Therefore Y should consist of distributive lattices $\mathcal{L}$ of subspaces of V. To get a poset we should order these by inclusion.

__Example:__ A chain is an obvious example of distributive lattice. (The chain should contain 0 and V.) Thus Y contains the building.

See if I can describe Y in coset terms. We've seen that every $\mathcal{L}$ comes from a distributive lattice of subsets of a basis S of V. Thus every $\mathcal{L}$ comes from an isom $k[S] \xrightarrow{\sim} V$ and a surjection $S \twoheadrightarrow J$ where $\mathbf{J}$ is a poset. The map $S \twoheadrightarrow J$ is the same thing as a pre-ordering on S. Thus $\mathcal{L}$ comes from a basis S and a preordering on S.

Generalities: If $\mathcal{H}$ is a family of subgroups of G then I get the poset X of left cosets of members of $\mathcal{H}$. G acts on X and $G\backslash X \xrightarrow{\sim} \mathcal{H}$ and there is a section of the map $X \to \mathcal{H}$. This is what happens for the building.

But it does not seem that there is a section of $Y \to G\backslash Y$, $G = \mathrm{Aut}(V)$. Thus there seems to be no way to select from a G-orbit on Y a canonical member, as it can be done for chains.

Stabilizer of a distributive lattice $\mathcal{L}$ resembles a parabolic group. It contains a standard reductive group.

If we fix the axes $V = L_1 \oplus \cdots \oplus L_n$ and look at all associated distributive lattices, then we are looking at all pre-orderings on the set S of axes. If we want the unipotent radical we look at $s_1 < s_2$ but $s_2 \not< s_1$.

Suppose I have a flask sheaf over a poset J' with global sections V. Better: I have a lattice homom. $\theta : Cl(J') \longrightarrow Sub(V)$. I have seen there there exists then subspace $V_{\{x\}}$ of V such that $\theta(Z) = \bigoplus_{x \in Z} V_{\{x\}}$. This clearly depends only on those x in J' such that $V_{\{x\}} \neq 0$. So if I put $J = \{x \in J' \mid V_{\{x\}} \neq 0\}$, I have

$$\theta(Z) = \bigoplus_{x \in J \cap Z} V_{\{x\}}.$$

where $J \hookrightarrow \mathrm{Sub}(V)$ is a distributive lattice. Thus what seems to be the case is that a "decomposition" of V with respect to J' can be represented by a pair (J, i) where J is a distributive lattice in $\mathrm{Sub}(V)$ and where i is an embedding of J in J'.

Go over this carefully: Let J be a fin. poset and suppose to each closed set Z in J we give a subspace W_Z of V such that

$$\begin{aligned} \mathrm{Cl}(J) &\longrightarrow \mathrm{Sub}(V) \\ Z &\longmapsto W_Z \end{aligned}$$

is a lattice homom. (preserving $0, I$). Choose a complement V_j for $W_{\{<j\}}$ in $W_{\{\le j\}}$. Claim

$$W_Z = \bigoplus_{j \in Z} V_j$$

for any closed set Z. Argue by induction on card Z. If j_0 is a maximal element of Z then $Z' = Z - \{j_0\}$ is closed and

$$Z = Z' \cup \{\le j_0\}$$

$$\{< j_0\}$$

$$W_{Z'} +^{W_{\{<j_0\}}} W_{\{\le j_0\}} =$$

so

$$W_Z = W_{Z'} \oplus V_{j_0} = \bigoplus_{j \in Z} V_j .$$

Let $J_1 \subset J$ be the subset such that $V_j \neq 0$ for $j \in J_1$. Then

(*) $$W_Z = \bigoplus_{j \in J_1} V_{j1} .$$

In fact you should note that $\mathcal{L} = \{W_Z \in \mathrm{Sub}\, V \mid Z \in \mathcal{Cl}(J)\}$ is a distributive sublattice of $\mathrm{Sub}(V)$ since

$$W_Z + W_{Z'} = W_{Z \cup Z'}$$

$$W_Z \cap W_{Z'} = W_{Z \cap Z'}$$

etc. Let K be the poset of join-irreducibles in $\mathcal{L}$. We have a map

$$\mathcal{Cl}(J_1) \longrightarrow \mathcal{L}$$

$$Z \cap J_1 \longmapsto W_Z .$$

Better, the map $\mathcal{Cl}(J) \twoheadrightarrow \mathcal{L}$, $Z \longmapsto W_Z$ factors

$$\begin{array}{ccccc} \mathcal{Cl}(J) & \longrightarrow & \mathcal{Cl}(J_1) & \twoheadrightarrow & \mathcal{L} . \\ Z & \longmapsto & Z \cap J_1 & \longmapsto & W_Z \end{array}$$

The map $\mathcal{Cl}(J_1) \longrightarrow \mathcal{L}$ is clearly onto, and by (*) **it** is 1-1, hence it is a lattice isom. Now J_1 can be recovered as the join-irreducibles in $\mathcal{Cl}(J_1)$, so J_1 must $= K$.

So what I find is that any decomposition of V with respect to the poset J (this by defn. is a lattice homom. $Cl(J) \longrightarrow Sub(V)$) factors uniquely

$$Cl(J) \twoheadrightarrow Cl(J_0) \hookrightarrow Sub(V)$$

where J_0 is a subset of J.

Suppose now we have $Cl(J) \xrightarrow{\theta} Sub(V)$ and a map $f: J \to J'$ whence we get a decomposition of V relative to J':

$$\begin{tikzcd}
Cl(J') \arrow[r, "f^*"] \arrow[d, dashed, two heads] & Cl(J) \arrow[r, leftrightarrow, "\theta"] \arrow[d, dashed, two heads] & Sub(V) \\
Cl(J_0') \arrow[urr, dashed, hook] & Cl(J_0) \arrow[ur, dotted, hook] &
\end{tikzcd}$$

<u>October 31, 1975</u>

Let V be a vector space equipped with a decomposition relative to J:

$$w : Cl(J) \longrightarrow Sub(V)$$
$$Z \longmapsto w_Z .$$

Let $J_0 = \{ j \in J \mid w_{\{\leq j\}} / w_{\{<j\}} \neq 0 \}$. Then I know w factors

$$\begin{array}{ccccc} Cl(J) & \twoheadrightarrow & Cl(J_0) & \hookrightarrow & Sub(V) \\ Z & \longmapsto & Z \cap J_0 & \longmapsto & w_Z \end{array}$$

and these maps are lattice homos.

Suppose $f : J \longrightarrow J'$ is a map. Then I have an induced decomp. of V relative to J':

$$w' : Cl(J') \xrightarrow{f^{-1}} Cl(J) \xrightarrow{w} Sub(V)$$

$$w'_Y = w_{f^{-1}(Y)}$$

Suppose I've chosen a splitting: $w_Z = \bigoplus_{j \in Z} V_j$. Then I have

$$w'_{\{\leq k\}} / w'_{\{<k\}} = w_{f^{-1}\{\leq k\}} / w_{f^{-1}\{<k\}} = \bigoplus_{f(j)=k} V_j$$

i.e. I get a splitting: $w'_Y = \bigoplus_{k \in Y} V'_k$ where $V'_k = \bigoplus_{f(j)=k} V_j$.

Thus it's clear that $J_0' = \{k \in J' \mid V_k' \neq 0\} = fJ_0$.

[redacted]

Summary: [redacted] Any decomp. $\omega: Cl(J) \to Sub(V)$ of V rel. J has a unique decomposition

$$Cl(J) \twoheadrightarrow Cl(J_0) \hookrightarrow Sub(V).$$

If $f: J \to J'$, then the induced decomp. $f_*\omega$ of V relative to J has the decomposition

$$Cl(J') \twoheadrightarrow Cl(fJ_0) \hookrightarrow Cl(J_0) \hookrightarrow Sub(V).$$

The following picture might help

$$\begin{tikzcd}
Cl(J) \arrow[r, two heads] & Cl(J_0) \arrow[r, hook] & Sub(V) \\
Cl(J') \arrow[u, "f^*"] \arrow[r, two heads] & Cl(fJ_0) \arrow[u, hook] \arrow[ur, hook] &
\end{tikzcd}$$

It is essential[redacted] to work out some sorites about finite distributive lattices L. Recall ~~such~~ such an L is a poset having a greatest member I a least member O, and sup's & infs for subsets, and one has the distributive laws. The dual of a distributive lattice is distributive.

Given such an L, call $x \in L$ *irreducible* if $x = y_1 \cup y_2$ [redacted] $\Rightarrow x = y_1$ or $x = y_2$ and if $x \neq O$. Any x has a unique irredundant decomposition into irreducibles:

$$y = x_1 \cup \cdots \cup x_n$$

where irredundant means no x_i can be deleted. Let J be the poset of irreducibles. Then we can define a map

$$\varphi : \mathcal{L} \longrightarrow Cl(J)$$

$$y \longmapsto \{ j \in J \mid j \le y \}$$

and a map ψ the other way sending Z to $\bigcup_{j \in Z} j$. Now

$$\varphi(y_1 \cup y_2) = \varphi(y_1) \cup \varphi(y_2)$$

$$\varphi(y_1 \cap y_2) = \varphi(y_1) \cap \varphi(y_2)$$

$$\varphi(0) = \phi$$

$$\varphi(I) = \{J\}$$

($j \le y_1 \cup y_2 \Rightarrow j = (j \cap y_1) \cup (j \cap y_2) \Rightarrow \exists i \; j = j \cap y_i \Rightarrow \exists i \; j < y_i$).

$$\bigcup_{j \le y} j = y \quad \Rightarrow \quad \psi\varphi = id$$

$$j_0 \le \bigcup_Z j \Rightarrow \exists j \in Z, j_0 \le j \Rightarrow j_0 \in Z$$

$$\therefore \quad \varphi\Big(\bigcup_Z j\Big) = Z.)$$

Therefore $\mathcal{L} \xrightarrow{\sim} Cl(J)$

Assume now that I have a morphism

$$u : \mathcal{L} \longrightarrow \mathcal{L}'$$

of finite distributive lattices. Let J, J' be the irreducibles in $\mathcal{L}, \mathcal{L}'$ respectively. Notice first that $u\mathcal{L}$ is a sublattice of $\mathcal{L}'$.

~~Consider the case where u is onto. Given $k \in J'$ let $f(k) = \bigcap_{u(x)=k} x$. Then $uf(k) = k$ so $f(k)$ is the smallest member in the fibre of u over k. If $f(k) = y_1 \cup y_2$, then $u(y_1) \cup u(y_2) = k$ so $u(y_i) = k \Rightarrow y_i \geq f(k) \Rightarrow y_i = f(k)$. Thus $f(k)$ is irreducible.~~

~~$f(k)$ = least member of $u^{-1}(k)$~~

~~$f(k) \leq x$ if $u(x) = k$~~

Let us show adjoint functors $\exists$

$$\begin{tikzcd} \mathcal{L} \arrow[r, "u"] & \mathcal{L}' \arrow[l, bend right, "v"'] \arrow[l, bend left, "w"] \end{tikzcd}$$

Put $$v(y) = \overset{\min}{\inf} \{ x \in \mathcal{L} \mid y \leq u(x) \}.$$ Then

$$u(v(y)) = \inf \{ u(x) \mid x \in \mathcal{L},\ y \leq u(x) \} \geq y$$

so $$v(y) \leq x_0 \implies uv(y) \leq u(x_0) \implies y \leq u(x_0).$$

Conversely $$y \leq u(x_0) \implies v(y) \leq x_0 \text{ by defn.}$$

Thus $$\mathrm{Hom}(v(y), x) = \mathrm{Hom}(y, u(x)).$$

Similarly put
$$w(y) = \sup \{ x \in \mathcal{L} \mid u(x) \leq y \}$$

Then $$uw(y) = \sup \{ u(x) \mid x \in \mathcal{L}\ \ u(x) \leq y \} \leq y,$$

so

$$x \leq w(y) \implies u(x) \leq uw(y) \implies u(x) \leq y$$

Conversely $$u(x) \leq y \implies x \leq w(y) \text{ by defn.}$$

$\therefore$
$$\mathrm{Hom}(x, w(y)) = \mathrm{Hom}(u(x), y).$$

Note that v, being a left adjoint, will preserve $\cup$ and w, being a right adjoint will preserve $\cap$.

Now assume that u is <u>onto</u>. In this case for each $y \in \mathcal{L}'$, $\exists\ x \ni u(x) = y$, whence

$$u\,v(y) = \inf\{u(x) \mid x \in \mathcal{L},\ y \leq u(x)\} = y.$$

Then $v(y)$ is the least element of $u^{-1}(y)$ and similarly $w(y)$ is the greatest member of $u^{-1}(y)$.

If $y \in J'$ is irreducible and $v(y) = x_1 \cup x_2$ then $y = uv(y) = ux_1 \cup ux_2 \Rightarrow y = u(x_i) \Longrightarrow v(y) \leq x_i \Rightarrow v(y) = x_i$. Also $v(y) = 0 \Rightarrow uv(y) = y = 0$. Thus $v(y)$ is irreducible and v defines a map of posets.

$$i : J' \longrightarrow J$$

which is injective because v is injective as a map from $\mathcal{L}'$ to $\mathcal{L}$ $(uv = id)$. Now to compute u in terms of i.

$$u(x) = \bigcup_{j' \leq u(x)} j' = \bigcup_{\substack{v(j') \leq x \\ i(j') \in \{J \leq x\}}} j' = \bigcup_{j' \in i^{-1}\{J \leq x\}} j'.$$

Thus in terms of closed sets we see $u(Z) = i^{-1} Z$.

One sees the diagram at bottom page 10 is

$$\overline{iZ'} \longleftarrow\!\shortmid \; Z'$$

$$\begin{tikzcd} Cl(J) \arrow[r, shift left=3, leftarrow] \arrow[r, "i^*"] \arrow[r, shift right=3, leftarrow] & Cl(J') \end{tikzcd}$$

$$\text{largest closed set restricting to } Z' \longleftarrow\!\!\shortmid \; Z'$$

What I have to understand next are the injective maps of distributive lattices. Suppose $u: \mathcal{L} \hookrightarrow \mathcal{L}'$ is an embedding.

First look at the case $u = f^*: Cl(J) \to Cl(J')$ where $f: J' \to J$ is a map. If u is injective I want to prove f is onto. If Z is closure in J, then

$$ff^{-1}(Z) \subset \overline{ff^{-1}(Z)} \subset Z$$

$$\Rightarrow \quad f^{-1}(Z) \subset f^{-1}\left(\overline{ff^{-1}(Z)}\right) \subset f^{-1}(Z) \quad \text{so} \quad \overline{ff^{-1}(Z)} = Z$$

otherwise f^* would not be injective. Now let $y \in J$ and let $Z = \{J \le y\}$, whence $ff^{-1}(Z) = \{f(x) \mid f(x) \le y\}$. Since $\{J < y\}$ is a closed subset of $\{J \le y\}$, it follows that $\overline{ff^{-1}(Z)} = Z \Rightarrow y \in \{f(x) \mid f(x) \le y\} \Rightarrow y \in \mathrm{Im} f$. Thus f is onto.

Setup: I now know that those injective maps $Cl(J) \longrightarrow Cl(J')$ which arise from maps $f: J' \longrightarrow J$ arise actually from surjective maps f. Do all injective maps so arise? If not, can one characterize the ones that do?

Example: Let us consider $Cl(\{0<1\}) \hookrightarrow Cl(J)$, i.e. simply giving an x in $Cl(J)$, $0 < x < 1$.

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Let $u: \mathcal{L} \longrightarrow \mathcal{L}'$ be a morphism of finite distributive lattices. For each $k \in J' = \mathrm{Irr}(\mathcal{L}')$, consider $\{x \in \mathcal{L} \mid k \leq u(x)\}$. This is closed under intersection: $k \leq u(x_1) \cap u(x_2) = u(x_1 \cap x_2)$, hence it has a least element which I denote $f(k)$. $f(k)$ is irred: $f(k) = x_1 \cup x_2 \Rightarrow k \leq uf(k) = u(x_1) \cup u(x_2)$ $\Rightarrow k \leq u(x_i)$ some $i \Rightarrow f(k) \leq x_i \Rightarrow f(k) = x_i$; also $f(k) = 0 \Rightarrow k \leq u(0) = 0$ impossible. Thus u determines a map $f: J' \longrightarrow J$, which is a morphism of posets: $k_1 \leq k_2 \Rightarrow k_1 \leq k_2 \leq uf(k_2) \Rightarrow f(k_1) \leq f(k_2)$.

Claim

$$\begin{tikzcd} \mathcal{L} \arrow[r, "u"] \arrow[d, "\wr"] & \mathcal{L}' \arrow[d, "\wr"] \\ Cl(J) \arrow[r, "f^{-1}"] & Cl(J') \end{tikzcd}$$

commutes

$$\begin{tikzcd}
x \arrow[r, mapsto] \arrow[dd, mapsto] & u(x) \arrow[d, mapsto] \\
& \{k \leq u(x)\} = \{k \mid f(k) \leq x\} \arrow[d, phantom, "\shortparallel"] \\
\{j \leq x\} \arrow[r, mapsto] & f^{-1}\{j \leq x\}
\end{tikzcd}$$

So I have proved the following:

<u>Theorem</u>: The category of finite distributive lattices is equivalent to the dual of the category of finite posets.

So I now know that the factorization

$$\mathcal{L} \twoheadrightarrow \mathrm{Im}(u) \hookrightarrow \mathcal{L}'$$

corresponds to

$$Cl(J) \twoheadrightarrow Cl(\mathrm{Im} f) \hookrightarrow Cl(J')$$

so a map f is an immersion (resp. surjective) iff f^{-1} is surjective (resp. injective).

Consider next a specialization situation.

$$\begin{tikzcd}
Cl(J) \arrow[d, "t^{-1}"] \arrow[dr] & \\
Cl(A_r J) \arrow[r] & \text{Sub-}V \\
Cl(J) \arrow[u, "s^{-1}"] \arrow[ur] &
\end{tikzcd}$$

We know [redacted] the map $Cl(\mathrm{Ar}\,J) \to \mathrm{Sub}\,V$ factors

$$Cl(\mathrm{Ar}\,J) \twoheadrightarrow Cl(J') \subset \mathrm{Sub}(V).$$

Thus we get poset maps

$$\begin{tikzcd}
J & \\
\mathrm{Ar}\,J \arrow[u, "s"] \arrow[d, "t"'] & J' \arrow[l, hook', "h"'] \arrow[ul, dashed, "f"'] \arrow[dl, dashed, "g"] \\
J &
\end{tikzcd}$$

If we ignore h being injective, this diagram is the same thing as a pair $f, g : J' \rightrightarrows J$ such that $f(k) \leq g(k)$ for all $k \in J'$. (Recall $\mathrm{Ar}\,J \subset J \times J$).

<u>Question</u>: Given $Cl(J) \underset{v}{\overset{u}{\rightrightarrows}} \mathrm{Sub}(V)$ such that $u(x) \leq v(x)$ for all $x \in Cl(J)$, does it follow that $\exists$ factorization

$$Cl(J) \rightrightarrows Cl(K) \subset \mathrm{Sub}(V) \; ?$$

Assuming such a factorization exists, let $u = f^*$, $v = g^*$. Then I know for all $Z \in Cl(J)$ that

$$f^{-1}Z \subset g^{-1}Z$$

$$f^{-1}\{y \leq x\} \subset g^{-1}\{y \leq x\}$$

Take $x = f(k)$. $k \in f^{-1}\{j \leq f(k)\} \subset g^{-1}\{j \leq f(k)\}$
$\Rightarrow g(k) \leq f(k)$ for all k. Conversely assume $g \leq f$. Then for any $Z \in \mathcal{Cl}(J)$, $k \in f^{-1}(Z)$
$\Rightarrow f(k) \in Z \Rightarrow g(k) \in Z \Rightarrow k \in g^{-1}(Z)$, so $f^{-1}(Z) \subset g^{-1}(Z)$.

Thus I see that

$$\mathcal{Cl}(J) \underset{g^*}{\overset{f^*}{\rightrightarrows}} \mathcal{Cl}(K)$$

one has $f^* \leq g^* \iff f \geq g$. In this case we have a unique map $h: K \longrightarrow \mathrm{Ar}\, J$ such that $sh = g$, $th = f$. So we get:

<u>Prop:</u> Let $\mathcal{Cl}(J) \underset{v}{\overset{u}{\rightrightarrows}} \mathrm{Sub}(V)$ be lattice homomorphisms. Then v is a <u>specialization</u> of u iff i) $u(x) \leq v(x)$ for all x in $\mathcal{Cl}(J)$
ii) $\mathrm{Im}\, u$ and $\mathrm{Im}\, v$ generate a distributive sub-lattice of $\mathrm{Sub}(V)$.

Is ii) necessary? Is ii) unnecessary if J is a product of chains. ii) will be necessary when I consider V to be in an exact category.

<u>Definition:</u> The generalized building of a vector space V is the poset of distributive sub-lattices of $\mathrm{Sub}(V)$.

<u>Question:</u> Can you somehow represent the realization of this poset geometrically?

You might try to take the inverse limit of the various posets involved.

$$\begin{tikzcd} J & Cl(J) \arrow[d, hook] \arrow[r, hook] & \mathrm{Sub}(V) \\ J' \arrow[u, two heads] & Cl(J') \arrow[ur, hook] & \end{tikzcd}$$

The maximal distributive lattices correspond to breaking V into a direct sum of lines.

Further ideas: Go back to a situation where exact sequences don't split and try to piece together non-split extensions and filtrations.