1975 Notes on Buildings, Symmetric spaces, etc.

First section: Adjoint action of a compact Lie gp. Roots, Chambres, Weyl group.

Let K be a compact Lie gp, let $\mathfrak{k}$ be its Lie algebra, and let $(\ ,\ )$ be an inner product on $\mathfrak{k}$ invariant under the adjoint action of K.

Let $\xi \in \mathfrak{k}$ and identify the tangent space to $\mathfrak{k}$ at ξ with $\mathfrak{k}$ itself. If $X \in \mathfrak{k}$, $Ad(e^{tX})\xi$ is a path in $\mathfrak{k}$ starting with ξ; as

$$Ad(e^{tX})\xi = \xi + t[X,\xi] + \cdots$$

the tangent vector to this path is $[X,\xi]$. Thus the tangent space to the orbit $K\cdot\xi$ at ξ is $[\mathfrak{k},\xi]$. A vector η is perpendicular to the orbit at ξ iff

$$([X,\xi],\eta) = +(X,[\xi,\eta]) = 0$$

for all $X \in \mathfrak{k}$, i.e. if $\eta \in \mathfrak{k}_\xi = \{Y \in \mathfrak{k} \mid [Y,\xi] = 0\}$. Thus:

<u>Prop 1</u>: The normal space to the orbit $K\xi$ at ξ may be identified with the centralizer $\mathfrak{k}_\xi$ in $\mathfrak{k}$ of ξ.

From general facts about actions of compact groups, one knows that for a generic ξ in $\mathfrak{k}$

its stabilizer K_ξ acts trivially on the normal space to the orbit. Thus we get

<u>Prop 2:</u> If ξ is generic, then $\mathfrak{k}_\xi$ is abelian.

From now on ξ will denote a given point of $\mathfrak{k}$ such that $\mathfrak{k}_\xi$ is abelian.

On the orbit $K\eta$ we consider the function

$$f(k\eta) = \frac{1}{2}|k\eta - \xi|^2 = \text{const} - (k\eta, \xi).$$

Since $K\eta$ is compact this function has critical points; $k\eta$ is a critical point of f iff

$$0 = ([X, k\eta], \xi) = (X, [k\eta, \xi])$$

for all $X \in \mathfrak{k}$, i.e. iff $k\eta \in \mathfrak{k}_\xi$. Thus we have

<u>Prop. 3.</u> $K\eta \cap \mathfrak{k}_\xi \neq \emptyset$, i.e. every element of $\mathfrak{k}$ is K-conjugate to an element of $\mathfrak{k}_\xi$.

Let N be the subgroup of K normalizing $\mathfrak{k}_\xi$. It is clear that $K\eta \cap \mathfrak{k}_\xi$ is stable under N. I claim N acts transively on this intersection. To see this I can suppose $\eta \in \mathfrak{k}_\xi$. Let $x \in K$ be such that $x\eta \in \mathfrak{k}_\xi$. Let us consider the compact group $K_{x\eta}$ whose Lie algebra contains $\xi, x\xi$. Applying Prop. 3 to $K_{x\eta}$ we see $\exists\, z \in K_{x\eta}$ such that $zx\xi \in \mathfrak{k}_\xi$. Then

$z x \mathfrak{k}_\xi = \mathfrak{k}_\xi$ and $zx \in N$ while $zxy = xy$. Thus

<u>Prop. 3'</u>: $Ky \cap \mathfrak{k}_\xi$ is an N-orbit in $\mathfrak{k}_\xi$.

Thus $K\backslash\mathfrak{k} \xleftarrow{\sim} N\backslash\mathfrak{k}_\xi$.

<u>Remark</u>: So far I have not assumed K is connected. If ξ is generic in the sense that K_ξ acts trivially on $\mathfrak{k}_\xi$, then K_ξ is the centralizer of $\mathfrak{k}_\xi$, hence the N-action factors through the quotient group

$$W = N/K_\xi = \text{the gp of autos of } \mathfrak{k}_\xi \text{ produced by elts of } K.$$

Now we come to roots. Let A be an abelian subspace of $\mathfrak{k}$. The operator $\mathrm{ad}(X)$ $X \in \mathfrak{k}$ is skew-symmetric, hence has eigenvalues $\pm i\lambda$, $\lambda \in \mathbb{R}$. $\mathfrak{k}$ can be decomposed into a direct sum of irreducible modules for the $\mathrm{ad}(A)$ action. If V is an irreducible non-trivial submodule, then there exists an isom. $\theta: V \xleftarrow{\sim} \mathbb{C}$ and a linear function $\alpha: A \to \mathbb{R}$ such that

$$[a, \theta(z)] = \theta(i\alpha(a) z) \qquad z \in \mathbb{C}.$$

Replacing θ by $\theta\sigma$ $(\sigma z = \bar{z})$ has the effect of changing α to $-\alpha$. The functions $\alpha \in \mathrm{Hom}(A, \mathbb{R})$ obtained in this way are called the <u>roots</u> of $\mathfrak{k}$ with respect to A; denote this set by Φ, Let us choose an element $a_0 \in A$ such that $\alpha(a_0) \neq 0$ for all $\alpha \in \Phi$, and let $\Phi^+ = \{\alpha \in \Phi \mid \alpha(a_0) > 0\}$. Then in each pair $\{\alpha, -\alpha\}$ we have selected one member, so each irreducible non-trivial submodule of k for the $ad(A)$-action has a definite complex structure. We have a direct sum

$$(1) \qquad k = k_A \oplus \sum_{\alpha \in \Phi^+} k^\alpha$$

where k_A is the centralizer of A, and k^α is isomorphic to a direct sum of copies of $\mathbb{C}$ with a acting as $i\alpha(a)$.

If A is a maximal abelian subspace of k, i.e. $A = k_A$, then from (1) we see that $k_A = k_x$ if x is a generic element of A. Since x is conjugate to an element of k_ξ we get

<u>Prop. 4</u>. **All maximal abelian subspaces of k are conjugate under K.**

Next let us apply the root space decomposition to $A = k_\xi$ with the generic element ξ. We get a direct sum decomposition

$$(2) \qquad k = k_\xi \oplus \sum_{\alpha \in \Phi^+} k^\alpha$$

where $\Phi^+ \subset \mathrm{Hom}(k_\xi, \mathbb{R})$ is a finite set of linear functions which are > 0 on ξ. I will denote by

$pr^\alpha : k \longrightarrow k^\alpha$ the projection onto k^α relative to this decomposition, and by $pr^0 : k \longrightarrow k^0 = k_\xi$ the projection on $k^0 = k_\xi$.

Let's consider again the function

$$f(k\eta) = \frac{1}{2}|k\eta - \xi|^2 = \text{const} - (k\eta, \xi)$$

on $K\eta$. Assuming η is a critical point, i.e. $\eta \in k_\xi$, I want the Hessian at this critical point. A point on $K\eta$ near to η can be represented $e^{\operatorname{ad} X}\eta$ where $X \in k \ominus k_\eta$. Since

$$f(e^{\operatorname{ad} X}\eta) = \text{const} - (e^{\operatorname{ad} X}\eta, \xi)$$

$$= \text{Const} - \underset{\substack{\| \\ (X, [\eta, \xi]) \\ \| \\ 0}}{([X, \eta], \xi)} - \frac{1}{2}((\operatorname{ad} X)^2\eta, \xi) - \cdots$$

the Hessian is the quadratic function of $X \in k \ominus k_\eta$

$$-\frac{1}{2}((\operatorname{ad} X)^2\eta, \xi) = \frac{1}{2}([X, \eta], [X, \xi])$$

$$= \frac{1}{2}([\eta, X], [\xi, X]).$$

Let $X = pr^0 X + \sum_{\alpha \in \Phi^+} pr^\alpha X$ be the decomposition of X relative to (2). Then

$$[\eta, X] = \sum_\alpha i\alpha(\eta)\, pr^\alpha(X) \quad , \quad [\xi, X] = \sum_\alpha i\alpha(\xi)\, pr^\alpha(X)$$

so

(3) $\quad \frac{1}{2}([\eta,X],[\xi,X]) = \frac{1}{2}\sum_{\alpha\in\Phi^+} \alpha(\eta)\alpha(\xi)\,|pr^\alpha(X)|^2$

where I have used that multiplication by i on k^α preserves norm.

As $k\ominus k_\eta = \bigoplus_{\alpha(\eta)\neq 0} k^\alpha$ one sees that the Hessian is non-degenerate on $k\ominus k_\eta$. Moreover its index is the sum over α such that $\alpha(\xi)>0$ $\alpha(\eta)<0$ of $\dim k^\alpha$. Summarizing:

<u>Prop. 5:</u> Assuming ξ such that k_ξ is abelian, the function $k\zeta \mapsto \frac{1}{2}|k\zeta-\xi|^2$ on $K\zeta$ has critical points where the orbit intersects k_ξ. Each critical point is non-degenerate. The index of a critical point η is

$$i_\xi(\eta) = \sum_{\alpha(\eta)<0} \dim k^\alpha$$

where α runs over Φ^+ (= roots α such that $\alpha(\xi)>0$).

At this point we can apply Morse theory to deduce many results. First of all because k^α has a complex structure $i_\xi(\eta)$ is always even, hence

<u>Prop. 6:</u> Each orbit $K\zeta$ has the homotopy type of a finite CW complex with even-dimensional cells. Hence $H_*(K\zeta,\mathbb{Z})$ is free and it has a basis indexed by the points of $K\zeta\cap k_\xi$ (which is a W-orbit)

The zero cells of $K\xi$ are therefore in 1-1 correspondence with points of $K\xi \cap C_\xi$ where

$$C_\xi = \{\eta \in \mathfrak{k}_\xi \mid \alpha(\eta) \geq 0 \quad \text{for } \alpha \in \Phi^+\}.$$

But because ~~[illegible]~~ there are no 1-cells, there is exactly one zero-cell in each component of $K\xi$. A zero-cell corresponds to a critical point of index 0 i.e. a local minimum for the function f. Thus we have:

<u>Prop. 7</u>: Assume K is connected. Then each orbit $K\xi$ intersects the chambre C_ξ in exactly one point, namely where f is minimum. Thus

$$C_\xi \xrightarrow{\sim} K\backslash \mathfrak{k} \simeq W\backslash \mathfrak{k}_\xi .$$

<u>Prop. 7'</u>: Let $\eta \in \mathfrak{k}_\xi$. Then

$$\eta \in C_\xi \iff |w\eta - \xi| > |\eta - \xi| \quad \text{all } w\eta \neq \eta.$$

Next, again assuming K connected, we see that because $K\eta$ has no one-cells, it is simply-connected. As $K/K_\eta \simeq K\eta$ this implies

<u>Prop. 8</u>. If K is connected, then the stabilizers K_η are connected. In particular K_ξ is a torus.

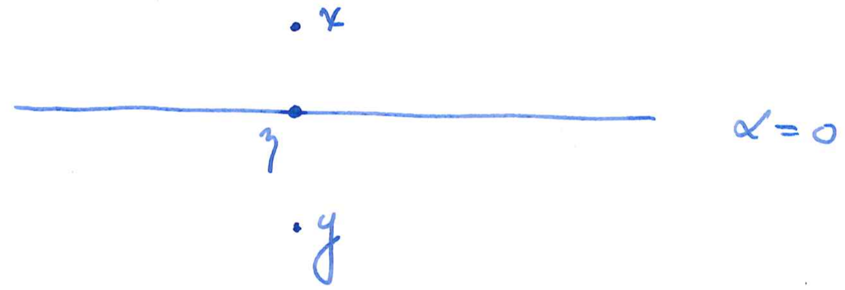
This result can be reformulated as follows

<u>Prop. 8'</u>: In a connected group K, the centralizer of a torus is connected.

Suppose from now on that K is connected. An element of k_ξ is called <u>regular</u> if it doesn't lie on any root hyperplane: $\alpha = 0$, $\alpha \in \Phi$. The closure of a component of the set of regular elements is called a <u>chambre</u>. If ξ' is any regular element, the chambre containing ξ' is $C_{\xi'} = \{x \mid \alpha(x) \geq 0 \text{ for all } \alpha \in \Phi \text{ with } \alpha(\xi') > 0\}$.

The Weyl group W carries Φ into itself, hence it permutes the chambres of k_ξ. Since any regular element can be moved by an elt. of W into C_ξ, W <u>acts transitively on the chambres</u>. In fact it acts simply-transitively, because C_ξ contains exactly one point of each W orbit, hence the stabilizer of C_ξ in W must be 1.

Consider next a root hyperplane $\alpha = 0$ in k_ξ and let η be a generic point of the wall (i.e. contained in no other root hyperplane). Let us form the line perpendicular to $\alpha = 0$ at η and let x, y be oppositely situated points:



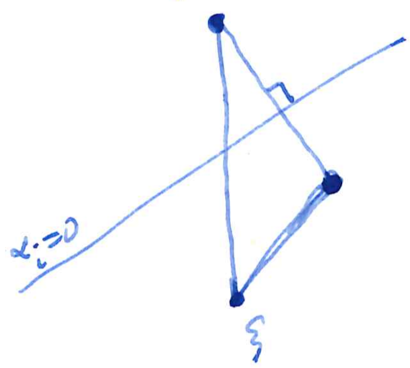
If x and y are close to η, then x and y are regular. Let s_α be the element of W such that $s_\alpha C_x = C_y$. Since $\eta, s_\alpha \eta \in C_y$ and C_y is a fundamental domain for the W-action, we have $s_\alpha(\eta) = \eta$. This will be true for all points of $C_x \cap C_y$ which contains an open set in $\alpha = 0$. Hence $s_\alpha = id$ on $\alpha = 0$, and we see that s_α is the reflection through the hyperplane $\alpha = 0$.

By a <u>wall</u> of a chambre C we mean a root hyperplane $\alpha = 0$ whose intersection with C contains an interior point of the hyperplane. (It comes to the same to say that $s_\alpha C \cap C = C \cap \{\alpha = 0\}$ is of codimension 1 in C.) Let $\alpha_1 = 0, \dots, \alpha_\ell = 0$ be the walls of the chambre C_ξ, and let W' be the subgroup of W generated by the reflections $s_{\alpha_1}, \dots, s_{\alpha_\ell}$. Put s_i for s_{α_i}.

I want to show $W' = W$. Given $w \in W$ we can choose $w' \in W'$ so that $w' w \xi$ is an element of $W' w \xi$ of minimum distance to ξ.

If $w'w\xi$ is not in C_ξ, then there is wall of C_ξ : $\alpha_i = 0$ separating ξ and $w'w\xi$. To see this choose a generic point ξ' of C_ξ so that the straight line joining ξ' to $w'w\xi$ avoid intersections of pairs of distinct root hyperplanes.

Then the first root hyperplane crossed in going from ξ' to $w'w\xi$ is a wall of C, because exactly one hyperplane passes through this point. So if $\alpha_i = 0$ is this wall in question $s_i w'w\xi$ would be closer to ξ:



contradicting choice of w'. Thus $w'w\xi \in C_\xi$, so $w'w = 1$ and $w \in W$!

Let's summarize the facts established so far: (Recall K is connected, $\xi \in k$ such that k_ξ is abelian, Φ = roots of k with respect to ξ, W = Weyl group of k_ξ).

<u>Prop. 9</u>: The Weyl group W acts simply-transitively on the chambres.

Recall that any chambre is a fund. domain for W (Prop. 7).

Prop. 11: For any root hyperplane $\alpha = 0$, the reflection s_α through this hyperplane belongs to W.

Prop. 12: W is generated by the reflections $s_1, \dots, s_\ell$ thru the walls of the fundamental chambre C_ξ.

Clarification: I am thinking always of the following situation: I have a Euclidean space E with a finite set of hyperplanes $\mathcal{H}$ acted on by a finite group W such that any chambre is a fundamental domain. Thus I wish momentarily to forget that $\mathcal{H}$ comes from a set Φ in E^*. So far I have established W is a reflection group. Conversely one can start with a reflection group and establish that any chambre is a fundamental domain (Borel's Notes). So what I am doing now is to develop the theory of reflection groups[*] in a special case. Notice that reflection groups are more general[*] than root systems, e.g. dihedral groups in $\mathbb{R}^2$ are not always root systems.

Next let's take up reduced decompositions in the envisaged situation.

Given an element of W written as a product of

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* It is more precise to say that a root system determines a reflection group. This "association" is neither 1-1 nor onto.

fundamental reflections

$$w = s_{i_1} \cdots s_{i_n}$$

I can associate a sequence of chambres

$$C_0 = C_\xi$$
$$C_1 = s_{i_1} C_\xi$$
$$\vdots$$
$$C_j = s_{i_1} \cdots s_{i_j} C_\xi$$

$$C_n = w\, C_\xi$$

such that C_{j-1}, C_j have a wall in common.
Such a sequence of chambres is called a <u>gallery</u>.
Conversely given a gallery $C_0, \dots, C_n$ starting with $C_0 =$ the fundamental chambre C_ξ, I get a unique sequence $s_{i_1}, \dots, s_{i_n}$ of fundamental reflections with $C_j = s_{i_1} \cdots s_{i_j} C_\xi$. In effect if I have $C_{j-1} = s_{i_1} \cdots s_{i_{j-1}} C_\xi$, then $C_\xi = s_{i_{j-1}} \cdots s_{i_1} C_{j-1}$ has a wall in common with $s_{i_{j-1}} \cdots s_{i_1} C_j$; hence if s_{i_j} is the reflection thru this wall, then $s_{i_{j-1}} \cdots s_{i_1} C_j = s_{i_j} C_\xi$.

Thus a sequence $s_{i_1}, \dots, s_{i_n}$ of fundamental reflections such that $w = s_{i_1} \cdots s_{i_n}$ is the same as a gallery starting with C_ξ and ending with $w\, C_\xi$.

Given a chambre $C = w C_\xi$ I consider those hyperplanes separating C and C_ξ: $\mathcal{H}_w = \{ W \in \mathcal{H} \mid W \text{ sep. } C_\xi, C_\xi \}$

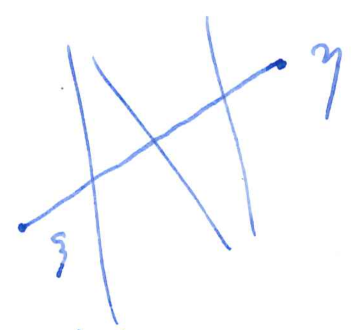
$= \{ W \in \mathcal{H} \mid \varphi(\xi) > 0,\ \varphi(w\xi) < 0 \quad \text{where } W = \operatorname{Ker}\varphi \}$.

Let $l_\xi(w)$ denote the number of these hyperplanes.

Suppose we have a decomposition $w = s_{i_1} \cdots s_{i_n}$, i.e. a gallery as above. As we pass from C_{j-1} to C_j we cross the hyperplane $s_{i_1} \cdots s_{i_{j-1}} W_{i_j}$ and no others. Thus we have

<u>Prop. 13</u>: If $w = s_{i_1} \cdots s_{i_n}$, then $n \geq l_\xi(w)$ = number of root hyperplanes separating C_ξ and wC_ξ.

On the other hand if we pick a generic point η in wC_ξ, the line joining ξ to η meets no intersection of distinct root hyperplanes.



Let $V_1, \dots, V_n$ be the root hyperplanes crossed as we go from ξ to η along this straight line. Then the V_i's are distinct and all all the root hyperplanes separating C_ξ and wC_ξ, so $n = l_\xi(w)$. Let C_j be the chambre containing the line segment of the line $\xi\eta$ between V_{j-1} and V_j. Then C_j is a gallery because V_j is a wall of both C_{j-1} and C_j.

Thus we get a decomposition $w = s_{i_1} \cdots s_{i_n}$ with $n = \ell_\xi(w)$.

<u>Prop 14</u>: If $n = \ell_\xi(w)$, there is a decomposition $w = s_{i_1} \cdots s_{i_n}$.

Such decompositions of w are called <u>reduced decompositions</u>. A decomposition is reduced iff the hyperplanes crossed, namely $s_{i_1} \cdots s_{i_{j-1}}(W_{i_j})$ are the hyperplanes separating C_ξ and wC_ξ without repetitions. So we should add the following to Prop. 13.

<u>Prop. 13</u>$^{\text{bis}}$: If $n = \ell_\xi(w)$, then the hyperplanes $s_{i_1} \cdots s_{i_{j-1}}(W_{i_j})$, $j = 1, \dots, n$ are distinct and they are exactly the hyperplanes separating C_ξ and wC_ξ.

Let us now consider a reduced decomp. $w = s_{i_1} \cdots s_{i_n}$ and fundamental reflection s_i. The chambres wC_ξ and ws_iC_ξ have the wall $w(W_i)$ in common. We have

$$\ell_\xi(ws_i) = \begin{cases} \ell(w) + 1 & \text{if } C_\xi, wC_\xi \text{ same side of } w(W_i) \\ \ell(w) - 1 & \text{if } C_\xi, wC_\xi \text{ opposite sides of } w(W_i) \end{cases}$$

In the latter case $w(W_i)$ separates C_ξ and wC_ξ hence by the above it is of the form $s_{i_1} \cdots s_{i_{j-1}}(W_{i_j})$ for some j. This means

$$w s_i w^{-1} = s_{i_1} \cdots s_{i_{j-1}} s_{i_j} s_{i_{j-1}} \cdots s_{i_1}$$

or

$$w s_i = s_{i_1} \cdots s_{i_{j-1}} s_{i_j} s_{i_{j-1}} \cdots s_{i_1} s_{i_1} \cdots s_{i_n}$$

$$s_{i_1} \cdots s_{i_{j+1}} s_{i_j} \cdots s_{i_n} s_i = s_{i_1} \cdots s_{i_{j-1}} \widehat{s_{i_j}} s_{i_{j+1}} \cdots s_{i_n}$$

and we have proved:

<u>Prop. 15</u>: (Exchange condition) If $w = s_{i_1} \cdots s_{i_n}$ is a reduced decomp., and s_i is a fund. reflection such that $\ell_\xi(w s_i) < \ell_\xi(w)$, then for some $j = 1, \ldots, n$ we have

$$s_{i_j} \cdots s_{i_n} s_i = s_{i_{j+1}} \cdots s_{i_n}$$

Easy consequence is that if $w = s_{i_1} \cdots s_{i_n}$ is any decomposition, then there is a reduced decomposition of the form $w = s_{i_{a_1}} \cdots s_{i_{a_m}}$ for some subset $\{a_1 < \cdots < a_m\}$ of $\{1, \ldots, n\}$.

Bernstein, Gelfand + Gelfand consider the following improvement. Suppose again $w = s_{i_1} \cdots s_{i_n}$ reduced, let V be any root hyperplane, and let s_V be the corresponding reflection. If V separates C_ξ and $w C_\xi$, then there is a unique

j such that $V = s_{i_1} \cdots s_{i_{j-1}}(W_{i_j})$ whence

$$s_V = s_{i_1} \cdots s_{i_{j-1}} s_{i_j} s_{i_{j-1}} \cdots s_{i_1}$$

and

$$s_V w = s_{i_1} \cdots s_{i_{j-1}} s_{i_{j+1}} \cdots s_{i_n} .$$

(Conversely $s_{i_1} \cdots \hat{s}_{i_j} \cdots s_{i_n}$ is obviously of the form $s_V w$.) Thus $l(s_V w) < l(w)$ if V separates C_ξ and $w C_\xi$. If V doesn't separate, then it does separate $s_V w C_\xi$ and C_ξ, so $l_\xi(s_V w) > l_\xi(s_V s_V w) = l_\xi(w)$. So

<u>Prop. 16</u>: For any root hyperplane V we have

$$l_\xi(s_V w) < l_\xi(w) \iff V \text{ separates } C_\xi, \, w C_\xi$$

$$l_\xi(s_V w) > l_\xi(w) \iff V \text{ doesn't sep. } C_\xi, \, w C_\xi$$

In the former case $s_V w = s_{i_1} \cdots \hat{s}_{i_j} \cdots s_{i_n}$ for a unique $j = 1, \ldots, n$, assuming $w = s_1 \cdots s_n$ is reduced.

Suppose again we have the situation of a finite group W acting on a Euclidean space E equipped with a set of "root" hyperplanes $\mathcal{H}$ such that a chambre C_ξ is a fundamental domain. For each V in $\mathcal{H}$ choose a vector e_V perpendicular to V such that $(e_V, \xi) > 0$ and $|e_V|^2 = 2$. Thus

$$s_V(x) = x - (e_V, x) e_V .$$

Call a V in $\mathcal{H}$ <u>simple</u> if e_V can not be written as $c_1 e_{V_1} + c_2 e_{V_2}$ with $V_1, V_2 \in \mathcal{H}$, and c_1, c_2 real nos. > 0. Because $\mathcal{H}$ is finite, every e_V can be expressed as a linear combination

$$e_V = c_1 e_{V_1} + \dots + c_m e_{V_m}$$

where e_{V_i} are simple and $c_i \geq 0$.

Let Σ be the set of simple $V \in \mathcal{H}$. Let $V_1, V_2 \in \Sigma$ and suppose $(e_{V_1}, e_{V_2}) > 0$, and that $V_1 \neq V_2$, whence e_{V_1}, e_{V_2} are independent. Since $\mathcal{H}$ is stable under W we know that

$$s_{V_2}(e_{V_1}) = e_{V_1} - (e_{V_1}, e_{V_2}) e_{V_2}$$

is of the form $c\, e_{V_3}$ with $c \neq 0$ and where $V_3 \in \mathcal{H}$ is different from V_1, V_2. If $c > 0$, then

$$e_{V_1} = c\, e_{V_3} + (e_{V_1}, e_{V_2}) e_{V_2}$$

contradicting $V_1 \in \Sigma$. If $c < 0$, then

$$(e_{V_1}, e_{V_2}) e_{V_2} = (-c) e_{V_3} + e_{V_1}$$

contradicting $V_2 \in \Sigma$. Thus we conclude $(e_{V_1}, e_{V_2}) \leq 0$

for $V_1 \neq V_2$ in Σ. (Could also reduce to 2 plane spanned by e_{V_1}, e_{V_2} - see Borel notes).

Suppose we try to show the $\{e_V, V \in \Sigma\}$ are linearly dependent. A relation between these can be written

$$\sum{}' c_V e_V = \sum c_{V'} e_{V'}$$

where $c_V, c_{V'} > 0$ and where V, V' run over disjoint subsets of Σ. If $\lambda = \sum c_V e_V$, then

$$(\lambda, \lambda) = \sum c_V c_{V'} (e_V, e_{V'}) \leq 0$$

hence $\lambda = 0$. But $0 = (\lambda, \xi) = \sum c_V (e_V, \xi)$ and all the $(e_V, \xi) > 0$ forces $c_V = 0$; similarly $c_{V'} = 0$ for all V'.

Let $V_1, \ldots, V_\ell$ be the elements in Σ. Then $e_{V_1}, \ldots, e_{V_\ell}$ are independent and any e_V is a linear combination of these with positive (≥ 0) coefficients. It follows that $C_\xi = \{x \mid (e_{V_i}, x) \geq 0\}$ and that the V_i are the walls of C_ξ.

Summarizing:

<u>Prop. 17</u>: For each V in $\mathcal{H}$ let e_V be the vector of length 2 perpendicular to V pointing in the direction of C_ξ (i.e. $(e_V, \xi) > 0$). Let $V_1, \dots, V_\ell$ be the walls of C_ξ. Then

i) $(e_{V_i}, e_{V_j}) \leq 0$ $\quad i \neq j$ (i.e. the angle between e_{V_i}, e_{V_j} is $\leq 90°$.

ii) the elements e_{V_i} are linearly independent.

iii) any e_V, $V \in \mathcal{H}$ is a linear combination with coefficients ≥ 0 of the e_{V_i}.

(Good proof: i) by reduction to $\mathbb{R}^2$; ii) as given; iii): First note that because V_i are the walls of C_ξ, $(e_{V_i}, x) \geq 0 \Rightarrow x \in C_\xi \Rightarrow (e_V, x) \geq 0$. In particular $(e_{V_i}, x) = 0 \ \forall i \Rightarrow (e_V, x) = 0 \Rightarrow e_V$ is a linear combination of the e_{V_i}. Now use ii) to get all coefficients ≥ 0.)

Let me now leave reflection groups and return to $K, \mathfrak{k}, \Phi$. Let α be a root in Φ^+, let $V_\alpha = \operatorname{Ker} \alpha$. ~~Consider the centralizer K_{V_α} which is connected (it is K_η where η is a generic point of V_α)~~ Let η be a point of V_α which is generic in the sense that it lies on no other root hyperplane.

The group K_η is connected and

$$\mathfrak{k}_\eta = \mathfrak{k}_\xi + \sum_{\substack{\beta \in \Phi^+ \\ \beta(\eta)=0}} \mathfrak{k}^\beta .$$

By the choice of η, $\beta(\eta)=0 \iff \beta$ is proportional to α. I am now going to review the proof that β has to $=\alpha$ and the $\mathfrak{k}^\alpha$ has dimension 2.

It's clear that the center of $\mathfrak{k}_\eta$ is V_α. If I divide out by the ~~connected~~ identity component of the center of K_η I then reach a group which has maximal abelian subspaces of dim 1.

So let's suppose K is a connected group with a maximal abelian subspace $\mathfrak{k}_\xi$ of $\mathfrak{k}$ of dimension 1. I suppose K non-abelian so that there is a root α and all other roots are proportional to α. Let $L \subset \mathfrak{k}^\alpha$ be a complex line ~~[illegible]~~ for the complex structure on $\mathfrak{k}^\alpha$. Then $[L,L]$ is a quotient of $\Lambda^2 L$ hence of real dimension 1, and as it is invariant under $\mathfrak{k}_\xi$, we must have $[L,L] \subset \mathfrak{k}_\xi$, in fact $[L,L] = \mathfrak{k}_\xi$ or $\mathfrak{k}$ would have abelian subspaces of dim. 2. Clearly $\mathfrak{k}_\xi \oplus L$ is a Lie subalgebra of $\mathfrak{k}$.

I assume now that one can easily identify $\mathfrak{k}_\xi \oplus L$ with su_2. Then $\mathfrak{k}$ becomes a su_2-module and is a sum of irreducibles whose structure one knows. In particular, one knows that the weights of such a module with respect to $\mathfrak{k}_\xi$ must be ~~[illegible]~~ of the form $k\frac{\alpha}{2}$ where k is an integer ≥ 0.

So the roots of k are integer multiples of $\frac{\alpha}{2}$, and α being an arbitrary root, one sees there are only two possibilities: a unique root α or two roots $\alpha, 2\alpha$. (I choose α to be the smallest root)

We also know that an irreducible su_2-module M with the weight $k\alpha$, k integral > 0, has a 1-dimensional 0 weight space, and that it is generated by M^α. Since k_ξ has dimension 1 this forces $\dim k^\alpha = 2$. But then $[k^\alpha, k^\alpha] \subset k_\xi$ and so k^α couldn't generate a $k^{2\alpha}$ if it were to exist.
Thus we obtain

Prop. 18: Let K be a connected (non-abelian) group of rank 1, i.e. $\dim k_\xi = 1$. Then $k = su_2$ and so $K = SU_2$ or $SU_2/\{\pm 1\}$.

Prop. 19: Let K be connected (non-abelian) such that k_ξ has a single root hyperplane. Then there is a unique root α, k^α is 2 dimensional and $k = k_\xi \oplus k^\alpha \simeq (\mathrm{Ker}\,\alpha) \oplus su_2$.

One should go on and classify rank 2 groups to have a reasonably complete theory. However at this point I wish to do the examples of the classical compact groups.

$K = SU(n)$, $\mathfrak{k}$ = skew-hermitian matrices of trace 0. A max. torus is given by diagonal matrices

$$T: \quad \begin{pmatrix} e^{i\theta_1} & & \\ & \ddots & \\ & & e^{i\theta_n} \end{pmatrix}$$

where $\theta_1 + \cdots + \theta_n = 0$. For each $1 \le i < j \le n$ we get a 2-diml subspace of $\mathfrak{k}$ consisting of

$$x E_{ij} - \bar{x} E_{ji} = \begin{pmatrix} \ddots & & x \\ & \ddots & \\ -\bar{x} & & \ddots \end{pmatrix}$$

x in the (i,j)-th position, $x \in \mathbb{C}$

which is stable under conjugation by diagonal matrices.

$$\begin{pmatrix} e^{i\theta_1} & \\ & e^{i\theta_n} \end{pmatrix} (x E_{ij} - \bar{x} E_{ji}) \begin{pmatrix} e^{-i\theta_1} & \\ & e^{-i\theta_n} \end{pmatrix}$$

$$= e^{i(\theta_i - \theta_j)} x E_{ij} - e^{i(\theta_j - \theta_i)} \bar{x} E_{ij}$$

Thus if we identify $\mathrm{Lie}(T)$ with $\{(\theta_1, \ldots, \theta_n) \mid \Sigma\, \theta_i = 0\}$, then the roots are $\theta_i - \theta_j$ $1 \le i, j \le n$, $i \ne j$. Take

$$\xi = \begin{pmatrix} i\xi_1 & \\ & i\xi_n \end{pmatrix} \qquad \xi_1 > \cdots > \xi_n$$

the positive roots are $\theta_i - \theta_j$ $\quad 1 \le i < j \le n$.

Reflection thru $\theta_i - \theta_j = 0$ is given by interchanging θ_i and θ_j. The Weyl group is Σ_n.

The chambre C_ξ consists of $(\theta_1, \ldots, \theta_n)$ such that $\theta_1 \geq \theta_2 \geq \cdots \geq \theta_n$. The simple roots are $\theta_1 - \theta_2, \ldots, \theta_{n-1} - \theta_n$. The angles between two adjacent simple roots is $\frac{2\pi}{3} = 120°$ for

$$\cos = \frac{(\theta_1-\theta_2)\cdot(\theta_2-\theta_3)}{|\theta_1-\theta_2|\,|\theta_2-\theta_3|} = \frac{-1}{2}$$

Thus the Dynkin diagram for this root system is

(Type A_{n-1}) $\quad \bullet\!-\!\!-\!\!-\bullet\!-\!\!-\!\!-\bullet\!-\!\!-\!\!- \cdots -\!\!-\!\!-\bullet \qquad n-1$ vertices

(Recall in the Dynkin diagram, the vertices are simple roots, one puts 0, 1, 2, 3 edges between vertices depending if the angle is 90°, 120°, 135°, 150°.
$\pi/2 \quad 2\pi/3 \quad 3\pi/4 \quad 5\pi/6$
Over the vertices, one puts scalars to indicate the relative squared lengths of the simple roots.)

$K = SO(2m)$. k consists of skew symmetric matrices. A maximal torus is

$$T: \begin{pmatrix} \begin{pmatrix} \cos\theta_1 & -\sin\theta_1 \\ +\sin\theta_1 & \cos\theta_1 \end{pmatrix} & & \\ & \ddots & \\ & & \begin{pmatrix} \cos\theta_m & -\sin\theta_m \\ +\sin\theta_m & \cos\theta_m \end{pmatrix} \end{pmatrix}$$

$$k_t = \mathrm{Lie}(T): \begin{pmatrix} \begin{pmatrix} & -\theta_1 \\ \theta_1 & \end{pmatrix} & & \\ & \ddots & \\ & & \begin{pmatrix} & -\theta_m \\ \theta_m & \end{pmatrix} \end{pmatrix}$$

To calculate the roots I can suppose $m=2$ whence $k\ominus k_\xi$ can be identified with $M_2(\mathbb{R}) = \mathrm{Hom}_{\mathbb{R}}(\mathbb{C},\mathbb{C})$ with $\begin{pmatrix} r_{\theta_1} & 0 \\ 0 & r_{\theta_2}\end{pmatrix}$ acting on A sending it to $r_{\theta_1} A r_{-\theta_2}$. Since $M_2(\mathbb{R}) = \mathbb{C}\cdot id \oplus \mathbb{C}\cdot\sigma$ we get the two roots $\theta_1 \pm \theta_2$. Thus the roots of $SO(2m)$ are $\pm\theta_i \pm \theta_j$, $i \neq j$. The Weyl group permutes the θ_i and changes θ_i into $-\theta_i$ with the requirement that an even number of signs be changed:

$$W \simeq \Sigma_m \ltimes (\mathbb{Z}/2\mathbb{Z})^{m-1}$$

Let ξ be an element in k_ξ with $\xi_1 > \cdots > \xi_m > 0$. Then C_ξ consists of $(\theta_1, \ldots, \theta_m)$ such that $\theta_i \pm \theta_j$ has the same sign as $\xi_i \pm \xi_j$. Thus C_ξ is described by

$$\theta_i \geq \theta_j \qquad 1 \leq i < j \leq m$$

$$\theta_i + \theta_j \geq 0 .$$

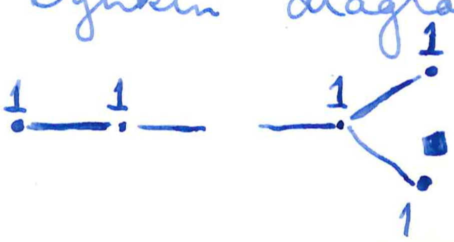
But in the presence of the first inequalities, one has $\theta_i + \theta_j \geq \theta_{m-1} + \theta_m$, so C_ξ is given by

$$\theta_1 \geq \theta_2 \geq \cdots \geq \theta_m \geq -\theta_{m-1}$$

the simple roots are

$$\theta_1 - \theta_2, \ \ldots, \ \theta_{m-1} - \theta_m, \ \theta_{m-1} + \theta_m$$

and the Dynkin diagram is



(m vertices)

(Type D_m)

$K = SO(2m+1)$. Same max. torus and the same roots $\pm\theta_i \pm \theta_j$ $1 \leq i,j \leq m$, but there are new roots corresponding to the extra row at the end. The critical case is $m = 1$.

$$\begin{pmatrix} 0 & & a \\ & 0 & b \\ -a & -b & 0 \end{pmatrix}$$

The new roots space may be identified with $\mathbb{C}$ with r_θ acting as $e^{i\theta}$. Thus the new roots are $\pm\theta$.

The roots of $SO(2m+1)$ are therefore

$$\pm\theta_i \pm \theta_j \qquad i \neq j$$

$$\pm\theta_i$$

The Weyl group permutes the θ_i and changes their signs:

$$W \simeq \Sigma_m \times (\mathbb{Z}/2\mathbb{Z})^m$$

A chambre is clearly $\theta_1 \geq \cdots \geq \theta_m \geq 0$. Thus the simple roots are

$$\theta_1 - \theta_2, \cdots, \theta_{m-1} - \theta_m, \theta_m$$

and the Dynkin diagram is

(m vertices)

(Type B_m).

$K = Sp(n)$. Recall $\mathbb{H} = \mathbb{C} + \mathbb{C}j$ where $jz = \bar{z}j$ and $j^2 = -1$. Identify $\mathbb{H}^n$ with $\mathbb{C}^{2n}$ using the basis $e_1, \dots, e_n, je_1, \dots, je_n$. If J is the linear operator on $\mathbb{C}^{2n}$ given by

$$\begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix},$$

and if σ is complex conjugation on $\mathbb{C}^{2n}$, then the operator on $\mathbb{C}^{2n}$ given by j-multiplication is clearly:

$$j = \sigma J.$$

$K = Sp(n)$ is the subgroup of $U(2n)$ commuting with j. It consists of $A = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$ such that $\sigma J A = A \sigma J$, i.e.

$$\cancel{\sigma J A = A J}$$

$$\cancel{J A J^{-1} = \sigma A}$$

$$\cancel{\begin{pmatrix} \alpha & -\beta \\ -\gamma & \delta \end{pmatrix} = \begin{pmatrix} \bar{\alpha} & \bar{\beta} \\ \bar{\gamma} & \bar{\delta} \end{pmatrix}}$$

~~i.e.~~ $\cancel{Sp(n) = \{(}$

$$\begin{pmatrix} -\bar{\gamma} & -\bar{\delta} \\ \bar{\alpha} & \bar{\beta} \end{pmatrix} = \sigma J A = A J = \begin{pmatrix} \beta & -\alpha \\ \delta & -\gamma \end{pmatrix}$$

$$Sp(n) \doteq \left\{ \begin{pmatrix} \alpha & \beta \\ -\bar{\beta} & \bar{\alpha} \end{pmatrix} \in U(2n) \right\}.$$

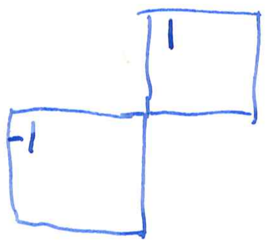
Thus $Sp(1) = SU(2)$.

The Lie alg. $\mathfrak{k}$ consists of $\begin{pmatrix} \alpha & \beta \\ -\bar{\beta} & \bar{\alpha} \end{pmatrix}$ which are skew hermitian $\Rightarrow$ α skew-herm, β symmetric.

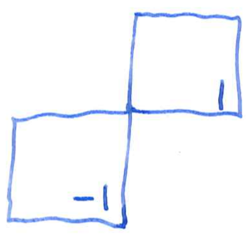
Maximal torus is

$$T: \begin{pmatrix} e^{i\theta_1} & & & & & \\ & \ddots & & & & \\ & & e^{i\theta_n} & & & \\ & & & e^{-i\theta_1} & & \\ & & & & \ddots & \\ & & & & & e^{-i\theta_n} \end{pmatrix}$$

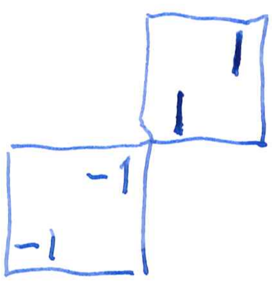
To determine the roots take $n=2$. The β-part in $\mathfrak{k}$ may be identified with symmetric complex 2×2 matrices



gives character $2\theta_1$

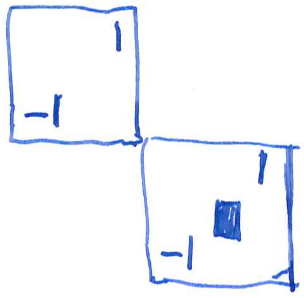


" " $2\theta_2$



gives character $\theta_1 + \theta_2$

Also we have the off diagonal α part:



gives $\theta_1 - \theta_2$

Thus the roots of Sp(n) are

$$\theta_i - \theta_j \qquad i \neq j$$
$$\pm(\theta_i + \theta_j) \qquad i \neq j$$
$$\pm 2\theta_i$$

The Weyl group permutes the θ_i and changes signs:

$$W = \Sigma_n \ltimes (\mathbb{Z}/2)^n$$

A chambre is $\theta_1 \geq \cdots \geq \theta_n \geq 0$, the simple roots are

$$\theta_1 - \theta_2, \cdots, \theta_{n-1} - \theta_n, 2\theta_n$$

and the Dynkin diagram is

(n vertices)

(Type C_n)