<u>September 7, 1975</u>: Discrete series + cusp forms.

In the following G will be a finite group like $GL_n(\mathbb{F}_q)$ so we have a building attached to G. I will be interested in finite dimensional complex representations of G.

Given $f \in C(G)$ $\quad (=$ functions $G \to \mathbb{C})$, one calls it a <u>cusp form</u> if for any parabolic $P < G$ one has

$$f_P(g) = \int_U f(gu)\,du = 0$$

where $U = P^u$. An irred. rep. V of G is said to be in the discrete series if all its matrix elements are cusp forms.

If $v \in V$, $\lambda \in V^*$ let $f(g) = \langle gv, \lambda \rangle$

$$f_P(g) = \int_U \langle guv, \lambda \rangle\,du = \langle g \int_U u \cdot v\,du\,,\, \lambda \rangle.$$

~~If $V^U \neq 0$ then taking $v \in V^U$ & λ suitable, this will be $\neq 0$, so V is not in the discrete series!~~

~~If V is not in the discrete series~~

Thus $V^U = 0 \iff f_P = 0$ all matrix elements f of V.

Consequently one has

<u>Assertion:</u> An irreducible repn V of G is in the discrete series $\Longleftrightarrow$ [redacted] for every parabolic $P \neq G$ one has $V^{U} = 0$ where $U = P^{u}$. (Enough to test this for the maximal parabolics, i.e. vertices of the building)

P, U as before, put $M = P/U$. $V^{U} = 0$ is the same as

$$\mathrm{Hom}_{P}\left(\mathbb{C}[P/U],\, V\right) = 0$$

since $\mathrm{Res}_{P \to M}(\mathbb{C}[M]) = \mathbb{C}[P, U]$, this means that

$\mathrm{Res}_{P \to G}(V)$ is orthogonal to $\mathrm{Res}_{P \to M}(W)$ for any repn. W of M because any irreducible representation is a direct factor of $\mathbb{C}[M]$. By Frobenius recip. we get.

<u>Assertion:</u> [redacted] An irreducible repn. V of G is in the discrete series iff its class [redacted] in $R(G)$ is orthogonal to the image of

$$(*) \qquad R(M) \xrightarrow{\text{res}} R(P) \xrightarrow{\text{ind}} R(G)$$

for any parabolic P of G, $P \neq G$.

Let V be an irreducible representation of G which is not in the discrete series. This means that there exists a parabolic $P \neq G$ an an irreducible repn. W of P/P^u such that V occurs in

$$\mathrm{Ind}_{P \to G}\ \mathrm{Res}_{P \to P/P^u}(W).$$

Let us assume P is minimal such that V "comes from" P/P^u in this sense. I want to show that W is in the discrete series of P/P^u.

If not, W would come from a parabolic of P/P^u. Such a parabolic is of the form Q/P^u where Q is a parabolic of G with $Q \subset P$; in fact $P^u \subset Q^u \subset Q \subset P$, and $Q/Q^u = (Q/P)/(Q/P)^u$.

$$\begin{tikzcd}
Q \arrow[r, hook] \arrow[d, two heads] \arrow[dr, phantom, "*"] & P \arrow[r, hook] \arrow[d, two heads] & G \\
Q/P^u \arrow[r, hook] \arrow[d, two heads] & P/P^u & \\
Q/Q^u & &
\end{tikzcd}$$

The square $(*)$ is OKAY for restriction + induction, so one sees that

$$\mathrm{Ind}_{Q\to G}\,\mathrm{Res}_{Q\to Q/Q^u} = \mathrm{Ind}_{P\to G}\,\mathrm{Res}_{P\to P/P^u}\,\mathrm{Ind}_{Q/P^u\hookrightarrow P/P^u}\,\mathrm{Res}_{Q/P^u\to Q/Q^u}$$

Thus V comes from a small parabolic if W is not in the discrete series. So we get

<u>Assertion</u>: If V is an irreducible representation of G, there exists a parabolic subgp. P of G and an irreducible discrete series representation W of P/P^u such that V occurs in $\mathrm{Ind}_{P\to G}\,{}'\mathrm{Res}_{P\to P/P^u} W$.

<u>Sept. 10, 1975:</u>

Let's review some facts about induction in the case of finite groups. Let $f: G \longrightarrow G'$ be a homomorphism, whence we have adjoint functors

$$\begin{array}{ccc} & \xrightarrow{\quad f_! \quad} & \\ \mathrm{Mod}_G & \xleftarrow{\quad f^* \quad} & \mathrm{Mod}_{G'} \\ & \xrightarrow[\quad f_* \quad]{} & \end{array}$$

Notice in the case of semi-simple $k[G]$-modules, the dimensions of $\mathrm{Hom}_G(X,Y)$ and $\mathrm{Hom}_G(Y,X)$ are the same, and this dimension is the intertwining number of X, Y, denoted $\langle x, y\rangle$ $x = cl(X)$ $y = cl(Y)$ in $R(G)$. From the adjunction formulas we see that $\langle x, f^* y \rangle = \langle f_! x, y \rangle$

$$\langle f^* y, x \rangle = \langle y, f_* x \rangle$$

for $x \in R(G)$, $y \in R(G')$, so $f_* = f_! = (f^*)^t$ on the $R(G)$ level.

In fact the functors $f_*, f_!$ for finite groups and semi-simple representations are isomorphic.

$$f_! X = \mathbb{Z}[G'] \otimes_{\mathbb{Z}[G]} X \;, \quad f_* X = \mathrm{Map}_G(G', X)$$

Treat separately the case where f is onto: $G' = G/N$ whence

$$f_! X = X_N \;, \quad f_* X = X^N$$

and by semi-simplicity $X^N \xrightarrow{\sim} X_N$. When f is injective $\mathrm{Map}_G(G', X)$ has a system of imprimitivity given by functions with support in each element of G/G', hence one gets an isom $f_! X \xrightarrow{\sim} f_* X$.

We have shown an irreducible repn. V is in the discrete series if for any parabolic $P < G$ and representation W of P/P^u, one has V orthogonal to $i_* j^* W$ where

$$P/P^u \xleftarrow{j} P \xrightarrow{i} G$$

This is completely equivalent to

$$j_* \imath^* V = 0$$

(i.e. $V^{P^u} = 0$).

Suppose we try now to calculate the intertwining number of two G-modules of the form $\imath_* j^* W$ where W is a discrete series repn. of P/P^u. Notation: $\Omega(P, W) = \imath_* j^* W$ where W is a repn. of P/P^u and

$$P/P^u \xleftarrow{j} P \xrightarrow{\imath} G$$

Use the Mackey formula

$$\langle \Omega(P_1, W_1), \Omega(P_2, W_2) \rangle_G$$

$$= \sum_{P_1 x P_2 \in P_1 \backslash G / P_2} \langle \mathrm{Res}_{P_1 \cap {}^xP_2 \hookrightarrow P_1} j_1^* W, \ \mathrm{Res}_{P_1 \cap {}^xP_2 \hookrightarrow {}^xP_2} {}^x(j_2^* W) \rangle_{P_1 \cap {}^xP_2}$$

Here ${}^xg = xgx^{-1}$; if $\theta : H \xrightarrow{\sim} H'$ and Z is a repn of H, I denote by θZ, the representation of H' given by the space Z with h' acting as $\theta^{-1} h'$.

So now what I want to compute is

(1) $$\langle \mathrm{Res}_{P_1\cap P_2 \to P_1} j_1^* W_1 , \mathrm{Res}_{P_1\cap P_2 \to P_2} j_2^* W_2 \rangle_{P_1 \cap P_2}$$

where P_1, P_2 are parabolic. Say that $P_1 = B_\xi$, $P_2 = B_\eta$.

Let's calculate the image of $P_1 \cap P_2$ in P_1/P_1^u. Recall B_ξ / B_ξ^u can be identified with G_ξ.

$$\mathfrak{b}_\xi \cap \mathfrak{b}_\eta = \mathfrak{h} + \sum_{\substack{\alpha(\xi) \geq 0 \\ \alpha(\eta) \geq 0}} \mathfrak{g}^\alpha \qquad \mathfrak{b}_\xi^u = \sum_{\alpha(\xi) > 0} \mathfrak{g}^\alpha$$

$$\mathfrak{g}_\xi = \mathfrak{h} + \sum_{\alpha(\xi) = 0} \mathfrak{g}^\alpha$$

Thus $\mathrm{Im}\{\mathfrak{b}_\xi \cap \mathfrak{b}_\eta \to \mathfrak{g}_\xi\} = \mathfrak{h} + \sum_{\substack{\alpha(\xi)=0 \\ \alpha(\eta)\geq 0}} \mathfrak{g}^\alpha$. So we get

<u>Lemma</u>: If we identify B_ξ / B_ξ^u with G_ξ, then the image of $B_\xi \cap B_\eta$ in B_ξ / B_ξ^u is $G_\xi \cap B_\eta$. More precisely

$$(B_\xi \cap B_\eta)\cdot B_\xi^u / B_\xi^u \xleftarrow{\sim} G_\xi \cap B_\eta$$

So with the new notation, the term (1) becomes:

(2) $$\left\langle \operatorname{Res}_{B_\xi \cap B_\eta \twoheadrightarrow G_\xi \cap B_\eta \subset G_\xi} W_1 \, , \, \operatorname{Res}_{B_\xi \cap B_\eta \twoheadrightarrow B_\xi \cap G_\eta \subset G_\eta} W_2 \right\rangle_{B_\xi \cap B_\eta}$$

Diagram

$$\begin{tikzcd}
B_\xi \cap B_\eta \arrow[dr, two heads] & & & G_\eta \\
& B_\xi \cap B_\eta / B_\xi^u \cap B_\eta^u \arrow[r, two heads, "p_1'"] \arrow[d, "p_2'"'] \arrow[dr, phantom, "\text{cart.}"] & B_\xi \cap G_\eta \arrow[ur, hook, "i_2"'] \arrow[d, "p_2"] & \\
& G_\xi \cap B_\eta \arrow[r, two heads, "p_1"'] \arrow[dl, hook, "i_1"] & G_\xi \cap G_\eta & \\
G_\xi & & &
\end{tikzcd}$$

Notice for a surjection $G \xrightarrow{p} G/N$ that

$$\langle p^* x, p^* y \rangle_G = \langle p_* p^* x, y \rangle_{G/N} = \langle x, y \rangle$$

because $$p_* p^* X = X^N = X, \text{ for } X \in \mathrm{Mod}_{G/N}.$$

(2) becomes

$$\langle p_2'^* i_1^* W_1, p_1'^* i_2^* W_2 \rangle$$

$$= \langle p_{1*}' p_2'^* i_1^* W_1, i_2^* W_2 \rangle$$

$$= \langle p_2^* p_{1*} i_1^* W_1, i_2^* W_2 \rangle$$

$$= \langle p_{1*} i_1^* W_1, \; p_{2*} i_2^* W_2 \rangle .$$

<u>Formula</u>: Let B_ξ, B_η be two parabolics of G and let $V_1 \in R(G_\xi)$, $V_2 \in R(G_\eta)$. Then the intertwining number of $\Omega(B_\xi, W_1)$ and $\Omega(B_\eta, W_2)$ is a sum taken over double cosets $B_\xi g B_\eta$. Recall that such a $B_\xi g B_\eta$ can be identified with a point of $W_\xi \backslash W / W_\eta$ i.e. with an orbit of W_ξ on the orbit $W\eta$. Let $w\eta \in W\eta$. The corresponding contribution to the sum is the intertwining number of two representations of $G_\xi \cap G_{w\eta}$, namely

$$(+) \quad \mathrm{Ind}_{G_\xi \cap B_{w\eta} \twoheadrightarrow G_\xi \cap G_\eta} \; \mathrm{Res}_{G_\xi \cap B_{w\eta} \subset G_\xi} \; V_1$$

$$\mathrm{Ind}_{B_\xi \cap G_{w\eta} \twoheadrightarrow G_\xi \cap G_{w\eta}} \; \mathrm{Res}_{B_\xi \cap G_{w\eta} \subset G_{w\eta}} \; {}^{w}V_2$$

Suppose next that the representation V_1 belongs to the discrete series of G_ξ. Then $(+)$ will be zero in $R(G_\xi \cap G_\eta)$ unless $G_\xi \cap B_{w\eta}$ (which is a parabolic in G_ξ) is all of G_ξ, i.e. $G_\xi \subset B_{w\eta}$, which means that the W_ξ orbit of $w\eta$ is a point.

Consequently the term (2) will be $\neq$ zero when V_1, V_2 are discrete series representations, iff $G_\xi = G_{w\eta}$ and $V_1 \cong {}^w V_2$. So

$\langle \Omega(B_\xi, V_1), \Omega(B_\eta, V_2) \rangle =$ number of $w\eta \in W\eta$ such that $G_\xi = G_{w\eta}$ and $V_1 \cong {}^w V_2$.

Corollary of this calcullation is

(4) $\quad G_\xi \cong G_\eta \implies \Omega(B_\xi, V) \cong \Omega(B_\eta, V)$.

because one can apply the following

<u>Lemma</u>: If $\langle V, W \rangle_G = \langle V, V \rangle_G = \langle W, W \rangle_G$, then V, W are isomorphic representations

<u>Proof</u>: Suppose $[V] = \sum m_\chi [\chi]$ where χ runs over the irred. reps., $W = \sum n_\chi \chi$. Then the hypothesis gives

$$\sum m_\chi n_\chi = \sum m_\chi^2 = \sum n_\chi^2$$

$$\implies \sum (m_\chi - n_\chi)^2 = 0 \implies m_\chi = n_\chi \text{ so } V \cong W.$$

I want a direct proof of (4).

Sept. 12, 1975:

Let P, Q be two parabolics having a common reductive factor M:

$$P = M \ltimes P^u \quad , \quad Q = M \ltimes Q^u$$

and let V be a representation of M. I want to prove that the two representations

$$\mathbb{Z}[G] \otimes_{\mathbb{Z}[P]} V \quad , \quad \mathbb{Z}[G] \otimes_{\mathbb{Z}[Q]} V$$

are isomorphic. To do this I must find a system of imprimity in $\mathbb{Z}[G] \otimes_{\mathbb{Z}[Q]} V$ with stability group P and repres. V.

Since Q^u acts trivially on V,

$$\mathbb{Z}[G] \otimes_{\mathbb{Z}[Q]} V = \mathbb{Z}[G/Q^u] \otimes_{\mathbb{Z}[M]} V$$

where M acts on G/Q^u by right mult.

$$gQ^u/Q^u \cdot x = gxQ^u/Q^u .$$

I want to find a subspace V' invariant under P and isomorphic to V. If V were irreducible as a rep. of M, it would be irred. over P, so the subspace V' sought for, would be in one of the factors of the decomposition:

$$\mathbb{Z}[G/Q^u] \otimes_{\mathbb{Z}[M]} V = \bigoplus_{\substack{PgQ^u/Q^u \\ \in P\backslash G/Q^u}} \mathbb{Z}[PgQ^u/Q^u] \otimes_{\mathbb{Z}[M]} V$$

So we look inside the factor corresponding to PQ^u/Q^u.

$$PQ^u/Q^u = M \times P^u Q^u/Q^u$$
$$\simeq M \times P^u/P^u \cap Q^u$$

Since M normalizes P^u, Q^u it is clear that M acts freely on the right of $PQ^u/Q^u \simeq P/P^u \cap Q^u$. So

$$\mathbb{Z}[PQ^u/Q^u] \otimes_{\mathbb{Z}[M]} V \cong \mathbb{Z}[P^u/P^u \cap Q^u] \otimes V$$

As a $P = M \times P^u$ module, the action is given by

$$(mu)(z \otimes v) = {}^m(uz) \otimes mv .$$

Next I need to associate to each element $v \in V$ an element $\varphi(v) \in \mathbb{Z}[P^u/P^u \cap Q^u] \otimes V$ invariant under P^u, i.e.

$$\varphi(v) = \sum_{x \in P^u/P^u \cap Q^u} x \otimes v$$

Let's try to show this works:
Then we have this map

$$\psi : V \longrightarrow \mathbb{Z}[G]\otimes_{\mathbb{Z}[Q]} V$$

$$\psi(v) = \sum_{x(P^u\cap Q^u)\in P^u/P^u\cap Q^u} x\otimes v$$

This is compatible with action of $P = P^u \rtimes M$
i.e.

$$\psi(mv) = \sum_{x\in P^u/P^u\cap Q^u} x\otimes mv = m\sum m^{-1}xm\otimes v$$
$$= m\psi(v)$$

$$u\psi(v) = \sum ux\otimes v = \sum x\otimes v = \psi(v)$$

Thus ψ induces a map

$$\mathbb{Z}[G]\otimes_P V \longrightarrow \mathbb{Z}[G]\otimes_Q V,$$

which I want to show is an isomorphism for all M-modules V. Enough to show for $V = \mathbb{Z}[M]$, so we want the induced map

$$\mathbb{Z}[G/P^u] \longrightarrow \mathbb{Z}[G/Q^u].$$

This is clearly the map given by the element

$$\sum_{xQ^u \in P^uQ^u/Q^u} xQ^u \quad \in \quad \mathbb{Z}[G/Q^u]^{P^u}$$

This is the characteristic function of the orbit $P^uQ^u/Q^u \subset G/Q^u$. To calculate the composition

$$\mathbb{Z}[G/P^u] \longrightarrow \mathbb{Z}[G/Q^u] \longrightarrow \mathbb{Z}[G/P^u]$$

$$\langle P^u \rangle \longmapsto \sum_{x \in P^u/P^u \cap Q^u} \langle xQ^u \rangle \longmapsto \sum_{x(P^u \cap Q^u) \in P^u/P^u \cap Q^u} x \sum_{y(P^u \cap Q^u) \in Q^u/P^u \cap Q^u} yP^u$$

$$x \in P^u/P^u \cap Q^u = B^u_\xi / B^u_\xi \cap B^u_\eta \cong B^u_\xi \cap B^u_{-\eta} \qquad \alpha(\xi) > 0, \quad \alpha(\eta) < 0$$

$$y \in Q^u/P^u \cap Q^u \cong B^u_{-\xi} \cap B^u_\eta . \qquad \alpha(\xi) < 0, \quad \alpha(\eta) > 0$$

xyP^u ranges over $\quad B^u_\xi \cap B^u_{-\eta} \cdot B^u_{-\xi} \cap B^u_\eta \cdot P^u/P^u$

Certainly this is not a multiple of $\langle P^u \rangle$ because of the positivity of the terms. So the the inverse of map $\mathbb{Z}[G/P^u] \to \mathbb{Z}[G/Q^u]$ is not the same sort of map.

It's clear now that I have to understand the Hecke algebra and similar things. Suppose G is a finite group, S and T are finite G-sets. Then

$$\mathrm{Hom}_G(\mathbb{Z}[T], \mathbb{Z}[S]) = \mathrm{Map}_G(T, \mathbb{Z}[S])$$

If $T = G/H$, this becomes

$$\mathrm{Map}_G(G/H, \mathbb{Z}[S]) = \mathbb{Z}[S]^H$$

and ~~the~~ one has an isomorphism $\mathbb{Z}[S]^H \cong \mathbb{Z}[H\backslash S]$ given by associating to an orbit Hs the sum of the elements of this orbit. Since $G\backslash(G/H \times S) = H\backslash S$, one ~~sees~~ gets an isom.

$$\mathrm{Hom}_G(\mathbb{Z}[T], \mathbb{Z}[S]) \cong \mathbb{Z}[G\backslash S \times T].$$

Perhaps a more straight-forward derivation would be to use the fact that $\mathbb{Z}[T]$ is isomorphic as a G-set to its dual $\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}[T], \mathbb{Z}) = \mathrm{Map}(T, \mathbb{Z})$, the isom being

$$t \longmapsto \delta_t \qquad \delta_t(x) = \begin{cases} 1 & x = t \\ 0 & x \neq t. \end{cases}$$

or

$$\sum a_t t \longmapsto \{t \mapsto a_t\}.$$

Thus

$$\operatorname{Hom}_G(\mathbb{Z}[T], \mathbb{Z}[S]) = \operatorname{Hom}_G(\mathbb{Z}[T], \operatorname{Hom}(\mathbb{Z}[S], \mathbb{Z}))$$
$$= \operatorname{Hom}_G(\mathbb{Z}[S\times T], \mathbb{Z})$$
$$= \operatorname{Hom}(\mathbb{Z}[G\backslash S\times T], \mathbb{Z})$$

Specifically we get a canonical map of $G\backslash S\times T$ into $\operatorname{Hom}_G(\mathbb{Z}[T], \mathbb{Z}[S])$ by associating to an orbit $\mathcal{O}$ the map

$$t \longmapsto \sum_{(s,t)\in\mathcal{O}} s$$

(this is the char. function of $\{s \mid (s,t)\in\mathcal{O}\}$.)

The traditional formulas identify $\mathbb{Z}[T]$ with $\operatorname{Map}(T, \mathbb{Z})$. In this case the formula becomes

$$\operatorname{Map}(G\backslash S\times T, \mathbb{Z}) \longrightarrow \operatorname{Hom}_G(\mathbb{Z}[T], \mathbb{Z}[S])$$
$$k(s,t) \longmapsto \left(f(t) \longmapsto \int_{t\in T} k(s,t) f(t)\,dt\right)$$

This is a G-map because

$$[g\cdot(k*f)](s) = (k*f)(g^{-1}s)$$
$$= \int_{t\in T} k(g^{-1}s, t) f(t)$$
$$= \int_{t\in T} k(g^{-1}s, g^{-1}t) f(g^{-1}t) = (k * gf)(s).$$

Suppose $T = G/P$, $S = G/Q$.

$$G\backslash(G/Q \times G/P) = \Delta G \backslash G \times G / Q \times P$$

But $\Delta G\backslash G \times G \xrightarrow{\sim} G$, $\quad g_1, g_2 \mapsto g_1^{-1} g_2$,

so

$$G\backslash(G/Q \times G/P) \cong Q\backslash G/P.$$

Then given $k : Q\backslash G/P \longrightarrow \mathbb{Z}$, $f : G/P \longrightarrow \mathbb{Z}$

we have

$$(k * f)(x) = \int_{y \in G/P} k(x^{-1}y) f(y).$$

Check: $\quad (k*f)(xq) = (k*f)(x)$

$$(k * gf)(x) = \int_{y \in G/P} k(x^{-1}y) f(g^{-1}y)$$

$$= \int_{y \in G/P} k(x^{-1}g \cdot g^{-1}y) f(g^{-1}y)$$

$$= (g \cdot k * f)(x).$$

In terms of δ-functions

$$(k * \delta_{yP})(x) = k(x^{-1}y)$$

$$(\delta_{QgP} * \delta_{yP})(x) = \delta_{QgP}(x^{-1}y) = \begin{cases} 1 & \text{if } x^{-1}yP \in QgP \\ 0 & \text{if } \quad \notin \end{cases}$$

Sept 13, 1975

Above formulas are somewhat complicated. To simplify let us work with

$$C(P\backslash G) = \mathrm{Maps}_P(G, \mathbb{Z})$$

where g acts as the transposes of right mult:

$$(gf)(x) = f(xg).$$

Next note that any $k \in C(Q\backslash G/P)$ gives rise to an operator

$$f \longmapsto k \underset{P}{*} f\,, \quad (k \underset{P}{*} f)(x) = \int_{y \in P\backslash G} k(xy^{-1}) f(y)$$

from $C(P\backslash G)$ to $C(Q\backslash G)$ which is compatible the G-operations:

$$[g(k*f)](x) = (k*f)(xg) = \int_{y \in P\backslash G} k(xgy^{-1}) f(y)$$

$$= \int_{y \in P\backslash G} k(xy^{-1}) f(yg) = k * gf.$$

<u>Assertion</u>: The map

$$C(Q\backslash G/P) \longrightarrow \mathrm{Hom}_G\big(C(P\backslash G), C(Q\backslash G)\big)$$
$$k \longmapsto (k \underset{P}{*}\ )$$

is an isomorphism.

<u>Proof</u>: Let S, T be finite right G-sets. Recall we make $C(T) = Map(T, \mathbb{Z})$ into a G-mod by $(gf)(t) = f(tg)$, ~~and $\mathbb{Z}[T]$ into a G-module~~ ~~by $g \cdot [t] = [tg^{-1}]$~~ and we make T into a left G-set by $g \cdot t = tg^{-1}$. Note that

$$\begin{array}{ccc} Hom(C(T), V) & = & Map(T, V) \\ \varphi & \longmapsto & (t \mapsto \varphi(\delta_t)) \\ \left(f \mapsto \sum f(t)\psi(t)\right) & \longleftarrow & \psi \end{array}$$

and these isomorphisms commute with G-action:

$$\begin{aligned} (t \mapsto (g\varphi)(\delta_t)) &= (t \mapsto g\varphi(g^{-1}\delta_t)) \\ &= (t \mapsto g^{-1}t \mapsto g\varphi(\delta_{g^{-1}t})) \\ &= g(t \mapsto \varphi(\delta_t)) \end{aligned}$$

whence we have

$$Hom_G(C(T), V) \simeq Map_G(T, V).$$

Apply this to $V = C(S)$ and we get

$$\begin{array}{ccccc} Hom_G(C(T), C(S)) & \simeq & Map_G(T, C(S)) & = & Map_G(S \times T, \mathbb{Z}) \\ \left(f \mapsto \sum_t f(t)(s \mapsto k(s,t))\right) & \longleftarrow & (t \mapsto (s \mapsto k(s,t))) & \longleftarrow & k \\ \| & & & & \\ f \mapsto k * f & & & & \end{array}$$

where $(k*f)(s) = \int_t k(s,t) f(t)$.

Thus we have established an isom.

$$\boxed{\begin{array}{l} \mathrm{Map}_G(S\times T, \mathbb{Z}) \xrightarrow{\ \sim\ } \mathrm{Hom}_G(C(T), C(S)) \\ \quad k \longmapsto (f \mapsto k*f) \\ \text{where } (k*f)(s) = \int_{t\in T} k(s,t)f(t) \end{array}}$$

Now take $S = Q\backslash G$, $T = P\backslash G$.

$$(Q\backslash G \times P\backslash G)_G = Q\times P\backslash G\times G/\Delta G \xrightarrow{\sim} Q\backslash G/P$$

$$(Qg_1, Pg_2)G \longrightarrow Qg_1g_2^{-1}P$$

so any G-invariant function on $Q\backslash G \times P\backslash G$ is of the form $(g_1,g_2) \longmapsto k(g_1g_2^{-1})$ where $k\in C(Q\backslash G/P)$. Therefore we get the assertion on pg 18.

Composition

$$C(R\backslash G/Q)\otimes C(Q\backslash G/P) \longrightarrow C(R\backslash G/P)$$

is given by $\qquad k_1, k_2 \longmapsto k_1 \underset{Q}{*} k_2$

$$(k_1 * k_2)(x) = \int_{y\in Q\backslash G} k_1(xy^{-1})\,k_2(y)$$

Alternative way of viewing the pairing

$$C(Q\backslash G/P) \otimes C(P\backslash G) \longrightarrow C(Q\backslash G)$$

has basis χ_{QgP} has bases $\chi_{Pg'}$

$$\left(\chi_{QgP} \overset{P}{*} \chi_{Pg'}\right)(x) = \int_{y \in P\backslash G} \chi_{QgP}(xy^{-1})\chi_{Pg'}(y)$$

$$= \chi_{QgP}(xg'^{-1}) = \begin{cases} 1 & \text{if } x \in QgPg' \\ 0 & \text{if } x \notin QgPg' \end{cases}$$

$$\chi_{QgP} * \chi_{Pg'} = \chi_{QgPg'}$$

More generally,

$$C(R\backslash G/Q) \otimes C(Q\backslash G/P) \longrightarrow C(R\backslash G/P)$$

$$\chi_{RgQ} \otimes \chi_{QhP} \longmapsto \chi_{RgQ} \overset{Q}{*} \chi_{QhR}$$

and $\left(\chi_{RgQ} \overset{Q}{*} \chi_{QhR}\right)(x) =$ number of elements of $RgQ \times^{Q} QhP$ over x.

I want now to calculate the algebra [redacted] $\mathrm{End}(\mathbb{Z}[G/B])$. We have seen this is $C(B\backslash G/B)$ with $\overset{B}{*}$ product. This algebra has a $\mathbb{Z}$-basis consisting of [redacted] χ_{BgB} and by the Bruhat lemma

$$B\backslash G/B \xleftarrow{\sim} W$$

it has the basis χ_{BwB} indexed by W. We know that if $w = s_1 \cdots s_n$, $n = \ell(w)$ is a reduced decomposition, then

$$BwB = Bs_1B \times^B \cdots \times^B Bs_nB$$

and so

$$\chi_{BwB} = e_{s_1} \cdots e_{s_n} \quad \text{in } C(B\backslash G/B)$$

where we put $e_s = \chi_{BsB}$. Thus the e_s generate $C(B\backslash G/B)$.

I want next the relations. I know

$$BsB \cdot BsB = B \cup BsB$$

so $e_s^2 = ae_s + b$ where the constants a, b have to be determined. Note that there is an augmentation

$$C(B\backslash G/B) \longrightarrow \mathbb{Z}$$

[redacted]

$$k \longmapsto \int_{x \in B\backslash G} k(x)$$

which is a homomorphism since

$$\int_{x\in B\backslash G} k_1 * k_2(x) = \int_{x\in B\backslash G} \int_{y\in B\backslash G} k_1(xy^{-1})\, k_2(y)$$

$$= \int_{y\in B\backslash G} k_2(y) \int_{x\in B\backslash G} k_1(xy^{-1})$$

$$= \int_{y\in B\backslash G} k_2(y) \int_{x\in B\backslash G} k_1(x).$$

$$\int_{x\in B\backslash G} \chi_{BsB}(x) = \operatorname{card} B\backslash BsB$$

$$= q$$

when we are in the finite field $G = GL_n(\mathbb{F}_q)$ situation. Thus from $e_s^2 = ae_s + b$ we will get by applying the augmentation

$$q^2 = aq + b$$

(which leads one to expect $a = q-1,\ b = q$).

To compute a, b recall that $B_s/B = (BsB \cup B)/B$ can be identified with $\mathbb{P}^1$, and

$$B_s \times^B B_s/B$$

can be identified with pairs of points z_1, z_2 in

$\mathbb{P}^1$, (corresponding to the gallery ∞, z_1, z_2). $BsB \times^B BsB/B$ can be identified with pairs (z_1, z_2) such that $z_1 \neq \infty$ and $z_2 \neq z_1$. The map

$$BsB \times^B BsB/B \longrightarrow B_s/B$$

takes (z_1, z_2) to z_2. The point $z_2 = \infty$ can be obtained in 1 way for each $z_1 \in \mathbb{P}^1 - \{\infty\}$, i.e. q ways, so $b = q$. A point $z_2 \neq 0$, can be obtained from $z_1 \in \mathbb{P}^1 - \{\infty\} - \{z_2\}$, i.e. $(q-1)$ ways, so $a = q-1$. Thus

$$\boxed{e_s^2 = (q-1)e_s + q}$$

Next we want the relation between e_s and $e_{s'}$ corresponding to the relation $(ss')^m = 0$. Background:

The algebra $C(B\backslash G/B)$ has the $\mathbb{Z}$-basis χ_{BwB} and so it has an increasing filtration given by the length of w. This means

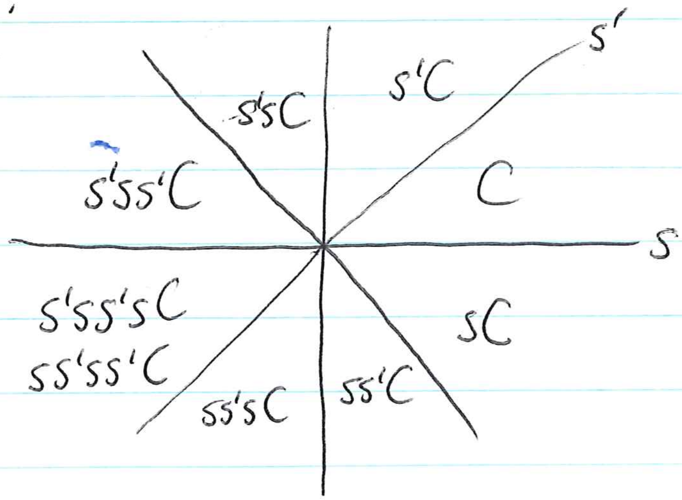
$$F_p\, C(B\backslash G/B) = \sum_{l(w) \leq p} \mathbb{Z} \cdot \chi_{BwB}.$$

In view of the fact that

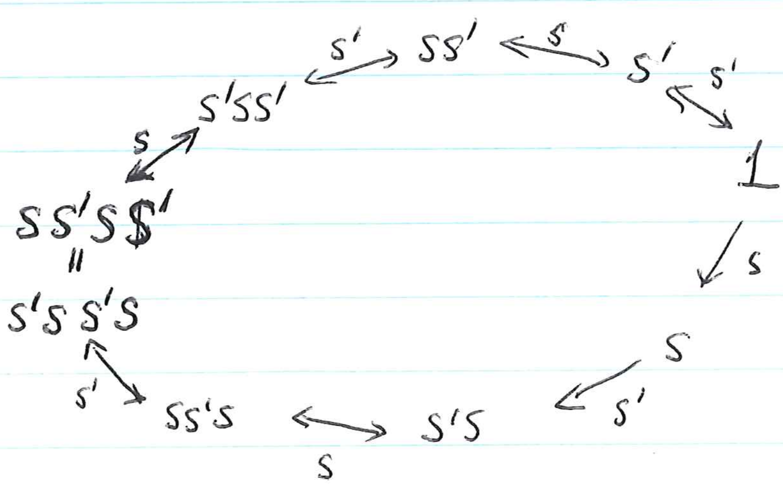
$$\chi_{BwB} \cdot \chi_{Bw'B} = \sum_{l(w'') \leq l(w) + l(w')} \chi_{Bw''B}$$

one does get an algebra filtration. Moreover one has $\chi_{BwB} \cdot \chi_{Bw'B} = \chi_{Bww'B}$ if $\ell(ww') = \ell(w) + \ell(w')$.

Consider the dihedral group generated by s, s' with the relations $s^2 = s'^2 = 1 = (ss')^m$. Take $m = 4$:



The reduced words together with multiplication by s, s' are represented:



It is clear that this group is described completely by the relations $s^2 = s'^2 = 1$ and

$$ss'ss' = s'ss's$$

which is another version of $(ss')^4 = 1$. In general, the group will be described by the relations

$$s^2 = s'^2 = 1$$

and $$\underbrace{ss's\cdots\cdots}_{m \text{ factors}} = \underbrace{s'ss'\cdots}_{m\text{-factors}}$$

It is clear that the missing relation between e_s and $e_{s'}$ is

$$\boxed{\underbrace{e_s e_{s'} e_s \cdots\cdots}_{m\text{-factors}} = \underbrace{e_{s'} e_s e_{s'} \cdots\cdots}_{m\text{-factors}}}$$

because if I take this relation along with $e_s^2 = (q-1)e_s + q$, and also for s', the algebra generated will have the right basis.

<u>Assertion:</u> $C(B|G/B)$ is isomorphic modulo $q-1$ to the group algebra $\mathbb{Z}[W]$.

This is clear because of the fact W has generators s_i subject to relations $s_i^2 = 1$, $(s_i s_j)^{m_{ij}} = 1$.

Let's return to the problem of showing that $C(Q|G)$ and ~~$C(Q\backslash P)$ are~~ $C(P\backslash G)$ are isomorphic G modules, when P, Q are parabolics with a common reductive part M:

$$P = M \ltimes P^u, \quad Q = M \ltimes Q^u$$

My idea was to use the operator

$$\begin{aligned} C(P\backslash G) &\longrightarrow C(Q\backslash G) \\ f &\longmapsto k \overset{P}{*} f \end{aligned}$$

where $k = \chi_{QP}$, and show this map is an isom.

Consider the case where $P = B$ and $Q = wBw^{-1}$ with $w \in W$. Then we have

$$C(B\backslash G) \longrightarrow C(wBw^{-1}\backslash G) \simeq C(B\backslash G)$$

$$f \longmapsto (g \mapsto f(wg))$$

$$f \longmapsto \int_{B\backslash G} \chi_{QP}(xy^{-1}) f(y) \mapsto \int_{B\backslash G} \chi_{QP}(wxy^{-1}) f(y)$$

$$\chi_{QP}(wx) = \begin{cases} 1 & wx \in wBw^{-1}B \\ 0 & \notin \end{cases}$$

$$= \chi_{Bw^{-1}B}(x).$$

So now I want to see if the operator of convolution by $e_{w^{-1}} = \chi_{Bw^{-1}B}$ on $C(B\backslash G)$ is an isomorphism.

As $e_{w^{-1}}$ has augmentation $q^{l(w)}$, it is clear that $e_{w^{-1}}$ is a unit in $C(B\backslash G/B)$ iff q is invertible (in the ground ring). Conversely as

$$e_s^2 - (q-1)e_s = e_s(e_s - q+1) = q,$$

if q is invertible, then e_s is a unit. This implies that $e_{w^{-1}}$, which is the product of $l(w)$ e_s's, is also a unit.

<u>September 14, 1975</u>

Now that I have understand the isomorphism $e_w : C(B\backslash G) \longrightarrow C(B\backslash G)$, it would be nice to understand the situation on the level of $C(B^u\backslash G)$.

To be more direct, I would like to show that given two parabolics P, Q with common reductive factor M, one has an isomorphism

$$C(P^u\backslash G) \xrightarrow{\sim} C(Q^u\backslash G)$$

given by $\chi_{QP^u} * (?)$. It should be possible to understand the case $P = B$, $Q = w^{-1}Bw$ easily. But

$$C(B^u \backslash G) \longrightarrow C(w^{-1}B^u w \backslash G) \simeq C(B^u \backslash G)$$

$$B^u g \longmapsto \overbrace{w^{-1}B^u w}^{Q^u} \overbrace{B^u}^{P^u} g \xmapsto{B^u w} B^u w B^u g$$

is multiplication by $\chi_{B^u w B^u}$. Thus it seems desirable to calculate the Hecke algebra $C(B^u \backslash G / B^u)$.

Assertion: $B^u \backslash G / B^u \simeq N$ (= normalizer of T).

We know G breaks up into double cosets BwB indexed by $w \in W$. Put $P = B$, $Q = w^{-1}Bw$ so that P, Q are parabolics with common reductive factor $M = T$. Then

$$QP/P^u = QP^u/P^u = Q/Q \cap P^u$$

$$\Longrightarrow \quad Q^u \backslash QP/P^u = Q^u \backslash Q/Q \cap P^u = Q^u \backslash Q = M$$

because $Q \cap P^u \subset Q^u \triangleleft Q$. Thus

$$w^{-1}B^u w \backslash w^{-1}BwB/B^u = T$$

i.e. $\quad B^u \backslash BwB/B^u \xleftarrow{\sim} wT \subset N$. The above assertion follows easily.

So we know now that $C(B^u|G/B^u)$ has a basis $e_n = \chi_{B^u n B}$ indexed by $n \in N$. I want the relations giving the multiplicative structure.

Assume now to simplify that $N = W \ltimes T$ as in GL_n. Even without this assumption consider the elements e_t $t \in T$. e_t belongs to the coset $B^u t B^u = tB^u$ and one has

$$B^u t B^u \times^{B^u} B^u t' B^u = tB^u \times^{B^u} t'B^u = tt'B^u$$

Thus $e_t e_{t'} = e_{tt'}$ and so the subalgebra of $C(B^u|G/B^u)$ generated by the e_t is isomorphic to the group algebra of T. Note also that

$$B^u n B^u \times^{B^u} B^u t B^u = B^u n t B^u$$

so that we have

$$e_t e_n = e_{tn}, \quad e_n e_t = e_{nt}.$$

Let's now calculate e_s^2 in GL_2 where $s = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. One has $B^u = \begin{pmatrix} 1 & * \\ & 1 \end{pmatrix}$ and $B^u s B^u \simeq B^u \times B^u$. Thus

$$B^u s B^u \times^{B^u} B^u s B^u \simeq B^u \times sB^u s \times B^u$$

and we want the multiplication map of this to G, and to calculate this fibres of this map.

$$\begin{pmatrix} 1 & x \\ & 1 \end{pmatrix}\begin{pmatrix} 1 & \\ y & 1 \end{pmatrix}\begin{pmatrix} 1 & z \\ & 1 \end{pmatrix} = \begin{pmatrix} 1+xy & x \\ y & 1 \end{pmatrix}\begin{pmatrix} 1 & z \\ & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1+xy & z+xyz+x \\ y & yz+1 \end{pmatrix}$$

If $= \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, then $ad-bc=1$

and

$c \neq 0 \Rightarrow \exists$ unique solution for x, y, z

$c=0,\ a=1,\ d=1 \Rightarrow \exists\ q$ solutions for x, y, z

otherwise no solutions.

Now $c \neq 0$ describes elements in

$$B^u\, s\, T^{1} B^u$$

and those with determinant 1 means the diagonal matrix has determinant 1. So the formula I want seems to be

$$e^2_{\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}} = \sum_{a\in\mathbb{F}_q^*} e_{\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}} e_{\begin{pmatrix} a & \\ & a^{-1} \end{pmatrix}} + q$$

$$= \sum_{a\in\mathbb{F}_q^*} e_{\begin{pmatrix} a & \\ & a^{-1} \end{pmatrix}} e_{\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}} + q$$

So we get

$$e_{\left(\begin{smallmatrix}0&1\\1&0\end{smallmatrix}\right)}^2 = \alpha\, e_{\left(\begin{smallmatrix}0&1\\1&0\end{smallmatrix}\right)} + q$$

where α commutes with $e_{\left(\begin{smallmatrix}0&1\\1&0\end{smallmatrix}\right)}$, so if q is invertible, $e_{\left(\begin{smallmatrix}0&1\\1&0\end{smallmatrix}\right)}$ is a unit.

Suppose we try this more generally. Let $\tilde s \in N$ map onto $s \in W$. To compute $e_{\tilde s} e_{\tilde s^{-1}}$ we need the fibres of the mult. map

$$B^u \tilde s B^u \times^{B^u} B^u \tilde s^{-1} B^u \longrightarrow G$$

Now if α is the root of B such that $s_\alpha = s$, then the corresponding 1-parameter subgp U_α is such that

$$\begin{aligned} B^u \tilde s B^u &= B^u \times^{B^u \cap \tilde s B^u \tilde s^{-1}} \tilde s B^u \\ &= U_\alpha \times (B^u \cap {}^{\tilde s}B^u) \times^{(B^u \cap {}^{\tilde s}B^u)} \tilde s B^u \\ &\cong U_\alpha \times \tilde s B^u . \end{aligned}$$

Thus

$$\begin{aligned} B^u \tilde s B^u \times^{B^u} B^u \tilde s^{-1} B^u &\cong U_\alpha \times \tilde s U_\alpha \tilde s^{-1} \times B^u \\ &\cong U_\alpha \times U_{-\alpha} \times B^u \end{aligned}$$

and so we need the fibres of the multiplication map $U_\alpha \times U_{-\alpha} \times B^u \longrightarrow G$.

Suppose to start with that [redacted] $G = SL_2$. I have seen that the map

$$U_\alpha \times U_{-\alpha} \times U_\alpha \longrightarrow G$$

over [redacted] $B^u \tilde{s} B = \bigcup_{t \in T} B^u \tilde{s} t B^u$ is an isomorphism, whereas over U_α it is q-to-1.

So notice that for $G = SL_2$

$$U_\alpha \times U_{-\alpha} - \{1\} \times U_\alpha \hookrightarrow G$$

In the case of PSL_2 this will be a double covering of its image, I think.

<u>Lemma:</u> For SL_2 one has

[redacted]

$$B^u \times (B^u_- - \{1\}) \times B^u \xrightarrow{\sim} BsB$$

and for PSL_2 this map is a double covering.

In general we let G_α be the [redacted] subgroup generated by U_α, $U_{-\alpha}$ in G. For a simply-connected group one knows $G_\alpha = SL_2$; [redacted] this is clear for GL_n and SL_n. [redacted]

$$U_\alpha \times U_{-\alpha}^{-\{1\}} \times B^u = U_\alpha \times U_{-\alpha}^{-\{1\}} \times U_\alpha \times (B^u \cap {}^s B^u)$$

$$\xrightarrow{\sim} U_\alpha \times \tilde{s}^{-1}\theta_\alpha(\mathbb{F}_q^*) \times U_\alpha \times (B^u \cap \tilde{s} B^u \tilde{s}^{-1})$$

$$\xrightarrow{\sim} U_\alpha \times \tilde{s}^{-1}\theta_\alpha(\mathbb{F}_q^*) \times B^u$$

$$\simeq \theta_\alpha(\mathbb{F}_q^*) \times U_\alpha \tilde{s}^{-1} B^u \simeq \theta_\alpha(\mathbb{F}_q^*) \times B^u \tilde{s}^{-1} B^u.$$

Here θ_α is the 1-parameter subgroup $\mathbb{G}_m \longrightarrow T$ corresponding to coroot vector H_α. So this gives the formula

$$\boxed{e_{\tilde{s}}\, e_{\tilde{s}^{-1}} = \sum_{x \in \mathbb{F}_q^*} e_{\theta_\alpha(x)}\, e_{\tilde{s}^{-1}} + q}$$

It would seem this formula holds in general.

Granted this we see that with q invertible the element $e_{\tilde{s}}$ and hence e_n are invertible in $C(B^u\backslash G)$ for all $n \in N$.

So the next thing is to go back and try to work out the Q, P situation.

Let's begin by describing the algebra $C(P\backslash G/P)$. We have a map $C(P\backslash G/P) \hookrightarrow C(B\backslash G/B)$ which views a P-biinvariant function as a B-biinvariant one. This map is not an algebra homomorphism because

$$k_1 \overset{P}{*} k_2 = \int_{y \in P \backslash G} k_1(xy^{-1}) k_2(y) = \frac{|B|}{|P|} k_1 \overset{B}{*} k_2$$

rel. prime to q

So it is clear that working modulo torsion an element of $C(P\backslash G/P)$ is a unit provided it is a unit in $C(B\backslash G/B)$.

Recall

$$B_\xi \backslash G / B_\xi \xrightarrow{\sim} W_\xi \backslash W / W_\xi$$

September 17, 1975

Summary. Let $G = GL_n(\mathbb{F}_q)$ to fix the ideas, and let P, Q be two parabolics in G having common reductive factor M: $P = M \ltimes P^u$, $Q = M \ltimes Q^u$. Let W be a repn. of M. One knows by intertwining number calculations that the induced repn. $\Omega(P, W) = \mathrm{Ind}_{P\to G}\, \mathrm{Res}_{P\to P/P^u \simeq M} W$ is isomorphic to $\Omega(Q, W)$.

One gets a candidate for such an isomorphism as follows. Recall

$$\Omega(P, W) = \mathrm{Map}_P(G, W) = \{f: G\to W \mid f(px) = \bar{p}f(x)\} \quad \bar{p} = \text{image of } p \text{ in } M.$$

$$\cong \mathrm{Map}_M(P^u|G, W).$$

Let $\chi_{Q^uP^u}$ be the characteristic function of Q^uP^u in G. Then for $f \in \Omega(P, W)$

$$\left(\chi_{Q^uP^u} \overset{P^u}{*} f\right)(x) = \int_{y \in P^u|G} \chi_{Q^uP^u}(xy^{-1})\, f(y)$$

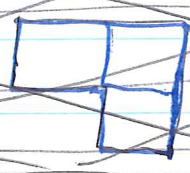
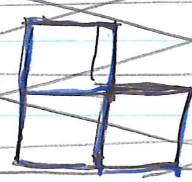
$$= \int_{y \in P^uQ^ux} f(y)$$

is in $\Omega(Q, W)$: $\displaystyle\int_{y\in P^uQ^umx} f(y) = \int_{y \in P^uQ^ux} f(my) = m\int_{y\in P^uQ^ux} f(y)$

The conjecture is that this map given by $\chi_{Q^uP^u} *$ is an[redacted] isomorphism of $\Omega(P,W)$ with $\Omega(Q,W)$.

I have checked this conjecture where $P = B$, $Q = {}^{w}B$ and found that the map is an isomorphism "over $\mathbb{Z}[\frac{1}{q}]$".

~~Go on to the cases of $G = GL_n$ with P, Q, M as follows~~

~~$P =$~~  , ~~$Q =$~~  , ~~$P \cap Q = M$~~

Another method for showing $\Omega(P,W)$ and $\Omega(Q,W)$ are isomorphic is to compute characters.

Recall that if $i : H \hookrightarrow G$, and if χ is the character of a repn. W of H, then the character $i_*\chi$ of i_*W is

$$i_*(\chi)(g) = \sum_{\substack{xH \in (G/H)^g \\ (\text{i.e. } x^{-1}gx \in H)}}' \chi(x^{-1}gx)$$

Thus the character of $\Omega(P,W)$ at g will be a sum over fixpts. $(G/P)^g$, i.e. flags left fixed by

g of type P, of terms giving the W-character of the effect of g on the quotients of the flag.

For example, let P be the parabolic associated to $0 < L_1 < \cdots < L_p = V$ with $\dim L_i = n_1 + \cdots + n_i$. Let c_i denote a conjugacy class in GL_{n_i} and c a conjugacy class in GL_n. Fix $x \in C$. Let

$$g^{c}_{c_1, \ldots, c_p}$$

the number of flags $0 < L_1 < \cdots < L_p = V$ normalized by x such that the effect of x on L_i/L_{i-1} belongs to c_i. If W is the representation of $P/P^u \cong GL_{n_1} \times \cdots \times GL_{n_p}$ with character $\chi_1 \otimes \cdots \otimes \chi_p$, then

$$\mathrm{ch} \text{ of } \Omega(P,W)(c) = \sum g^{c}_{c_1 \cdots c_p} \, \chi_1(c_1) \cdots \chi_p(c_p)$$

where the sum is taken over all $c_1, \ldots, c_p$.

So the isomorphism $\Omega(P,W) \simeq \Omega(Q,W)$ will result formally from knowing that if $n = p+q$ and if c is a class in GL_n, c_1 in GL_p, c_2 in GL_q, then

$$g^{c}_{c_1, c_2} = g^{c}_{c_2, c_1}$$

To prove this we want to replace a matrix x

by its transpose. Precisely: $g^c_{c_1,c_2}$ is the number of subspaces $W \subset V$ normalized by x such that $x|W \simeq x_1$, $x|(V/W) \simeq x_2$ (here x, x_1, x_2 are fixed elts of c, c_1, c_2 resp.). By duality this is the same as the number of subspaces $L \subset V^*$ normalized by ${}^t x$ such ${}^t x|L \simeq {}^t x_2$, ${}^t x|(V^*/L) \simeq x_1^t$. Thus one has by duality

$$g^c_{c_1,c_2} = g^{{}^t c}_{{}^t c_2, {}^t c_1}.$$

where ${}^t c$ is the image of c under the transpose map on GL_n. However calculation with Jordan forms shows that ${}^t c = c$.

Green's algebra: Put $GL_n(\mathbb{F}_q) = G_n$ and let

$$R = \bigoplus_{n \geq 0} R(G_n)$$

with the following algebra structure. Given $\chi_i \in G_{n_i}$

$$\chi_i \cdot \chi_j = \operatorname{Ind}_{\begin{pmatrix} G_{n_1} & * \\ & G_{n_2} \end{pmatrix} \hookrightarrow G_{n_1+n_2}} \operatorname{Res}_{\begin{pmatrix} \ & \ \\ & \ \end{pmatrix} \to G_{n_1} \times G_{n_2}} \chi_1 \otimes \chi_2$$

i.e. by what we've seen above

$$(\chi_i \cdot \chi_j)(c) = \sum g^c_{c_1,c_2} \chi_1(c_1) \chi_2(c_2)$$

This algebra is commutative as $g^{c}_{c_1, c_2} = g^{c}_{c_2, c_1}$, and associative as can be seen easily.

What is the decomposable subspace of $R(G_n)$? One takes a partition $n = n_1 + \cdots + n_p$ and considers

$$R(G_{n_1}) \otimes \cdots \otimes R(G_{n_p}) \longrightarrow R(G_n)$$

The image consists of representations $\mathrm{Ind}_{P \to G}\, \mathrm{Res}_{P \to P/P^u} W$ as P runs over the proper parabolics of G. The decomposable space of $R(G_n)$ contains the discrete series representations, but could conceivably be larger. In fact it should be clear that as soon as $\Omega(P, \omega)$, ω discrete series rep. of P/P^u, fails to be irreducible, then indecomposables exceed the discrete series repns.

September 19, 1975.

According to Springer's paper pg C-12, Iwahori + Tits have proved $C(B|G/B) \simeq \mathbb{C}[W]$. For SL_2 I know $C(B\backslash G/B)$ is

$$\mathbb{C}[e_s] \big/ e_s^2 - (q-1)e_s - q$$

$$\shortparallel$$

$$(e_s - q)(e_s + 1).$$

which is isomorphic to $\mathbb{C} \times \mathbb{C}$ as a $\mathbb{C}$-algebra, which

is in turn isomorphic to the grp. alg. $\mathbb{C}[s]/(s^2-1)$.
An explicit isomorphism is given by

$$s \longmapsto \frac{2e_s}{q+1} - \frac{q-1}{q+1} \qquad \text{or} - \text{this}$$

In the general case, I [redacted] would need to know this formula is compatible with the relations

$$\underbrace{(ss'\cdots\cdots) = (s's\cdots\cdots)}_{m_{ss'}\text{-factors}}$$