~~July 3, 1975~~: stratifying K action on $\mathfrak{k}$
more on ~~[illegible]~~ good K-spaces.
self-normalizing subgps.

Let ~~[illegible]~~ K be a connected compact Lie group acting on its Lie algebra $\mathfrak{k}$. I wish to define a stratification of $\mathfrak{k}$.

Given X, Y in $\mathfrak{k}$ we say they are in the same stratum if there is a path X_t in $\mathfrak{k}$ ~~[illegible]~~ going from X to Y such that

$$\mathfrak{k}_{X_t} = \mathfrak{k}_X \qquad \text{for all } t.$$

Suppose X belongs to the max. abelian subspace E. Then $E \subset \mathfrak{k}_X = \mathfrak{k}_{X_t}$, so $[X_t, E] = 0 \implies X_t \in E$. Let $\Phi \subset E^*$ be the roots of $\mathfrak{k}$ wrt E. Then

$$\mathfrak{k}_{X_t} = E + \sum_{\alpha(X_t)=0} \mathfrak{k}^\alpha \qquad \alpha \in \Phi^+$$

so $\mathfrak{k}_X = \mathfrak{k}_{X_t}$ means that $\alpha(X) = 0 \iff \alpha(X_t) = 0$. By continuity $\alpha(X_t)$ has the same sign (I mean $+, -$, or 0) as $\alpha(X)$ for all t. Thus the stratum of X is the subset of E consisting of all Z such that $\alpha(X)$ has the same sign as $\alpha(Z)$ for all roots α.

Let C be a chamber in E. If $X \in C$, then the stratum of X is the "open" face of X in C. Precisely if $\alpha_1, \cdots, \alpha_\ell$ are simple roots, ~~[illegible]~~

so that $C \xrightarrow{\sim} (\mathbb{R}^+)^k$, then the stratum of X is described by the "open" face

$$\alpha_i(Z) = 0 \qquad \text{if } i \in I = \{ j \mid \alpha_j(X) = 0 \}$$
$$\alpha_i(Z) > 0 \qquad \text{if } i \notin I.$$

Thus K-orbits on the strata of k are the "open" faces of C.

In general suppose K is a compact group acting on a manifold X. Then we can define strata in the same way as a connected component of the space of points with the same isotropy group.

Let x_0 be generic so that K_{x_0} acts trivially on the normal space to Kx_0. The normal tube around the orbit is isomorphic to the disk bundle of the normal bundle (isomorphic as K-manifolds). Thus the stratum thru x_0 will coincide near x_0 with the submanifold $X^{K_{x_0}}$. If $N =$ normalizer of $K_{x_0} = H$, and N/H is discrete the stratum will coincide with, the exponential of the normal space.

In fact given any x and x' near x we know

$K_{x'}$ is conjugate to a subgroup of K_x. Thus if x' is fixed by K_x we have $K_x = K_{x'}$. Therefore the stratum thru x coincides with X^{K_x} near x.

The really good case is where N_x = normalizer of K_x is such that N_x/K_x is discrete. For in this case, the stratum will be transversal to the orbit.

Now because derivations of compact groups are inner (essentially) the Lie algebra of N_x/K_x ought to be filled out by $\mathfrak{k}_x$ and the centralizer of $\mathfrak{k}_x$. ~~This means (generally) K_x has to contain a maximal torus of K.~~

So what seems to be very interesting are self-normalizing subalgebras $\mathfrak{h}$ of $\mathfrak{k}$. The corresp. connected group H is necessarily closed hence compact.

Suppose H is a compact connected subgroup of K, let N be the normalizer of H. N is closed and $\mathfrak{h}$ is an ideal in $\mathfrak{n}$. By complete reducibility $\mathfrak{n} = \mathfrak{h} \oplus \mathfrak{a}$ where $\mathfrak{a}$ is stable under $Ad(H)$. Hence $[\mathfrak{h}, \mathfrak{a}] \subset \mathfrak{a} \cap \mathfrak{h} = 0$. Thus $\mathfrak{h} = \mathfrak{n}$ iff the centralizer of $\mathfrak{h}$ in $\mathfrak{k}$ is contained in $\mathfrak{h}$, i.e.

<u>Prop</u>: K compact, $\mathfrak{k} = \mathrm{Lie}(K)$, $\mathfrak{h}$ a Lie subalgebra of $\mathfrak{k}$. Then $\mathfrak{h}$ is its own normalizer $\Longleftrightarrow$ the centralizer of $\mathfrak{h}$ in $\mathfrak{k}$ is contained in $\mathfrak{h}$.

<u>Proof</u>. $\Longrightarrow$ obvious

$\Longleftarrow$ Let $\mathfrak{n} = \{X \in \mathfrak{k} \mid [X, \mathfrak{h}] \subset \mathfrak{h}\}$ be the normalizer of $\mathfrak{k}$. Let $(\, , \,)$ be an invariant inner product on $\mathfrak{k}$. Let $\mathfrak{a} = \mathfrak{n} \ominus \mathfrak{h}$. Then $\mathfrak{a}$ is stable under $\mathfrak{h}$, so $[\mathfrak{h}, \mathfrak{a}] \subset \mathfrak{a}$, and $\subset \mathfrak{h}$ because $\mathfrak{h}$ is an ideal in $\mathfrak{n}$; hence $[\mathfrak{h}, \mathfrak{a}] = 0$. By assumption $\mathfrak{a} \subset \mathfrak{h}$ so $\mathfrak{n} = \mathfrak{h}$.

<u>Cor</u>. If $\mathfrak{h}_1 \subset \mathfrak{h}_2$ and $\mathfrak{h}_1$ is self-normalizing, then so is $\mathfrak{h}_2$.

Because the centralizer of $\mathfrak{h}_2$ is contained in the centralizer of $\mathfrak{h}_1$.

But symmetric spaces give us examples where the action is good but such that stabilizers are not self-normalizing infinitesimally, e.g. K^σ can be abelian. So we don't yet have the good geometric condition.

Go back and define the strata of $\mathfrak{k}^-$ under K^σ.

Let $X \in k^-$ and put it inside a max. abelian subspace E^-. Then we have the root decomp.

$$k = k_{E^-} \oplus \sum_{\alpha \in \Phi^+} k^\alpha$$

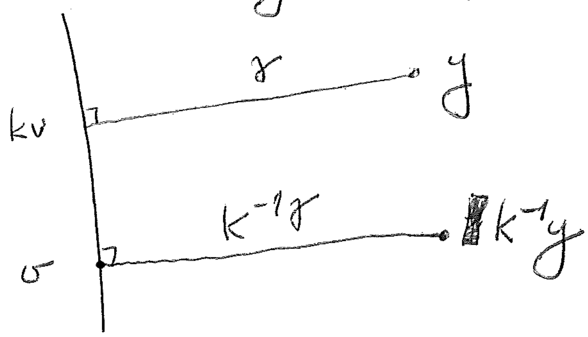
$$k = k_X \oplus \sum_{\alpha(X) \neq 0} k^\alpha$$

$$k_X = k_{E^-} \oplus \sum_{\alpha(X) = 0} k^\alpha$$

Now let $k^+_Y = k^+_X$, $Y \in k^-$. Does it follow $[Y,X] \in 0$? NO.

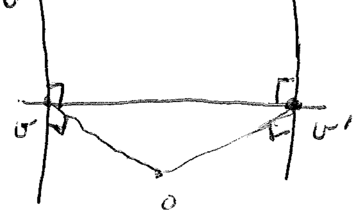
Let K acts on a vector space V with inner product. I have decided that the good geometric situation occurs when at each generic point v (i.e. K_v acts trivially on $(kv)^\perp$) one has that $(kv)^\perp$ is perpendicular to the orbits of each of its points.

In fact suppose this is true for ones generic point v. Given $y \in V$ draw a minimum geodesic γ from y to the orbit Kv. If the end of γ is kv, replacing γ by $k^{-1}\gamma$, we get a geodesic from v to $k^{-1}y$ perpendicular to Kv



This argument shows that each orbit Ky meets $(kv)^\perp$. Thus if I have another generic point v' I can move it into $(kv)^\perp$. Because I am assuming $(kv)^\perp$ is perpendicular to the K-orbit of each of its points, I know $(kv)^\perp \subset (kv')^\perp$. But v' generic $\Rightarrow$ $K_{v'}$ acts trivially on $(kv')^\perp$ $\Rightarrow$ $K_{v'} \subset K_v$. Similarly because K_v acts trivially on $(kv)^\perp$ we have $K_v \subset K_{v'}$. Thus $K_v = K_{v'}$ and so the orbits Kv, Kv' have the same dimension $\Rightarrow$ $(kv)^\perp = (kv')^\perp$. Thus I have proved that if for one generic point v, $(kv)^\perp$ is $\perp$ to the orbits thru its points, it will be true for all generic points.

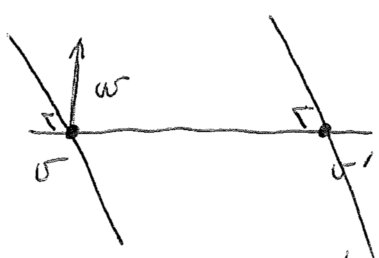
Start again. Let v be generic. Then I know every K orbit meets $(kv)^\perp$. Moreover if $v' \in (kv)^\perp$, then the line vv' has to be $\perp$ ~~to~~ both Kv and Kv' so $v \in (kv')^\perp$



As K_v acts trivially on $(kv)^\perp \Rightarrow K_v \subset K_{v'}$.
If also v' is generic, then $K_{v'} \subset K_v$, so $K_v = K_{v'}$.
All this is true with no assumptions. $\therefore$

<u>Prop</u>: Let K act on the vector space V with inner product. Let v be a point of V. Then every orbit of K intersects $(kv)^\perp$. If K_v acts trivially on $(kv)^\perp$ (this is the case when v is generic), then $K_v \subset K_{v'}$ for any $v' \in (kv)^\perp$. If $K_{v'}$ also acts trivially on $(kv')^\perp$, then $K_v = K_{v'}$. If $(kv)^\perp$ is perpendicular to the K-orbits of each of its points, then for v, v' generic & $v' \in (kv)^\perp$, we have $(kv)^\perp = (kv')^\perp$.

The problem in general is that although vv' is $\perp$ to both orbits Kv, Kv', there may



exist vectors w in $(kv)^\perp$ which are not in $(kv')^\perp$.

Reduction: Let v be generic and $H = K_v$ and let $N = \mathrm{Norm}_K(H)$. Then $(kv)^\perp \subset V^H$, and

$G = N/H$ acts on V^H, + the action is free at v. So if I want an example to show that not all actions are good I can look for one where $K_v = 1$ for K generic.

Suppose $K = S^1$ acting on $\mathbb{C}^2$ by characters z^{n_1}, z^{n_2}, where n_1, n_2 are relatively prime so the action is free generically on the sphere. Let $v = ae_1 + be_2$. Then

$$z \cdot v = z^{n_1} a e_1 + z^{n_2} b e_2 \qquad z \in S^1$$

$$e^{i\theta} \cdot v = e^{in_1\theta} a e_1 + e^{in_2\theta} b e_2$$

so

$$\mathfrak{k} v = \mathbb{R}(in_1 a e_1 + in_2 b e_2)$$

$$(\mathfrak{k} v)^\perp = \mathbb{R} e_1 + \mathbb{R} e_2 + \mathbb{R}(in_2 e_1 - in_1 e_2)$$

Take

$$v' = ae_1 + be_2 \qquad a, b \text{ real} \neq 0.$$

$$(\mathfrak{k} v') = \mathbb{R}(in_1 a e_1 + in_2 b e_2)$$

$$(\mathfrak{k} v')^\perp = \mathbb{R} e_1 + \mathbb{R} e_2 + \mathbb{R}(in_2 b e_1 - in_1 a e_2)$$

$v' \in (\mathfrak{k} v)^\perp$ is a generic element, and we see that

$$(\mathfrak{k} v')^\perp \neq (\mathfrak{k} v)^\perp$$

~~Note: Let V be a complex representation of K. One knows V contains a dominant weight vector v. If the dominant weight is regular, the centralizer of v in $k \otimes \mathbb{C} = g$ is the group $n = \sum_{\alpha \in \Phi^+}$ Thus~~

Given any K, one can equivariantly embed K in the sphere of a representation V, hence there are many reps. of K such that $K_v = 1$ for v generic.

July 3, 1975. Buildings

Previously I considered an orbit $K\eta$, $\eta \in k$, and used the function

$$|k\eta - \xi|^2 = \text{const} - 2(k\eta, \xi)$$

on the orbit where ξ is regular. In this case critical points were non-degenerate. Now I want to look at the case where ξ is not regular, in which case the critical points should fall into non-degenerate critical submanifolds.

For $\eta \in k$ to be a critical point of $|k\eta - \xi|^2$ (means) $\forall X \in k$ that

$$0 = ([X, \eta], \xi) = (X, [\eta, \xi])$$

i.e. that $[\eta, \xi] = 0$. Thus the group K_ξ acts on the critical points.

When $\eta\ \xi$ commute we can choose a maximal abelian subspace E containing them and calculate the Hessian

$$-((\mathrm{ad}\, X)^2 \eta, \xi) = ([\eta, X], [\xi, X])$$

Suppose $k = E \oplus \sum_{\alpha \in \Phi^+} k_\alpha$, is the root decomposition.

(Recall this means that each k_α is equipped with a complex structure such that an element b in E

acts on k_α by multiplying by $\alpha(b)i$.
Expand

$$X = b + \sum p_\alpha(X)$$

$$[\eta, X] = \sum \alpha(\eta) i \, p_\alpha(X)$$

$$[\xi, X] = \sum \alpha(\xi) i \, p_\alpha(X)$$

$$([\eta, X], [\xi, X]) = \sum_{\alpha \in \Phi^+} \alpha(\eta)\alpha(\xi) |p_\alpha(X)|^2$$

The tangent space to $K\cdot\eta$ at η is $k/k_\eta = \sum_{\substack{\alpha(\eta)\neq 0 \\ \alpha\in\Phi^+}} k_\alpha$. Inside this we have the tangent space to $K_\xi\cdot\eta$ which is $k_\xi + k_\eta / k_\eta$. Thus the normal space to the orbit $K_\xi\cdot\eta$ inside $K\cdot\eta$ is

$$k/k_\xi + k_\eta = \bigoplus_{\substack{\alpha(\eta)\neq 0 \\ \alpha(\xi)\neq 0}} k_\alpha \qquad \alpha \in \Phi^+$$

It is clear that the **Hessian** is non-degenerate on this space. Thus the K_ξ**-orbits** are non-degenerate critical submanifolds for the function $|k\eta - \xi|^2$ on $K\eta$. The index of this manifold is the number of roots α such that $\alpha(\xi) > 0$ and $\alpha(\eta) < 0$, i.e. the number of hyperplanes crossed in going from ξ to η.

Question: If M is a compact manifold with a Morse function f, is M a CW complex with cells indexed by the critical points of f?

~~What I want to do next is to show~~

Suppose now I try to show $K\eta$ is a CW complex. Morse theory only tells me it has the homotopy type of a CW complex with cells indexed by the critical points.

So want I want to do is to show that the P_ξ-orbits of $K\eta$ can be compactified over the K_ξ orbits of $K\eta \cap \mathfrak{k}_\xi$.

Let ξ be regular first, whence $\mathfrak{k}_\xi = E$ meets $K\eta$ at $W\eta$. We want to compactify the cell $P_\xi^u \eta$. By compactify I mean to find a compact manifold with a divisor of normal crossings such that the cell is the complement.

Method. Start with the straight line joining ξ to η. I will enumerate the points of the line where roots vanish, excluding η. Suppose these points are

$$\eta_{(1)}, \dots, \eta_p = \eta$$

and put $\xi = \eta_0$. Thus the segment η_i, η_{i+1} crosses no walls.

~~Now the basic idea is to introduce the stabilizers~~ $\cancel{K_0 = T \subset K_1 \subset \cdots \subset K_{p-1}}$.

Assume ξ regular and let C_0 be the chambre containing it. Consider the line

$$\eta_t = \xi + t(\eta - \xi) \qquad 0 \le t \le 1.$$

For most t, η_t is regular. Let $t_1, t_2, \ldots, t_m$ be those $t < 1$ such that η_t is singular arranged in order. We get a sequence of chambres in $E = \mathfrak{p}_\xi$

$$C_0, C_1, \ldots, C_m$$

such that $\eta_t \in C_i$ for $t \in [t_i, t_{i+1}]$. Let $s_1, \ldots, s_m$ be the elements of W such that

$$C_1 = s_1 C_0$$

$$C_2 = s_1 s_2 C_0$$

$$C_m = s_1 s_2 \cdots s_m C_0 .$$

Thus $s_i C_0 = s_{i-1}^{-1} \cdots s_1^{-1} C_i$.

$$\cancel{s_{i-1}^{-1} \cdots s_1^{-1} \eta_t \in C_0 \quad t_{i-1} < t < t_i}$$

$$t \in [t_{i-1}, t_i] \implies \eta_t \in C_{i-1} = s_1 \cdots s_{i-1} C_0$$

$$\implies s_{i-1}^{-1} \cdots s_1^{-1} \eta_t \in C_0$$

$$t \in [t_i, t_{i+1}] \Rightarrow \eta_t \in C_i = s_1 \cdots s_i C_0$$
$$\Rightarrow s_{i-1}^{-1} \cdots s_1^{-1} \eta_t \in s_i C_0$$

Thus $s_{i-1}^{-1} \cdots s_1^{-1} \eta_t$ is a line running from the interior of C_0 to the interior of $s_i C_0$. So one considers the roots vanishing on $s_{i-1}^{-1} \cdots s_1^{-1} \eta_{t_i}$ then s_i reverses exactly these roots. Thus s_i is the Coxeter element of the subgroup of W fixing $s_{i-1}^{-1} \cdots s_1^{-1} \eta_{t_i}$. $\therefore$ s_i is of order 2.

It would be more precise to let I_i be the set of simple roots vanishing on $s_{i-1}^{-1} \cdots s_1^{-1} \eta_{t_i}$, and put $s_i = s_{I_i}$.

Compactification of $P_\xi^u \eta$ can now be constructed as follows. ~~[illegible]~~ I consider all sequences of chambres in $\mathfrak{p}$

$$C_0 = \gamma_0, \ldots, \gamma_m$$

such that γ_i, γ_{i+1} have the I_i-th face in common. (If I want to I can think of a chambre as a point of $K \cdot \xi$). Then

$$\gamma_1 = k_1 C_0 \qquad\qquad \gamma_1 = k_1 C_0$$
$$C_0 = k_1^{-1}\gamma_1, \quad k_1^{-1} \gamma_2 = k_2 C_0 \qquad\qquad \gamma_2 = k_1 k_2 C_0$$
$$\cdots$$
$$k_{m-1}^{-1} \cdots k_1^{-1} \gamma_m = k_m C_0 \qquad \gamma_m = k_1 \cdots k_m C_0$$

and ~~[illegible]~~

$$k_1 \in K_{I_1}$$

$$k_2 \in K_{I_2}$$

so therefore it is clear that the sequence $(\gamma_0, \ldots, \gamma_m)$ is the same as a point in

$$G(I_1, \ldots, I_m) = K_{I_1} \times^T K_{I_2} \times^T \cdots \times^T K_{I_m} / T .$$

We have a map ~~[illegible]~~

$$G(I_1, \ldots, I_m) \longrightarrow K\eta$$

$$(k_1, \ldots, k_m) \longmapsto k_1 \cdots k_m \eta$$

and what I want to prove is that this map is a resolution of the closure of $P^u_\xi \cdot \eta$.

Suppose ξ, η are arbitrary elements of E. I have seen that the orbit $P^u_\xi \cdot \eta$ is isomorphic to the unipotent group normalized by T having the roots $\{\alpha \mid \alpha(\xi) > 0,\ \alpha(\eta) < 0\}$. I claim ξ can be perturbed to a regular element ξ' in E such that $P^u_{\xi'} \eta = P^u_\xi \eta$. Thus I want a regular element ξ such that

$$\{\alpha \mid \alpha(\xi) > 0,\ \alpha(\eta) < 0\} = \{\alpha \mid \alpha(\xi') > 0,\ \alpha(\eta) < 0\}.$$

Let η' be a regular element close to η, and let $\xi' = \xi + \varepsilon\eta'$

where $\varepsilon > 0$ is so small that $\alpha(\xi) > 0 \Rightarrow \alpha(\xi') > 0$. If α is a root such that $\alpha(\eta) < 0$, $\alpha(\xi) > 0$, then clearly $\alpha(\eta) < 0$ and $\alpha(\xi') > 0$. If conversely $\alpha(\eta) < 0$ and $\alpha(\xi') > 0$, then because η' is close to η we have $\alpha(\eta') < 0$, so

$$0 < \alpha(\xi') = \alpha(\xi) + \varepsilon\,\alpha(\eta') < \alpha(\xi)$$

Finally ξ' is regular, because $\alpha(\xi) > 0 \Rightarrow \alpha(\xi') > 0$ and $\alpha(\xi) = 0 \Rightarrow \alpha(\xi') = \varepsilon\,\alpha(\eta') \neq 0$, hence no root vanishes on ξ'.

What the above shows is that the cells $P^u_\xi \cdot \eta$ are Schubert cells, i.e. that their normalizers are parabolic.

Ugly formulas: Suppose again ξ regular, and let $0 < t_1 < \cdots < t_m < 1$ be those points t such that $\alpha(\eta_t) = 0$ for some $\alpha \ni \alpha(\xi) > 0$ but $\alpha(\eta) < 0$. Following Bott-Samelson one considers the broken geodesic

$$\begin{array}{ll} \eta_t & t \in [0, t_1] \\ k_1 \eta_t & t \in [t_1, t_2] \\ k_1 \cdots k_i \eta_t & t \in [t_i, t_{i+1}] \end{array}$$

where $k_i \in K_i$ = stabilizer of η_{t_i}. Then we get a map

(1) $$K_1 \times^T \cdots\cdots \times^T K_m / T \longrightarrow \mathcal{P}$$
$$(k_1, \ldots\ldots, k_m) \longmapsto k_1 \cdots k_m \eta$$

On the other hand we define a sequence of elements $s_1, \ldots, s_m$ in the Weyl group by

$$s_1 \cdots s_i C_0 = \text{the chambre containing } \eta[t_i, t_{i+1}]$$

and a sequence of groups

$$K_{I_i} = \text{stabilizer of } s_{i-1}^{-1} \cdots s_1^{-1} \eta_{t_i}.$$

Thus as t passes thru t_i, $s_{i-1}^{-1} \cdots s_1^{-1} \eta_t$ goes from C_0 to $s_i C_0$. Then I wanted to consider sequences of chambres

$$\gamma_0, \ldots, \gamma_m$$

starting with the fundamental chambre such that γ_i and γ_{i+1} have the I_i-th face in common. We then had

$$\gamma_i = x_1 \cdots x_i \gamma_0 \qquad x_i \in K_{I_i}$$

and so got a map

(2) $$K_{I_1} \times^T \cdots\cdots \times^T K_{I_m} / T \longrightarrow \mathcal{P}$$
$$(x_1, \ldots\ldots, x_m) \longmapsto x_1 \cdots x_m s_m^{-1} \cdots s_1^{-1} \eta$$

~~Thus~~ The maps (1) and (2) are related as follows.

$$K_{I_i} = \text{st. of } s_{i-1}^{-1} \cdots s_1^{-1} \eta_{t_i}$$
$$= s_{i-1}^{-1} \cdots s_1^{-1} K_i \, s_1 \cdots s_{i-1}.$$

$$x_1 \cdots x_i \gamma_0 = \gamma_i = \text{chambre containing } k_1 \cdots k_i \eta_{[t_i, t_{i+1}]}$$

$$\eta_{[t_i, t_{i+1}]} \subset s_1 \cdots s_i C_0$$

$$x_1 \cdots x_i = k_1 \cdots k_i \, s_1 \cdots s_i$$

$$\text{or} \quad \boxed{x_i = s_{i-1}^{-1} \cdots s_1^{-1} k_i \, s_1 \cdots s_i}$$

Thus we have the comm. diagram

$$(k_1, \ldots, k_m) \longmapsto k_1 \cdots k_m \eta$$

$$\begin{tikzcd}
K_1 \times^T \cdots \times^T K_m / T \arrow[d] \arrow[dr] & \\
K_{I_1} \times^T \cdots \times^T K_{I_m} / T \arrow[r] & \mathcal{K}
\end{tikzcd}$$

$$(k_1, \ldots, k_m) \mapsto (x_1 s_1, s_1^{-1} x_1 s_1 s_2, \ldots)$$

$$(x_1, \ldots, x_m) \longmapsto x_1 \cdots x_m s_m^{-1} \cdots s_1^{-1} \eta$$

Think of the geodesic η_t as being a geodesic in C_0 ~~...~~ reflecting off the walls.

Let's use the Bott-Samelson formulas, but no longer supposing ξ to be regular. Again we let $t_1 < \cdots < t_m$ be the points t such that $\exists$ a root α with $\alpha(\xi) > 0$, $\alpha(\eta_t) = 0$, $\alpha(\eta) < 0$, and let K_i = stabilizer of η_{t_i}. Put $Z = K_\xi \cap K_\eta$; it has roots those α such that $\alpha(\xi) = \alpha(\eta) = 0$. Z stabilizes η_t for all t. If α is not a root of Z and $0 < t < 1$ is such that

$$\alpha(\eta_t) = \alpha(\xi) + t\left[\alpha(\eta) - \alpha(\xi)\right] = 0$$

then $\alpha(\xi)$ and $\alpha(\eta)$ are non-zero and have opposite sign; hence t must be one of the t_i. Thus it is clear that

$$\dim\left(K_1 \times^Z K_2 \times^Z \cdots \times^Z K_m / Z\right) = \sum \dim K_i/Z$$

is twice the number of roots α such that $\alpha(\xi) > 0$ and $\alpha(\eta) < 0$.

Now the thing to prove is that the map

$$\begin{aligned} K_1 \times^Z \cdots \times^Z K_m / Z &\longrightarrow \mathfrak{p} \\ (k_1, \ldots, k_m) &\longmapsto k_1 \cdots k_m \eta \end{aligned}$$

is a compactification of the orbit $P_\xi^u \cdot \eta$. Note that then

$$K_\xi \times^Z K_1 \times^Z \cdots \times^Z K_m / Z \longrightarrow \mathfrak{p}$$

will be a compactification of the orbit $P_\xi \cdot \eta$.

Consider the case where ξ is regular. To each element of $K_1 \times^T \cdots \times^T K_m / T$ I have associated a broken geodesic in the building, namely

$$\begin{array}{ll} \eta_t & t \in [0, t_1] \\ k_1 \eta_t & t \in [t_1, t_2] \\ k_1 \cdots k_i \eta_t & t \in [t_i, t_{i+1}]. \end{array}$$

Let us consider the retraction of the building back to E which associates to a point p the unique pt. in $P^u_\xi p \cap E$. This retraction is a simplicial mapping, hence it carries the broken geodesic into a broken geodesic in E. In fact it is clear that there are unique* elements $w_1, \ldots, w_m$ of W such that the image is the broken geodesic

* because ξ is regular

$$\begin{array}{ll} \eta_t & t \in [0, t_1] \\ w_1 \eta_t & t \in [t_1, t_2] \\ w_1 \cdots w_i \eta_t & t \in [t_i, t_{i+1}]. \end{array}$$

where $w \in W_i$ = stabilizer of η_{t_i}.

Point: $P^u_\xi k_1 \cdots k_m \eta = P^u_\xi w_1 \cdots w_m \eta$ has the dimension

$$l(w_1 \cdots w_m) \leq l(w_1) + \cdots + l(w_m) \leq l(s_1) + \cdots + l(s_m)$$

~~if dim $P^u_{\xi}(k_1 \dots k_m, \eta) = \dim P^u_{\xi}$~~

Point: Put $\ell(\xi) = \dim (P^u_{\xi} \cdot \xi)$. Then for $t \in (t_i, t_{i+1})$ η_t is in the chambre $s_1 \cdots s_i C_0$, hence

$$\ell(w_1 \cdots w_i \eta_{t_i+\varepsilon}) = \ell(w_1 \cdots w_i s_1 \cdots s_i \xi)$$

Put $v_i = s_{i-1}^{-1} \cdots s_1^{-1} w_i s_1 \cdots s_i$ so that

$w_1 \cdots w_i \eta_{t_i+\varepsilon}$ is in the chambre $v_1 \cdots v_i C_0$.

and $w_1 \cdots w_i s_1 \cdots s_i = v_1 \cdots v_i$

Then

$$\begin{aligned} \dim P^u_{\xi} k_1 \cdots k_m \eta &= \ell(w_1 \cdots w_m \eta) \\ &= \ell(v_1 \cdots v_m) \\ &\leq \ell(v_1) + \cdots + \ell(v_m) \\ &\leq \ell(s_1) + \cdots + \ell(s_m) = \dim P^u_{\xi} \eta \end{aligned}$$

with equality $\iff$ $v_i = s_i$ $\iff$ $w_i = id$. (The reason $\ell(v_i) \leq \ell(s_i)$ is because v_i is in W_{I_i} = stabilizer in W of $s_{i-1}^{-1} \cdots s_1^{-1} \eta_{t_i}$ and s_i is the Coxeter element.)

~~It follows that there is a great deal of ... the only ... points~~

General construction. Suppose η_t $0 \le t \le 1$ is a path is a space on which the group acts, and suppose given a subdivision

$$0 < t_1 < \cdots < t_m < 1 \qquad \text{put } t_0 = 0,\ t_{m+1} = 1.$$

Then we can consider modifications of η_t of the following form

$$k_0 \eta_t \qquad t \in [0, t_1] \qquad k_0 \in K_{\eta_0}$$

$$k_0 k_1 \eta_t \qquad t \in [t_1, t_2] \qquad k_1 \in K_{\eta_{t_1}}$$

etc.

The space of these modified paths is clearly

$$K_{\eta_{t_0}} \times^{K_{\eta[t_0,t_1]}} K_{\eta_{t_1}} \times^{K_{\eta[t_1,t_2]}} \cdots \times K_{\eta_{t_m}} / K_{\eta[t_m,t_{m+1}]} .$$

Note that refining the subdivision by putting in a point $t_i < u < t_{i+1}$ will not change the space of modifications provided

$$K_{\eta_u} = K_{\eta[t_i,t_{i+1}]}$$

In good cases there exists a subdivision fine enough so that the stabilizer of an interior point of a segment is the stabilizer of the whole segment. Thus we get a canonical space of modifications.

July 5, 1975

I continue with the compactification of the orbit $P^u_\xi * \eta$. The procedure is as follows:

Let $\lambda(t) = \xi + t(\eta - \xi) \qquad 0 \leq t \leq 1$.

Thus λ is a path in $\mathfrak{p}_\xi$. I will now form a manifold of modifications of λ, denote it $M(\lambda)$. An element φ of $M(\lambda)$ is a path $\varphi(t) \quad 0 \leq t \leq 1$ such that $\exists$ a subdivision

$$1) \qquad 0 = t_0 < t_1 < \cdots < t_m < 1$$

and elements $k_i \in K_{\lambda(t_i)} \qquad i = 0, \ldots, m$ such that

$$\varphi(t) = k_0 \cdots k_i \lambda(t) \qquad \text{for } t \in [t_i, t_{i+1}].$$

Consider the choice of k_i. We have $\lambda(t)$ for $t \leq t_i$ continued by $k_i \lambda(t)$ for $t \geq t_{i+1}$. If it should happen that $K_{\lambda(t_i)} \subset K_{\lambda(t)}$ for $t > t_i$ close to t_i, then $k_i \lambda(t) = \lambda(t)$ for $t > t_i$ close to t_i; thus the choice of k_i is irrelevant. What this means is the subdivision 1) need only include points t such that $K_{\lambda(t)} > K_{\lambda(t+\varepsilon)}$ where ε is small and >0. ~~[illegible]~~

Choose E to contain both ξ, η. Then $K_{\lambda(t)}$ is the connected group containing $\exp(iE) = T$ with those roots α such that $\alpha(\lambda(t)) = 0$. Consequently for $K_{\lambda(t)} > K_{\lambda(t+\varepsilon)}$ means that there exists a root α such that

$$\underset{(1-t)\alpha(\xi) + t\,\alpha(\eta)}{\alpha(\lambda(t)) = 0} \qquad \underset{(1-t-\varepsilon)\alpha(\xi) + (t+\varepsilon)\,\alpha(\eta).}{\alpha(\lambda(t+\varepsilon)) \neq 0}$$

i.e. $\exists$ root α such that $\alpha(\lambda(t)) = 0$ but such that $\alpha(\lambda(t+\varepsilon)) < 0 \leftarrow$ for $\varepsilon > 0$.

Therefore given the path $\lambda(t)$ we need only consider the points $0 \leq t < 1$ such that $K_{\lambda(t)} > K_{\lambda(t+\varepsilon)}$ for ε small and >0. So I consider all the roots α such that $\alpha(\xi) \geq 0$ and $\alpha(\eta) < 0$ and introduce for each such root α the t such that $\alpha(\lambda(t)) = 0$ into the subdivisions. ~~[illegible]~~ This allows me to identify $M(\lambda)$ with the manifold

$$K_0 \times^Z K_1 \times^Z \cdots \times^Z K_m / Z$$

where $\quad K_i = K_{\lambda(t_i)} \quad , \quad Z = K_\xi \cap K_\eta = K_\lambda$

[redacted] Generalization: Let $\lambda(t)$, $0 \le t \le 1$, be a piecewise-linear path in [redacted] $\mathfrak{p}$ such that any linear [redacted] piece consists of mutually commuting elements. (In other words $\lambda(t)$ is a broken geodesic in $\mathfrak{p}$ everywhere perpendicular to K-orbits: $(k\cdot\xi, \dot\eta - \xi) = (k, [\xi, \eta - \xi]) = 0 \iff [\xi, \eta] = 0$.)

Question: Start with a linear path $\lambda(t) = \xi + t(\eta - \xi)$ as before and let $\varphi(t)$ be one of its modifications. Then $e^{-\infty\xi} * \varphi(t) = \psi(t)$ is a path (not-necessarily continuous) in $\mathfrak{p}_\xi$. Is $\psi(t)$ a broken geodesic perpendicular to K-orbits?

Proposition: Let C be a chambre in the building (C is a chambre in some maximal abelian [redacted] subspace of $\mathfrak{p}$). Let T_ξ be the retraction of $\mathfrak{p}$ onto $\mathfrak{p}_\xi$ which sends a point η to $e^{-\infty\xi} * \eta$ = unique ξ-fixpt of $P^u_\xi * \eta$. Then T_ξ restricted to C is an isomorphism of C with a chambre of $\mathfrak{p}_\xi$ which is induced by an element of K.

Let $C = k \cdot C_0$ where C_0 is a chambre containing ξ, and let ξ_0 be an interior point of C_0. Then $T(k\xi_0) \in K\xi_0 \cap \mathfrak{p}_\xi$. Now suppose that $\xi_0, k_0\xi_0 \in \mathfrak{p}_\xi$. Then if $E = \mathfrak{p}_{\xi_0}$, both E and k_0E contain ξ, hence $E, k_0E \subset \mathfrak{p}_\xi$ and so $\exists$ $z \in K_\xi$ such that $z^{-1}k_0E = E$, hence $k_0 = zw$ with $w \in N = \mathrm{Norm}_K(E)$. Thus $T(k\xi_0) = zw\xi_0$ with $z \in K_\xi$, $w \in W$; and so

$$k\xi_0 = uzw\xi_0 \qquad \text{some } u \in P_\xi^u$$

or

$$k = uzwv \qquad \text{some } v \in P_{\xi_0}.$$

However P_{ξ_0} acts trivially on C_0, hence we find for any $\eta \in C_0$ that

$$T(k\eta) = T(uzwv\eta)$$

$$= T(zw\eta) = zw\cdot\eta$$

Therefore $T : C \longrightarrow \mathfrak{p}_\xi$ is given by $\xi \mapsto zwk^{-1}\xi$ which proves the proposition.

Description of $K\eta \cap \mathfrak{p}_\xi$. Suppose $\eta, k\eta \in \mathfrak{p}_\xi$ and let E be a maximal abelian subspace containing η, ξ. We can find $z \in K_\xi$ such that $z^{-1}k\eta \in E$. Then $\eta, z^{-1}k\eta$ are two K-conj. points of E, so $\exists$ $n \in \mathrm{Norm}_K(E)$ such that

$$n\eta = z^{-1}k\eta$$

Thus $k\eta = zn\eta$ for some $z \in K_\xi$, $n \in N$. Conversely any point of this form is in $K\eta \cap \mathfrak{p}_\xi$.

Clearly n can be changed by multiplying on the right by an elt of $N_\xi = N \cap K_\xi$ and on the left by an element of $N_\eta = N \cap K_\eta$. Thus we get a map

$$W_\xi \backslash W / W_\eta \longrightarrow K_\xi \backslash K\eta \cap \mathfrak{p}_\xi = G_\xi \backslash K\eta \cap \mathfrak{p}_\xi$$

$$= G_\xi \times P_\xi^u \backslash K\eta \cong P_\xi \backslash G / P_\eta$$

which is surjective. I want to show it is bijective. So let $n_1, n_2 \in N$ be such that $\exists$ $z \in K_\xi$ such that

$$zn_1\eta = n_2\eta.$$

But then $n_1\eta$, $n_2\eta$ are two points of the max. ab. subspace E of $\mathfrak{p}_\xi$ which are K_ξ-conjugate $\Rightarrow$ they are N_ξ-conjugate. Thus $\exists$ $n \in N_\xi$ such that $nn_1\eta = n_2\eta$, whence

$\eta_2 = n n_1 n'$ with $n \in N_\xi$ and $n' \in N_\eta$, proving the desired injectivity.

Return to the problem of compactifying the orbit $P_\xi \eta$ in $K\eta$. (Here $\xi, \eta \in E$). The idea was to introduce the geodesic

$$\lambda(t) = \xi + t(\eta - \xi) \qquad 0 \le t \le 1$$

and to consider the space $M(\lambda)$ of modifications defined as follows. Let $0 = t_0 < t_1 < \cdots < t_m < t_{m+1} = 1$ be a subdivision such that $\lambda([t_i, t_{i+1}])$ is contained in a chambre of E for each i (this means no root changes sign in the interior of this segment). Put

$$K_i = K_{\lambda(t_i)} \qquad i = 0, \ldots, m$$

$$K_{i,i+1} = K_{\lambda[t_i, t_{i+1}]} \qquad i = 0, \ldots, m \,.$$

Then

$$M(\lambda) = K_0 \times^{K_{01}} K_1 \cdots\cdots \times^{K_{m-1,m}} K_m / K_{m,m+1}$$

and we map $M(\lambda)$ to $K\eta$ by sending $(k_1, \ldots, k_m)$ to $k_1 \cdots k_m \eta$.

Instead of the linear path $\lambda(t) = \xi + t(\eta - \lambda)$ I can use a broken geodesic in $\mathfrak{p}_\xi$ such that for every t, $0 \le t < 1$ one has:

1) $$\alpha(\lambda(t)) = 0 \;,\quad \underset{\varepsilon \text{ small} > 0}{\alpha(\lambda(t+\varepsilon)) < 0} \implies \alpha(\xi) > 0.$$

(this makes sense because $\lambda(t)$ being a broken geodesic in $\mathfrak{p}_\xi$ means that $\xi, \lambda(t), \lambda(t+\varepsilon)$ are contained in a Cartan subalg. E). The real way to state this condition however is that

$$\dim K_{\lambda(t)}/K_{\lambda[t,t+\varepsilon]} = \dim P_\xi^u \cdot \lambda(t+\varepsilon)/P_\xi^u \cdot \lambda(t)$$

1) should be changed to $\forall\, t \in [0,1]$ one has:

$$\left[\begin{array}{lll} \alpha(\lambda(t)) = 0 & \alpha(\lambda(t-\varepsilon)) < 0 \implies & \alpha(\xi) < 0 \\ \alpha(\lambda(t)) = 0 & \alpha(\lambda(t+\varepsilon)) < 0 \implies & \alpha(\xi) > 0 \end{array}\right.$$

The former implies $\dim P_\xi^u \lambda(t-\varepsilon) = \dim P_\xi^u \lambda(t)$, hence $\dim P_\xi^u \lambda(t)$ is an increasing function of t. The latter shows that

$$\dim K_{\lambda(t)}/K_{\lambda[t,t+\varepsilon]} = \dim P_\xi^u \lambda(t+\varepsilon)/P_\xi^u \lambda(t)$$

Suppose now we try to understand the function $\dim P^u_\xi \varphi(t)$ where $\varphi \in M(\lambda)$.

<u>Question</u>: If $\varphi \in M(\lambda)$, then is the function $e^{-\infty \xi} \varphi(t)$ also in $M(\lambda)$?

Suppose $\varphi(t) = k_1 \cdots k_i \lambda(t)$ for $t \in [t_i, t_{i+1}]$ and that we have found elements $k_1', \ldots, k_j'$ such that

$$e^{-\infty \xi} \varphi(t) = k_1' \cdots k_j' \lambda(t) \qquad t \in [t_j, t_{j+1}] \quad \text{for } j < i$$

Now we consider the i-th segment $k_1 \cdots k_i \lambda(t)$ for $t \in [t_i, t_{i+1}]$. We have seen that the operation of going from a point γ in a chamber C in Ξ to $k\gamma$ then to $e^{-\infty \xi} k\gamma$ is the same as multiplying by zn with $z \in K_\xi$, $n \in N$. Consequently there exists a $h \in K$ such that

$$e^{-\infty \xi} k_1 \cdots k_i \lambda(t) = h \lambda(t) \qquad \text{for } t \in [t_i, t_{i+1}].$$

It therefore follows we can find the required k_i' by:
$$h = k_1' \cdots k_i' .$$

Therefore if I am interested in $\dim P^u_\xi \varphi(t)$, I can suppose $\varphi(t)$ is a broken geodesic in $\mathfrak{p}_\xi$ everywhere K-conjugate to $\lambda(t)$. What I want to prove is that if $\dim P_\xi^u \cdot \varphi(1) = \dim P_\xi^u \cdot \lambda(1)$, then φ and λ are K_ξ-conjugate.

~~So proceeding by induction, I can suppose that $\dim P_\xi^u \varphi(t_i)$~~

Let us consider the segment $t \in [t_i, t_{i+1}]$ where

$$\varphi(t) = h\lambda(t)$$

$h = k_1 \cdots k_i$. Since both φ and λ are geodesics in chambres of $\mathfrak{p}_\xi$ we know h can be taken in the form $z n$ with $z \in K_\xi$ and $n \in N$. I want to calculate the quantity

$$\dim P_\xi^u \varphi(t_i + \varepsilon) - \dim P_\xi^u \varphi(t_i)$$

and so I might as well replace φ by $w\lambda$.

$$\dim P_\xi^u (w\lambda(t_i+\varepsilon)) - \dim P_\xi^u (w\lambda(t_i))$$

$$= \operatorname{card}\{\alpha \mid \alpha(\xi) > 0,\ \alpha(w\lambda(t_i+\varepsilon)) < 0,\ \alpha(w\lambda(t_i)) = 0\}$$

$$\leq \operatorname{card}\{\alpha \mid \alpha(w\lambda(t_i+\varepsilon)) < 0,\ \alpha(w\lambda(t_i)) = 0\}$$

$$= \dim K_{w\lambda(t_i)} / K_{w\lambda(t_i+\varepsilon)} = \dim K_{\lambda(t_i)} / K_{\lambda(t_i+\varepsilon)}.$$

(Here I am using complex dimension)

Thus

$$\dim P^u_\xi\varphi(t_{i+1}) - \dim P^u_\xi\varphi(t_i) \leq \dim K_{\lambda(t_i)}/K_{\lambda[t_i,t_{i+1}]}$$
$$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad i \qquad\quad i,i+1$$

with equality $\Longleftrightarrow$

i) $\dim P^u_\xi\varphi(t_i+\varepsilon) = \dim P^u_\xi\varphi(t_{i+1})$ i.e.

$$\left.\begin{array}{l}\alpha(w\lambda(t_{i+1}-\varepsilon)) < 0 \\ \alpha(w\lambda(t_{i+1})) = 0\end{array}\right\} \Longrightarrow \alpha(\xi) < 0$$

ii) $\alpha(w\lambda(t_i+\varepsilon)) < 0,\ \alpha(w\lambda(t_i)) = 0 \Longrightarrow \alpha(\xi) > 0.$

So

$$\dim P^u_\xi\varphi(1) - \dim P^u_\xi\varphi(0)$$
$$\leq \dim K_0 \times K_{01} \dots \times K_m/K_{m,m+1}$$
$$\leq \dim M(\lambda)$$

with equality iff for $i = 0,\dots,m$ the roots α such that $\alpha(\varphi(t_i)) = 0$ satisfy

$$\alpha(\varphi(t_i-\varepsilon)) > 0 \Longrightarrow \alpha(\xi) > 0$$
$$\alpha(\varphi(t_i+\varepsilon)) < 0 \Longrightarrow \alpha(\xi) > 0$$

Go back: φ is a modification of λ, and I want the function $\dim P^u_\xi\varphi(t)$. Without changing this function I can replace φ by $e^{-\infty\xi}\varphi$, and so assume $\varphi(t) \in P_\xi$. But also multiplying by an element of K_ξ doesn't change the dimension of the P^u_ξ orbit. So we can modify $\varphi(t)$ using elements of K_ξ without changing the dimension of the P^u_ξ-orbit function, and so suppose φ is a modification of λ lying in E.

Now I would like to show that $\dim P^u_\xi \varphi(1) = \dim P^u_\xi \lambda(1)$ implies $\varphi = \lambda$.

~~Suppose we first try to show $\dim P^u_\xi \varphi(0) = \dim P^u_\xi \lambda(0) = 0 \Longrightarrow \varphi(0) = \lambda(0) = \xi$. Now $\varphi(0) = w\xi$, so we have~~

$$\cancel{P^u_\xi w\xi = w\xi}$$

~~If $w\xi \neq \xi$, then if C is a chambre containing ξ, we know $w\xi \notin C$, hence $\exists\, \alpha \ni \alpha(\xi) > 0$ and $\alpha(w\xi) < 0$; this implies $P^u_\xi w\xi$ is of positive dimension, so we see $w\xi = \xi$.~~

Next suppose $\lambda(t)$ is contained in a single chambre C whence $\varphi(t) = w\lambda(t)$ for some w. Assume $\dim P^u_\xi \lambda(1) = \dim P^u_\xi \varphi(1)$. Now $\dim P^u_\xi \lambda(1)$ is the number of α such that $\alpha(\xi) > 0$ and $\alpha(\lambda(1)) < 0$. Since $\lambda(t) = \xi + t(\lambda(1) - \xi)$ is contained in a chambre, there is no such α, hence $\dim P^u_\xi \lambda(1) = 0$.

~~Hence dim $P_\xi^u(w\lambda(1)) = 0$ which means~~
~~that no root α hyperplane separates~~
~~ξ and $w\lambda(1)$.~~

Suppose first that $\lambda(t)$ is contained in a single chambre, whence $\varphi(t) = w\lambda(t)$ where $w\xi = \xi$. Then $P_\xi^u w\lambda(1) = w\lambda(1)$ has the same dimension as $P_\xi^u \lambda(1)$, so the conjecture on page 24 about
$\dim P_\xi^u \varphi(1) = \dim P_\xi^u \lambda(1) \Longrightarrow \varphi = \lambda$ is nuts.

July 7, 1975

Compactification of $P_\xi \eta$. $\qquad \xi, \eta \in E$.

Review: I considered the function $|k\eta - \xi|^2$ on the orbit $K\eta$. Its critical points fall into non-degenerate critical submanifolds. The critical points for this function are elements of $K\eta \cap \mathfrak{p}_\xi$; the K_ξ-orbits on $K\eta \cap \mathfrak{p}_\xi$ are non-degenerate critical submanifolds, and they are in 1-1 correspondence with elts of

$$K_\xi \backslash K\eta \cap \mathfrak{p}_\xi = G_\xi \backslash K\eta \cap \mathfrak{p}_\xi$$

$$= G_\xi \backslash (P_\xi^u \backslash K\eta) = P_\xi \backslash K\eta$$

$$\simeq W_\xi \backslash W\eta$$

Presumably $P_\xi \eta$ is the decreasing submanifold for critical submanifold $K_\xi \eta$.

I wish to compactify $P_\xi \eta$ via the Bott-Samelson method. Let $\lambda(t) = \xi + t(\eta - \xi)$ be the line joining ξ to η, and $M(\lambda)$ the space of modifications of λ.

$$M(\lambda) = K_0 \times^Z K_1 \times^Z \cdots \times K_m / Z \qquad Z = K_\xi \cap K_\eta$$

$P_\xi^u \cdot K_\xi \eta$
$\parallel$

Note that both $M(\lambda)$ and $P_\xi \eta$ are fibred over $K_0/Z = K_\xi / K_\xi \cap K_\eta = K_\xi \eta$, hence to show that $M(\lambda)$ is a compactification of $P_\xi \eta$ it should be enough to show that the map

$$K_1 \times^Z \cdots \times^Z K_m / Z \longrightarrow K\eta$$
$$(k_1, \ldots, k_m) \longmapsto k_1 \cdots k_m \eta$$

is a compactification of $P_\xi^u \eta$.

Consider K_i/Z, which has roots those α such that $\alpha(\xi) \neq 0$, $\alpha(\lambda(t_i)) = 0$; here $i = 1, \ldots, m$. Inside K_i we have the unipotent group U_i with roots α such $\alpha(\xi) > 0$ and $\alpha(\lambda(t_i)) = 0$. The orbit $U_i Z/Z$ is open and dense in K_i/Z. So what I want to prove now is that

$$\begin{array}{c} (U_1 Z) \times^Z (U_2 Z) \times^Z \cdots \times^Z (U_m Z/Z) \\ \uparrow \wr \\ U_1 \times U_2 \times \cdots \times U_m \end{array}$$

gets mapped isomorphically onto $P_\xi^u \eta$. But this is clear because

$$U_1 \times U_2 \times \cdots \times U_m \xrightarrow{\sim} P_\xi^u \cap P_{-\eta}^u .$$

<u>Prop:</u> If $gP^u_\xi g^{-1} \subset P_\xi$, then $g \in P_\xi$.

<u>Proof:</u> Since $P_\xi N P_\xi = G$, we can suppose $g \in N$. The hypothesis implies $P^u_\xi \cdot g^{-1}\xi = g^{-1}\xi$, hence for any root α such that $\alpha(\xi) > 0$ we must have $\alpha(g^{-1}\xi) \geq 0$.

<u>Lemma:</u> If $w \in W$ and $w\xi \neq \xi$, then $\exists\, \alpha \in \Phi$ such that $\alpha(\xi) > 0$ and $\alpha(w\xi) < 0$

Assuming this we see $g^{-1}\xi = \xi$ i.e. $g \in P_\xi$ as claimed.

Proof of lemma: Assume that no α with $\alpha(\xi) > 0$, $\alpha(w\xi) < 0$ exists, that is, that ξ and $w\xi$ are not separated by a root hyperplane. Let C be a chambre containing the midpoint of $\xi, w\xi$; clearly C contains ξ and $w\xi$. But this implies $w\xi = \xi$, for we know C is a fundamental domain for the action of W.

<u>Corollary:</u> P_ξ is the normalizer of P^u_ξ; P_ξ is its own normalizer.

~~Variant: Assume $\xi, \eta \in E$ are in the same chambre (not separated by a root hyperplane). If $g P^u_\xi g^{-1} \subset P_\eta$, then $g \in P_\eta$. Note~~

Equivalent conditions for $\xi, \eta \in E$

i) $\exists$ chambre C containing ξ and η

$\Downarrow$ ii) $\nexists$ no root hyperplane separating ξ and η

$\Downarrow$ iii) $P^u_\xi \subset P_\eta$ (i.e. $\alpha(\xi) > 0 \Longrightarrow \alpha(\eta) \geq 0$)

The implication ii) $\Longrightarrow$ i) is done above: let C be a chambre containing the midpoint of $[\xi, \eta]$.

Prop: Suppose ξ, η contained in the same chambre. Then

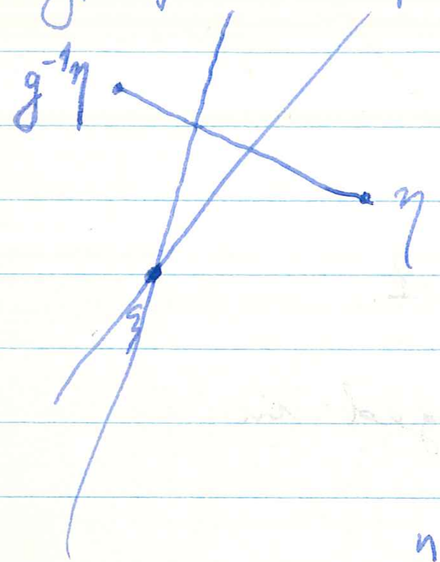
$$g P^u_\xi g^{-1} \subset P_\eta \Longleftrightarrow g \in P_\eta N_\xi$$

Proof: $\Longleftarrow$ Clear.

$\Longrightarrow$ Since $G = P_\eta N P_\xi$ we can suppose $g \in N$. Since

$$P^u_\xi \subset g^{-1} P_\eta g = P_{g^{-1}\cdot\eta}$$

we know that $g^{-1}\eta$ and ξ are not separated by a root hyperplane. It follows that any root hyperplane separating η and $g^{-1}\eta$ contains ξ.



If $n \in N_\xi$ is such that $ng^{-1}\eta$ is an element of $W_\xi \cdot g^{-1}\eta$ of minimum distance to η, then clearly no root hyperplane separates η and $ng^{-1}\eta$. Then [redacted] $\eta = ng^{-1}\eta$ so

$ng^{-1} \in P_\eta$, hence $g \in P_\eta \cdot N_\xi$. QED.

The preceding holds [redacted] even if ξ, η do [redacted] not belong to E since every pair can be put into an apartment.

Note that i) ii) iii) above are equiv. to
iv) $P_\xi \cap P_\eta$ contains a [redacted] P_ζ
v) $P_\xi^u, P_\eta^u \subset P_\zeta^u$ for some ζ.

In effect, [redacted] if we have $P_\xi, P_\eta \supset P_\zeta$ [redacted] then choosing a max. torus contained in P_ζ both ξ, η lie in the corresponding apartment, and they lie in any chambre containing ζ.

The point really is that P_ξ^u [redacted] $\subset P_\zeta^u \Longleftrightarrow P_\xi \supset P_\zeta$.

In effect, one can suppose both ξ and ζ are in E. Then $P^u_\xi \subset P^u_\zeta$ means $\alpha(\xi) > 0 \Rightarrow \alpha(\zeta) > 0$, which is equivalent to $\alpha(\zeta) \leq 0 \Rightarrow \alpha(\xi) \leq 0$ or $\alpha(\zeta) \geq 0 \Rightarrow \alpha(\xi) \geq 0$ which means $P_\zeta \supset P_\xi$.

July 9, 1975.

It is now time to work out the theory for the big building $\mathcal{X}$.

First topic is Iwasawa decomposition. Let $\xi \in \mathcal{X}$ and define

$$\mathcal{P}_\xi = \{ g \in \mathcal{G} \mid \xi^{-1} g \xi \text{ converges in } G \text{ as } \operatorname{Im} t \to -\infty \}$$

(Note $\operatorname{Im} t \to -\infty$ means $z = e^{2\pi i t} \to +\infty$). Assuming $\xi \in \tilde{E}$, ~~[illegible]~~ i.e. $\xi = e^{2\pi i x t}$ with $x \in E$, then as

$$\xi^{-1} \exp(f(z) X_\alpha) \xi = \exp(e^{-2\pi i \alpha(x) t} f(z) X_\alpha)$$

we see that $\exp(z^{-n} X_\alpha) \in \mathcal{P}_\xi$ iff

$$e^{-2\pi i \alpha(x) t} z^{-n} = e^{-2\pi i (n + \alpha(x)) t}$$

converges as $\operatorname{Im} t \to -\infty$, i.e. $n + \alpha(x) \geq 0$.

Example: $G = SL_n$. Let x be diagonal with entries $x_1 > \cdots > x_n > x_1 - 1$. Then the roots are pairs (i,j) $i \neq j$. So if $\alpha(x) = x_i - x_j$ with $i < j$, $1 > x_i - x_j > 0$, so $n \geq 0$. If however $i > j$ then $-1 < x_i - x_j < 0$, so $n \geq 1$. Thus P_x is the Iwahori subgrp.

$$P_\xi^u = \{ g \mid \xi^{-1} g \xi \longrightarrow 1 \quad \text{as } \operatorname{Im} t \to -\infty \}.$$

If $g \in P_\xi$, then

$$l = \lim_{\operatorname{Im} t \to -\infty} \xi(t)^{-1} g(z) \xi(t)$$

$$= \lim_{\operatorname{Im} t \to -\infty} \xi(t+1)^{-1} g(z) \xi(t+1)$$

$$= \xi(1)^{-1} l \, \xi(1)$$

so $l \in G_{\xi(1)}$. Conversely if $\gamma \in G_{\xi(1)}$, then

$$\xi \gamma \xi^{-1} = f(z) e^{tX} \gamma e^{-tX} f(z)^{-1}$$

$$= f(z) e^{tX} e^{-\gamma . X} \gamma f(z)^{-1} \quad \in \mathcal{G}$$

is evidently in P_ξ. Put

$$\mathcal{G}_\xi = \xi G_{\xi(1)} \xi^{-1}.$$

Then
$$\mathcal{P}_\xi = \mathcal{G}_\xi \ltimes \mathcal{P}^u_\xi .$$

Calculate $\mathcal{K}_\xi$ = stabilizer of ξ in $\mathcal{K}$:

$$k\,\xi\,k(1)^{-1} = \xi \iff \xi(1) \in K_{k(1)}$$

and $k = \xi\, k(1)\, \xi^{-1}$.

Thus
$$\mathcal{K}_\xi = \xi\, K_\xi\, \xi^{-1} = \mathcal{K} \cap \mathcal{G}_\xi .$$

~~Calculate $\mathcal{K}\cap P_\xi$. If $g \in \mathcal{K} \cap P_\xi$, then $\xi^{-1} g\, \xi$ is holomorphic in t with values in K for t real. The same is true for~~

$$\cancel{\left(\xi^{-1}(\bar t)\, g(\bar t)\, \xi(\bar t)\right)^{*-1}}$$

Calculate $\mathcal{K}\cap P_\xi$. If $g \in \mathcal{K}$, then
$$\left[\xi^{-1}(t)\, g(z)\, \xi(t)\right]^{-1} = \left[\xi^{-1}(\bar t)\, g\!\left(\tfrac{1}{\bar z}\right) \xi(\bar t)\right]^{*}$$

because both sides are holomorphic in t and they coincide for t real. If $g \in P_\xi$ the left side is periodic and bounded as $\operatorname{Im} t \to -\infty$, whereas the right side is bounded as $\operatorname{Im} t \to +\infty$. Thus the holomorphic function must be constant i.e. in K_ξ. Thus

$$K \cap P_\xi = K_\xi .$$

The Iwasawa decomposition says

$$G = K P_\xi$$

$$= K \times^{K_\xi} P_\xi = K \times \mathcal{X}_\xi \times P_\xi^u$$

where $\mathcal{X}_\xi = \xi X_{\xi(1)} \xi^{-1} \subset G_\xi$. $\quad X_{\xi(1)} = e^{i k_{\xi(1)}}$

Next topic is the Bruhat decomposition, or more generally the classification of P_ξ^u-orbits. We first need a set of ξ-fixpts:

<u>Lemma</u>: Let $\xi \in \mathcal{X}$, and let $Y \in k_{\xi(1)}$ (i.e. $Y \in k$ and $Ad(\xi(1))Y = Y$). Then

$$\xi e^{tY}$$

is a special path in K.

<u>Proof</u>: Because K_Y

is connected, I know that $\xi(1) = e^X$ where $[X, Y] = 0$. Then $\xi(t) = f(z)e^{tX}$ some $f \in \mathcal{K}'$, so

$$\xi(t)e^{tY} = f(z)e^{t(X+Y)}$$

Q.E.D.

My aim is to prove that each $\mathcal{P}^u_\xi$-orbit contains a unique element of the form ξe^{tY} with $Y \in k_{\xi(1)}$.

Consider **uniqueness** in the special case where $\xi = \mathbf{1}\ (= \tilde{o})$. Let $\eta_i = e^{tY_i}$ $Y_i \in k$ and let $g \in \mathcal{P}^u_\xi$ be such that $g * \eta_1 = \eta_2$, which means that

$$\eta_2^{-1} g \eta_1 \quad \text{converges in } G \text{ as } \operatorname{Im} t \to -\infty.$$

Replace t by $t+a$:

$$\eta_2^{-1}(t+a)\, g(e^{2\pi i(t+a)})\, \eta_1(t+a) = e^{-aY_2}e^{-tY_2} g(e^{2\pi i a}z) e^{tY_1}e^{aY_1}$$

So we see that $g(e^{2\pi i a}z) * \eta_1 = \eta_2$, hence

$$g(z)^{-1} g(e^{2\pi i a}z) \in \mathcal{P}_{\eta_1}$$

But $g \in \mathcal{P}^u_\xi$ which means that it is holomorphic at $z = \infty$, with $g(\infty)=1$. Thus if we let $e^{2\pi i a} \to +\infty$ and use the fact that $\mathcal{P}_{\eta_1}$ is closed (look at GL_n)

then we get $g^{-1} \in \mathcal{P}_{\eta_1}$, hence $\eta_1 = \eta_2$.

Let's next consider uniqueness in general. Let $g \in \mathcal{P}^u_\xi$, $\eta_i = \xi e^{tY_i}$ where $Y_i \in \mathfrak{k}_{\xi(1)}$, be such that $g * \eta_1 = \eta_2$. This means

$$\eta_2^{-1} g \eta_1 = e^{-tY_2} \xi^{-1} g \xi e^{tY_1}$$

converges as $\operatorname{Im} t \longrightarrow -\infty$. Suppose $\xi = e^{tX}$, then

$$e^{-(t+a)Y_2} e^{-(t+a)X} (g) (e^{2\pi i a} z) e^{(t+a)X} e^{(t+a)Y_1}$$

$$= e^{-aY_2} \left[\eta_2^{-1} \left\{ e^{-aX} g(e^{2\pi i a} z) e^{aX} \right\} \eta_1 \right] e^{aY_1}$$

converges as $\operatorname{Im} t \longrightarrow -\infty$. Thus

$$g_a(z) = e^{-aX} g(e^{2\pi i a} z) e^{aX}$$

satisfies $g_a * \eta_1 = \eta_2$, whence

$$g(z)^{-1} g_a(z) \in \mathcal{P}_{\eta_1}.$$

However $g_a(z) = \xi(a)^{-1} g(z \cdot e^{2\pi i a}) \xi(a)$ converges to 1 as $\operatorname{Im} a \longrightarrow -\infty$, because

$$\xi(a)^{-1} g(e^{2\pi i(a)}z)\, \xi(a+t)$$

$$= \xi(t)\, \xi(a+t)^{-1} g(e^{2\pi i(a+t)})\, \xi(a+t)\, \xi(t)^{-1}$$

$$\longrightarrow \xi(t)\, \xi(t)^{-1} = 1.$$

Thus $g^{-1} g_a \in P_{\eta_1} \implies g^{-1} \in P_{\eta_1}$.

In general suppose $\xi = k(z)e^{tX}$, $k(1) = 1$.

Then

$$\eta_2^{-1} g \eta_1 = e^{-tY_2} \xi^{-1} g \xi e^{tY_1} \quad \text{conv. as } \mathrm{Im}\, t \to \infty$$

$$\implies \underbrace{e^{-tY_2} \xi^{-1}(t)}\ \xi(t)\, \xi(t+a)^{-1} g(e^{2\pi i a} z)\, \xi(t+a)\, \underbrace{\xi(t)^{-1}}_{\xi(t) e^{tY_1}}$$

converges as $\mathrm{Im}(t) \to -\infty$. Thus if

$$g_a = k(z) e^{tX} e^{-tX} e^{-aX} g(e^{2\pi i a} z).$$

$k \cdot \xi' = \xi$.

In general let $\xi = k \cdot \xi'$, say $k \in K'$, whence $\xi = k\xi'$ and

$$\eta_i = \xi e^{tY_i} = k \xi' e^{tY_i} = k \eta_i' \qquad \eta_i' = \xi' e^{tY_i}$$

so our hyp. $\eta_2^{-1} g \eta_1$ converges translates to $\eta_2'^{-1} k^{-1} g k \eta_1'$ converges. But $g \in P_\xi^u \implies k^{-1} g k \in P_{\xi'}$, so the above proof shows that $k^{-1}gk \in P_{\eta_1'}$, hence $g \in P_{\eta_1}$ and $\eta_1 = \eta_2$.

$$\eta_2(a+t)^{-1} g(e^{2\pi i a}z)\,\eta_1(a+t) =$$

$$= e^{-aY_2} e^{-tY_2} \xi(a+t)^{-1} g(e^{2\pi i a}z)\, \xi(a+t) e^{tY_1} e^{aY_1}$$

$$= e^{-aY_2} \underbrace{\left[e^{-tY_2} \xi(t)^{-1}\right]}_{\eta_2(t)^{-1}} \xi(t)\xi(a+t)^{-1} g(e^{2\pi i(a+t)})$$

$$\xi(a+t)\xi(t)^{-1} \underbrace{\left[\xi(t) e^{tY_1}\right]}_{\eta_1(t)} e^{aY_1}$$

The fact this converges as $t \to -i\infty$ means that if

$$g_a(e^{2\pi i t}) = \xi(t)\xi(a+t)^{-1}(g)(e^{2\pi i(a+t)})\xi(a+t)\xi(t)^{-1}$$

(note: $g_a \in \mathcal{G}$ as $\xi(t)\xi(a+t)^{-1}$ is periodic), then

$g_a * \eta_1 = \eta_2$ hence $g^{-1} \cdot g_a \in P_{\eta_1}$. But as $a \to -i\infty$

$$g_a(e^{2\pi i t}) = \xi(t)\,\xi(a+t)^{-1} g(e^{2\pi i(a+t)})\,\xi(a+t)\,\xi(t)^{-1}$$

$$\to \xi(t)\xi(t)^{-1} = 1.$$

Thus $g \in P_{\eta_1}$.

July 10, 1975:

Problem: Show each P_ξ^u-orbit on X has a unique point of the form ξe^{tY} where $Y \in k_{\xi(1)}$.

In the spherical building, we were able to give a formula for this "center" of $P_\xi^u * \eta$ namely

$$\eta_0 = \lim_{s\to\infty} e^{-s\xi} * \eta$$

We want an analogue of this idea for X.

Take $\xi = 0$. To any $\eta \in X$ I wish to associate something of the form $\eta_0 = e^{tX}$ in the same P_ξ^u-orbit. $P_\xi^u = \{g \mid g \text{ holom at } z=\infty\}$. Note that

$$\eta_0(t+a) = \eta_0(t)\,\eta_0(a),$$

hence X_0 consists of η_0 fixed under

$$\eta_0 \longmapsto \eta_0(t+a)\,\eta_0(a)^{-1}.$$

Now we ought to be able to make an action of $\mathbb{C}^*$ on the building corresponding to the action $g(z) \xrightarrow{b} g(bz)$ on G. This is because the building ought to be intrinsically associated to G.

Recall the following variant of the Mumford definition of the building $\mathfrak{p}$: One considered all elements X of $\mathfrak{g}$ which are conjugate to elements of $\mathfrak{p}$, and one introduced the equivalence relation $X \sim Y \iff$

$$e^{-tY} e^{tX} \text{ converges as } t \to +\infty.$$

In each class there is a unique representative in $\mathfrak{p}$. The G-action on the building corresponds then to conjugation.

The analogue for $\mathcal{X}$ is as follows. I consider ~~[illegible]~~ holomorphic maps of $\{\operatorname{Im} t \ll 0\}$ into G of the form

$$\xi(t) = f(z)\, e^{tZ}$$

where $f \in \mathcal{G}$ is meromorphic at ∞, and where Z is conjugate to an element of $\mathfrak{k}$. Call ξ_1, ξ_2 equivalent if $\xi_2^{-1}\xi_1$ converges as $\operatorname{Im} t \to -\infty$. Claim that there is a unique element ξ_1 of $\mathcal{X}$ in each equivalence class.

Uniqueness results from the fact that if $\xi, \eta \in \mathcal{X}$, then because $\xi^{-1}\eta$ has ~~[illegible]~~ values in

K for real t, we have

$$\left[\xi^{-1}\eta(\bar{t})\right]^* = \left[\xi^{-1}\eta(t)\right]^{-1}$$

so if $\xi^{-1}\eta$ converges as $\operatorname{Im} t \to -\infty$, then it is a bounded analytic function, hence constant, hence 1.

Existence: Suppose $\zeta = f(z)e^{tZ}$ given and we wish to produce $\xi \in \mathfrak{X}$ such that $\xi^{-1}\zeta$ converges at $-i\infty$. By assumption $\exists g \in G$ such that $gXg^{-1} = Z$ with $X \in k$. Thus $\zeta = f(z)ge^{tX}g^{-1}$ and we reduce to the case where $Z = X \in k$. Let $\eta = e^{tX}$; using the Iwasawa decomposition $\mathcal{G} = KP_\eta$, we can factor $f = kh$, whence

$$\zeta = khe^{tX} = \underbrace{k\cdot e^{tX}k(1)^{-1}}_{=\xi}\ \underbrace{k(1)e^{-tX}he^{tX}}_{\text{converges}}$$

QED.

Next point: Given $\xi(t) = f(z)e^{tX}$ we can consider

$$\xi(t+a) = f(e^{2\pi i a}z)e^{aX}e^{tX}$$

which yields another element of $\mathfrak{X}$. To

[$\mathcal{G}$ action on $\mathcal{X}$ in these new terms is
$$g * cl(\xi_1) = cl(g\xi_1)$$
]

find it we have only to factor
$$f(e^{2\pi i a} z) = k(z)\, h(z)$$
with $h \in \mathcal{P}_\eta$, $\eta = e^{tX}$, and $k \in \mathcal{K}$. Then
$$cl\left(\xi(t+a)\right) = cl\Big(k(z)\, \underbrace{h(z)e^{aX}}_{\in \mathcal{P}_\eta}\, \underbrace{e^{tX}}_{\eta}\Big)$$
$$= cl(k\, e^{tX})$$
$$= k(z)\, e^{tX}\, k(1)^{-1}.$$

So we define in this way an operation T_a on $\mathcal{X}$:
$$T_a \xi = cl(\xi(t+a)).$$
If $T_a \xi = \xi$ for all a
$$\xi(t)^{-1}\xi(t+a) = e^{-tX} f(z)^{-1} f(e^{2\pi i a} z)\, e^{tX} e^{aX}$$
is convergent for each a. But if a is taken to be real, then $cl(\xi(t+a)) = \xi(t+a)\,\xi(a)^{-1}$. Thus
$$\xi(t+a) = \xi(t)\,\xi(a)$$
for a real, hence identically, showing that $\xi(t) = e^{tX}$ for some X.

July 11, 1975

I recall from yesterday that I can think of the building as equivalence classes of functions of the form

$$\gamma(t) = g(z)e^{tX}$$

with $g \in \mathcal{G}$ and X conjugate to an elt. of $\mathfrak{k}$. The $\mathcal{G}$-action on the building is given by left multiplication. In addition I have the translation operation

$$(T_a \gamma)(t) = \gamma(t+a)$$

One has

$$\big(T_a(g\cdot\gamma)\big)(t) = (g\gamma)(t+a) = g(e^{2\pi i a}z)\,\gamma(t+a)$$

$$= (T_a g \cdot T_a \gamma)(t).$$

where $(T_a g)(z) = g(e^{2\pi i a}z)$. The fixpts for the group T_a are the geodesics e^{tX}.

Take $\xi = \tilde{0} = 1$, whence $P_1 = \{g \in \mathcal{G} \mid g(\infty) \; \exists\}$. $P_1^u = \{g \mid g(\infty) = 1\}$. I wish to show that any P_1^u-orbit is stable under the group T_a. But if $h \in P_1^u$ then

$$T_a(h\cdot\gamma) = T_a h \cdot T_a \gamma$$

and $T_a h \in P_1^u$; hence if $T_a\gamma = \gamma$ the orbit is stable.

Incidently we see that if ζ is fixed under T_a then

$$\lim_{a\to -i\infty} T_a(h\cdot\zeta) = \lim_{a\to -i\infty} h(e^{2\pi i a}z)\,\zeta = \zeta$$

so we have this nice procedure for finding the fixpt for the P_1^4-orbit.

Next we wish to generalize this for arbitrary ξ. Notice that if $\zeta = g e^{tZ}$ is in our building and $\xi = f e^{tX}$, then

$$\left(\xi T_a \xi^{-1}\zeta\right)(t) = f(z)e^{tX}\, T_a\left(e^{-tX} f(z)^{-1} g(z) e^{tZ}\right)$$
$$= f(z)\, e^{-aX}(f^{-1}g)(e^{2\pi i a}z)\, e^{tZ} e^{aZ}$$

is again in the building. In fact

$$\xi\left(T_a\left(\xi^{-1}\zeta\right)\right) = \xi(T_a\xi)^{-1}(T_a\zeta)$$

is in the building because

$$\xi(T_a\xi)^{-1} = f(z)e^{tX} e^{-(t+a)X} f(e^{2\pi i a}z)^{-1}$$

is in G. (Note that $\xi = fe^{tX}$ should not be allowed to vary in its equivalence class for $\xi \sim \eta \not\Rightarrow \xi(T_a\xi)^{-1}\gamma \sim \eta(T_a\eta)^{-1}\gamma$.)

~~If ζ is fixed under $\xi T_a \xi^{-1}$ for all a, then for a real, ξ and ζ in X~~

Suppose a real. Then

$$(\xi T_a \xi^{-1} \zeta)(t) = \xi(t)\,\xi(t+a)^{-1}\zeta(t+a)$$

has values in K; hence to normalize it we right multiply by the inverse of its value at $t=0$.

$$(\xi T_a \xi^{-1} \zeta)(t) = \xi(t)\,\xi(t+a)^{-1}\zeta(t+a)\,\zeta(a)^{-1}\xi(a)$$

This equals $\zeta(t) \iff \xi^{-1}\zeta$ is a 1 parameter subgroup, i.e.

$$\zeta = \xi e^{tY}. \qquad Y \in k$$

So I have to know when $e^{tX}e^{tY}$ is a special path where $X, Y \in k$. Put $h(t) = e^{tX}e^{tY}$. Then assuming h is special we have

$$h(t+1) = h(t)\,e^{X}e^{Y}$$

But we have $h(t+1) = e^{X}h(t)e^{Y}$. Thus e^{X} commutes with $h(t)$, hence with e^{tY}. This proves that

$$\zeta \in \mathcal{X}_\xi \iff \zeta \text{ fixed under } \xi T_a \xi^{-1} \text{ for all } a.$$

Note we have proved:

<u>Lemma</u>: $e^{tX}e^{tY}$ special $\iff [e^{X}, Y] = 0$.