June 17, 1975. Morse theory + Lie groups.

Let K be a compact connected group with Lie algebra $\mathfrak{k}$, let $(\,,)$ be an invariant inner product on $\mathfrak{k}$, let E be a maximal abelian subspace of $\mathfrak{k}$ and T the corresponding maximal torus, and $\Phi \subset \mathrm{Hom}(E,\mathbb{R})$ the set of roots of $\mathfrak{k}$ with respect to T. Let $\mathcal{O}$ be an orbit of K in $\mathfrak{k}$: $\mathcal{O} = K\cdot\eta$.

Choose a regular element ξ of E (i.e. $\alpha(\xi)\neq 0$ all $\alpha\in\Phi$) and consider the function on $\mathcal{O}$

$$f(k\eta) = |k\eta-\xi|^2 = \underbrace{|\eta|^2+|\xi|^2}_{\text{const.}} - 2(k\eta,\xi)$$

Calculation shows f has $\mathcal{O}\cap E$ for its set of critical points, that each critical point $k\eta$ is non-degenerate and has index equal to twice the number of root hyperplanes crossed in going along the straight line from ξ to $k\eta$.

Consequences from Morse theory:

1) $K\eta \cap E \neq \phi$. This implies conjugacy of maximal abelian subspaces of $\mathfrak{k}$, and also that if W = group of autos of E induced by elements of K, then $K\eta\cap E$ is a W-orbit on E.

$$\boxed{W\backslash E = K\backslash \mathfrak{k}}$$

2) $K\eta$ has a cell decomposition indexed by points of $W\eta$ (say $\eta \in E$), the dimension of the cell indexed by $w\eta$ being twice the number of root hyperplanes separating $w\eta$ and ξ. This implies the homology is free over $\mathbb{Z}$, 0 in odd dimensions, and a basis is given by the cells.

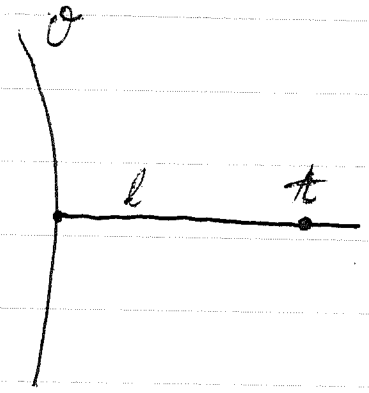
As $K\eta$ is connected, f has a unique minimum value $\Rightarrow$ $W\eta$ meets cone $C_\xi = \{x \mid \alpha(x) \geq 0$ for $\alpha(\xi) > 0\}$ in exactly one pt:

$$C_\xi \xrightarrow{\sim} W\backslash E.$$

Further $\eta \in C_\xi \iff |\eta - \xi| < |w\eta - \xi|$ for $w\eta \neq \eta$.

Since $K\eta$ has no 1-cells, $\pi_1(K\eta) = 0$ $\Rightarrow$ stabilizer of η in K is connected. Thus centralizers of tori are connected, and T is self-centralizing. (These facts are usually proved by showing every k is conjugate to an element of T, and that a group gen. by a torus and a centralizing element is gen. by a single element).

Next I wish to consider the group K acting by conjugation on itself. Bott – Samelson consider geodesics ℓ starting perpendicular to an orbit:



They prove the geodesic is perpendicular to all orbits it crosses, that a point t is a conjugate point if $\dim K_t > \dim K_\ell$, in which case the Jacobi fields along ℓ vanishing at t all arise from $\mathrm{Lie}(K_t)/\mathrm{Lie}(K_\ell)$. This result is "variational completeness" of the conjugation action of K on itself.

Suppose t is a regular element of T (I mean that the centralizer of t in K is E). We consider the space $\Lambda = \Lambda(K; t, \mathcal{O})$ of paths joining t to $\mathcal{O}$. Critical points for the energy function on Λ are geodesics ℓ joining t to $\mathcal{O}$ and perpendicular to $\mathcal{O}$. ℓ has to be perpendicular to the orbit $K \cdot t$ $\Rightarrow$ $\ell \subset T$; ~~Bott-Samelson show that the critical points of~~ conversely $\ell \subset T \Rightarrow \ell \perp \mathcal{O}$. Thus the critical points are geodesics in T from t to $O \cap T$, which may be identified with lines l joining ξ to a point of $p^{-1}(O \cap T)$, where $p : E \longrightarrow T$ is the exp. map and ξ is a given point in $p^{-1}(t)$. These critical pts turn out to be non-degenerate, and the index of l is twice the number of hyperplanes of the form $\alpha(x) = n$, $\alpha \in \Phi$, $n \in \mathbb{Z}$ crossed in going along l.

Consequences from Morse theory:

1) $O \cap T \neq \emptyset$. This means every element is conjugate to an element of T, and implies $O \cap T$ is a W-orbit of T:

$$W \backslash T \xrightarrow{\sim} K \backslash K$$

2) $\Lambda(K; t, O)$ has the homotopy type of a CW complex with even-dimensional cells, so these cells have to be a basis for the integral homology.

Take $O = 1$ and look at $H_0(\Lambda) =$ free abelian group gen. by $\pi_0 \Lambda = \pi_1 K$. Thus we find $|\pi_1 K| =$ number of points of $p^{-1}(1)$ contained in the small chambre C'_ξ. Choose ξ to be an interior point of fundamental cone close to 0, whence

$$C'_\xi = \{ x \in E \mid 0 \leq \alpha(x) \leq 1 \quad \text{all } \alpha \in \Phi^+ \}.$$

Assume now that K is simply-connected. Λ has the type of the fibre of the inclusion $\mathcal{O} \hookrightarrow K$ over t_0, hence $\pi_1 K = \pi_0 \mathcal{O} = 0 \implies \pi_0 \Lambda = 0$. Consequently there is a unique point of $p^{-1}(\mathcal{O} \cap T)$ contained in C'_ξ. So:

$$C'_\xi \xrightarrow{\sim} W|T \xrightarrow{\sim} K\backslash K.$$

Next note that because Λ has no 1-cells Λ is simply-connected $\implies \mathcal{O}$ is simply-connected. Therefore the centralizer of any element of K is connected.

Next we consider the symmetric space situation: K compact connected Lie group and σ is an involution. $X = K/K^\sigma$ is the symmetric space. We can identify X with the component of $\{x \mid \sigma x = x^{-1}, x \in K\}$ containing the identity, or with the orbit of 1 under the action $k \cdot y = k y (\sigma k)^{-1}$.

For the linear situation, we look at the action of K^σ on the tangent space to X at 1 which is $k_- = \{\xi \in k \mid \sigma\xi = -\xi\}$.

Let E_0 be a maximal abelian subspace of $\mathfrak{k}_-$, and S the corresponding torus. Let Φ_0 be the set of roots of $\mathfrak{k}$ with respect to S. Let $\xi \in E_0$ be regular, i.e. $\alpha(\xi) \neq 0$ for $\alpha \in \Phi_0$, let $K^\sigma \cdot \eta$ be a K^σ orbit in $\mathfrak{k}_-$, and consider the function

$$f(k\eta) = |k\eta - \xi|^2 = |\eta|^2 + |\xi|^2 - 2(k\eta, \xi)$$

Suppose $k\eta$ is a critical point. Then for $X \in \mathfrak{k}_-$

$$f(\exp(X)k\eta) = |\eta|^2 + |\xi|^2$$

$$-2((I + \mathrm{ad}\,X + (\mathrm{ad}\,X)^2 + \dots)k\eta, \xi)$$

hence $0 = ((\mathrm{ad}\,X)k\eta, \xi) = ([X, k\eta], \xi) = (X, [k\eta, \xi])$. Since $[k\eta, \xi] \in \mathfrak{k}_-$ this implies $[k\eta, \xi] = 0$, so $k\eta \in E_0$. Thus $K^\sigma\eta \cap E_0$ is the set of critical points.

The Hessian at a critical point $k\eta$ is

$$-2((\mathrm{ad}\,X)^2 k\eta, \xi) = +2([k\eta, X], [\xi, X]).$$

I need to understand the representations of $S' = \mathbb{Z}/2\mathbb{Z} \ltimes S$, where the $\mathbb{Z}/2\mathbb{Z}$ acts as -1. Let V be an irreducible complex repn. of S'; if S acts trivially, then it is a character of $\mathbb{Z}/2\mathbb{Z}$. Otherwise if the character $\chi : S \to S^1$ occurs, so does χ^{-1}, and as $\chi \neq \chi^{-1}$ for $\chi \neq 0$, V is 2-dimensional. V is induced from a non-trivial character χ of S. V^* is induced from χ^{-1} so $V^* \simeq V$. But S' can't have non-trivial quaternion characters as only -1 is of order 2 in S^3. So V must be the complexification of a real 2-dim'l representation. In fact one takes $\chi : S \longrightarrow S^1$ and lets S^1 act on $\mathbb{C}$ thru χ and σ act via $-$. This gives us the irreducible real repns. V of S' on which S acts non-trivially.

(Actually S' is a generalized dihedral group so we know its representations are all defined over $\mathbb{R}$)

So therefore we know that as an S-module k is the direct sum of its centralizer $m \oplus E_0$ plus root spaces $k_{\pm\alpha}$ indexed by each pair $\pm\alpha$ in Φ_0. If I select one of this pair, then $k_{\pm\alpha}$ gets a complex structure such that $\exp(u)$ is multiplication by $e^{2\pi i \alpha(u)}$. Moreover σ is a conjugation for this complex structure.

Let X have the component X_α in $k_{\pm\alpha}$, X_0 in m. Then

$$[k\eta, X_\alpha] = 2\pi i\, \alpha(k\eta)\, X_\alpha$$
$$[\xi, X_\alpha] = 2\pi i\, \alpha(\xi)\, X_\alpha$$

and their inner product is $(4\pi)^2 \alpha(k\eta)\alpha(\xi)\,|X_\alpha|^2$. Thus the Hessian is:

$$-2\big((\operatorname{ad} X)^2 k\eta, \xi\big) = (8\pi^2) \sum_{\alpha \in \Phi_0^+} \alpha(k\eta)\,\alpha(\xi)\,|X_\alpha|^2$$

For the critical point to be non-degenerate this form must be non singular on $k \ominus (k)_{k\eta}$, that is, for X such that $X_\alpha = 0$ for $\alpha(k\eta) = 0$, and $X_0 = 0$. Clear.

Finally the index of the critical point is the number of α such that $\alpha(k\eta) < 0$ $\alpha(\cdot\xi) > 0$ each α counted with multiplicity $\frac{1}{2} \dim k_{\pm\alpha}$. Note that $\frac{1}{2}\dim k_{\pm\alpha} = \dim(k_{\pm\alpha} \cap k_+)$ is the excess over $\dim m$ of the dim. of the stabilizer in K^σ of a general point of $\alpha = 0$ in E_0.

So Morse theory tells us that the orbit $K^\sigma\eta$ has a cell decomposition with cells indexed by points of $K\eta \cap E_0$ with indices given by the number of root hyperplanes crossed counted with multiplicities.

From the fact that $K^\sigma\eta \cap E_0$ is non-empty we get the K^σ-conjugacy of maximal abelian subspaces of k_-; moreover $K^\sigma\eta \cap E_0$ is a W_0-orbit of E_0, where $W_0 =$ subgroup of $\mathrm{Aut}(E_0)$ consisting of θ induced by elts of K^σ.

Now we can not at this point argue that the points of index 0 corresponds to components of $K\eta$. For this one shows that the closure of each cell in $K^\sigma\eta$ is the image of a birational map from a closed manifold. This will show that the cells give a basis for the mod 2 homology of $K^\sigma\eta$.

Let's admit the fact that the cells in K^σ_η give a basis for the mod 2 homology. I will replace K^σ by its connected component K_0; this won't affect any of the calculations made so far.

So now $K_0\eta$ is connected, hence there is exactly one zero-cell. This implies that there is exactly one point of the W_0-orbit $W_0\eta = K_0\eta \cap E_0$ (say $\eta \in E_0$) which is in the chambre $C_0 = \{x \in E_0 \mid \alpha(x) \geq 0 \text{ if } \alpha(\xi) > 0\}$ containing ξ. And this point is where $|k\eta - \xi|$ is minimum:

$$\eta \in C_0 \iff |k\eta - \xi| > |\eta - \xi| \quad \text{all } k \in K_0 - (K_0)_\eta$$

Thus $C_0 \simeq W_0 \backslash E_0 \simeq K_0 \backslash k_-$.

Digression: If K is 1-connected, then the group of lattice points in E (those in $p^{-1}(1)$, $p : E \twoheadrightarrow T$) is generated by the root vectors H_α. In effect, given a lattice point λ, one can move it by reflections into C'. But zero is the only lattice point in C'. Thus λ is in the orbit of 0 under the group $\tilde{W}$ gen. by reflections thru $\alpha \in \mathbb{Z}$, and $\tilde{W}$ preserves the lattice $\sum \mathbb{Z} H_\alpha$, so $\lambda \in \sum \mathbb{Z} H_\alpha$. ~~An analogous argument shows that if K is simple,~~

Note that C' is described by $C' = \{x \mid 0 \le \alpha(x) \le 1$ for all $\alpha \in \Phi^+\}$. Suppose K is simple, i.e. $\mathfrak{k}$ is an irreducible module, hence has a! maximal root $\psi \ni [X_\alpha, X_\psi] = 0$ all $\alpha \in \Phi^+$. ~~Known that if $\alpha_1, \alpha_2, \ldots, \alpha_\ell$ are simple roots~~ Let $\alpha_1, \alpha_2, \ldots, \alpha_\ell$ be the simple roots. If $\alpha \in \Phi$, then there is a sequence $1 \le i_1, \ldots, i_m \le \ell$ such that

$$\alpha,\ \alpha + \alpha_{i_1},\ \alpha + \alpha_{i_1} + \alpha_{i_2},\ \ldots,\ \alpha + \alpha_{i_1} + \cdots + \alpha_{i_m} = \psi$$

is a sequence of roots. (This results from the fact that the weight vector X_α has to be of the form

$$X_\alpha = \overset{\neq 0}{c}\ \underbrace{\mathrm{ad}(X_{-\alpha_{i_1}}) \cdots \underbrace{\mathrm{ad}(X_{-\alpha_{i_m}})\, X_\psi}_{X_{\psi - \alpha_{i_m}}}}_{X_{\psi - \alpha_{i_m} - \cdots - \alpha_{i_1}}}\ )$$

Thus it is clear that $\alpha_1(x) \ge 0, \ldots, \alpha_\ell(x) \ge 0$, $\psi(x) \le 1$ forces $\alpha(x) \le 1$ for all $\alpha \in \Phi^+$. We conclude therefore that C' is the simplex described by

$$\alpha_1, \ldots, \alpha_\ell \ge 0 \quad , \quad \psi \le 1$$

in the irreducible case.

Suppose again K simple and 1-connected, and let K_s be the centralizer of $s \in T$. I have seen K_s is the connected group containing T with roots $\Phi(\xi) = \{\alpha \mid \alpha(\xi) \in \mathbb{Z}\}$ where $p(\xi) = s$. The question is whether K_s can be different from the centralizer of a torus, i.e. whether $\Phi(\xi) = \{\alpha \mid \alpha(\eta) = 0\}$ for some $\eta \in E$.

Suppose $\xi \in C'$ whence $0 \leq \alpha(\xi) \leq 1$ for $\alpha \in \Phi^+$. If $\psi(\xi) < 1$, then $\Phi(\xi) = \{\alpha \mid \alpha(\xi) = 0\}$, and we are OK. Let then $\psi(\xi) = 1$ and arrange the simple roots in order so that

$$\alpha_1(\xi) = \cdots = \alpha_r(\xi) < \alpha_{r+1}(\xi) \leq \cdots \leq \alpha_\ell(\xi)$$

Let $\alpha \in \Phi(\xi)^+$. If $\alpha(\xi) = 0$, then α is a linear combination of $\alpha_1, \dots, \alpha_r$, (In general $\alpha = \sum n_i \alpha_i$ uniquely with $n_i \geq 0$), and conversely. Call $\Delta \subset \Phi$ the set of roots which are lin. combinations of $\alpha_1, \dots, \alpha_r$. If $\alpha(\xi) = 1$, we have $\alpha = \psi + \alpha_{i_1} + \cdots + \alpha_{i_p}$ and it is clear that $\cancel{\alpha_{i_1}, \dots, \alpha_{i_p} \in \Delta}$ $1 \leq i_1, \dots, i_p \leq r$

Assume that $\Phi(\xi) = \{\alpha \mid \alpha(\eta) = 0\}$ for some η. Then $\alpha_i(\eta) = 0$ $i = 1, \dots, r$. If $\varphi = \sum_{i=1}^{\ell} m_i \alpha_i$ is any pos. root not in Δ, then $\varphi(\xi) = 1 \Longleftrightarrow \varphi(\eta) = 0$. So there's a problem if when we write $\psi = n_1\alpha_1 + \cdots + n_\ell \alpha_\ell$, some $n_i > 1$.

<u>Example</u>: Suppose $\psi = n_1\alpha_1 + \cdots + n_\ell \alpha_\ell$ where say $n_\ell > 1$. (Then $\ell \geq 2$) Take ξ to be the point of C' where $\alpha_1(\xi) = \cdots = \alpha_{\ell-1}(\xi) = 0$ and $n_\ell \alpha_\ell(\xi) = 1$.

Suppose $\exists\, \eta$ such that $\alpha(\xi) \in \mathbb{Z}$ $\Longleftrightarrow \alpha(\eta) = 0$. Then $\alpha_1(\eta) = \cdots = \alpha_{\ell-1}(\eta) = \psi(\eta) = 0$, so η would have to be zero, and we get a contradiction.

B_2 has $\psi = \alpha_1 + 2\alpha_2$
G_2 has $\psi = 2\alpha_1 + 3\alpha_2$.

Examples of symmetric spaces:

Start by classifying the symmetric spaces arising from involutions on U_n and SU_n.

~~First look involutions arising from inner autos~~ $\cancel{\sigma(x) = yxy^{-1},\ y^2 \in \text{center}}$.

First some general considerations. Suppose σ is a given involution on K. Then I might look for involutions τ of the form

$$\tau(x) = y\,\sigma(x)\,y^{-1}.$$

For this to be an involution means $y\,\sigma(y) \in$ center. Involutions τ and $x \longmapsto z\,\tau(z^{-1} x z)\,z^{-1}$ have conjugate fixpt. groups, hence isomorphic symmetric spaces. Since

$$z\tau(z^{-1}xz)z^{-1} = zy(\sigma z)^{-1}\sigma x \ \sigma z y^{-1} z^{-1}$$

we see that the different kinds of symmetric spaces we get from involutions of the form $\tau x = y\sigma x y^{-1}$ are described by:

(*) $\quad \{yZ \in K/Z \mid y\cdot\sigma y \in \text{center}\} \big/ \text{action: } z * y = zy(\sigma z^{-1}).$

Now look at U_n and first take σ to be complex conjugation (or $x \mapsto (x^t)^{-1}$). If z is a scalar matrix then $z\bar{z}^{-1} = z^2$ and so given y with $y\bar{y} \in$ Center we can modify it so that $y\bar{y} = 1$. Let V be the eigenspace of y with eigenvalue λ. Then

$$yv = \lambda v \Rightarrow \bar{y}\bar{v} = \bar{\lambda}\bar{v} \Rightarrow \bar{v} = \bar{\lambda} y\bar{v}$$

$$\Rightarrow y\bar{v} = \bar{\lambda}^{-1}\bar{v} = \lambda\bar{v}$$

so V is stable under $-$. Thus if I select z so as to have the same eigenspaces as y and also $z^2 = y$, then $\sigma z = z^{-1}$, and so we conclude (*) is a single point. Corresp. symm. space is U_n/O_n.

Take σ to be the identity in $U(n)$. If $y^2 \in$ center, then multiplying y by a scalar, we can suppose $y^2 = 1$; so we are classifying involutions in U_n up to conjugation, and multiplication by ± 1. So the type of symmetric spaces obtained are the Grassmannians

$$U(n)/U(p) \times U(n-p) \qquad 0 \leq p \leq \left[\tfrac{n}{2}\right]$$

Next consider SU_n with $\sigma = \text{id}$. whence $y^2 = \zeta$, $\zeta^n = 1$. So y has two eigenvalues, and we get the Grassmannians again for the symmetric spaces.

~~Consider SU_n with $\sigma x = \bar{x}$, and let $y\bar{y} = \zeta$, $\zeta^n = 1$. If n is odd, then we can change y so that $y\bar{y} = 1$, whence the eigenspaces of y are stable under σ. Let W be an eigenspace of y. By what we know about U_n we can find a z in $W^\perp \ni z\sigma z^{-1} = y$, and we can extend z to an element of SU_n. Thus we can suppose $y = 1$ in $W^\perp$, $y = \lambda$ in W so~~

Consider SU_n with $\sigma x = \bar{x}$ and let $y\bar{y} = \zeta$ be a scalar matrix. Then $\bar{\zeta} = \bar{y}y$ and $(y,\bar{y}) = 1$ as ζ is in the center) so $\bar{\zeta} = \zeta$ whence $\zeta = \pm 1$. Note that $\zeta^n = \det y \det \bar{y} = 1$, so $\zeta = 1$ if n odd.

If V_λ is where $y = \lambda$, then $\overline{V_\lambda}$ is where $\bar{y} = \bar{\lambda}$ or where $y = \zeta \bar{y}^{-1} = \zeta\lambda$. Thus $\overline{V_\lambda} = V_{\zeta\lambda}$.

If $\zeta = +1$, then $\overline{V_\lambda} = V_\lambda$ and so we can find a line L with $\bar{L} = L$ stable under y. By what we've seen for U_n, there exists z in $L^\perp$ such that $z\bar{z}^{-1} = y$ in $L^\perp$, hence extending z to an element of SU_n, we can arrange y to be 1 in $L^\perp$, whence $y = 1$. Thus if $\zeta = 1$, we get only the symm. space:

$$SU_n/SO_n$$

~~[illegible] If there is a proper subspace W stable under σ and y, then we can arrange that y be 1 [illegible] on $W^\perp$, whence $\zeta = 1$.~~

~~So if $\zeta = -1$, so we have $V = V_\lambda \oplus V_{-\lambda}$ where these are interchanged under σ. $n = 2m$. Suppose m is even, whence $\lambda^m(-\lambda)^m = (-1)^m\lambda^{2m} = 1$, so we can arrange $\lambda = 1$.~~

Consider U_n or SU_n with $\sigma x = \bar{x}$. If $y\bar{y}$ is in the center, say $y\bar{y} = \zeta$, then $y, \bar{y}$ commute and $\bar{\zeta} = (y\bar{y})^{-} = \bar{y}y = y\bar{y} = \zeta$, so $\zeta = \pm 1$.

Now Recall that if we interpret elements as transf. of $\mathbb{C}^n$, then $\sigma(x) = \sigma \circ x \circ \sigma^{-1}$, where $\sigma =$ conjugation on $\mathbb{C}^n$. Thus our involution is

$$x \longmapsto y\,\sigma(x)\,y^{-1} = (y\sigma)\circ x \circ (y\sigma)^{-1}$$

But $y\sigma$ is an anti-linear transf. of $\mathbb{C}^n$ such that

$$y\sigma \circ y\sigma = y\bar{y} = \zeta$$

Thus if $\zeta = 1$ we get a real structure on $\mathbb{C}^n$, whereas if $\zeta = -1$, we get a symplectic structure. Thus the symmetric spaces in question are

$$SU_n/SO_n \qquad SU_n/Sp_{[\frac{n}{2}]}$$

$$U_n/O_n \qquad U_n/Sp_{[\frac{n}{2}]}$$

n has to be even

It is desirable to work out the Grassmannian examples carefully. Take the case where σ is conjugation on U_n by:

$$\sigma = \left(\begin{array}{c|c} -I_p & 0 \\ \hline 0 & I_q \end{array}\right) \qquad p+q=n$$

Then $K^\sigma = U_p \times U_q$.

First analyze $\{y \in U_n \mid \sigma(y) = y^{-1}\} = \tilde{X}$.

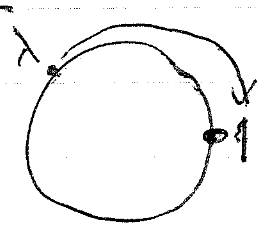
Let V_λ be an eigenspace of $y \ni \sigma y = y^{-1}$. Then

$$y v = \lambda v \implies \underset{\shortparallel}{\sigma(y)} \sigma v = \lambda \sigma v$$

$$y^{-1}(\sigma v) = \lambda \sigma v$$

$$\implies y(\sigma v) = \lambda^{-1} \sigma v$$

So $\sigma V_\lambda = V_{\lambda^{-1}}$. If $\lambda \neq -1$, we can move λ to 1



doing the same for λ^{-1}. This deforms y into those y such that $\sigma y = y$ and $y^2 = 1$.

Note: $\det(y) = \det(\sigma y) = \det(y^{-1})$ so $\det(y) = \pm 1$. This is an invariant of the component of $\tilde{X}$ to which y belongs. ~~So it's~~ Therefore $\tilde{X}$ is not connected.

Consider the group generated by σ and y. It is a dihedral group so its irreducible complex representations are as follows

dim = 2 with y acting as λ, λ^{-1}, $\lambda \neq \pm 1$.

dim = 1 with

$$\begin{array}{ll} y=-1 & \sigma=+1 \\ y=-1 & =-1 \\ y=+1 & \sigma=-1 \\ y=1 & \sigma=1 \end{array}$$

Letting λ move around the upper half of the unit circle we can specialize into the sums of the two characters.

Thus as y varies over $\{y \mid \sigma y = y^{-1}\}$ the difference between the dim of the $\sigma=+1, y=-1$ eigenspace and the $\sigma=-1, y=-1$ eigenspaces doesn't change. If this is zero we can deform y to 1. Thus the components are indexed by:

$$\begin{array}{lcc} \sigma: & \overbrace{-1 \cdots -1}^{p} & \overbrace{1 \cdots 1}^{q} \\ y: & \underbrace{-1 \cdots -1}_{p'} & \underbrace{-1 \cdots -1}_{q'} \end{array}$$

so the invariant is $p' - q'$.

Let σ be an involution on K, given by an inner automorphism: $\sigma x = \sigma_0 x \sigma_0^{-1}$ where $\sigma_0^2 \in Z$. Then

$$\{y \mid y\cdot\sigma(y) \in Z\} \xleftarrow{\ \sim\ } \{u \mid u^2 \in Z\}$$

$$u\sigma_0 \longleftarrow u$$

(for $(u\sigma_0)\sigma_0(u\sigma_0)\sigma_0^{-1} = u^2\sigma_0^2 \in Z \iff u^2 \in Z$). And moreover if $\sigma_0^2 = 1$, then

$$\{y \mid y\cdot\sigma(y) = 1\} \xleftarrow{\ \sim\ } \{u \mid u^2 = 1\}$$

$$u\sigma_0 \longleftarrow u$$

Furthermore conjugation action on the right corresponds to twisted conjugation on the left:

$$z(u\sigma_0)\sigma(z^{-1}) = zu\sigma_0\sigma_0 z^{-1}\sigma_0^{-1} = zuz^{-1}\sigma_0$$

Therefore when σ is the inner auto. produced by σ_0 of order 2, the set of y such that $\sigma(y) = y^{-1}$ is nothing but the different elements of order 2 in K. In U_n the elements of order 2 fall into conjugacy classes according to the number of -1 eigenvalues.

Given an involution σ on K, the symmetric spaces associated to involutions of the form $x \mapsto y\,\sigma(x)\,y^{-1}$ (related symm. spaces to K/K^{σ}) are the K-orbits under σ-twisted conjugation on the set $\{yZ \in K/Z \mid y\cdot\sigma(y) \in Z\}$

~~So~~ if I want the symmetric spaces associated to U_n and the involutions related to $\sigma = id$, I look at the conjugacy classes of involutions, and these are the Grassmannians.

If $\sigma = -$ on the other hand, and $y\bar{y} = \xi$ I have seen that $\xi = \pm 1$. If $\xi = +1$, $i_y\sigma$ is a conjugation and if $\xi = -1$, $i_y\sigma$ is a symplectic structure. One gets one U_n-orbit, when n is odd, and 2 if n is even.

Time now to understand the Grassmannians as symmetric spaces in detail. So start with an involution σ_0 in U_n with p eigenvalues -1. If T is a maximal torus normalized by σ_0, then T corresponds to a decomposition of $\mathbb{C}^n$ into lines which are permuted by σ_0. It is clear that T contains a maximal reversed torus S iff there p orbits of order 2. Here I suppose $2p \leq n$. So I can divide $\mathbb{C}^n$ into

$$\mathbb{C}^n = \begin{matrix} L_1 \oplus L_2 \oplus \cdots \oplus L_p \\ L_1' \oplus L_2' \oplus \cdots \oplus L_p' \end{matrix} \oplus \Gamma$$

where each L_i, L_i' is a line and $\sigma L_i = L_i'$, whence if S acts on L_i thru χ_i, then it acts on L_i' thru χ_i^{-1}. Γ is a trivial representation of S and σ.

The good way to do the above is to choose a max. σ_0-reversed torus S, then to consider the "dihedral" group $\mathbb{Z}/2 \ltimes S$ acting on $\mathbb{C}^n$, and to decompose it into irreducible representations

$$(L_1 \oplus L_1') \oplus \cdots \oplus (L_p \oplus L_p') \oplus \Gamma$$

where Γ is a trivial representation (as $2p \leq n$). The group M is the centralizer of this dihedral group, hence consists of scalars in each $L_i \oplus L_i'$ and any auto of Γ:

$$M = (S^1) \times \cdots \times (S^1) \times \mathrm{Aut}(\Gamma).$$

~~[illegible]~~

June 20, 1975: The Grassmannians as symmetric spaces.

Start with $\mathbb{P}^1(\mathbb{C}) = S^2 = \dfrac{U(2)}{U(1) \times U(1)}$ which is the conjugation class of the involution

$$\sigma = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$$

in $U(2)$. $\mathbb{C}^2 = \mathbb{C}e_1 + \mathbb{C}e_2$, a line will contain a vector $x_1 e_1 + x_2 e_2$ unique up to a non-zero scalar, and we associate to it the point $z = \dfrac{x_2}{x_1}$ of the sphere. Hence

$$\mathbb{C}e_1 \longleftrightarrow 0$$
$$\mathbb{C}e_2 \longleftrightarrow \infty.$$

Maximal flat submanifolds of S^2 are the great circles. So our reversed torus S will be a double covering of $\mathbb{P}^1(\mathbb{R}) \subset \mathbb{P}^1(\mathbb{C})$. Thus S is the torus consisting of the rotations

$$r_\theta : \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \qquad 0 \leq \theta \leq 2\pi$$

and r_θ maps $\binom{1}{0}$ to $\binom{\cos\theta}{\sin\theta}$ which corresponds to $\tan\theta \in P^1(\mathbb{R})$.

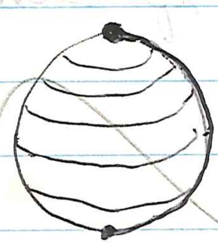
$$K^\sigma = U(1) \times U(1)$$
$$M = \Delta U(1) .$$

Note that if we identify $S \subset U(2)$ properly with a flat submanifold of X, then we map s to $s^{1/2}\cdot o$, o = basept of X. Thus r_θ corresp. to $\tan\frac{\theta}{2}$.

The group $W_0 = \mathbb{Z}/2\mathbb{Z}$ and is generated by σ. The effect of σ on S^2 is given by

$$\sigma(z) = -z .$$

K^σ orbits on S^2 are lattitude lines and



$\tan(\frac{\theta}{2})$, $0 \leq \theta \leq \frac{\pi}{2}$ is a fundamental domain

---

Next generalize to $X = \dfrac{U(2n)}{U(n) \times U(n)}$.

$$\sigma = \left(\begin{array}{c|c} -I_n & \\ \hline & I_n \end{array}\right) \qquad K^\sigma = U(n) \times U(n)$$

$$S : \left(\begin{array}{ccc|ccc} \cos\theta_1 & & & -\sin\theta_1 & & \\ & \cos\theta_2 & & & -\sin\theta_2 & \\ & & \ddots & & & \ddots \\ \hline \sin\theta_1 & & & \cos\theta_1 & & \\ & \sin\theta_2 & & & \cos\theta_2 & \\ & & \ddots & & & \ddots \end{array}\right)$$

The Grassmannian we think of as n-planes is $\mathbb{C}^n \oplus \mathbb{C}^n$, and the orbit of S is X contains those n-planes A which decompose

$$A = L_1 \oplus \cdots \oplus L_n$$

where $\quad L_i \in \mathbb{C}e_i \oplus \mathbb{C}e_{n+i}$.

$$M = \left(\begin{array}{ccc|ccc} \lambda_1 & & & & & \\ & \lambda_2 & & & & \\ & & \ddots & & & \\ \hline & & & \lambda_1 & & \\ & & & & \lambda_2 & \\ & & & & & \ddots \end{array}\right) = \Delta T \subset U(n)\times U(n)$$

The orbit of S in X will be denoted $\bar{S}$; it consists of planes stabilized by M. Of course $\bar{S} \simeq (\mathbb{P}_1\mathbb{R})^n$. The Weyl group W_0 is a semi-direct product $\Sigma_n \ltimes (\mathbb{Z}/2)^n$.

Next I want to compute the diagram of this symmetric space. It is first necessary to transform S into the diagonal matrices. In this case S will appear as matrices of the form

$$\left(\begin{array}{ccc|ccc} e^{i\theta_1} & & & & & \\ & \ddots & & & & \\ & & e^{i\theta_n} & & & \\ \hline & & & e^{-i\theta_1} & & \\ & & & & \ddots & \\ & & & & & e^{-i\theta_n} \end{array}\right)$$

and σ interchanges the i-th and $(n+i)$-th entries. The group W_0 permutes the angles θ_i and changes their signs; it is therefore $\simeq$ subgroup of Σ_{2n} ~~preserving the~~ centralizing σ.

T consists of matrices

$$\begin{pmatrix} e^{i\theta_1} & & & & & \\ & \ddots & & & & \\ & & e^{i\theta_n} & & & \\ & & & e^{i\theta_{n+1}} & & \\ & & & & \ddots & \\ & & & & & e^{i\theta_{2n}} \end{pmatrix}$$

$\theta_i\sigma = \theta_i'$

and it might be convenient to put $\theta_{n+i} = \theta_i'$ for $i = 1, \dots, n$. Then S consists of those elements of T such that $\theta_i' = -\theta_i$.

I should maybe think of the angle θ_i as being a linear function on $\mathrm{Lie}(T)$: $\theta_i(x)$ is the i-th entry of the element x.

Positive region C_0 of E_0 will be described by $\theta_1 \geq \cdots \geq \theta_n \geq 0$, and the positive region C will be:

$$\theta_1 \geq \cdots \geq \theta_n \geq \theta_n' \geq \theta_{n-1}' \geq \cdots \geq \theta_1'$$

Suppose now I see what happens to a pos. root α of T when it's restricted to S.

<u>Case 1</u>: $\alpha = \theta_i - \theta_j$ $\quad i < j$. This restricts to the root $\theta_i - \theta_j$ on E_0. Note that $\alpha\sigma = \theta_i' - \theta_j' \neq -\alpha$.

<u>Case 2</u>: $\alpha = \theta_j' - \theta_i'$ $\quad i < j$. This restricts to the root $\theta_i - \theta_j$ on E_0. Note that $\alpha\sigma = \theta_j - \theta_i \neq -\alpha$.

<u>Case 3</u>: $\alpha = \theta_i - \theta_j'$. This restricts to the root $\theta_i + \theta_j$ on E_0. Note that $\alpha\sigma = \theta_i' - \theta_j \neq -\alpha$ when $i \neq j$, but $= -\alpha$ when $i = j$.

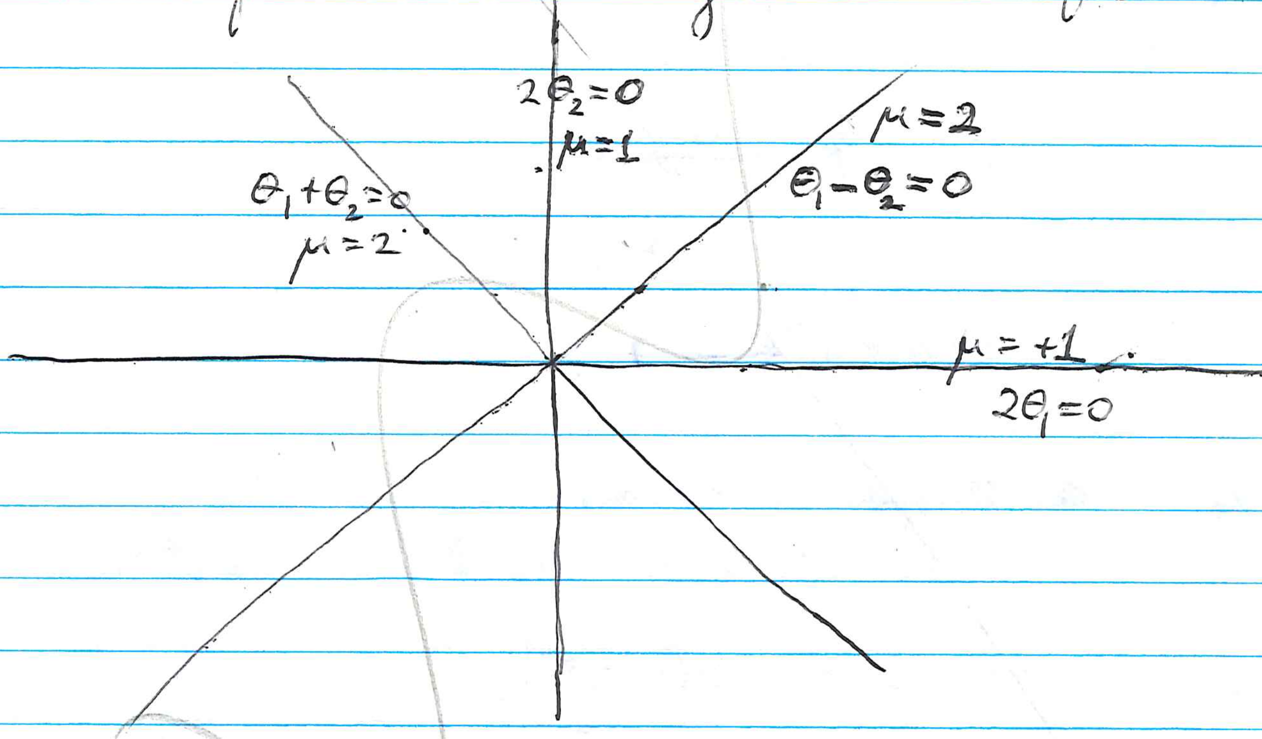
Thus the positive roots on E_0 are:

$\theta_i - \theta_j \quad i<j \quad$ multiplicity 2

$\theta_i + \theta_j \quad i<j \quad$ mult. 2

$2\theta_i \quad$ mult. 1.

So the infinitesimal diagram is for $n=2$

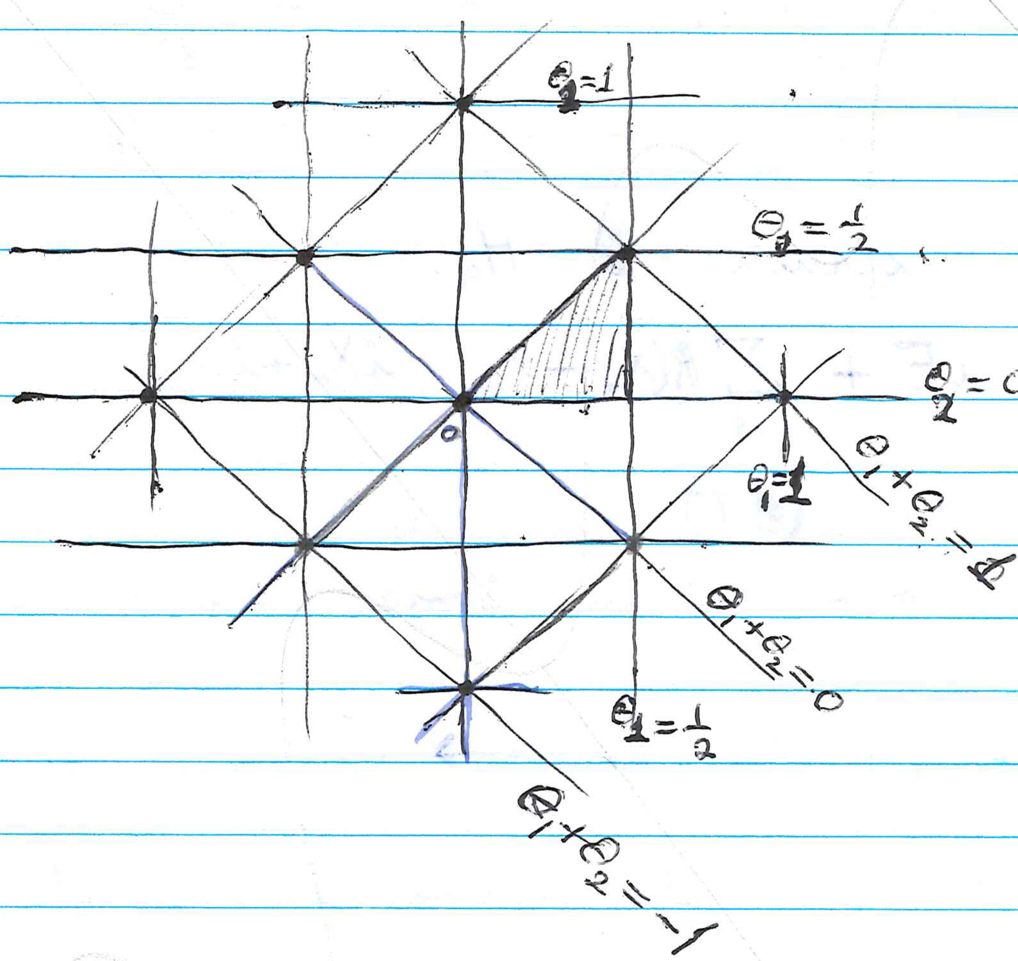


The Weyl group reflects around the lines; it is the dihedral group of order 8. Incidently this diagram is the same as for $SO(4)$ except that the multiplicities in the latter are all 2.

$$\dim(K/M) = 2+1+2+1 = 6 \overset{?}{=} \dim \frac{U(2)\times U(2)}{T} = 4+4-2$$

YES.

The big diagram will have in addition the planes where the roots have integral values (I replace θ_i by $(2\pi)^{-1}\theta_i$ say): For $n=2$



For $n=1$. $2\theta_1$ is the only root and it has mult. 1.
Diagram is

I get a fundamental domain using $\begin{pmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{pmatrix}$ with 2π $0\leq\theta\leq\frac{1}{2}\cdot 2\pi=\pi$

It is necessary to work out the relation between my building approach to ΩX and the Bott-Samelson approach.

In my approach I have a building $\mathfrak{X}^\sigma$, an action of $\mathcal{G}^\sigma$ on $\mathfrak{X}^\sigma$, with quotient C_0', and the choice of C_0' gives me a min. parahorique $\mathcal{B}_0$ such that

$$\mathcal{B}_0 \backslash \mathfrak{X}^\sigma \leftarrow E_0$$

Thus quite canonically points of E_0 are to be interpreted as cells in $\mathfrak{X}^\sigma$. The basic CW complexes in my theory are the $\mathcal{G}^\sigma$-orbits in $\mathfrak{X}^\sigma$; there is one for each $\eta \in C_0'$. The CW complex $\mathcal{K}^\sigma\eta \doteq \mathcal{G}^\sigma\eta$ has the homotopy type of the space of paths starting at ~~some~~ any point t_0 of $p(C_0') \subset \mathfrak{S}$ and ending on the orbit $K^\sigma\bar{\eta}$, $\bar{\eta} = p(\eta)$.

In the path interpretation we have given to elements of $\mathfrak{X}^\sigma$, $\mathcal{K}^\sigma\eta$ is the space of special paths h in K such that $\overline{h(t)} = h(-t)$ and $h(1) \in K^\sigma_{\bar{\eta}}$.

In the Bott-Samelson approach one uses ~~paths~~ broken geodesics in X perpendicular to K^σ orbits from t_0 to $K^\sigma\bar{\eta}$.

Conclusion: Do not think of $\mathcal{K}^\sigma\eta$ as paths in the symmetric space, but rather as an orbit in the building which intersects E_0 in a $\widetilde{W}_0$-orbit

$\tilde{W}_0\eta$ and which has a cell decomposition with cells having their centers in $\tilde{W}_0\eta$. (The point is that X^σ is just a contractible fibre space over X, hence not necessarily canonically identifiable with a space of paths.)

So the spaces that make sense algebraically are ~~paths~~ of the homotopy type of the spaces considered by Bott–Samelson, namely, paths from a point t_0 to a K^σ-orbit. (Recall that $P(X, t_0, K^\sigma\eta)$ has the homotopy type of the fibre of the inclusion $K^\sigma\bar{\eta} \subset X$.)

So the next point is that if we want the loop space of the symmetric space, then we we want to choose the orbit $K^\sigma\bar{\eta}$ to be a point. So I want all roots α to have integral values at the point η.

June 21, 1975:

Recall that the building approach gives you the spaces $K^\sigma\tilde{\eta}$, $\eta \in C_0'$ which have the homotopy of the fibre of the inclusion $K^\sigma\bar{\eta} \subset X$. Thus if $\bar{\eta}$ is fixed under K^σ, the space $K^\sigma\tilde{\eta}$ has the homotopy type of ΩX.

I am interested in the inclusion

$$K^\sigma \tilde{\eta} \subset \mathcal{K}^\sigma \tilde{\eta}.$$

The former space is isomorphic to $K^\sigma \cdot \eta \subset k_-$, and has a cell decomposition indexed by $W_0 \eta$, whereas $\mathcal{K}^\sigma \tilde{\eta}$ has a cell decomposition indexed by $\tilde{W}_0 \eta$. It seems clear that since $\eta \in C_0'$, the multiplicities are the same so in fact $K^\sigma \tilde{\eta}$ should be a subcomplex of $\mathcal{K}^\sigma \tilde{\eta}$.

I wanted to do the example of $X = G_n(\mathbb{C}^{2n})$. Here

$$\sigma = \left(\begin{array}{c|c} -I & \\ \hline & I \end{array}\right) \qquad K^\sigma = \left(\begin{array}{c|c} U(n) & 0 \\ \hline 0 & U(n) \end{array}\right)$$

and

$$k_- = \left(\begin{array}{c|c} 0 & C \\ \hline -C^* & 0 \end{array}\right)$$

Take

$$\eta = \left(\begin{array}{c|c} & I \\ \hline -I & \end{array}\right). \quad \text{Then}$$

$$e^{2\pi t \eta} = \begin{pmatrix} \cos 2\pi t & \sin 2\pi t \\ -\sin 2\pi t & \cos 2\pi t \end{pmatrix}$$

so if $t = \frac{1}{2}$ we get

$$e^{\pi \eta} = \begin{pmatrix} -I & 0 \\ 0 & -I \end{pmatrix}$$

which is centralized by K^σ.

$$\left(\begin{array}{c|c} A & \\ \hline & B \end{array}\right)\left(\begin{array}{c|c} 0 & 1 \\ \hline -1 & 0 \end{array}\right)\left(\begin{array}{c|c} A^{-1} & \\ \hline & B^{-1} \end{array}\right) = \left(\begin{array}{c|c} 0 & AB^{-1} \\ \hline -BA^{-1} & 0 \end{array}\right)$$

So the orbit $K^\sigma \eta$ is isomorphic to

$$U(n)\times U(n)/\Delta U(n) \simeq U(n) \qquad \left(\begin{array}{c|c} A & \\ \hline & B \end{array}\right) \mapsto AB^{-1}$$

Take the inner product $\operatorname{tr} A^*A = \sum |a_{ij}|^2$ on $\mathfrak{U}_{2n}$. On $\mathfrak{k}_-$ it becomes

$$\left\langle \left(\begin{array}{c|c} & C \\ \hline -C^* & \end{array}\right), \left(\begin{array}{c|c} & C \\ \hline -C^* & \end{array}\right)\right\rangle = 2 \operatorname{tr} C^*C$$

whence on polarizing

$$\left\langle \left(\begin{array}{c|c} & C_1 \\ \hline -C_1^* & \end{array}\right), \left(\begin{array}{c|c} & C_2 \\ \hline -C_2^* & \end{array}\right)\right\rangle = \operatorname{tr}\left(C_1^*C_2 + C_2^*C_1\right)$$

The Morse function on $K^\sigma\eta$ we use is $k\eta \longmapsto -(k\eta, \xi)$ where ξ is a generic pt of C_0. Take

$$\xi = \left(\begin{array}{c|c} & \Lambda \\ \hline -\Lambda & \end{array}\right) \qquad \Lambda = \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{pmatrix}$$

where the λ_i are real positive and decreasing $\lambda_1 > \dots > \lambda_n > 0$.

Then

$$-\left(\left(\begin{array}{c|c} 0 & A \\ \hline -A^{-1} & 0 \end{array}\right), \left(\begin{array}{c|c} & \Lambda \\ \hline -\Lambda & \end{array}\right)\right) = -\mathrm{tr}\,(A^*\Lambda + \Lambda A)$$

is the Morse function to consider; call it $f(A)$.

$$f(A(I+\varepsilon B)) = -\mathrm{tr}\,((I-\varepsilon B)A^*\Lambda + \Lambda A(I+\varepsilon B))$$
$$= -\mathrm{tr}\,(A^*\Lambda + \Lambda A) + \varepsilon\,\mathrm{tr}\,(BA^*\Lambda - \Lambda AB).$$

So a critical point is where

$$\mathrm{tr}\,(BA^*\Lambda - \Lambda AB) = 0$$

for all B.

$$\mathrm{tr}\,B \cdot \underbrace{(A^*\Lambda - \Lambda A)}_{} = 0$$

So A critical $\iff$ $A^*\Lambda = \Lambda A$ $\iff$ ΛA is hermitian.

Now if $\Lambda A = H$ hermitian, then

$$(\Lambda A)(\Lambda A)^* = \Lambda A A^* \Lambda = \Lambda^2 = H^2$$

so therefore H will be a square root of Λ^2 i.e.:

$$H = \begin{pmatrix} \pm\lambda_1 & & \\ & \ddots & \\ & & \pm\lambda_n \end{pmatrix} \qquad 2^n$$

It follows that A is a diagonal matrix with ± 1 entries.

June 22, 1975 (35 yrs old)

I still don't understand the Bruhat decomposition in the symmetric space case.

K compact connected with involution σ, S a maximally-reversed torus, T a max. torus containing S, Φ = roots of K with respect to T, Φ_0 = roots of K with respect to S.

Basic point was to look at $\mathfrak{k}$ as a representation of $S' = \{\sigma\} \ltimes S$. It then splits as a direct sum according to the diff. reps of S':

$$\mathfrak{k} = \underset{\substack{\sigma = 1 \quad\ \ -1 \\ S \text{ trivial}}}{(\mathfrak{m} \oplus E_0)} \oplus \sum_{\beta \in \Phi_0^+} \mathfrak{k}_\beta$$

where $\mathfrak{k}_\beta$ is isomorphic to a direct sum of the representation χ_β of S' on $\mathbb{C}$ where

$$\begin{cases} \sigma(z) = \bar{z} \\ \exp(\eta) \cdot z = e^{2\pi i \beta(\eta)} z \end{cases}$$

I am going to try to prove the Bruhat decomposition. The method goes like this: Suppose given the group G with subgroups B and N with $B \cap N \triangleleft N$ and $W = N/B\cap N$ a Coxeter group gen. by $s \in S$. Then I have to prove

a) $BsBwB = BswB$ if $\ell(sw) = \ell(w)+1$

b) $BsBsB = B \amalg BsB$

The general case goes like this:

$\ell(sw) = \ell(w) - 1$, then $\ell(w) = \ell(sw)+1$ so

$$\begin{aligned} BsBwB &= BsB\,BsB\,BswB \\ &= (B \cup BsB)\,BswB \\ &= BswB \cup BswB. \end{aligned}$$

<u>Example</u>. Recall that I can identify the spherical building of G^σ with the sphere in $\mathfrak{k}$ (or $\mathfrak{p} \subset \mathfrak{g}^\sigma$). I want to describe the action of G^σ on $S(\mathfrak{p})$.

The idea is as follows. To an element ξ of $\mathfrak{p}$ I can associate the geodesic $e^{t\xi}\cdot O$ in the symmetric spaces. Given $g \in G^\sigma$, the geodesic

$ge^{t\xi}.o$ should be asymptotic to a unique geodesic of the $e^{t\eta}o$. Then $g * \xi = \eta$.

Recall the group K^σ is the fixpt group for the Cartan involution which I will denote by $g \longmapsto (g^*)^{-1}$. Then I can embed the symmetric space inside G^σ via $g.o \longmapsto gg^*$, (and we get the positive component of those $h \in G \ni h^* = h$.)

So now η is defined by

$$ge^{2t\xi}g^* \sim e^{2t\eta} \qquad \text{as } t \to \pm\infty.$$

Let $P_\xi = \{g \in G^\sigma \mid e^{-t\xi} g e^{t\xi}$ is bounded as $t \to \pm\infty\}$.

Suppose $G = GL(n, \mathbb{C})$ with $\sigma = id$, $* =$ conjugate transpose. If

$$\xi = \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{pmatrix} \qquad \text{with } i<j \Rightarrow \lambda_i \geq \lambda_j$$

then $(ge^{t\xi}g^*)_{ik} = \sum_j g_{ij} e^{t\lambda_j} \bar{g}_{kj} \sim e^{t\lambda_i}\delta_{ik}$ implies that

$$\lambda_j > \lambda_i \Longrightarrow |g_{ij}|^2 = 0 \Longrightarrow g_{ij} = 0.$$

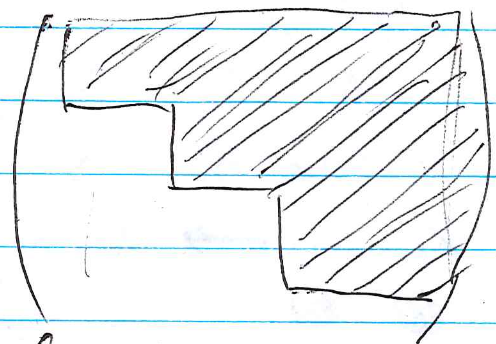
~~$\sum_{j:\lambda_j=\lambda_i} g_{ij}\bar{g}_{kj} = \delta_{ik}$~~

~~Thus if I group arrange the λ's in decreasing order.~~

Then $\left(e^{-t\xi} g e^{t\xi}\right)_{ij} = e^{t(\lambda_j-\lambda_i)} g_{ij}$. So [redacted]

[redacted] $\lambda_i < \lambda_j \implies g_{ij} = 0$

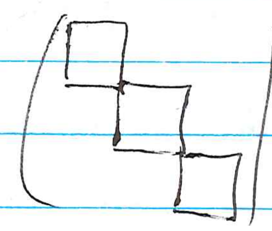
Thus P_ξ is the parabolic



Note that the limit

$$\lim_{t\to+\infty} e^{-t\xi} g e^{t\xi} = g^0$$

is just the reductive part of g : and it centralizes ξ.



So now I can understand the asymptotic behavior of $g e^{2t\xi} g^*$ when $g \in P_\xi$. I [redacted] I pull this back [redacted] via $e^{-t\xi}$. Better I have $g e^{t\xi}.\theta$ which I pull back to $e^{-t\xi} g e^{t\xi}\theta$ which converges to $g^0.\theta$.

So at least when $g \in P_\xi$ I see that the asymptotic behavior is

$$g e^{t\xi} \cdot o \sim e^{t\xi} \cdot g^\circ o$$

where $\sim$ means the distance between these points approaches zero.

<u>General case</u>: Because P_ξ is parabolic I know $K/K_\xi = G/P_\xi$, hence $\exists\, k \in K$ such that $g^{-1}k \in P_\xi$. Then

$$k^{-1} g e^{t\xi} \cdot o \sim e^{t\xi} (k^{-1}g)^\circ o$$

or

$$g e^{t\xi} \cdot o \sim e^{t(k\cdot\xi)}\, k (k^{-1}g)^\circ o$$

Thus I see that the action of G^σ on $\mathfrak{p}$ is described by the asymptotic behavior of geodesics.

It should be possible to compactify G^σ/K^σ by adding limit points corresponding to $S(\mathfrak{p})$.

Existence of asymptotics:

Assertion: For any $g \in G^\sigma$ and $\xi \in \mathfrak{p} = i\mathfrak{k}_-$, there is a unique $\eta = g * \xi$ in $\mathfrak{p}$ such that

$$e^{-t\eta}\, g\, e^{t\xi}$$

converges as $t \to +\infty$.

First we want to establish the uniqueness. Thus if

$$e^{-t\eta_1} g e^{t\xi} \quad \text{and} \quad e^{-t\eta_2} g e^{t\xi}$$

are convergent so is $e^{-t\eta_1} e^{t\eta_2}$. Hence I want to prove:

Lemma: If $\eta, \xi \in \mathfrak{p}$ and $e^{-t\eta} e^{t\xi}$ is convergent, then $\xi = \eta$. (can replace convergent by polynomial growth see below)

Proof: If I embed K inside $U(n)$, then I can reduce to the case where $G^\sigma = Gl_n(\mathbb{C})$ and η, ξ are hermitian matrices.

If $\quad e^{-t\eta} e^{t\xi} \longrightarrow g \quad$ as $t \to +\infty$

then

$$e^{-s\eta} g e^{s\xi} = g \quad \text{for any } s$$

or

$$g e^{s\xi} g^{-1} = e^{s\eta} \quad \text{or} \quad g \cdot \xi = \eta.$$

So we can rewrite:

$$g e^{-t\xi} g^{-1} e^{t\xi} \longrightarrow g$$

or

$$e^{-t\xi} g e^{t\xi} \longrightarrow 1.$$

~~[illegible]~~ Since I am trying to show $g=1$, I can suppose ~~by~~ conjugating with a unitary matrix, then ξ is diagonal

$$\xi = \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{pmatrix}$$

with $\lambda_1 \geq \cdots \geq \lambda_n$. Then

$$\left(e^{-t\xi} g e^{t\xi}\right)_{ij} = e^{t(\lambda_j - \lambda_i)} g_{ij}$$

so $\lambda_i \leq \lambda_j \Rightarrow g_{ij} = \delta_{ij}$. Thus if $\mu_1, \dots, \mu_r$ are the distinct eigenvalues, g is of the form

$$g : \left(\begin{array}{c|c|c} I & * & * \\ \hline 0 & I & * \\ \hline 0 & 0 & I \end{array}\right)$$

But consider $g \xi g^{-1} = \eta \Rightarrow g^{*-1} \xi g^{*} = \eta \Rightarrow g^{*}\eta g^{*-1} = \xi$
$\Rightarrow g^{*} g \xi = \xi.$

If
$$g = \left(\begin{array}{cc|c} I & A & * \\ & I & \\ \hline & O & * \end{array}\right)$$

then
$$g^* g = \left(\begin{array}{cc|c} I & O & \\ A^* & I & O \\ \hline & * & * \end{array}\right)\left(\begin{array}{cc|c} I & A & * \\ O & I & \\ \hline & O & * \end{array}\right)$$

$$= \left(\begin{array}{cc|c} I & A & \\ A^* & A^*A+I & * \\ \hline & * & * \end{array}\right)$$

and this can centralize ξ only if $A = 0$.

$$g^* g = \left(\begin{array}{c|c} I & O \\ \hline A_2^* & \\ & * \\ A_n^* & \end{array}\right)\left(\begin{array}{c|c} I & A_2 \quad A_n \\ \hline & \\ O & * \end{array}\right)$$

$$= \left(\begin{array}{c|c} I & A_2 \cdots A_n \\ \hline A_2^* & \\ \vdots & * \\ A_n^* & \end{array}\right)$$

This can centralize ξ iff $A_2, \dots, A_n = 0$. Thus we see $g = \underline{1}$ proving the lemma.

~~Another proof~~ So next we want to establish the existence of an η such that

$$e^{-t\eta}\, g\, e^{t\xi}$$

converges. Enough to do for a sets of g which generate G^σ (with all ξ for each g), [redacted] because

$$e^{-t g_1^*(g_2^* \xi)}\, g_1 g_2\, e^{t\xi} = e^{-t g_1^*(g_2^*\xi)}\, g_1\, e^{t g_2^*(\xi)}\; e^{-t g_2^* \xi}\, g_2\, e^{t\xi}.$$

So I can suppose g is in the neighborhood of 1 [redacted].

Next we can use Lie algebra theory to show us that any g near [redacted] 1 can be [redacted] represented $g = k \cdot u$ where $k \in K^\sigma$ and where $u \in P_\xi = \{u \mid e^{-t\xi}\, u\, e^{t\xi}$ [redacted] converges$\}$. Then I have

$$g e^{t\xi} = k e^{t\xi}\, e^{-t\xi}\, u e^{t\xi}$$

$$= e^{t\, \mathrm{Ad}(k)\cdot\xi}\, e^{-t\xi}\, u e^{t\xi}$$

[redacted]

and we win.

— ~~Notice that the eigenvalue~~

[redacted] Lemma: $\eta, \xi \in \mathfrak{p}$, $e^{-t\eta} e^{t\xi}$ bounded $t \to +\infty$
$\Rightarrow$ $\eta = \xi$.

Proof: Enough to do for GL_n. The

point is that the matrices η, ξ have real eigenvalues, ~~[illegible]~~ hence the entries of $e^{-t\eta}, e^{t\xi}$ will be linear combinations of exponentials: $e^{\alpha t}$ $\alpha \in \mathbb{R}$. Such a function if bounded as $t \to +\infty$ will be convergent, so we can use the preceding proof.

Consider the asymptotic behavior of $e^{t\xi}$ where ξ is any element of $\mathfrak{g}^\sigma$. First take $G^\sigma = GL_n\mathbb{C}$. If $n = 1$, then

$$e^{-t\,\mathrm{Re}\,\xi}\, e^{t\xi} = e^{it\,\mathrm{Im}(\xi)}$$

is bounded. Write $\xi = \xi_s + \xi_n$ Jordan decomposition.

$$e^{t\xi} = e^{t\xi_s} \underbrace{e^{t\xi_n}}_{\text{polynomial in } t.}$$

Basic asymptotic behavior result is:

<u>Prop</u>: For any $\xi \in \mathfrak{g}^\sigma$, there is a unique $\eta \in \mathfrak{p}$ such that

$$e^{-t\eta} e^{t\xi}$$

has polynomial growth as $t \to +\infty$

Uniqueness: If $e^{-t\eta_1} e^{t\eta_2}$ is of polynomial then because we know its entries are linear combinations of real exponentials, we have that polynomial growth $\Longrightarrow$ convergence, whence preceding arguments show that $\eta_1 = \eta_2$.

Existence: Using a Jordan decomposition $\xi = \xi_s + \xi_n$ we have

$$e^{t\xi} = e^{t\xi_s}\underbrace{e^{t\xi_n}}_{\text{polynomial function of } t}$$

hence we can suppose that ξ is semi-simple.

I claim I can find a g such that if $Ad(g)\cdot\xi$ is split according to $\mathfrak{g}^\sigma = \mathfrak{k}^\sigma \oplus \mathfrak{p}$, then the two components commute. Assume this for the moment and let $Ad(g)\xi = \xi' + \eta$ be this decomposition. Then

$$g^{-1}e^{t\xi}g = e^{t\eta}\underbrace{e^{t\xi'}}_{\text{bounded } (\in K^\sigma)}$$

$$e^{t\xi} = g e^{t\eta} e^{t\xi'} g^{-1}$$

$$\sim e^{t(g*\eta)} \underbrace{\qquad}_{\text{bounded}}$$

so we win. As for the claim, it should be known that any semi-simple ξ is contained in a Cartan subalgebra fixed under the Cartan involution σ.

$$e^{t\xi} = e^{t\xi_s} e^{t\xi_n}$$

$$= g e^{t\eta} e^{t\xi'} g^{-1} e^{t\xi_n}$$

$$= e^{t\, g*\eta} \underbrace{e^{-t(g*\eta)} g e^{t\eta}}_{\text{fast convergence}} \underbrace{e^{t\xi'}}_{\text{oscillatory}} \underbrace{g^{-1} e^{t\xi_n}}_{\text{polynom. growth}}$$

A way of describing the points of building without using a specific K is to consider all ξ inside of σ which are ~~semi~~ simple and have only real eigenvalues.* Put an equivalence relation on by saying $\xi_1 \sim \xi_2$ iff

$$e^{-t\xi_1} e^{t\xi_2}$$

is convergent. (* conjugate to an element of E_0, or generating a split torus in G.)

~~[illegible]~~

Review: K compact connected with involution σ, G ~~the~~ complexification of K, $*$ the Cartan involution of G with respect to K. $\tau x = (x^*)^{-1}$
One defines σ on G so that $\sigma \circ$ [redacted] $\tau = \tau \circ \sigma$; is the $\mathbb{C}$-linear extension of σ to G; thus σ is [redacted] extended anti-linearly to G.

Basic facts about G, K. G is connected. Put [redacted] $X = G/K$, and identify it with the identity component of $\{g \mid g^* = g\}$ via $gK \mapsto g$ [redacted] g^*.

~~We know from looking at Lie algebras that for g small we have $g = e^{\eta}k$ with $k \in K$, $\eta \in i\mathfrak{k}$, hence $gg^* = e^{2\eta}$.~~ [redacted] ~~which implies that $\exp : i\mathfrak{k} \to X$, $\eta \mapsto e^{\eta}K \mapsto e^{2\eta}$ is~~

~~One has the map~~

$$\cancel{i\mathfrak{k} \longrightarrow X}$$

$$\cancel{\eta \mapsto e^{\eta}K \longrightarrow e^{2\eta}}$$

~~and one can compute its differential. I can do this in $GL_n\mathbb{C}$, and I find the differential is non-singular because η is hermitian, hence its eigenvalues are real, so are not identified by $e^{2\lambda}$. Thus the map $i\mathfrak{k} \to X$ is etale.~~

~~Next for g small the etaleness implies~~

~~Can solve $g e^{2\eta} g^* = e^{2\xi}$ for ξ near η~~

I want to show that the map
$$i\mathfrak{k} \longrightarrow X$$
$$\eta \longmapsto e^{\eta}$$
is a diffeomorphism. Possible proof: Verify for $K = U(n)$, $G = GL_n$ and descend via a $K \longrightarrow U(n) \rightrightarrows U(m)$ presentation. This will prove bijectivity, and the rest is clear because the map is etale. Incidentally this shows G is connected if K is.

Next one has obviously $G^\sigma/K^\sigma \hookrightarrow (G/K)^\sigma$, ~~Hence~~ and $(i\mathfrak{k})^\sigma = \mathfrak{p}$, whence
$$\mathfrak{p} \xrightarrow{\sim} X^\sigma = G^\sigma/K^\sigma,$$
so in particular $\pi_0 K^\sigma = \pi_0 G^\sigma$.

Next I need to show the G^σ-orbits on $\mathfrak{p}$ are the same as the K^σ-orbits. Go back over the action of G^σ on $\mathfrak{p}$, which is defined as follows. Given $g \in G^\sigma$ and $\xi \in \mathfrak{p}$, $g * \xi$ is the unique element of $\mathfrak{p}$ such that

$$e^{-t(g*\xi)}\, g\, e^{t\xi}$$

converges. I shall recall the proof that $g*\xi$ exists.

First suppose g is near the identity, in which case $g = e^{\xi}$ and we can work in the Lie algebra. So I look at the action of $\mathrm{Ad}(e^{t\xi})$ on $\mathfrak{g}^\sigma = \mathfrak{k}^\sigma \oplus \mathfrak{p}$.

Suppose first I try to understand $\mathfrak{g} = \mathfrak{k} \oplus i\mathfrak{k}$. Let $\mathfrak{t}$ be the Lie algebra of $T \subset K$, $\mathfrak{a} = i\mathfrak{t}$. Choose usual basis

$$\mathfrak{g} = \mathfrak{h} + \sum_{\alpha \in \Phi} \mathbb{C} X_\alpha \qquad X_\alpha^* = X_{-\alpha}$$

$$\mathfrak{h} = \mathfrak{t} \oplus \mathfrak{a} \qquad \mathfrak{a} \text{ contains the } H_\alpha.$$

$\alpha(\xi)$ is real if $\xi \in \mathfrak{a}$. Then

$$\mathrm{Ad}(e^{t\xi}) = 0 \text{ on } \mathfrak{h}$$
$$= e^{t\alpha(\xi)} \text{ on } \mathbb{C}X_\alpha,$$

and so the roots divide up into those such that $\alpha(\xi)$: >0, $=0$, <0.

I let P_ξ be the Lie subgroup of G with the Lie algebra

$$\mathrm{Lie}(P_\xi) = \mathfrak{h} + \sum_{\alpha(\xi)\geq 0} \mathbb{C}X_\alpha$$

$$= \left\{ \eta \in \mathfrak{g} \;\middle|\; \mathrm{Ad}(e^{-t\xi})\eta \text{ converges as } t \to \infty \right\}$$

Since $\mathfrak{g} = \mathfrak{k} + \mathrm{Lie}(P_\xi)$ any element of g near the identity is a product $g = ku$ where $k \in K$ and $u \in P_\xi$. If $u = \exp(\eta)$ then

$$e^{-t\xi}ue^{t\xi} = \exp(\mathrm{Ad}(e^{-t\xi})\eta)$$

converges, so

$$ge^{t\xi} = ke^{t\xi}\, e^{-t\xi}ue^{t\xi}$$

$$= e^{t\,\mathrm{Ad}(k)\xi}\, ke^{-t\xi}ue^{t\xi}$$

showing that $g * \xi = \mathrm{Ad}(k)\xi$.

Next I want to show $G = KP_\xi$. First remark is that KP_ξ contains an open nbd$^{\text{d}}$ of O which can be chosen K-invariant; this follows from the above Lie algebra study plus the implicit function thm. Now suppose $g_2 \in KP_\xi$ and $g_1 \in \mathcal{U}$. Then

$$g_2 = k_2 p_2 \qquad k_2 \in K,\ p_2 \in P_\xi$$

$$g_2 * \xi = k_2 \cdot \xi$$

Because $\mathcal{U}$ is K-invariant

$$\mathcal{U} \subset KP_\xi \cdot k \qquad \text{for any } k$$

and $k P_\xi k^{-1} = P_{k \cdot \xi}$, so choosing $k = k_2$ we have

$$g_1 = k_1 p_1 \quad \text{where } p_1 \in P_{k_2 \cdot \xi} = k_2 P_\xi k_2^{-1}$$

$$k_2^{-1} p_1 k_2 \in P_\xi$$

Thus

$$g_1 g_2 = k_1 p_1 k_2 p_2 = \underbrace{k_1 k_2}_{\in K} \underbrace{k_2^{-1} p_1 k_2 p_2}_{\in P_\xi}$$

It follows that KP_ξ is stable under mult. by $\mathcal{U}$, which generates G.

(Precisely: $\bigcap_{k \in K} KP_\xi k^{-1} = \bigcap_{k \in K} KP_{k \cdot \xi} = \mathcal{U}$ contains an open nbd of K, hence generates G.)

Suppose now that $g * \xi = \xi$, and let $g = kp$ with $k \in K$ and $p \in P_\xi$. Then

$$g * \xi = k \cdot \xi = \xi$$

whence k centralizes ξ. But the centralizer K_ξ of ξ in K is connected, hence $K_\xi \subset P_\xi$. Thus $g \in P_\xi$. So we have proved

$$K/K_\xi = G/P_\xi .$$

In particular if I take ξ to be a regular element of $\mathfrak{a} = i\mathfrak{t}$, then I get

$$K/T = G/B$$

or the Iwasawa decomposition

$$G = K \times^T B .$$

June 25, 1975

$\mathbb{C}P^n$ as a symmetric space:

Continue with the Grassmannians $U(p+q)/U(p)\times U(q)$ where $p \leq q$.

$$K^\sigma = \left(\begin{array}{c|c} U(p) & 0 \\ \hline 0 & U(q) \end{array}\right) \qquad \sigma = \left(\begin{array}{c|c} -1 & 0 \\ \hline 0 & 1 \end{array}\right)$$

The maximal reversed torus can be taken to be

$$S: \left(\begin{array}{cc|c} \begin{matrix} \cos\theta_1 & & \\ & \ddots & \\ & & \cos\theta_p \end{matrix} & \begin{matrix} \sin\theta_1 & & \\ & \ddots & \\ & & \sin\theta_p \end{matrix} & \\ \hline \begin{matrix} -\sin\theta_1 & & \\ & \ddots & \\ & & -\sin\theta_p \end{matrix} & \begin{matrix} \cos\theta_1 & & \\ & \ddots & \\ & & \cos\theta_p \end{matrix} & 0 \\ \hline & 0 & I \end{array}\right)$$

The space Lie(S) is ~~generated then by~~

$$\mathrm{Lie}(S) = \left(\begin{array}{cc|c} 0 & \begin{matrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_p \end{matrix} & \\ \hline \begin{matrix} -\lambda_1 & & \\ & \ddots & \\ & & -\lambda_p \end{matrix} & 0 & 0 \\ \hline & 0 & 0 \end{array}\right)$$

To see what the roots are and their multiplicities, it is more convenient to ~~use~~ use a basis consisting of the eigenvectors of S.

$$S: \begin{pmatrix} e^{i\theta_1} & & & & & & \\ & \ddots & & & & & \\ & & e^{i\theta_p} & & & & \\ & & & e^{-i\theta_1} & & & \\ & & & & \ddots & & \\ & & & & & e^{-i\theta_p} & \\ & & & & & & I \end{pmatrix}$$

The Weyl group permutes the angles θ_i and changes signs.

$Lie(T)$ consists of

$$\begin{pmatrix} i\lambda_1 & & & & & & & & \\ & \ddots & & & & & & & \\ & & i\lambda_p & & & & & & \\ & & & i\lambda_1' & & & & & \\ & & & & \ddots & & & & \\ & & & & & i\lambda_p' & & & \\ & & & & & & i\lambda_{2p+1} & & \\ & & & & & & & \ddots & \\ & & & & & & & & i\lambda_n \end{pmatrix}$$

where $\lambda_i' = \lambda_{p+i}$. σ acts on $Lie(T)$ by interchanging λ_i and λ_i'. Thus

$$\lambda_i \circ \sigma = \lambda_i' \qquad 1 \le i \le p$$

as functions on $Lie(T)$.

Let us now compute the restrictions of the roots of k wrt T to ~~the~~ Lie(S).

$$\lambda_i \longrightarrow \theta_i \qquad i = 1, \dots, p$$

$$\lambda_{p+i} \longrightarrow -\theta_i \qquad i = 1, \dots, p$$

$$\lambda_i \longrightarrow 0 \qquad i > 2p.$$

Fundamental chamber C_0 in Lie(S) is given by $\theta_1 \geq \theta_2 \geq \dots \geq \theta_p \geq 0$. This is contained in the chambre

$$\lambda_1 \geq \dots \geq \lambda_p \geq \lambda_{p+p} \geq \dots \geq \lambda_{p+1} \geq \lambda_{2p+1} \geq \dots \geq \lambda_n.$$

Now take $\alpha = \lambda_i - \lambda_j$ ~~to be a positive root.~~ with $i < j$.

Case 1: $1 \leq i < j \leq p$. $\left(\begin{array}{l} \alpha|\text{Lie}(S) = \theta_i - \theta_j \\ -\alpha \longrightarrow \theta_j - \theta_i \end{array}\right.$

Case 2: $1 \leq i \leq p$, $p+1 \leq j \leq 2p$. $\left(\begin{array}{l} \alpha \longmapsto \theta_i + \theta_{j-p} \\ -\alpha \longmapsto -\theta_i - \theta_{j-p} \end{array}\right.$

Case 3: $1 \leq i \leq p$, $2p < j$ $\left(\begin{array}{l} \alpha \longmapsto \theta_i \\ -\alpha \longmapsto -\theta_i \end{array}\right.$

Case 4: $p+1 \leq i < j \leq 2p$ $\left(\begin{array}{l} \alpha \longmapsto \theta_{j-p} - \theta_{i-p} \\ -\alpha \longmapsto \theta_{i-p} - \theta_{j-p} \end{array}\right.$

Case 5: $p < i \leq 2p < j$ $\left(\begin{array}{l} \alpha \longmapsto -\theta_{i-p} \\ -\alpha \longmapsto \theta_{i-p} \end{array}\right.$

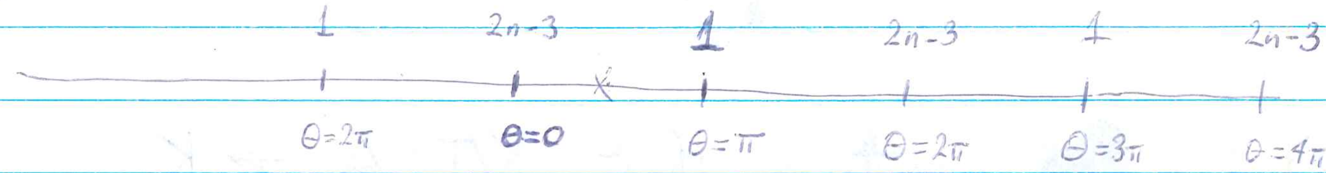
Thus the positive roots of the Grassmannian are the following

$\theta_i - \theta_j$	$1 \le i < j \le p$	multiplicity $= 2$
$\theta_i + \theta_j$	$1 \le i < j \le p$	multiplicity $= 2$
$2\theta_i$	$1 \le i \le p$	$= 1$
θ_i	$1 \le i \le p$	$= 2(n-2p) = 2(q-p)$

Suppose $p = 1$. This becomes

2θ	mult.	1
θ	mult	$2(n-2)$

Picture:



Here the symmetric space is $\mathbb{C}P^{n-1}$. The reversed torus S corresponds to the circle

$$\mathbb{R}P^1 \subset \mathbb{C}P^1 \subset \mathbb{C}P^{n-1}$$

The stabilizer of this circle is $M = \Delta S^1 \times U(n-2) \subset U(2) \times U(n-2)$. The special points are $0, \infty \in \mathbb{R}P^1$. The stabilizer in K^σ of 0 is K^σ; the stabilizer in K^σ of ∞ is $S^1 \times S^1 \times U(n-2)$. Thus

$$K^\sigma/M = \frac{U(1) \times U(n-1)}{\Delta S^1 \times U(n-2)} = S^{2n-3}$$

When we compute the loop space, we want the new directions we can head from a conjugate point, which are perpendicular to the K^{σ}-orbit.

$$\text{at } \otimes : \quad K^{\sigma}/M = S^{2n-3}$$

$$\text{at } \infty : \quad S^1 \times S^1 \times U(n-2)/M = S^1$$

So the Poincaré series is

$$\frac{(1+t)}{1-t^{2n-2}}$$

which corresponds to

$$\Omega S^{2n-1} \longrightarrow \Omega \mathbb{C}P^{n-1} \longrightarrow S^1$$

Features of this example:

i) both Θ, 2Θ roots

ii) high multiplicities for a root.

Consider next the sphere S^n $n \geq 2$. Here I can calculate the roots and the multiplicities without going to the group. Maximal flat submanifolds are the great circles. So

$$K^{\sigma}/M = \text{[redacted]}\ S^{n-1}$$

and it's clear the diagram has to be

$$\begin{array}{ccccccccc} n-1 & & n-1 & & n-1 & & n-1 & & \\ | & \text{———} & | & \text{———} & | & \text{———} & | & \text{———} & \\ \theta=-\pi & & \theta=0 & & \theta=\pi & & \theta=2\pi & & \cdots \end{array}$$

Roots for $O(2m)$: The Lie algebra ~~of the group~~ consists of skew-symmetric matrices. A maximal torus is

$$\begin{pmatrix} \begin{pmatrix} \cos\theta_1 & -\sin\theta_1 \\ +\sin\theta_1 & \cos\theta_1 \end{pmatrix} & & \\ & \ddots & \\ & & \begin{pmatrix} \cos\theta_m & -\sin\theta_m \\ +\sin\theta_m & \cos\theta_m \end{pmatrix} \end{pmatrix}$$

To see what the roots are I can suppose $m=2$. ~~Then~~ The part outside $\mathfrak{t}$ may be identified with ~~$M_2(R)$ acted on by multiply~~

$$M_2(\mathbb{R}) = \mathrm{Hom}_{\mathbb{R}}(\mathbb{C}, \mathbb{C})$$

with $\begin{pmatrix} r_{\theta_1} & \\ & r_{\theta_2} \end{pmatrix}$ acting as $A \mapsto e^{i\theta_1} A e^{-i\theta_2}$. $M_2(\mathbb{R})$ breaks up into $\mathbb{C}\cdot id \oplus \mathbb{C}\cdot\sigma$ $\quad \sigma =$ complex conjugation, so we get the roots

$$\theta_1 \pm \theta_2.$$

Thus the positive roots for $O(2m)$ are the functions $\theta_i \pm \theta_j$ for $1 \le i < j \le m$. The Weyl group permutes the θ_i and changes their signs (only an even number can be changed.

$$W = \Sigma_m \ltimes \{\alpha \in \mathbb{Z}/2^n \mid \Sigma \alpha_i = 0\}$$

To get the sphere we consider the involution $\ni$ $\theta_1 \mapsto -\theta_1$, $\theta_2, \dots, \theta_m$ fixed. The only **root** is θ_1; its multiplicity is $2m-2 = (2m-1)-1$.

Calculate Dynkin diagram of $O(2m)$. Choose generic point in $Lie(T)$ say $\xi = (\xi_1, \dots, \xi_m)$ and suppose it lies in the open region $\xi_1 > \dots > \xi_m > 0$. Then the **chambre** containing ξ is described by $\theta_i \pm \theta_j$ has same sign as $\xi_i \pm \xi_j$ or zero. Positive roots are therefore

$$\theta_i + \theta_j \qquad 1 \le i < j \le m$$

$$\theta_i - \theta_j . \qquad 1 \le i < j \le m$$

and the fundamental chambre is described by

$$\theta_i \ge \theta_j \qquad i < j$$

$$\theta_i + \theta_j \ge 0$$

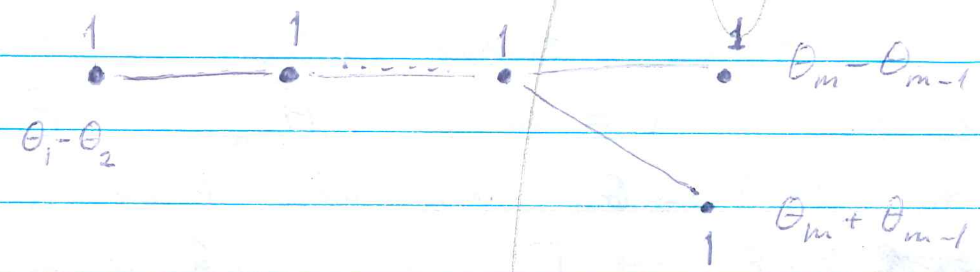
But $\theta_i + \theta_j$ is minimum for $\theta_{m-1} + \theta_m$. So the fundamental chambre is

$$\theta_1 \geq \theta_2 \geq \cdots \geq \theta_m \geq -\theta_{m-1}$$

and the fundamental roots are

$$\theta_1 - \theta_2, \quad \ldots, \quad \theta_{m-1} - \theta_m, \quad \theta_{m-1} + \theta_m$$

so we get the Dynkin diagram



(one line between vertices means the angle is $\frac{2\pi}{3}$)
over vertex goes something proportional to length)

$O(2m+1)$: Same torus and the same roots $\theta_i \pm \theta_j$ $1 \leq i \leq j \leq m$, But we have some more roots.
Critical case $m = 1$.

$$\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \quad \begin{matrix} * \\ * \\ 1 \end{matrix}$$

Extra root is just θ. So the roots are

$\pm(\theta_i - \theta_j)$, $\pm(\theta_i + \theta_j)$ $\qquad 1 \leq i < j \leq m$

and $\pm\theta_i$ $\qquad 1 \leq i \leq m$

The Weyl group is clearly $\Sigma_m \ltimes (\mathbb{Z}/2)^m$, so ~~the~~ a fundamental chamber is clearly

$$\theta_1 \geq \theta_2 \geq \cdots \geq \theta_m \geq 0$$

and the simple roots are

$$\theta_1 - \theta_2, \ldots, \theta_{m-1} - \theta_m, \theta_m.$$

Thus the Dynkin diagram is

Sp(2m). Subgroup of U_{2m} commuting with j.

First take $m = 1$. Let j denote the auto of $\mathbb{C} \oplus \mathbb{C}j$ such that $j(\alpha + \beta j) = \bar{\alpha} j - \bar{\beta}$. Then I am after those unitary matrices commuting with j. Let J be the linear operator with $J(1) = j$, $J(j) = -1$. Then $j = \sigma J$ where $\sigma(\alpha, \beta) = (\bar{\alpha}, \bar{\beta})$.

~~Thus~~ In general I work in $\mathbb{C}^{2m}$ with basis $e_1, \ldots, e_{2m}$ and $J e_i = e_{i+m}$, $J^2 = -1$. Thus

$$J = \left(\begin{array}{c|c} 0 & 1 \\ \hline -1 & 0 \end{array}\right)$$

and I want those unitary matrices A such that

$$\sigma J A = A \sigma J$$

i.e. such that $J A J^{-1} = \bar{A}$ ~~For example~~

Since

$$\begin{pmatrix} & 1 \\ -1 & \end{pmatrix}\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}\begin{pmatrix} & -1 \\ 1 & \end{pmatrix} = \begin{pmatrix} \gamma & \delta \\ -\alpha & -\beta \end{pmatrix}\begin{pmatrix} & -1 \\ 1 & \end{pmatrix}$$

$$= \begin{pmatrix} \delta & -\gamma \\ -\beta & \alpha \end{pmatrix}$$

this means we want all unitary matrices ~~such that~~ of the form

$$\begin{pmatrix} \alpha & \beta \\ -\bar{\beta} & \bar{\alpha} \end{pmatrix}$$

Thus $Sp_1 = SU_2$.

The Lie algebra consists of matrices of the above form which are skew-hermitian $\Rightarrow \alpha$ skew-hermitian, β symmetric. Maximal torus

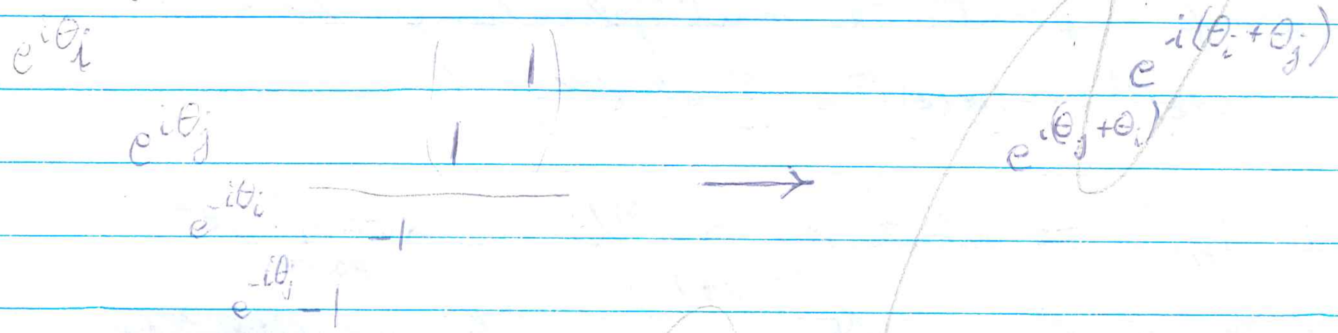
$$\begin{pmatrix} e^{i\theta_1} & & & & & \\ & \ddots & & & & \\ & & e^{i\theta_n} & & & \\ & & & e^{-i\theta_1} & & \\ & & & & \ddots & \\ & & & & & e^{-i\theta_n} \end{pmatrix}$$

~~roots~~ $\cancel{\alpha, \beta}$ ~~roots~~ $\cancel{\alpha+\beta}$

$\cancel{H_{\alpha+\beta} = H_\alpha + H_\beta}$

$\cancel{[[X_\alpha, X_\beta], [X_{-\alpha}, X_{-\beta}]]}$

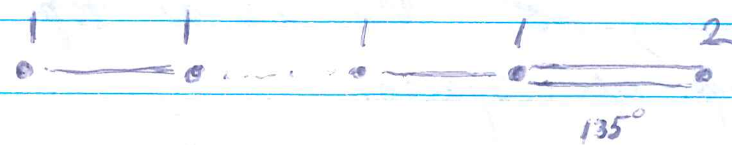
Pos. Roots include $\theta_i - \theta_j$ ~~$i<j$~~ $i<j$, $2\theta_i$, and $\theta_i + \theta_j$ for $i<j$.



Weyl group is clearly $\Sigma_n \ltimes (\mathbb{Z}_2)^n$, hence a fundamental chamber is given by $\theta_1 \geq \cdots \geq \theta_m \geq 0$, so the simple roots are

$$\theta_1 - \theta_2, \ldots, \theta_{m-1} - \theta_m, 2\theta_m,$$

and the Dynkin diagram is



Basic proof technique in the case of a group G is to consider for any root α, the map $SL_2 \longrightarrow G$ associated to the triple $X_\alpha, Y_\alpha, H_\alpha$. This map is unique up to picking out X_α. Also if G is simply-connected it is an embedding, because one can suppose α simple, and the simple H_{α_i} form a $\mathbb{Z}$-lattice for the lattice points when $\mathbb{Z}$ is 1-connected

What this technique means is that the centralizer of the hyperplane $\alpha = 0$ in $\mathfrak{h}$ is $\mathfrak{h} + \mathfrak{g}^{\alpha} + \mathfrak{g}^{-\alpha}$. In effect if ξ is generic such that $\alpha(\xi) = 0$, then $\beta(\xi) = 0 \Rightarrow$ β proportional to α whence $\beta = \pm\alpha$. These centralizers of hyperplanes are exactly the sort ~~[illegible]~~ of things one considers in the Morse theory.

I want to look at the analogous things in the symmetric space situation. Thus given a reduced root $\alpha \in \Phi_0$ I consider the torus killed by α and its centralizer which will be

$$(*) \qquad \mathfrak{m} + E_0 + k_\alpha + k_{2\alpha}$$

because only roots prop. to α are $\pm\alpha, \pm 2\alpha$.

and whose complexification is

$$\mathfrak{g}^{-2\alpha} \oplus \mathfrak{g}^{-\alpha} \oplus (\mathfrak{m} + \mathfrak{a}) \oplus \mathfrak{g}^{\alpha} \oplus \mathfrak{g}^{2\alpha}$$

It is clear that E_0 is still a maximal reversed abelian subspace of $(*)$, because the σ-minus space is contained in $E_0 + k_\alpha + k_{2\alpha}$. Next take ~~commutator subalgebra~~ the center of $(*)$ which ~~$\mathfrak{m} + \mathbb{R} i H_\alpha + k_\alpha + k_{2\alpha}$~~ has to be in $\mathfrak{m} + E_0$, in fact it is contained in $\mathfrak{m}$ + the part of E_0 killed by α. So therefore it is clear that the semi-simple part of (*) is going to be associated to a rank 1 symmetric space.

Rank 1 symmetric spaces are spheres, complex proj. spaces, quaternionic projective spaces, and there is an F_4 one. It is the octonion projective planes. (Therefore rank 1 symm. spaces are projective spaces).

Just to be complete I should work out the diagrams and multiplicities of the quaternionic projective spaces.

We had the max. torus in $Sp(m)$

$$\begin{pmatrix} e^{i\theta_1} & & & & & \\ & \ddots & & & & \\ & & e^{i\theta_m} & & & \\ & & & e^{-i\theta_1} & & \\ & & & & \ddots & \\ & & & & & e^{-i\theta_m} \end{pmatrix}$$

~~One takes the reflection in the θ_1 angle for involution, so the restriction to E_0 sends θ_1 to θ and $\theta_2, \dots, \theta_m$ to 0. Then we get~~

~~θ~~	~~mult.~~	~~$2(m-1)$~~
~~2θ~~	~~mult.~~	~~1~~

For involution we take interchange of θ_1, θ_2
i.e. conjugation by

$$\sigma : \left(\begin{array}{ccccc|ccccc} 0 & 1 & & & & & & & & \\ +1 & 0 & & & & & & & & \\ & & 1 & & & & & & & \\ & & & \ddots & & & & & & \\ & & & & 1 & & & & & \\ \hline & & & & & 0 & 1 & & & \\ & & & & & 1 & 0 & & & \\ & & & & & & & 1 & & \\ & & & & & & & & \ddots & \\ & & & & & & & & & 1 \end{array}\right)$$

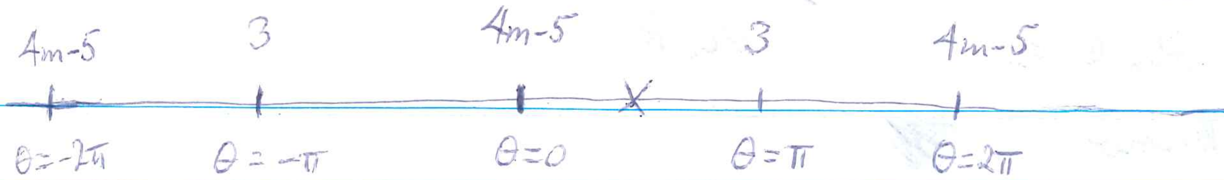
$\theta_1 \mapsto \theta \qquad \theta_2 \mapsto -\theta, \qquad \theta_3, \dots, \theta_m \mapsto 0$

Roots are:

2θ mult. 3 (Comes from $\theta_1 - \theta_2$, $2\theta_1$, $-2\theta_2$)

θ mult.
$$\begin{array}{rl} m-2 & (\theta_1 - \theta_j \quad j > 2) \\ m-2 & (-\theta_2 + \theta_j \quad j > 2) \\ m-2 & (\theta_1 + \theta_j \quad j > 2) \\ + \; m-2 & (-\theta_2 - \theta_j \quad j > 2) \\ \hline 4(m-2) & \end{array}$$

So the diagram is:



so the Poincaré series of the loop space is

$$1 + t^3 + t^{4m-2} + t^{3+4m-2} + \cdots$$

$$= (1+t^3)/(1-t^{4m-2})$$

which agrees with the fibring

$$S^3 \longrightarrow S^{4m-1} \longrightarrow \mathbb{H}P^{m-1}$$

$$\Omega S^{4m-1} \longrightarrow \Omega \mathbb{H}P^{m-1} \longrightarrow S^3$$

For a rank 1 symmetric space, the spherical building is a sphere, namely $S(\mathfrak{p})$, and the apartment is a pair of anti-podal points. It is a zero-dimensional complex.

We can ~~[illegible]~~ think of a rank 1 symmetric space as a projective space, in which closed geodesics ~~are~~ live in projective lines. Observe that projective lines are spheres.

$\mathbb{R}P_1 = S^1$ $\qquad S^0 \longrightarrow S^1 \longrightarrow S^1$

$\mathbb{C}P_1 = S^2$ $\qquad S^1 \longrightarrow S^3 \longrightarrow S^2$

$\mathbb{H}P_1 = S^4 \qquad S^3 \to S^7 \to S^4$

$\mathbb{O}P_1 = S^8 \qquad S^7 \to S^{15} \to S^8$

$K_\square^\sigma$ - orbits on S.

From Bott-Samelson we know that the space of paths in X from a generic point $t_0 = p_0(\xi_0)$ to the orbit $K^\sigma s$ is described by geodesics joining ξ_0 to $p_0^{-1}(K^\sigma s \cap S) = p_0^{-1}(W_0 s)$; here described means we get a CW complex whose cells form a basis for the mod 2 homology. In particular π_0 of this space of paths is the number of points of $p_0^{-1}(W_0 s)$ in the same chamber as ξ_0 in diagram. Suppose $0 < \alpha(\xi_0) < 1$ for any positive root $\alpha \in \Phi_0^+$, whence $C_0' = \{\xi \mid 0 \le \alpha(\xi) \le 1$ for all $\alpha \in \Phi_0^+\}$.

Suppose K simply-connected. Taking $s = 1$, we find $\pi_1 X = \pi_0 \Omega X$ is in one-one correspondence with $C_0' \cap p_0^{-1}(1)$. But as $S \subset T$ and $C_0' \subset C'$ and C' contains a unique lattice pt, it follows $\pi_1 X = 0$. Hence K^σ is connected. Thus the orbits $K_\sigma s$ are connected and so from the fibration

$$\text{paths}(t_0, K_\sigma s) \longrightarrow K_\sigma s \longrightarrow X$$

we see that $\pi_0 \, \text{path}(t_0, K_\sigma s) = 1$, hence

C_0' contains exactly one point of $p_0^{-1}(W_0 s)$. Thus

$$C_0' \xrightarrow{\sim} W_0 \backslash S \xrightarrow{\sim} K^\sigma \backslash X.$$

Go back to Lie alg. case: suppose K compact connected, let K_0 = identity component of K^σ. Apply Morse theory to the orbit $K_0 \eta$ and the function $|k\cdot\eta - \xi|^2$ where ξ is regular in E_0.

Consequence: Because this function has a minimum $K_0\eta \cap E_0 \neq \phi$, whence K_0-conjugacy for maximal subspaces of k_-, and $K_0\eta \cap E_0 = W_0 \eta$ (for $\eta \in E_0$).

Next you have to show critical points of index 1 are realized by circles, so that $\exists!$ critical points of index 0 in each component. This shows that C_0 contains exactly one point of each W_0-orbit.

This means W_0 is a reflection group: First — only the identity carries a chamber into itself. Next given hyperplane $\alpha = 0$, pick two chambres C_1, C_2 have this hyperplane as common walls. Then there is some element w such that $wC_1 = C_2$. If $\xi \in C_1 \cap C_2$, then both $\xi, w\xi \in C_2 \Rightarrow w\xi = \xi$. Thus w has to be the reflection thru $\alpha = 0$. Rest is standard.

Next let's try to understand the K^σ-orbit ~~[illegible]~~ $K^\sigma\eta$, where $\eta \in C_0$. ~~[illegible]~~ ~~[illegible]~~ I claim that every element of K^σ/K_0 is represented by a $k \in K^\sigma$ which normalizes C_0; ~~[illegible]~~ this is clear. The induced transf. of C_0 is of finite order, so has a fixpt ξ in the interior. The centralizer of ξ in K is connected, ~~[illegible]~~ and has Lie algebra $\mathfrak{m} + E_0$, so the centralizer of ξ centralizes E_0. Thus any element k of K^σ normalizing C_0 centralizes it, hence is in M. $\therefore K^\sigma/M$ is connected, so all K^σ-orbits are connected.

June 27, 1975:

~~[illegible]~~ Let θ be an automorphism of a compact conn. Lie group U. Let σ be the autom. of $K = U \times U$ given by

$$\sigma(x,y) = (\theta^{-1}y, \theta x)$$

Then $\sigma^2(x,y) = (\theta^{-1}(\theta x), \theta(\theta^{-1}y)) = (x,y)$, so σ is an involution. Clearly $K^\sigma = \{(x,\theta x) \mid x \in U\} = \Gamma_\theta$.

$K/K^\sigma \xrightarrow{\sim} U \qquad (x,y) \mapsto x(\theta^{-1}y)^{-1}$ and the K-action on U becomes

$$(x,y)\cdot u = x u\, \theta^{-1}y^{-1}.$$

~~Thus we see that the twisted conjugation action of K on itself associated to an auto. θ^{-1} is a special case of K^σ acting on X.~~

I was hoping that the $K^\sigma = U$ orbits on X would include the homogeneous space U/U^θ. Thus I want to find a $u \in U$ such that

$$(x, \theta x)\cdot u = u \qquad \Longleftrightarrow \qquad \theta x = x$$

But $(x,\theta x)\cdot u = x u x^{-1}$, so this means U^θ is the centralizer of x. Thus nothing new arises.

Let θ be an autom. of the compact connected group K. Let K act on itself by twisted conjugation $x * y = x y\, \theta x^{-1}$, whence the orbit through 1 is K/K^θ.

Consider the local situation first, that is, in a tubular nbd. of of $K*1$. This is completely described by K^θ acting on $\mathfrak{k}/\mathfrak{k}^\theta$. Now this action has to preserve the different eigenspaces of θ on $\mathfrak{k}/\mathfrak{k}^\theta$.

Let K be a connected compact Lie group acting linearly on a Euclidean space V. Let ξ be a generic point of V. A tubular nbd. of $K\xi$ is a disk bundle in $K \times^{K_\xi} W$ over K/K_ξ. Because ξ is generic, it must act trivially on W. Thus the orbit $K\cdot\xi$ meets the fixpoint set V^{K_ξ} transversally at ξ. (This is what one means by a principal orbit type). ~~Put K_ξ~~

~~Recall that there are~~

Put $H = K_\xi$ and $E = V^{K_\xi}$. ~~I recall that there are only finitely many orbit types of the K action on~~

Question: Given any point η of E, is the orbit $K\eta$ perpendicular to E at η? More precisely, is $X\cdot\eta$ perpendicular to E for any $X \in \mathfrak{k}$?

Idea might be that this is true if $K_\eta = H$, and that the set of such η in E is dense. Consider the map

$$\mathfrak{k}/\mathfrak{h} \times E \longrightarrow V$$

$$X + \mathfrak{h},\ \eta \longmapsto X\cdot\eta$$

which is linear. The set of η in E for which $\mathfrak{k}/\mathfrak{h} \to V$ is injective is Zariski open and non-empty, hence dense.

Thus it seems that given any $\eta_0 \in E$ $X\cdot\eta_0$ will be the limit of $X\cdot\eta$ where η is such that $X\cdot\eta \perp E$, hence $X\cdot\eta_0 \perp E$.

Now fix $\xi \in E$ with centralizer exactly H, and consider the Morse function on an orbit Kv given by

$$|kv-\xi|^2 = |v|^2+|\xi|^2-2(kv,\xi).$$

This has inf. behavior

$$\begin{aligned}|v|^2+|\xi|^2-2(e^X\cdot kv,\xi) = \ &\text{constant}\\ &-2(X\cdot kv,\xi)\\ &-2(X^2\cdot kv,\xi)\end{aligned}$$

so it has critical points where

$$(X\cdot kv,\xi) = -(kv, X\cdot\xi) = 0$$

for all $X\in\mathfrak{k}$, i.e. where $kv\in E$. Thus every orbit meets E, which shows among other things that there is a unique principal orbit type.

Consider the Hessian on the tangent space to the orbit at the critical point $\eta = kv \in E$. The tangent space is $\simeq \mathfrak{k}/\mathfrak{k}_\eta$, where $\mathfrak{k}_\eta = \{X \mid X\cdot\eta = 0\}$ is a subalgebra containing $\mathfrak{h}$.

The Hessian is the bilinear form $(X,Y) \mapsto (XY\cdot\eta, \xi)$ which is symmetric since $([X,Y]\eta, \xi) = 0$ as we are at a critical point.

$$-(XY\eta, \xi) = (Y\eta, X\xi)$$

But we have seen that the vector $Y\eta$ for any $Y \in k/k_\eta$ is perpendicular to E, hence of the form $X\xi$ for some X. This shows the Hessian is non-degenerate.

Different version of p.72: ~~Take $v \in V$, and write $V = k\cdot v \oplus E$ where E is the orthogonal complement of kv, hence stable under the k-action. The stabilizer of $v + E$~~

Given $v \in V$, we know that for v' near v $\dim k_{v'} \leq \dim k_v$. Thus if we choose ξ so that $\dim k_\xi$ is minimal, then $\dim k_\xi = \dim k_v$ for all v in some nbd. of ξ. Let E be the orthogonal complement of $k\cdot\xi$. ~~As X ranges over unit vectors in $k \ominus k_\xi$, Xv is never zero hence $|Xv| > \varepsilon$. Let e be any element of E such that $|Xe| < \varepsilon$~~ First note that $k_\xi E \subset E$: $X \in k$, $Y \in k_\xi$, $e \in E$

$$(X\xi, Ye) = -(YX\xi, e) = -(XY\xi, e) + ([X,Y]\xi, e) = 0$$

Now let's calculate the stabilizer of $\xi + e$ when e is very small. Given $X \in k$ write it $X = X' \oplus X'' \in k_\xi \oplus k_\xi^\perp$. Then

$$X(\xi + e) = \{X''\xi + X''e\} + \underbrace{X'e}_{E}$$

Thus if pr_1 is the projection of V onto k_ξ we have

$$pr_1(X(\xi+e)) = X''\xi + pr(X''e)$$

Now because $X'' \in k_\xi^\perp$ we have an estimate

$$|X''\xi| > \varepsilon |X''|$$

and as we have an estimate

$$|pr(X''e)| \leq C|X''||e|$$

if e is suff. small we will have

$$X(\xi+e) = 0 \Rightarrow X'' = 0 \Rightarrow X\xi = 0.$$

By the choice of ξ, this means that $k_{\xi+e} = k_\xi$. As this holds for all small e we see that

$$k_\xi \cdot E = 0.$$

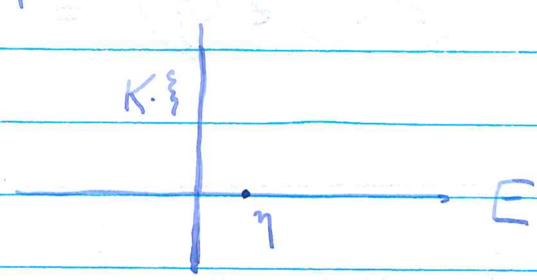
~~If $\eta \in E$, then $k_\xi \subset k_\eta$, so if $\dim k_\eta = \dim k_\xi$ (which is true for $\eta \in E$ near ξ), then $k_\xi = k_\eta$.~~

~~Next consider an element $u \in k^{\xi} \cap V$, say $u = \ldots$~~

You made an error: $(G/H)^H = N/H$ not H/H. Consequently we must see what can be salvaged.

Again let ξ be generic in V whence a tubular nbd of $K\cdot\xi$ is of the form $K \times^{K_\xi} W$ where K_ξ acts trivially on W. Put $H = K_\xi$, and let $N = N_K(H)$. Then N/H acts on V^H!

The problem is as follows: Choose ξ generic, then at ξ we know that the orbit $K\cdot\xi$ is perpendicular to E by definition of E. But we want $K\cdot\eta$ to be perpendicular to E for all η in E, and at least for η in E near to ξ.



But this doesn't always happen. For example assume K acts freely on SV, whence all ξ are generic. If the perpendicular to one orbit is perpendicular to nearby orbits, we get a flat connection on the principal bundle $S(V)$ over $K \backslash S(V)$.

So let's return to the auto θ of K and to the action of $\mathfrak{k}^\theta$ on $\mathfrak{k}/\mathfrak{k}^\theta$. The action preserves the eigenspaces of θ, so we shall let V be one of these eigenspaces.

Why do we get a good situation when $\theta^2 = I$? ~~Because~~ Let ξ be generic and E be the perp. space to $\mathfrak{k}^\theta \cdot \xi$. Then

$$-(\eta, X \cdot \xi) = (X \cdot \eta, \xi) = (X, [\eta, \xi])$$

so that $\eta \in E \iff \mathfrak{k}^\theta \perp [\eta, \xi]$. But when $\theta^2 = 1$, $[\eta, \xi] \in \mathfrak{k}^\theta$ so $[\eta, \xi] = 0 \iff \eta \in E$. Actually one proceeds as follows. One takes E to be a maximal abelian subspace of $\mathfrak{k}_-$, then one constructs the roots of $\mathfrak{k}$ with respect to E, and from this root decomposition one can see that $\mathfrak{k}^\theta \cdot E$ is the orthogonal complement to E.

So if K acts on a manifold X and $x \in X$ is of the principal type (this means that K_x acts trivial in the normal space to the orbit $K \cdot x$ at x) then the really good situation is where the a tubular neighborhood around the orbit is ~~[illegible]~~ not just topologically a ~~[illegible]~~ product of the orbit and the isotropy repn. but also preserving the metric.

~~Return~~ Return to loop space of a symmetric space and work out the theory.

K compact and connected with ~~involutions~~ involution σ. I used ~~the~~ the Bott–Samelson theory to understand the fibre of the inclusion $K^\sigma \cdot x \hookrightarrow X$ where $X = K/K^\sigma$. Choosing a generic element t_0 of X, one get S, E_0, W_0 as before. If $x = s \in S$, then the fibre $P(X; K^\sigma s, t_0)$ has the homotopy type of a CW cx. with cells indexed by points of $p^{-1}(W_0 s)$. Suppose $t_0 = p(\xi_0)$ with $\xi_0 \in C_0'$ $= \{\eta \in E_0 \mid \alpha \in \Phi_0^+ \Rightarrow 0 \le \alpha(\eta) \le 1\}$. The theory shows among other things that $\pi_0(P(X; K^\sigma s, t_0)) = p^{-1}(W_0 s) \cap C_0'$. (Helgason's method, ~~go~~ which is based on the ~~mapping~~ isom

$$K^\sigma/M \overset{W_0}{\times} S_r \xrightarrow{\sim} X_r$$

plus the fact that $\pi_1(X_r) = \pi_1(X)$ because the singular set has codim ≥ 2, also shows $p^{-1}(1) \cap C_0' = \pi_1(X)$.)

~~[illegible]~~

So by these style arguments I know that C_0' is a fundamental domain for (S, W_0) or (X, K^σ), when K is simply-connected. ~~[illegible]~~ ~~[illegible]~~ Thus C_0' will also be a fundamental domain for $\mathcal{K}^\sigma$ acting on X^σ.

I want to define an action of $\mathcal{G}$ on $\mathcal{X}$, following the procedure used for the G action on $i\mathfrak{k}$. Here K is assumed to be connected compact with given maximal torus T. Let me take an element $\tilde{x}$ in the apartment of $\mathcal{X}$; thus $x \in E = \frac{1}{2\pi i} \operatorname{Lie}(T)$ and

$$\tilde{x}(t) = e^{2\pi i t x}$$

I recall that $\mathcal{G}$ is generated by ~~[illegible]~~ the 1-parameter subgroups

$$x_\alpha(f) = \exp(f X_\alpha) \qquad f \in F$$

and also $H(F)$. ~~Suppose next~~

$$\tilde{x}^{-1} x_\alpha(f) \tilde{x} =$$

(My idea is to let F be ~~[illegible]~~ meromorphic near ∞: $f = \sum_{n \le N} a_n z^n$).

Convention: We will work near $z = 0$ to start with. Thus $\mathcal{f}$ will consist of series $\sum_{n \geq -N} a_n z^n$ which are convergent near zero. I can identify elements of $\mathcal{G}$ with certain holomorphic maps from punctured disks to $\mathbb{C}$. I should think of my objects as holomorphic fns. on $\operatorname{Im} t > \text{const.}$ with values in G.

Suppose given an element x of E and an element g of $\mathcal{G}$. Then

$$\tilde{x}^{-1} g \tilde{x} = e^{-2\pi i t x} g(e^{2\pi i t}) e^{2\pi i t x}$$

is a holom. function on a strip $\operatorname{Im} t > \text{const}$ with values in G. I want to consider those g such that this converges as $\operatorname{Im} t \to +\infty$.

Let $g(z) = \exp(f(z) X_\alpha) = x_\alpha(f)$. Then

$$\tilde{x}^{-1} g \tilde{x} = \exp\left(\operatorname{Ad}(e^{-2\pi i t x}) f(z) X_\alpha\right)$$

$$= \exp\left(f(z) e^{-2\pi i t \alpha(x)} X_\alpha\right)$$

since $f(z) = z^m f_1(z)$ with f_1 holom. and $\neq 0$ at 0, the above function will converges as $\operatorname{Im} t \to +\infty$ iff

$$e^{2\pi i t(m-\alpha(x))} \quad \text{converges}$$

which is the case if $m - \alpha(x) \geq 0$. Let P_x denote the set of $g \in G$ such that $\tilde{x}^{-1} g \tilde{x}$ has a limit as $\operatorname{Im} t \to +\infty$. Then clearly P_x is a subgroup of G. We see that it contains

$$\left[\begin{array}{ll} x_\alpha(f) & \text{for } \alpha(x) \leq \operatorname{ord} f \\ H(R) & \end{array}\right.$$

~~[illegible]~~

I want to show that given $\xi \in \mathfrak{X}$ and $g \in G$, there exists a unique $\eta \in \mathfrak{X}$ such that $\eta^{-1} g \xi$ has a value at $z = +\infty$, i.e. converges in G as $\operatorname{Im}(t) \to \infty$.

Uniqueness. If $\eta_1^{-1} g \xi$, $\eta_2^{-1} \cdot g \xi$ converge at $\operatorname{Im}(t) \to \infty$, then so does $\eta_2^{-1}\eta_1$. Thus we have to show that $\eta^{-1}\xi$ converges in G as $\operatorname{Im} t \to \pm\infty$ implies $\eta = \xi$.

Now ξ being an element of $\mathfrak{X}$ we have

$$\xi(t+1) = \xi(t)\,\xi(1)$$

and similarly for η, so

$$(\eta^{-1}\xi)(t+1) = \eta(1)^{-1}\cdot(\eta^{-1}\xi)(t)\cdot\xi(1)$$

so $\eta(1)T = T\xi(1)$ where $T = (\eta^{-1}\xi)(\mp i\infty)$

Choose X with $e^X = \xi(1)$, whence

$$\xi(t) = f(e^{2\pi i t})e^{tX}$$

with $f \in K'$. Then $e^{TXT^{-1}} = \eta(1)$, so

$$\eta(t) = f_1(z)\,e^{tTXT^{-1}}$$

with $f_1 \in K'$. Hence

$$(\eta^{-1}\xi)(t) = e^{-t(TXT^{-1})}(f_1^{-1}f)(z)\,e^{tX}$$

$$= T\,e^{-tX}\,T^{-1}(f_1^{-1}f)(z)\,e^{tX}$$

converges to T as $\mathrm{Im}(t) \to -\infty$. To show this is constant, I can suppose X diagonal with diagonal entries $i\lambda_j$ $\lambda_j \in \mathbb{R}$

$$T^{-1}(f_1^{-1}f)(z) = \sum_{|n|\le N} a_n z^n$$

~~with~~ So we have now

$$e^{-tX}\varphi(z)\,e^{tX} \quad \text{converging as } \mathrm{Im}\,t \to -\infty$$

where $\varphi(z) = T^{-1}(f_1^{-1}f)(z)$

Lemma: ξ, η are self-adjoint matrices such that $e^{-t\eta}e^{t\xi}$ converges in GL_n as $t \to +\infty$. Then $\xi = \eta$.

Proof: By converging in GL_n I mean that $T = \lim_{t\to+\infty} e^{-t\eta}e^{t\xi}$ is an invertible matrix. It follows that $T^{-1} = \lim_{t\to+\infty} e^{-t\xi}e^{t\eta}$. But because ξ and η are self-adjoint

$$T^* = \lim_{t\to+\infty} e^{t\xi}e^{-t\eta} = \lim_{t\to-\infty} e^{-t\xi}e^{t\eta}$$

Now the matrix $e^{-t\xi}e^{t\eta}$ has entries which are linear combinations of exponentials e^{at} with $a \in \mathbb{R}$. If such a function converges both ~~as~~ $t \to +\infty$ and as $t \to -\infty$ it is necessarily constant. Therefore $e^{-t\xi}e^{t\eta}$ is constant, so $\xi = \eta$.

Analogous lemma will be this:

L: Let $\xi, \eta \in X$ be such that $\eta^{-1}\xi$ converges in GL_n as $\operatorname{Im} t \to +\infty$. Then $\eta = \xi$.

Special case ~~[illegible]~~ $\eta = 1$. The matrix

$$\xi(t) = f(e^{2\pi i t})\, \cancel{\ } \, e^{2\pi i t X}$$

(X self-adjoint) has entries which are linear combinations of exponentials $e^{2\pi i t a}$ with $a \in \mathbb{R}$.

Thus if it is bounded as both $\mathrm{Im}\, t \to +\infty$ and $\mathrm{Im}\, t \to -\infty$, then it is constant. But $\xi(t) = (\xi(t)^*)^{-1}$ if t is real, so

$$\xi(t) = \left(\xi(\bar{t})^*\right)^{-1}$$

for all $t \in \mathbb{C}$. But as $\mathrm{Im}\, t \to -\infty$ $\mathrm{Im}(\bar{t}) \to +\infty$ and the right side converges, hence $\xi(t)$ converges in both directions, hence it is constant.

The same argument works with the function $(\eta^{-1}\xi)(t)$ which is a linear combination of exponentials $e^{2\pi i t a}$ with $a \in \mathbb{R}$.

I am still trying to show that for any $\xi \in \mathcal{X}$ and for any $g \in G$, there exists a unique η in $\mathcal{X}$ such that

$$\eta^{-1} g \xi$$

converges as $\mathrm{Im}(t) \to -\infty$. We have already seen the uniqueness holds, so it remains to prove the existence of η. Note that if $g \in K$, then $g \xi g(1)^{-1} \in \mathcal{X}$ so

$$\eta = g \xi g(1)^{-1}.$$

Thus $g * \xi = g \xi g(1)^{-1}$ if $g \in \mathcal{K}$.

Let $\mathcal{I} = \{ g \in \mathcal{G} \mid \forall \xi \in \mathcal{X}, \exists\, k \in \mathcal{K} \ni k^{-1} g \in P_\xi \}$.

1). $\mathcal{I}$ is a subgroup: Given $g_1, g_2 \in \mathcal{I}$ and $\xi \in \mathcal{X}$, choose k_2 so that

$$k_2^{-1} g_2 \in P_\xi$$

and choose k_1 so that

$$k_1^{-1} g_1 \in P_{k_2 \xi} = k_2 P_\xi k_2^{-1}.$$

Then $k_2^{-1} k_1^{-1} g_1 k_2 \in P_\xi$ so

$$(k_1 k_2)^{-1} \cdot g_1 g_2 = k_2^{-1} k_1^{-1} g_1 k_2 \cdot k_2^{-1} g_2 \in P_\xi .$$

QED.

~~2). $\mathcal{I} = \{ g \in \mathcal{G} \mid \forall \xi \in$ fund. domain for $\mathcal{K}$ on $\mathcal{X}$, $\exists\, k \in \mathcal{K} \ni k^{-1} g \in P_\xi \}$~~

~~Given $g \in \mathcal{G}$ and $\eta \in \mathcal{X}$ arbitrary, choose k_1 such that $k_1 \eta = \xi \in$ the fund. domain.~~

I want to return to the action of G on $\mathfrak{p} = i\mathfrak{k}$ and to see if I can define this by descent from GL_n. Let $\xi \in \mathfrak{p}$ and recall that we have defined

$$P_\xi = \{g \in G \mid e^{-t\xi} g e^{t\xi} \text{ converges}\}$$

$$P_\xi^u = \{g \in G \mid e^{-t\xi} g e^{t\xi} \longrightarrow 1\}$$

Let G_ξ be the centralizer of ξ in G. If $g \in P_\xi$, it is clear that if

$$(g)_0 = \lim_{t \to +\infty} e^{-t\xi} g e^{t\xi}$$

then $\qquad e^{-s\xi} (g)_0 e^{s\xi} = (g)_0$, hence $(g)_0 \in G_\xi$.
Thus

$$P_\xi = G_\xi \ltimes P_\xi^u .$$

~~[illegible]~~ Let $K_\xi = K \cap G_\xi$; it's clear that because $\xi^* = \xi$, G_ξ is stable under the Cartan involution, hence K_ξ is a maximal compact subgroup.

Let $X = \exp \mathfrak{p}$, $X_\xi = X \cap G_\xi$. Then

$$G_\xi = K_\xi \times X_\xi \qquad\qquad X_\xi = \exp(\mathfrak{p}_\xi).$$

(Take fixpoints under the group of inner autos $e^{it\xi}$ which preserves the involution.)

Now the claim is that

$$G = K \times^{K_\xi} P_\xi$$
$$= K \times^{K_\xi} (G_\xi \times P^u_\xi)$$
$$= K \times X_\xi \times P^u_\xi$$

and this can be proved by descent from GL_n. In fact, the only point really is that $G = K \cdot P_\xi$, for $K \cap P_\xi = K_\xi$:

$$e^{-t\xi} k e^{t\xi} = e^{-t\xi} e^{t k.\xi} k$$

which converges $\Longleftrightarrow$ $k.\xi = \xi$. For $GL(n)$, this results by Gram-Schmidt.

So at this point we can define the G-action on $\mathfrak{p}$ and we know that K acts transitively on each orbit. We can identify $S(\mathfrak{p})$ with the <u>spherical building</u>.

Next you want to generalize things to $\mathcal{G}$, $\mathfrak{X}$. Let $\xi \in \mathfrak{X}$.

$$\mathcal{P}_\xi = \{ g \in \mathcal{G} \mid \xi^{-1} g \xi \text{ converges as } \operatorname{Im} t \to -\infty \}$$
$$\mathcal{P}^u_\xi = \{ g \in \mathcal{G} \mid \xi^{-1} g \xi \to 1 \text{ as } \operatorname{Im} t \to -\infty \}$$

Because $\xi(t+1) = \xi(t)\,\xi(1)$ we have

$$\xi(t+1)^{-1} g(z)\, \xi(t+1) = \xi(1)^{-1}\xi(t)^{-1} g(z)\, \xi(t)\, \xi(1)$$

and so taking the limit as $\operatorname{Im} t \to -\infty$ we get

$$(g)_0 = \xi(1)^{-1} (g)_0\, \xi(1)$$

where $(g)_0$ is this limit. Thus we get a homomorphism $P_\xi \longrightarrow G_{\xi(1)}$ with kernel P_ξ^u.

Let $\gamma \in G_{\xi(1)}$. I want to manufacture an element of P_ξ mapping to γ. Suppose

$$\xi(t) = e^{2\pi i t X} \qquad X \in \mathfrak{p}.$$

Then

$$(\xi\gamma\xi^{-1})(t) = e^{2\pi i t X}\gamma\, e^{-2\pi i t X} = e^{2\pi i t X} e^{-2\pi i t \gamma(X)}\gamma$$

will be an element of $\mathcal{G}$ such that

$$\xi^{-1}(\xi\gamma\xi^{-1})\xi = \gamma$$

is constant, hence $\xi\gamma\xi^{-1} \in P_\xi$ maps to γ.

In general

$$\xi(t) = f(z)\, e^{2\pi i t X} \qquad f \in \mathcal{K}$$

so

$$(\xi\gamma\xi^{-1})(t) = f(z)\, e^{2\pi i t X}\gamma\, e^{-2\pi i t X} f(z)^{-1}$$
$$= f(z)\, e^{2\pi i t X} e^{-2\pi i \gamma\cdot X}\gamma\, f(z)^{-1}$$

is an element g of $\mathcal{G}$ such that $\xi^{-1} g \xi$ is constant.

~~[illegible]~~

Put $\mathcal{G}_\xi = \{ g \mid \xi^{-1} g \xi \text{ is constant} \}$, whence

$$\mathcal{G}_\xi \xrightarrow{\sim} G_{\xi(1)}.$$

and

$$\mathcal{P}_\xi = \mathcal{G}_\xi \ltimes \mathcal{P}_\xi^u.$$

Also ~~[illegible]~~ note that

$$\mathcal{K} \cap \mathcal{P}_\xi = \{ k \in \mathcal{K} \mid k * \xi = \xi \}.$$

~~[illegible]~~ Because if $k \in \mathcal{K} \cap \mathcal{P}_\xi$, then $\xi^{-1} k \xi = \xi^{-1}(k * \xi)\, k(1)$ converges as $\operatorname{Im} t \to -\infty$ implies $\xi^{-1}(k * \xi)$ converges $\Longleftrightarrow$ $k * \xi = \xi$. Thus if $\mathcal{K}_\xi = \{ k \mid k * \xi = \xi \}$, then

$$\mathcal{K} \cap \mathcal{P}_\xi = \mathcal{K}_\xi = \mathcal{K} \cap \mathcal{G}_\xi$$

and

$$\mathcal{K}_\xi \xrightarrow{\sim} K_{\xi(1)} \qquad k \mapsto k(1).$$

So now I want to prove the basic formula:

$$\mathcal{G} = \mathcal{K} \times^{\mathcal{K}_\xi} \mathcal{P}_\xi$$

$$= \mathcal{K} \times X_\xi \times \mathcal{P}_\xi^u.$$

Note that the isom

$$G_\xi \xleftarrow{\sim} G_{\xi(1)}$$

$$\xi\gamma\xi^{-1} \longleftarrow\!\mapsto \gamma$$

commutes with $*$ because ξ is unitary, so that X_ξ will be the image of $X_{\xi(1)}$. This formula obviously can be proved by descent from Gl_n.

Cells: Go back to the spherical building, and let $\xi \in \mathfrak{p}$. Consider the 1-parameter group of motions $\eta \longmapsto e^{t\xi} * \eta$ on $\mathfrak{p}$. Fixpts:

$$e^{t\xi} * \eta = \eta \iff e^{t\xi} \in P_\eta \cap X = X_\eta$$

$$\iff e^{t\xi} \text{ centralizes } \eta.$$

~~Because $\mathcal{X}(\mathfrak{p})$ is compact $\lim e^{t\xi}$~~

What I want to do is to retract the building for G onto the building for G_ξ with the fibres of the retraction being P_ξ^u-orbits. Thus you wish to show that each P_ξ^u-orbit has a unique fixpoint for the group $e^{t\xi}$.

Uniqueness: Consider a $\eta \in \mathfrak{p}_\xi$: $e^{t\xi} * \eta = \eta$ and $u \in P_\xi^u$. Then

$$e^{-t\xi} * (u * \eta) = e^{-t\xi} u e^{t\xi} * \eta \longrightarrow \eta$$

as $t \to +\infty$.

Existence: The idea might be to show that given any point η in the building, then $\lim_{t\to+\infty} e^{-t\xi} * \eta$ exists, say $= \eta_0$, and that $\eta \in P^u_\xi \eta_0$. But this won't be any good for descent purposes until we pin down the actually element g of P^u_ξ $\ni$ $g * \eta_0 = \eta$.

Note that $P^u_\xi = \exp(\mathrm{Lie}\, P^u_\xi)$ because you can see this in Gl_n. So we can certainly describe

$$P^u_\xi \eta_0 = P^u_\xi / P^u_\xi \cap P_{\eta_0}$$

in Lie algebra terms. ~~Thus I am interested in those elements of the Lie algebra such that~~

To really get a felling for what's going on I should think of ξ as a flag with real eigenvalues prescribed for each quotient, and the eigenvalues are in order. So therefore given two points ξ, η what are important are the two flags! Since these two flags are "refined" by a torus, this means I can find elements $p \in P^u_\xi$, $q \in P^u_\eta$ such that

$p\xi p^{-1}$ and $g\eta g^{-1}$ commute.

First suppose ξ commutes with $g\eta g^{-1}, g\in P^u_\eta$
Then

$$e^{-t\eta}e^{s\xi}e^{t\eta} = (e^{-t\eta}ge^{t\eta}g^{-1})e^{s\xi}(ge^{-t\eta}g^{-1}e^{t\eta})$$

$$\longrightarrow g^{-1}e^{s\xi}g \qquad \text{as } t\to+\infty.$$

showing that $e^{s\xi} * \eta = \eta.$

In general we know $p\xi p^{-1}$ commutes with $g\eta g^{-1}$ so we find

$$e^{-t\eta}pe^{s\xi}p^{-1}e^{t\eta} \longrightarrow g^{-1}pe^{s\xi}p^{-1}g.$$

i.e. $pe^{s\xi}p^{-1} * \eta = \eta$, or simply

$$e^{s\xi} * (p^{-1}*\eta) = (p^{-1}*\eta).$$

Therefore we ~~see~~ see that in the GL_n case that ~~[illegible]~~ in the orbit $P^u_\xi\eta$ there is a ~~[illegible]~~ ξ fixpoint, which is unique and given by

$$\lim_{s\to -\infty} e^{s\xi}*\eta.$$

But one can be a bit more specific. The idea is that η is semi-simple and $g*\eta$ is $g\eta g^{-1}$ written as

$$u\,(g*\eta)\,u^{-1}$$

with $u \in P^u_{g*\eta}$. ~~So in addition to having $G = K \times X_\xi \times P^u_\xi$, I want to know that any semisimple elt of P_ξ is of the form $u s u^{-1}$, $u \in P^u_\xi$, $s \in G_\xi$~~

Assertion: $\forall g \in G$, $\xi \in \mathfrak{p}$ we have

$$(*) \qquad g\cdot\xi g^{-1} = u(g*\xi)u^{-1}$$

with $u \in P^u_{g*\xi}$.

<u>Proof:</u> $G = K \times X_\xi \times P^u_\xi$. Let $g = kxu$ be the decomposition of g corresponding to this product. Then

$$g = tkx \qquad \text{where} \qquad t = kxu(kx)^{-1}$$

Now $u \in P^u_\xi \triangleleft P_\xi \supset X_\xi \implies xux^{-1} \in P^u_\xi$

$$\implies kxux^{-1}k^{-1} \in kP^u_\xi k^{-1} = P^u_{k\xi k^{-1}}.$$

Also $k\xi k^{-1} = g*\xi$, since $xu \in P_\xi$. Thus

$$g\xi g^{-1} = tk\xi k^{-1}t^{-1} = t(g*\xi)t^{-1} \qquad t \in P^u_{g*\xi}$$

QED.

Note that if (*) holds, then

$$\lim_{t\to+\infty} e^{-t(g*\xi)} g e^{t\xi} g^{-1} = \lim_{t\to+\infty} e^{-t(g*\xi)} u e^{t(g*\xi)} u^{-1} = u^{-1}$$

showing that u is unique.

July 1, 1975:

I am presently trying to establish the Bruhat decomposition for the spherical building. ξ is a fixed element of $\mathfrak{p}$ and I want to describe the P^u_ξ-orbits on $\mathfrak{p}$.

Assertion: Every P^u_ξ-orbit contains a unique point centralized by ξ:

$$P^u_\xi \backslash \mathfrak{p} \xrightarrow{\sim} \mathfrak{p}_\xi \quad = \{\eta \mid [\xi,\eta]=0\}$$

Furthermore if $\eta \in \mathfrak{p}_\xi$, then

$$P^u_\xi \cap P^u_{-\eta} \xrightarrow{\sim} P^u_\xi * \eta$$

$$u \longmapsto u*\eta$$

Also $P^u_\xi * \eta = \{\eta_1 \mid e^{-s\xi}\eta_1 \to \eta \text{ as } s\to+\infty\}$.

~~This is the~~ It should be enough to prove this for GL_n. Suppose we have ~~an embedding $G \hookrightarrow \bar{G}$ and the theorem is true for $\bar{G}$~~ the situation

$$G \hookrightarrow G' \underset{\beta}{\overset{\alpha}{\rightrightarrows}} G''$$

and the theorem is true for G' and G''. Given $\eta \in \mathfrak{p}$ we know that ~~[illegible]~~ the limit

$$\lim_{s \to +\infty} e^{-s\xi} * \eta$$

exists, call it η_0; ~~[illegible]~~ it is the unique ξ-fixpt in $P'^u_\xi \cdot \eta_0$. Moreover there is a unique

$$u \in P'^u_\xi \cap P'^u_{-\eta_0}$$

such that $\eta = u * \eta_0$. Clearly $\eta_0 \in \mathfrak{p}_{\xi'}$, and we must have by uniqueness $\alpha(u) = \beta(u)$, so $u \in P^u_\xi \cap P^u_{-\eta_0}$.

Suppose I want to prove ~~that~~

Lemma: $[\xi, \eta] = 0$. Then any element $u \in P^u_\xi$ can be unique written $u = u^- u^+$ where $u^- \in P^u_\xi \cap P_{-\eta}$ and $u^+ \in P^u_\xi \cap P_\eta$.

Consider $P^u_\xi * \eta$. This is stable under $e^{s\eta}$ because $e^{s\eta} P^u_\xi e^{-s\eta} = P^u_\xi$. What I want to show is that $P^u_\xi * \eta \subset P^u_{-\eta} * \eta$, or equivalently that

$$(e^{s\eta}\, u) * \eta \longrightarrow \eta \qquad s \to +\infty$$

for any $u \in P^u_\xi$. Now we know that

$$\lim_{s\to\infty} (e^{s\eta}\, u) * \eta = f(u * \eta)$$

exists for every element $u \in P^u_\xi$. Also

$$f(e^{t\xi}(u*\eta)) = e^{t\xi} f(u*\eta)$$

because ξ and η commute. Unfortunately, I can't argue that f is continuous.

~~The thing to consider is~~ $\cancel{e^{t\xi} e^{s\eta} * (u*\eta)}$

Possible argument:

- Argue that $P^u_{-\eta} * \eta$ is an open nbd of η in $G*\eta$. Thus for s large

$$e^{-s\xi} * (u*\eta) \in P^u_{-\eta} * \eta$$

But latter is invariant under $e^{s\xi}$ so $u*\eta \in P^u_{-\eta} * \eta$

So assume we know $P^u_\xi * \eta \subset P^u_{-\eta} * \eta$ which can be proved by descent because $P^u_{-\eta}\,\eta \cong P^u_{-\eta}$

i.e. $P^u_{-\eta} \cap P_\eta = 1$ (because if $x \in P^u_{-\eta} \cap P_\eta$, then $e^{-t\eta} x e^{t\eta}$ converges at both $t = \pm\infty$ hence is constant, so is 1, so x is 1).

Then I can write $u * \eta = u^- * \eta$, i.e. $u = u^- u^+$ with $u^- \in P^u_{-\eta}$, $u^+ \in P_\eta$. Next argue that this splitting has to commute with the ξ action. Since $e^{t\xi}(u^- * \eta) \to 0$ and the isom

$$P^u_{-\eta} * \eta \xleftarrow{\sim} P^u_{-\eta}$$

is topological, we get $e^{-t\xi} u^- e^{t\xi} \to 1$, hence $u^- \in P^u_\xi$, and also for u^+.

<u>Problem</u>: I know that $P^u_\xi \backslash K\eta$ can be identified with $K\eta \cap \mathfrak{p}_\xi$. When ξ is regular this is a W-orbit. What is its structure if ξ is not regular.

Take the case of GL_n. The orbit $K\eta$ can be identified with orthogonal splittings

$$V = W_1 \oplus \cdots \oplus W_m$$

where $\dim(W_i) = d_i > 0$, $d_1 + \cdots + d_m = n$, are fixed. We want the ξ-fixpoints where ξ gives a decomposition $V = U_1 \oplus \cdots \oplus U_\ell$. This means I want the set of decompositions

$$V = \bigoplus_{\substack{1\le i\le \ell \\ 1\le j\le m}} T_{ij}$$

such that $\quad W_j = \bigoplus_i T_{ij} \quad , \quad U_i = \bigoplus_j T_{ij}$

The obvious invariant of such a decomposition are the integers $t_{ij} = \dim T_{ij}$. Supposing these are fixed, then what I am looking at is a sequence of flags:

$$U_1 = T_{11} \oplus \cdots \oplus T_{1m}$$

$$U_\ell = T_{\ell 1} \oplus \cdots \oplus T_{\ell m}$$

such that $\dim(T_{ij}) = t_{ij}$. So in this case we see that each component of $(K/K_\eta)^\xi$ is a product of flag manifolds, and the different components are indexed by families $(t_{ij}) \ni \sum_i t_{ij} = \dim W_j$, $\sum_j t_{ij} = \dim U_i$.

In fact note that each component is an orbit for the group $\mathrm{Aut}(U_1) \times \cdots \times \mathrm{Aut}(U_\ell) = K_\xi$. So the conjecture will be that each component of $(K/K_\eta)^\xi$ is a K_ξ-orbit.

Because $P_\xi = G_\xi \ltimes P_\xi^u$, it follows that

$$P_\xi\backslash K_\eta = G_\xi\backslash(P_\xi^u\backslash K_\eta) \simeq G_\xi\backslash(K_\eta)^\xi$$

Now if I choose a Borel $B \subset P_\xi$ say $B = P_\zeta$ where ζ is a perturbation of ξ, then I know

$$P_\zeta\backslash K_\eta = W\eta \simeq W/W_\eta$$

(here I assume $\eta, \xi \in E$). Because $P_\xi = P_\zeta W_\xi P_\zeta$, it's more or less clear that

$$P_\xi\backslash K_\eta \xleftarrow{\sim} W_\xi\backslash W/W_\eta.$$

Concerning the action of K^σ on $\mathfrak{p}$: ξ is of principal orbit type iff K^σ_ξ acts trivially on the normal space to the orbit $K^\sigma\cdot\xi$ at ξ. This normal space may be identified with the space $\mathfrak{p}_\xi = \{\eta\in\mathfrak{p} \mid [\xi,\eta]=0\}$ because of the identity

$$([X,\xi],\eta) = (X,[\xi,\eta])$$

and the fact that $\xi,\eta\in\mathfrak{p} \Rightarrow [\xi,\eta]\in\mathfrak{k}^\sigma$. Thus ξ is of principal orbit type $\Longleftrightarrow$ K^σ_ξ acts trivially on $\mathfrak{p}_\xi$. But the really interesting point is that $\mathfrak{p}_\xi$

is abelian; for if $\eta_1, \eta_2 \in \mathfrak{p}_\xi$, then $[\eta_1,\eta_2] \in \mathfrak{k}^0_\xi$
hence

$$([\eta_1,\eta_2],[\eta_1,\eta_2]) = (\eta_1, \underbrace{[\eta_2,[\eta_1,\eta_2]]}_{=0})$$

(geometrically the significant point is that at any point $\eta_1 \in \mathfrak{p}_\xi$, the orbit $K^0\eta_1$ is perpendicular to $\mathfrak{p}_\xi$, i.e. $[\eta_1,\mathfrak{p}_\xi]=0$.)