May 1975

Lattices and Scattering Matrices

Let $\Delta = \mathbb{C}[z, z^{-1}]$ be the ring of Laurent polynomials $\sum a_m z^m$ with $a_n \in \mathbb{C}$ and only finitely many $a_m \neq 0$. Let S^1 denote the unit circle in $\mathbb{C}$. Δ may be viewed as a subring of continuous complex-valued functions on S^1. Conjugation on functions induces the involution:

1) $$p = \sum a_m z^m \longmapsto \bar{p} = \sum \bar{a}_m z^{-m}$$

on Λ. Let Δ^0 denote the subring of elements such that $p = \bar{p}$. Then

2) $$\mathbb{R}[x, y]/(x^2 + y^2 - 1) \xrightarrow{\sim} \Delta^0$$

where $x \longmapsto \frac{1}{2}(z + z^{-1}) = \cos\theta$, $y \mapsto \frac{1}{2i}(z - z^{-1}) = \sin\theta$ (if $z = e^{i\theta}$).

Let $g = (g_{ij})$, $g_{ij} \in \Delta$, $1 \leq i, j \leq n$ be a $(n \times n)$-matrix in Δ. Then $z \longmapsto g(z)$ is a map from S^1 to $n \times n$ matrices. For $g(z)$ to be a unitary matrix for each $z \in S^1$ means that

$$(g(z)^* g(z))_{ij} = \sum_k \overline{g_{ki}(z)}\, g_{kj}(z) = \delta_{ij}$$

for each z, or equivalently that in Δ we have

3) $$\sum_k \overline{g_{ki}}\, g_{kj} = \delta_{ij}.$$

In other words g maps S^1 into U_n iff g is a unitary matrix over the ring with involution Δ. Let $\mathcal{U}_n$ (or simply $\mathcal{U}$) denote the group of such matrices.

We can give another interpretation of $\mathcal{U}$ as follows. We equip Δ with the hermitian inner product which is the restriction of the L^2-inner product for functions on S^1:

$$4) \qquad \langle p_1, p_2 \rangle = \int_{S^1} p_1(z)\overline{p_2(z)}\,\frac{dz}{2\pi i z}.$$

This inner product is the one such that $z^m, m \in \mathbb{Z}$, is an orthonormal basis for Δ.

Let Δ^n be the space of column vectors with entries in Δ, let e_i, $1 \le i \le n$, be the standard basis for Δ^n. We interpret matrices $g = (g_{ij})$ over Δ as endos. of Δ^n in the usual way:

$$5) \qquad g e_k = \sum_{k=1}^{n} g_{ki} e_k .$$

Extend $(,)$ to Δ^n in the obvious way, so that $z^m e_i$ is an orthonormal basis of Δ^n. For the matrix g to preserve the inner product $(,)$ means

$$\delta_{lm}\,\delta_{ij} = \left(g(z^l e_i), g(z^m e_j)\right)$$
$$= \left(\sum_k z^l g_{ki} e_k, \sum_k z^m g_{kj} e_k\right)$$

$$= \sum_k \left(z^{l-m}, \overline{g_{ki}}\, g_{kj} \right) = \left(z^{l-m}, \sum_k \overline{g_{ki}}\, g_{kj} \right)$$

i.e. that 3) holds. Therefore we see that $\mathcal{U}$ is the group of autos. of Δ^n preserving the Δ-module structure and the inner product.

In fact we see from the calculation just made that there is a 1-1 correspondence between elements $g \in \mathcal{U}$ and sequences $v_1, \ldots, v_n$ of elements of Δ^n such that $\{ z^m v_i,\ m \in \mathbb{Z},\ 1 \le i \le n \}$ is an orthonormal subsets of Δ^n; the correspondence sends g to the sequence $g e_i$, and to a sequence v_i the matrix (g_{ij}) such that $v_i = \sum_j g_{ji} e_j$.

(<u>Remark</u>: The above corresponds to the fact (proved in Scattering Theory) that unitary autos. of $L^2(S^1)^n$ commuting with multiplication by z are in one-one correspondence with measurable maps from S^1 to U_n modulo null-set equivalence.)

By a lattice for $\mathbb{C}[z^{-1}]$ in Δ^n we mean a $\mathbb{C}[z^{-1}]$-submodule L which is free of rank n. Since $\mathbb{C}[z^{-1}]$ is a PID, such an L is the same thing as a $\mathbb{C}$-subspace of Δ^n such that $z^{-1}L \subset L$ and such that

6) $\qquad z^{-N}\mathbb{C}[z^{-1}]^n \subset L \subset z^N \mathbb{C}[z^{-1}]^n$

for some N. Let $\mathcal{L}$ denote the set of $\mathbb{C}[z^{-1}]$-lattices.

Clearly we have

$$7) \qquad GL_n(\Lambda)/GL_n(\mathbb{C}[z^{-1}]) \xrightarrow{\sim} \mathcal{L}.$$

~~Put Λ_0 for the lattice $\mathbb{C}[z^{-1}]^n$, and let L be any lattice (for $\mathbb{C}[z^{-1}]$ is understood) such that $z^{-N}\Lambda_0 \subset L \subset z^N\Lambda_0$. Denote by $F_{pq}\Delta^n$ the $\mathbb{C}$-subspace of Δ^n with basis $z^m e_i$ with $p \le m \le q$. Let W denote the subspace of elements of $L \cap F_{-N,N}\Delta^n$~~

Put Λ_0 for the lattice (for $\mathbb{C}[z^{-1}]$ is to be unstood) $\mathbb{C}[z^{-1}]^n$. Note that $z^q\Lambda_0 \cap (z^p\Lambda_0)^\perp$ has basis $z^m e_i$ for $p < m \leq q$, hence for $p \leq q$

$$z^q\Lambda_0 = z^q\Lambda_0 \cap (z^p\Lambda_0)^\perp + z^p\Lambda_0$$

so if L is a lattice with $z^p\Lambda_0 \subset L \subset z^q\Lambda_0$ we have

$$L = L \cap (z^p\Lambda_0)^\perp + z^p\Lambda_0 .$$

If also $z^p\Lambda_0 \subset z^{-1}L$, we have

$$z^{-1}L = z^{-1}L \cap (z^p\Lambda_0)^\perp + z^p\Lambda_0 .$$

Let N be the orthogonal complement of $z^{-1}L \cap (z^p\Lambda_0)^\perp$ inside $L \cap (z^p\Lambda_0)^\perp$. Then we have

$$L \cap (z^p\Lambda_0)^\perp = N \oplus z^{-1}L \cap (z^p\Lambda_0)^\perp$$

so $\qquad L = N + z^{-1}L$. On the other hand N is perpendicular to $z^{-1}L \cap (z^p\Lambda_0)^\perp$ and $z^p\Lambda_0$, hence N is perpendicular to zL. Thus we have

8) $$L = N \oplus z^{-1}L$$

where $N = \{x \in L \mid x \text{ perpendicular to } z^{-1}L\}$. ~~It has~~ ~~dimension n as it is free of rank n.~~

The dimension of N is n as L is free of rank n over $\mathbb{C}[z^{-1}]$. Let $v_1, \dots, v_n$ be an orth. basis for N. Since the spaces $z^m N$ are mutually perpendicular, $\{z^m v_i\}$ is an orth. set, so $\exists!\, g \in \mathcal{U}$ such that $g(e_i) = z^m v_i$. It follows that $g\Lambda_0 = L$. Therefore $\mathcal{U}$ acts transitively on $\mathcal{L}$. If $g\Lambda_0 = \Lambda_0$ with $g \in \mathcal{U}$, then g preserves the orth. complement of $z^{-1}\Lambda_0$ in Λ_0 which is $\mathbb{C}e_1 + \dots + \mathbb{C}e_n$. Thus $g \in U_n$, where U_n is regarded as the subgroup of constant matrices. Thus we have

9) $$\mathcal{U}/U_n \xrightarrow{\sim} \mathcal{L}.$$

~~Combining 7) and 9) we find~~

Note that the homomorphism $\mathcal{U} \to U_n$ sending g to $g(1)$ is the identity on U_n. If $\mathcal{U}'$ be the kernel of this map, then we have

10) $$\mathcal{U} = U_n \ltimes \mathcal{U}'$$

and 9) says that $\mathcal{U}'$ acts simply-transitively on $\mathcal{L}$.

So we see that for each L in $\mathcal{L}$ there is a unique $g \in \mathcal{U}'$ such that $g\Lambda_0 = L$. We call g the <u>scattering matrix</u> associated to L. (The terminology comes from scattering theory. The closure of L in $L^2(S^1)^n$ is an "incoming space" for the unitary operator of multiplying by z, and g is its scattering operator. In general incoming spaces form a homogeneous space isomorphic to Measfns.$(S^1, U_n)/U_n$ for the group of unitary autos. of $L^2(S^1)^n$ commuting with z which is essentially Measfns.(S^1, U_n).)

The topology on $\mathcal{U}$: Let $F_{pq}\mathcal{U}$ denote the subset of $\mathcal{U}$ consisting of g such that $g_{ij} \in \sum_{p \le m \le q} \mathbb{C} z^m$. Then $F_{pq}\mathcal{U}$ is a closed subset of $\mathbb{C}^N$ for $N = n^2(q-p+1)$. Also it is a bounded subset, because $g(z) \in U_n \Rightarrow |g_{ij}(z)| \le 1$, and then one gets bounds on the coefficients of $g_{ij}(z)$ using the formulas

$$a_n = \frac{1}{2\pi i} \oint_{|z|=1} \frac{p(z)}{z^{n+1}} \, dz$$

for $p(z) = \sum a_n z^n \in \Lambda$. ~~Another way of seeing this~~

Therefore $F_{pq}\mathcal{U}$ is compact in the topology obtained ~~from its coefficients~~ by considering coefficients.

Another way of seeing $F_{pq}\mathcal{U}$ is compact is to note that $F_{pq}\mathcal{U}/U_n = F_{pq}\mathcal{U}'$ corresponds under the isomorphism $\mathcal{U}' \simeq \mathcal{L}$ to the set of lattices L such that $z^p\Lambda_0 \subset L \subset z^q\Lambda_0$. (In effect $z^{p-1}\Lambda_0 \subset z^{-1}L \Longrightarrow (z^{p-1}\Lambda_0)^\perp \supset (z^{-1}L)^\perp$

$$\Longrightarrow L \cap (z^{-1}L)^\perp \subset \sum_{p \le m \le q} z^m \mathbb{C}^n \Longrightarrow g \in F_{pq}\mathcal{U}'.$$

Conversely if $g \in F_{pq}\mathcal{U}'$, then $L \cap (z^{-1}L)^\perp$ is in $z^q\Lambda_0$ so $L \subset z^q\Lambda_0$; also $z^{-1}L^\perp \subset z^{-1}z^p\Lambda_0^\perp$ for the same reason, so $L \supset z^p\Lambda_0$.) On the other hand the set of L such that $z^p\Lambda_0 \subset L \subset z^q\Lambda_0$ is $\cong$ to a closed subset of the union of the Grassmannians of subspaces in $(z^q\Lambda_0/z^p\Lambda_0)$. Hence this set of L is compact because the Grassmannians are.

We can now define a topology on $\mathcal{U}$ by requiring a set to be closed if it intersection with any $F_{pq}\mathcal{U}$ is compact. (Thus $\mathcal{U} = \varinjlim F_{pq}\mathcal{U}$ as $q \nearrow \infty$, $p \searrow -\infty$ is given the inductive limit topology.) Clearly $\mathcal{U}$ is a compactly generated space.

Note that multiplication $F_{pq}\mathcal{U} \times F_{p'q'}\mathcal{U} \to F_{p+p',q+q'}\mathcal{U}$ is continuous and hence gives a continuous map $\mathcal{U}\times\mathcal{U} \to \mathcal{U}$ provided to product is taken in the category of compactly generated spaces. In this manner $\mathcal{U}$ becomes a topological group in the compactly generated category.

~~Suppose R is a ring containing $\mathbb{C}[\pi]$ and flat over $\mathbb{C}[\pi]$, where we put $\pi = z^{-1}$~~

Put $\pi = z^{-1}$. Let R be a ring flat over $\mathbb{C}[\pi]$ such that $\mathbb{C} \xrightarrow{\sim} R/\pi R$, and let $F = R[\pi^{-1}] = \mathbb{C}[z,z^{-1}]\otimes_{\mathbb{C}[z^{-1}]} R$. It is easy to see that

$$\mathbb{C}[\pi]/\pi^m\mathbb{C}[\pi] \xrightarrow{\sim} R/\pi^m R$$

for all m. Hence for all p,q

$$\pi^{-q}\mathbb{C}[\pi]^n/\pi^{-p}\mathbb{C}[\pi]^n \xrightarrow{\sim} \pi^{-q}R^n/\pi^{-p}R^n$$

and so there is a 1-1 correspondence between $\mathbb{C}[\pi]$-lattices in $\mathbb{C}[z,z^{-1}]^n$ and R-lattices in F^n given by $L \mapsto R\otimes_{\mathbb{C}[\pi]} L$. Here, by

R-lattice, I mean a free R-submodule of F^n of rank n. $GL_n(F)$ acts transitively on these R-lattices, so we ~~[illegible]~~ have:

11) $$\mathcal{L} = GL_n(\mathbb{C}[\pi,\pi^{-1}])/GL_n(\mathbb{C}[\pi]) \xrightarrow{\sim} GL_n(F)/GL_n(R).$$

Combining 7), 9), and 11) we get

12) $$\mathcal{U}_n/U_n \xrightarrow{\sim} GL_n(F)/GL_n(R)$$

or equivalently

13) $$\boxed{GL_n(F) = \mathcal{U}_n \times^{U_n} GL_n(R)}$$

Special paths in GL_n

As usual identify $\mathfrak{gl}_n =$ Lie$(GL_n\mathbb{C})$ with $M_n(\mathbb{C})$ ($n{\times}n$ matrices over $\mathbb{C}$), and $\exp : \mathfrak{gl}_n \to GL_n$ with $A \longmapsto e^A = \sum_{m\geq 0} \frac{1}{m!} A^m$. Denote by $GL_n(\Delta)'$ the subgroup of $g \in GL_n(\Delta)$ such that $g(1)=1$, whence $GL_n\Delta = GL_n \ltimes GL_n(\Delta)'$.

<u>Lemma</u>: Let $A, B \in \mathfrak{gl}_n$ be such that $e^A = e^B$. Then there is a unique $g \in GL_n(\Lambda)'$ such that

$$e^{\omega A} e^{-\omega B} = g(e^{2\pi i \omega})$$

for $0 \leq \omega \leq 1$.

The uniqueness is clear as a holomorphic function is determined by its values on the unit circle. To prove the existence of g we can conjugate A, B by the same matrix. We can suppose if we want, by splitting $\mathbb{C}^n$ according to the eigenvalues of $e^A = e^B$, that e^A has a single eigenvalue.

Write $A = A_s + A_n$ where A_s is semi-simple, A_n is nilpotent, and $[A_s, A_n] = 0$. Do similarly for B. Then

$$e^A = e^{A_s} e^{A_n} = e^{B_s} e^{B_n}.$$

By uniqueness of the decomposition of an element of GL_n as the product of a semi-simple + unipotent elements, we have $e^{A_s} = e^{B_s}$, $e^{A_n} = e^{B_n}$. But the exponential map is ~~a~~ bijective between ~~the~~ nilpotent and unipotent matrices, so $A_n = B_n$. Hence

$$e^{wA} e^{-wB} = e^{wA_s} e^{wA_n} e^{-wB_n} e^{-wB_s}$$
$$= e^{wA_s} e^{-wB_s}$$

so we have reduced to the case where A, B are semi-simple.

In this case $e^A = e^B = \lambda \cdot I$ so writing $\lambda = e^{\mu}$ and replacing A, B by $A - \mu I$, $B - \mu I$ we reduce to showing that $e^A = I \Longrightarrow e^{wA} = g(e^{2\pi i w})$ with $g \in GL_n(\Delta)'$. Decomposing over the eigenvalues of A, we can suppose $A = xI$, whence $e^x = 1$, so $x = 2\pi i n$ with $n \in \mathbb{Z}$. Then

$$e^{wA} = e^{(2\pi i n)w} I = g(e^{2\pi i w})$$

where $g(z) = z^n I$. QED.

<u>Corollary</u>: If $A, B \in \mathrm{Lie}(U_n) =$ skew-adjoint $(n \times n)$-matrices, and $e^A = e^B$, then there is a unique $g \in \mathcal{U}_n'$ such that for $0 \le w \le 1$, we have

$$e^{wA} e^{-wB} = g(e^{2\pi i w}).$$

In effect, g exists by the lemma, and it takes S^1 into U_n, hence it is in $\mathcal{U}_n$.

By a <u>special path</u> in GL_n I mean a map $h : [0,1] \longrightarrow GL_n$ which is of the form

1) $$h(w) = g(e^{2\pi i w})\, e^{wX} \qquad 0 \le w \le 1$$

for some $X \in gl_n$ and $g \in GL_n(\Delta)'$. ~~Such a map h extends uniquely to a holomorphic map of $\mathbb{C}$ into GL_n such that~~

~~$h(w+1) = h(w)\, e^X$~~

Let P_n be the set of special paths in GL_n. We have an action of $GL_n(\Delta)'$ on P_n given by $(gh)(w) = g(e^{2\pi i w})\, h(w)$; this is a free action. We have a map $P_n \longrightarrow GL_n$ given by $h \longmapsto h(1)$, which is constant on $GL_n(\Delta)'$-orbits. Suppose A is an element of GL_n. Because exponential is onto for GL_n, there exists an $X \in gl_n$ such that $e^X = A$; hence $e^{wX} \in P_n$ lies over A. If h is any element of P_n with $h(1) = A$, say $h(w) = g(e^{2\pi i w})e^{wY}$, then $h(w) = g'(e^{2\pi i w})e^{wX}$ where $g' = g\, e^{wY} e^{-wX}$ is in $GL_n(\Delta')$ by the lemma. Thus $GL_n(\Delta)'$ acts transitively on the fibres of P_n over GL_n and we have a principal bundle (at least on the level of sets)

2) $$GL_n(\Delta)' \longrightarrow P_n \longrightarrow GL_n$$

A special path 1) extends uniquely to a holomorphic map $w \longmapsto h(w)$ from $\mathbb{C}$ to GL_n satisfying

3) $$h(w+1) = h(w)e^X.$$

Suppose given a linear first order DE in $\mathbb{C}^*$:

4) $$\frac{dy}{dz} = P(z)y$$

where y is a column vector of length n and $P(z)$ is a $(n \times n)$-matrix of analytic functions in $\mathbb{C}^*$. Using $e^w = z$ this can be transformed into

5) $$\frac{dy}{dw} = Q(w)y$$

where Q is holomorphic on $\mathbb{C}$ and $Q(w+1) = Q(w)$. (In fact, $\frac{dy}{dw} = \frac{dy}{dz}\frac{dz}{dw} = zP(z)y$ so $Q(w) = e^w P(e^w)$.) The solution of 5) starting at v when $w = 0$ is

$$y = h(w)v$$

where h is the matrix function holomorphic in W such that

$$h'(w) = Q(w)h(w)$$

6) $$h(0) = 1.$$

Using $Q(w+1) = Q(w)$, one sees $h(w+1)h(1)^{-1}$ a solution matrix (i.e. satisfies 6)) so one sees that

7) $$h(w+1) = h(w)h(1).$$

Conversely given a holomorphic map $h : \mathbb{C} \to GL_n$

satisfying 7) one sees it is the solution matrix of the DE 5) with $Q = h' h^{-1}$.

Therefore a holomorphic map $h : \mathbb{C} \longrightarrow GL_n$ satisfying 7) is the same thing as a linear holomorphic first order DE 4) in $\mathbb{C}^*$. If we choose X so that $e^X = h(1)$, then $h(w) e^{-wX} = f(e^{w})$ where f is holomorphic in $\mathbb{C}^*$. By definition one says that the DE 4) has regular singular points at $0, \infty$ if f is meromorphic, i.e. if $f \in GL_n(\Delta)$. Thus we see that elements of P_n are the same thing as the solution matrices of DE's with regular **singular** points at $0, \infty$.

Continuation (June 1975 after 2 weeks interruption).

Let us consider the problem of putting a topology on $GL_n(\Delta)'$ and P_n so that P_n becomes a principal $GL_n(\Delta)'$ bundle over GL_n. First observe that $\exp : \mathfrak{gl}_n \longrightarrow GL_n$ is a covering in the following sense. I recall that exp is etale at those matrices A such that no two eigenvalues of A differ by $2\pi i n$ with n a non-zero integer. (In other words if λ_1, λ_2 are eigenvalues of A such that

$e^{\lambda_1} = e^{\lambda_2}$, then $\lambda_1 = \lambda_2$). ~~[illegible]~~

~~[illegible]~~ Given $B \in Gl_n$ recall how one ~~[illegible]~~ finds $A \in \mathfrak{gl}_n$ with $e^A = B$. One factors $B = B_s B_u$, puts $B_u = \exp(\log B_u)$, and lifts B_s to a semi-simple matrix A_s so that if λ_1, λ_2 are two eigenvalues of A_s with $e^{\lambda_1} = e^{\lambda_2}$, then $\lambda_1 = \lambda_2$. Thus any matrix commuting with B_s commutes also with A_s; ~~[illegible]~~ in particular $A_u = \log(B_u)$ commutes with A_s. Now put $A = A_s + A_u$, and note that exp is etale at A.

Therefore we see that exp maps the etale points of $\mathfrak{gl}_n$ onto GL_n, which is what I mean by exp being a covering. ~~[illegible]~~

Additions: 1) ~~[illegible]~~ Given B there is a unique solution of $e^A = B$ such that the eigenvalues λ of A satisfy $0 \leq \mathrm{Im}(\lambda) < 2\pi$. ~~[illegible]~~ The exponential map is etale at such a point A. (Proof goes as follows:)

2) Derivation of the formula for the differential of exp at a point A.

We identify the tangent space to $\mathfrak{gl}_n$ at A with $\mathfrak{gl}_n$ by associating to X the vector $A + \varepsilon X$ ($\varepsilon^2 = 0$ as usual). Under exp this vector goes to $e^{A+\varepsilon X}$ which is a tangent vector to Gl_n at e^A. We identify the tangent space to Gl_n at e^A with

gl_n by associating to Y the vector $e^A(I+\varepsilon Y)$. In terms of these identifications the differential of exp is $X \longmapsto$ coeff. of ε in $e^{-A}e^{A+\varepsilon X}$.

Recall

$$e^{-tA}Xe^{tA} = e^{-t\,\mathrm{ad}A}X \qquad \left(= \sum \frac{1}{n!}(-\mathrm{ad}A)^n X\right)$$

for both sides satisfy.

$$\phi'(t) = -(\mathrm{ad}\,A)\,\phi(t) \qquad \phi(0) = X.$$

Hence

$$\frac{d}{dt}\, e^{-tA}e^{t(A+\varepsilon X)} = e^{-tA}\left(-A + A+\varepsilon X\right)e^{t(A+\varepsilon X)}$$

$$= e^{-tA}\varepsilon X e^{tA} \qquad \text{as } \varepsilon^2 = 0$$

$$= \varepsilon \sum_{n\geq 0} \frac{t^n}{n!}(-\mathrm{ad}A)^n X$$

Integrating we get

$$e^{-tA}e^{t(A+\varepsilon X)} = I + \varepsilon \sum_{n\geq 0} \frac{t^n}{(n+1)!}(-\mathrm{ad}A)^n X$$

Thus

$$\text{coeff of } \varepsilon \text{ in } e^{-A}e^{A+\varepsilon X} = \sum_{n\geq 0} \frac{(-\mathrm{ad}A)^n}{(n+1)!} X$$

$$\boxed{\mathrm{dexp}_A(X) = \left(\frac{1-e^{-\mathrm{ad}A}}{\mathrm{ad}\,A}\right)(X)}$$

3) Since $\mathrm{ad}(A) = \mathrm{ad}(A_s) + \mathrm{ad}(A_n)$ is a Jordan decomposition of $\mathrm{ad}(A)$, one sees that the eigenvalues of $\mathrm{ad}(A)$ are ~~[illegible]~~ $\lambda_i - \lambda_j$ where $\lambda_1, \ldots, \lambda_n$ are the eigenvalues of A. Now $\frac{1-e^{-x}}{x}$ vanishes

exactly when x is of the form $2\pi i n$, n an integer $\neq 0$. Thus $\exp$ is etale at A exactly when no two eigenvalues of A differ by $2\pi i n$, n an integer $\neq 0$.

Let us now return to putting a topology on P_n. The pull-back of P_n with respect to $\exp : gl_n \longrightarrow GL_n$ is canonically a product $GL_n(\Delta)' \times gl_n$. Let $\mathcal{F} = gl_n \times_{GL_n} gl_n$; if $(X, Y) \in \mathcal{F}$, let $f_{X,Y}$ be the element of $GL_n(\Delta)'$ such that $f_{XY}(e^{2\pi i t}) = e^{tX}e^{-tY}$. Then $(X,Y) \longrightarrow GL_n(\Delta)'$ is a cocycle: $f_{XY} f_{YZ} = f_{XZ}$, which describes the twisting of P_n over GL_n. (Specifically P_n is the cokernel of the pair of arrows

$$GL_n(\Delta)' \times \mathcal{F} \rightrightarrows GL_n(\Delta)' \times gl_n \dashrightarrow P_n$$

$$\begin{aligned} (\alpha, (X,Y)) &\longmapsto (\alpha, X) \qquad (\mapsto \alpha e^{tX} \\ &\qquad\qquad\qquad\qquad\quad \shortparallel \\ &\longmapsto (\alpha f_{X,Y}, Y) \quad (\mapsto \alpha e^{tX} e^{-tY} e^{tY}) \end{aligned}$$

Now to topologize P_n so that it becomes a bundle over GL_n means we have to put a topology on $GL_n(\Delta)'$ such that the map

$$\begin{aligned} GL_n(\Delta)' \times \mathcal{F} &\longrightarrow GL_n(\Delta)' \\ (\alpha\ ,\ (X,Y)) &\longmapsto \alpha f_{XY} \end{aligned}$$

is continuous.

Review the nature of $f_{XY} = e^{tX}e^{-tY}$, and calculate its degree. f_{XY} depends only on the semi-simple parts of X and Y; ~~[illegible]~~ let λ_j and μ_j be the eigenvalues of X, Y respectively. We look at what happens in the eigenspace of $e^X = e^Y$ with eigenvalue α. We choose ε such that $e^{\varepsilon} = \alpha$, whence $f_{XY} = e^{t(X-\varepsilon)}e^{-t(Y-\varepsilon)}$. The eigenvalues of $X-\varepsilon$ are $\lambda_j - \varepsilon = 2\pi i\, n_j$ where $n_j \in \mathbb{Z}$, so $e^{t(X-\varepsilon)}$ has degree $\max |n_j|$. Assuming ε choosen with imaginary part in $[0, 2\pi)$ we see

$$|n_j| = \left|\frac{1}{2\pi}(\mathrm{Im}(\lambda_j) - \varepsilon)\right| < \frac{1}{2\pi}|\mathrm{Im}\,\lambda_j| + 1$$

Thus

$$\deg f_{XY} \leq \max\left(\frac{1}{2\pi}|\mathrm{Im}\,\lambda_j|\right) + \max\left(\frac{1}{2\pi}|\mathrm{Im}\,\mu_j|\right) + 2$$

and we obtain:

<u>Lemma</u>: The degree of f_{XY} is bounded if (X,Y) range over a bounded subset of $(\mathfrak{gl}_n)^2$.

~~This shows that~~ Filter $GL_n(\Delta)'$ by degree: $F_N GL_n(\Delta)' = \{ \sum_{|i| \leq N} a_i z^i \text{ in } GL_n(\Delta)' \}$; each $F_N GL_n(\Delta)'$ is an affine variety over $\mathbb{C}$, hence it has a natural topology.

Let $GL_n(\Delta)'$ be endowed with the inductive limit topology (this is clearly the finest topology we might want to consider on $GL_n(\Delta)'$).

Recall that for L locally compact we have

$$*) \qquad L \times \varinjlim_{\alpha} X_\alpha = \varinjlim_{\alpha} (L \times X_\alpha).$$

Thus the continuous maps

$$F_N\, GL_n(\Delta)' \times F_{N_1}\, GL_n(\Delta)' \longrightarrow F_{N+N_1}\, GL_n(\Delta)'$$

induces a continuous map in the limit as $N \to \infty$:

$$GL_n(\Delta)' \times F_{N_1} GL_n(\Delta)' \longrightarrow GL_n(\Delta)'.$$

Because of the lemma it follows therefore that

$$GL_n(\Delta)' \times \mathcal{F} \longrightarrow GL_n(\Delta)'$$
$$(\alpha\ ,\ (X,Y)) \longmapsto \alpha f_{XY}$$

is continuous for (X,Y) in a neighborhood of each point of $\mathcal{F}$, hence this map is continuous.

Proof of $*)$: This is a consequence of the fact that for a locally compact space L, there is a mapping space Y^L functor adjoint to the product $X \times L$. Hence

$$\mathrm{Hom}(L \times \varinjlim X_\alpha, Y) = \mathrm{Hom}(\varinjlim X_\alpha, Y^L) = \varprojlim \mathrm{Hom}(X_\alpha, Y^L)$$

$$= \varprojlim \mathrm{Hom}(L \times X_\alpha, Y) = \mathrm{Hom}(\varinjlim L \times X_\alpha, Y).$$

So we have seen that P_n becomes a principal $GL_n(\Delta)'$-bundle over GL_n. An intriguing point which might be useful goes as follows.

~~Suppose we consider in $\mathfrak{gl}_n$ only those matrices X whose eigenvalues λ_j are such that~~

$$\cancel{0 \leq \frac{1}{2\pi}\,\mathrm{Im}(\lambda_j) \leq 1.}$$

~~It should be the case that P_n is obtainable by descent from this set of X. Thus if I assume~~

For any $a \in \mathbb{R}$, let $U_a \subset \mathfrak{gl}_n$ be the subset consisting of X whose eigenvalues λ_j satisfy

$$a < \frac{1}{2\pi}\,\mathrm{Im}(\lambda_j) < a+1.$$

Then $\exp : U_a \xrightarrow{\sim} V_a$ where V_a is the open set in GL_n consisting of matrices having no eigenvalue on the ray: $\arg = 2\pi a$. The U_a cover GL_n (in fact any $n+1$ of them do).

~~Suppose $a<b<a+1$ and $X \in U_a$, $Y \in U_b$ are such that $e^X = e^Y$. We can decompose $\mathbb{C}^n$ into $V' \oplus V''$ where V' (resp. V'') is the sum of the generalized eigenspaces corresponding to the eigenvalues of X in the interval (a,b) (resp. $(b,a+1)$).~~

~~Let λ_j (resp. μ_j) be the eigenvalues of X (resp. Y); suppose them enumerated so that $e^{\lambda_j} = e^{\mu_j}$. If $a < \frac{1}{2\pi}\operatorname{Im}\lambda_j < b$ then because $\frac{1}{2\pi}\operatorname{Im}\mu_j$ lies in $(b, b+1)$ we must have~~

$$\cancel{\mu_j = \lambda_j + 2\pi i \quad \text{if} \quad a < \tfrac{1}{2\pi}\operatorname{Im}\lambda_j < b}$$
$$\cancel{ = \lambda_j \quad \text{if} \quad b < \tfrac{1}{2\pi}\operatorname{Im}\lambda_j < a+1.}$$

~~Thus it is clear that on V'~~

Let $X \in U_a$, $Y \in U_b$ be such that $e^X = e^Y$. Let λ be an eigenvalue of X and let W_λ be the corresponding generalized eigenspace. Then W_λ is the gen. eigenspace of e^X with eigenvalue e^λ, because distinct eigenvalues of X map to distinct eigenvalues of e^X. Similarly W_λ is the gen. eigenspace of Y with eigenvalue μ, μ being the unique eigenvalue of Y with $e^\mu = e^\lambda$. On W_λ, $X = \lambda I + N$, N nilpotent, and $Y = \mu I + N = X + 2\pi i n I$, where $n \in \mathbb{Z}$.

Suppose now that $a < b < a+1$, and let V' (resp. V'') be the sum of the W_λ such that $a < \frac{1}{2\pi}\operatorname{Im}\lambda < b$ (resp. $b < \frac{1}{2\pi}\operatorname{Im}\lambda < a+1$). Then on V'' we have $Y = X$ and on V' we have $Y = X + 2\pi i$. Therefore

$$f_{XY} = e^{tX}e^{-tY} = z^{-1}\,\mathrm{Id}_{V'} \oplus \mathrm{Id}_{V''}.$$

Suppose next that $a < b < c < a+1$ and we have $X \in \mathcal{U}_a$, $Y \in \mathcal{U}_b$, $Z \in \mathcal{U}_c$ such that $e^X = e^Y = e^Z$. Let $\mathbb{C}^n = V_1 \oplus V_2 \oplus V_3$ where V_1 (resp. V_2, V_3) is the sum of the gen. eigenspaces of X corresp. to λ with $a < \frac{1}{2\pi}\operatorname{Im}(\lambda) < b$ (resp. in (b,c), in $(c, a+1)$). Then

$$f_{XY} = z^{-1} I_{V_1} \oplus I_{V_2} \oplus I_{V_3}$$

$$f_{YZ} = z^{-1} I_{V_2} \oplus I_{V_3} \oplus I_{V_1}$$

$$f_{XZ} = z^{-1} I_{V_1} \oplus z^{-1} I_{V_2} \oplus I_{V_3}$$

What's intriguing about this is that we see the cocycle ~~takes~~ on the family $\{\mathcal{U}_a \mid 0 < a < 1\}$ will take values in the partial monoid of projectors in $\mathbb{C}^n$. (The operation is $(E_1, E_2) \mapsto E_1 + E_2$ and is defined on pairs E_1, E_2 such that $E_1 E_2 = E_2 E_1 = 0$.)

Special Paths in U_n and SU_n

$\mathcal{U}_n$ = group of maps $S^1 \to U_n$ given by Laurent polynomials topologized with the inductive limit topology.

$\mathcal{U}'_n$ = subgroup consisting of f such that $f(1) = 1$.

Then $\mathcal{U}_n = U_n \ltimes \mathcal{U}'_n$ where U_n is identified with the subgroup of constant maps.

Call a path $h : [0,1] \to U_n$ special if it is of the form

$$h(t) = f(e^{2\pi i t}) \exp(tX)$$

where $f \in \mathcal{U}'_n$ and $X \in \mathrm{Lie}(U_n)$ = skew-hermitian matrices. Let X_n be the set of special paths. As for GL_n we get a principal bundle

$$\mathcal{U}'_n \longrightarrow X_n \xrightarrow{\ \phi\ } U_n$$

where $\phi(h) = h(1)$.

Suppose R, F as on page 8. Let X_n be the simplicial complex whose vertices are R-lattices L in F^n and whose simplices are chains

$$L_0 < L_1 < \dots < L_g$$

such that $\pi L_g \subset L_0$. Our aim is to construct a bijection $|X_n| \longrightarrow \mathcal{X}_n$, that is, to triangulate $\mathcal{X}_n$ via X_n.

We represent elements of $\mathrm{Lie}(U_n)$ in the form $2\pi i A$ where A is a hermitian matrix. Any unitary matrix Θ can be uniquely represented $\Theta = e^{2\pi i A}$ where A is hermitian and its eigenvalues are in $[0,1)$; notation: $0 \leq A < 1$. We begin by triangulating the set D of hermitian matrices A with $0 \leq A \leq 1$.

Given A in D let $\lambda_1, \dots, \lambda_g$ be the eigenvalues of A not 0 or 1 arranged in decreasing order. Then we have an orthogonal decomposition

$$\mathbb{C}^n = W_0 \oplus W_1 \oplus \cdots \oplus W_{g+1}$$

where $A = \lambda_i$ on W_i, where $1 = \lambda_0 > \lambda_1 > \cdots > \lambda_{g+1} = 0$, and where $W_1, \dots, W_g$ are $\neq 0$ but W_0, W_{g+1} may be zero. Let Y be the simplicial complex whose simplices are the chains of subspaces of $\mathbb{C}^n$. We associate to A the point of $|Y|$

$$(\lambda_0 - \lambda_1) V_0 + (\lambda_1 - \lambda_2) V_1 + \cdots + (\lambda_g - \lambda_{g+1}) V_g$$

where

$$\begin{aligned} V_0 &= W_0 \\ V_1 &= W_0 + W_1 \\ &\vdots \\ V_g &= W_0 + \cdots + W_g . \end{aligned}$$

(Think of $\lambda_i - \lambda_{i+1} \nearrow 1$ to get V_i).

Conversely given a point $\sum_{i=0}^{q} t_i V_i$ of $|Y|$ with $V_0 < V_1 < \cdots < V_q$ and $\sum t_i = 0$, $t_i > 0$ we associate to this point to operator $A = \sum t_i \, \mathrm{proj}_{V_i}$ which has eigenvalue $\lambda_i = t_i + t_{i+1} + \cdots + t_q$ on $V_i \ominus V_{i-1} = W_i$.

These two constructions are inverse to each other and so give a bijection $|Y| \longrightarrow D$. Actually if one puts the usual topology on simplices of Y, this becomes a homeom. by compactness. Note that the condition $A < 1$ means that $V_0 = W_0 = 0$, and that $0 < A$ means that $V_q = \mathbb{C}^n$. Thus if from Y we delete chains with $V_0 = 0$, $V_q = \mathbb{C}^n$, the resulting simplicial complex, which is the suspension of the Tits complex made out of proper subspaces, has realization the boundary of D, which $\sim S^{n^2-1}$; (D is a closed convex body with non-empty interior).

We identify R-lattices L such that $R^n \subset L \subset \pi^{-1} R^n$ with subspaces V of $\mathbb{C}^n$ via the formulas: $L = R^n + \pi^{-1} V$, $V = L \ominus R^n$. In this way Y becomes identified with the subcomplex of X_n made up of lattices between R^n and $\pi^{-1} R^n$.

On the other hand D maps to X_n by sending A to $e^{2\pi i t A}$. So we have maps

$$\begin{tikzcd} \mathcal{U}_n' \times |Y| \arrow[r, "\sim"] \arrow[d] & \mathcal{U}_n' \times D \arrow[d] \\ |X_n| & \mathfrak{X}_n \end{tikzcd}$$

~~The vertical maps are onto. For $\mathfrak{X}_n$ this is clear,~~

where the vertical maps are defined using the $\mathcal{U}_n'$ action; note $\mathcal{U}_n \subset GL_n F$. Now given a simplex $\sigma : L_0 < \cdots < L_q$ of X_n there is a unique element f of $\mathcal{U}_n'$ such that $fL_0 = R^n$. Let $|Y|^*$ be the open star of the vertex O; then $|Y|^* \simeq D^* = \{A \in D \mid A < 1\}$. We see then that any $\xi \in |X_n|$ is congugate under $\mathcal{U}_n'$ to a unique point of $|Y|^*$; since the analogous thing is so for $\mathfrak{X}_n, D^*$ we get the desired byjection

$$|X_n| \simeq \mathcal{U}_n' \times |Y|^* \xrightarrow{\sim} \mathcal{U}_n' \times D^* \xrightarrow{\sim} \mathfrak{X}_n .$$

Formulas for $S\mathcal{U}_n$. In this case special paths may be represented

$$f(e^{2\pi i t})\, e^{tX}$$

where X is skew-hermitian of trace 0 and $f \in S\mathcal{U}_n'$, i.e. $\det(f) = 1$.

June 6, 1975. On symmetric spaces.

Start with the simplest example:

$$\begin{array}{ccc} SL_2(\mathbb{R}) & \subset & SL_2(\mathbb{C}) \\ \cup & & \cup \\ SO_2 & \subset & SU_2 \end{array}$$

The non-compact symm. space is $SL_2(\mathbb{R})/SO_2 =$ upper half plane, the compact form is $SU_2/SO_2 \cong S^2$.

~~[illegible]~~ Maximal flat submanifolds of S^2 are the great circles; the stabilizer of any one in SO_2 is cyclic of order 4, because SU_2 acts on S^2 thru SO_3.

~~[illegible]~~ Consider broken geodesics starting from the north pole in S^2. The possible directions are described by ~~[illegible]~~ by points of the equator which is $\mathbb{P}_1(\mathbb{R})$. It is clear that broken geodesics ending at the north or south pole which do not cancel are the same as geodesics in the building of SL_2 over $F = \mathbb{R}[[\pi]][\pi^{-1}]$.

Consider next the situation:

$$\begin{array}{ccc} SL_n(\mathbb{R}) & \subset & SL_n(\mathbb{C}) \\ \cup & & \cup \\ SO_n & \subset & SU_n \end{array}$$

and the building of SL_n over $\mathbb{R}[[\pi]][\pi^{-1}] = F$. The vertices of the same type as the 0-point may be identified with lattices L of index 0 wrt R^n. Such lattices can be complexified and they give rise to algebraic loops $f: S^1 \longrightarrow SU_n$ such that if

$$f(z) = \sum a_m z^m$$

then a_m is a real matrix. Hence $\overline{f(z)} = f(\bar{z})$.

If $f(z)$ is projected into the symmetric space SU_n/SO_n, then $z \longmapsto f(z)\cdot SO_n$ for $z = e^{2\pi i t}$, $0 \leq t \leq \frac{1}{2}$ is a loop in the symmetric space. Thus we get a map from vertices of the building into loops in SU_n/SO_n.

Notice that if U/K is a compact symmetric space, $K =$ fixpoints of an involution on U, then a map $f: S^1 \longrightarrow U$ such that $f(\bar{z}) = \overline{f(z)}$ is the same thing as a path $[0,1] \xrightarrow{f} U$ such that $f(0) = 1$, $f(1) \in K$. The space of these is the homotopy-fibre of $K \hookrightarrow U$, which is also $\Omega(U/K)$.

Here's the idea: We have the situation

$$\begin{tikzcd} G_0 \arrow[r] \arrow[d, dash] & G_{\bullet} \arrow[d, dash] \\ K_0 \arrow[r] & K \end{tikzcd}$$

where G_0 is a real semi-simple $\overset{\text{connected}}{\wedge}$ alg. group, with maximal compact subgroup K_0 and complexification $G_{\bullet}$; K is a maximal compact of $G_{\bullet}$. The building for G_0 over $F_0 = \mathbb{R}[[\pi]][\pi^{-1}]$ should be the fixpts under conjugation for the building of G over $F = \mathbb{C}[[\pi]][\pi^{-1}]$. Vertices of the latter have been identified with alg. maps $S^1 \longrightarrow K$, so vertices of the former building are alg. maps $S^1 \longrightarrow K$ compatible with conjugation. We hope this would be a heq.

Go back to $SO_n \subset SU_n$ example. We have found inside of $SL_n(\mathbb{R}[z, z^{-1}])$ the group 2 of Laurent polynomial matrices

$$f(z) = \sum_m a_m z^m$$

such that $f(S^1) \subset SU_n$ and $f(\bar{z}) = \overline{f(z)}$. 2 acts on the building X_0 of SL_n over F, and $2'$ acts simply-transitively on the vertices of index 0. We want to understand the orbit structure of 2 on X_0. $2/2' = SO_n$.

Is it possible to exhibit the building $\mathcal{I}_0$ of SL_n over F_0 as a bundle over the symmetric space SU_n/SO_n? So start with a point in the building given by a simplex

$$L_0 < \cdots < L_q \qquad \text{and} \qquad t_i > 0 \ , \quad \sum_{i=0}^{q} t_i = 1 \, .$$

I have to recall the formulas relating SL_n building over F to special paths.

If $\theta \in SU_n$ its eigenvalues may be represented uniquely in the form

$$e^{2\pi i t_1} \ \cdots \ e^{2\pi i t_n}$$

where $\quad t_1 \geq \cdots \geq t_n \geq t_1 - 1 \ , \quad \sum t_i = 0.$
Thus I can lift θ to a self-adjoint X of trace 0 whose eigenvalues have spread ≤ 1.

Given such an X let

$$t_1 > t_2 > \cdots > t_q > t_{q+1} = t_1 - 1$$

be its eigenvalues and let $V = W_1 \oplus \cdots \oplus W_q \oplus W_{q+1}$ be the corresponding eigenspaces. Here $W_1, \ldots, W_q \neq 0$ but W_{q+1} might be 0. The simplicial coords of X are then $t_1 - t_2, \ldots, t_q - t_{q+1}$ and the vertices are the subspaces

$$zW_1 \oplus W_2 \oplus \cdots \oplus W_{g+1}$$
$$zW_1 \oplus zW_2 \oplus W_3 \oplus \cdots \oplus W_{g+1}$$
$$\vdots$$
$$zW_1 \oplus \cdots \oplus zW_g \oplus W_{g+1}$$

which correspond to lattices

$$R^n < L_1 < \cdots < L_g \subset zR^n.$$

Note: SU_n acts on the X which are self-adjoint of trace 0 and of spread ≤ 1. The orbit space is an $(n-1)$-simplex. If we require spread$(X) = 1$, then we are getting the realization of the Tits complex. This is the same as the rays in the space of self-adjoint matrices of trace 0, so the realization of the Tits complex is the unit sphere in the space of self-adjoint matrices.

Special paths: The tangent space to K/K_0 at origin can be identified with the space $k^- =$ -1 eigenspace of k under the involution. Note that if $X, Y \in k^-$ and $\exp(X) = \exp(Y)$

then $\exp(tX)\exp(-tY) = f(e^{2\pi i t})$ $\quad f \in \mathcal{K}'$
is algebraic loop such that

$$\overline{f(e^{2\pi i t})} = \exp(-tX)\exp(tY) = f(e^{-2\pi i t}).$$

So denote by ${}_{\circ}\mathcal{K}$ the group of alg. free loops in K $\ni \overline{f(z)} = f(\bar{z})$, ${}_{\circ}\mathcal{K}'$ the ones preserving basepoint, and define a special path to be one in K of the form

$$(*) \qquad h(t) = f(e^{2\pi i t}) \exp(tX)$$

with $f \in {}_{\circ}\mathcal{K}'$ and $X \in k_{-}$.

Endpoint map. Given the special path $(*)$ we associate to it the right coset $K_0 \cdot h(\tfrac{1}{2})$. Because $\overline{f(z)} = f(\bar{z})$, it follows that $\overline{f(-1)} = f(-1)$ so $f(-1) \in K_0$. Thus

$$K_0 \cdot h(\tfrac{1}{2}) = K_0 \exp(\tfrac{1}{2}X)$$

Let Y be another element of k_{-} such that

$$K_0 \exp(\tfrac{1}{2}X) = K_0 \exp(\tfrac{1}{2}Y)$$

i.e. $\exp(\tfrac{1}{2}X)\exp(-\tfrac{1}{2}Y) \in K_0$. Applying the involution we get

$$\exp(-\tfrac{1}{2}X)\exp(\tfrac{1}{2}Y) = \exp(\tfrac{1}{2}X)\exp(-\tfrac{1}{2}Y)$$

or $\qquad \exp(Y) = \exp(X).$

which means we can write $h(t)$ in the form

$$h(t) = f(e^{2\pi i t}) \underbrace{\exp(tX)\exp(-tY)}_{\in\, {}_0\mathcal{X}'} \exp(tY).$$

Therefore set-theoretically at least we get the principal bundle

$${}_0\mathcal{X}' \longrightarrow {}_0\mathcal{X} \xrightarrow{\ \phi\ } K/K_0$$

where ${}_0\mathcal{X}$ is the set of special paths and

$$\phi(h) = h(\tfrac{1}{2})^{-1} K_0$$

Action of ${}_0\mathcal{K}$ on ${}_0\mathcal{X}$:

$$(f.h)(t) = f(e^{2\pi i t})\, h(t)\, f(+1)^{-1}$$
$$= f(e^{2\pi i t})\, f(+1)^{-1} \exp(t \operatorname{Ad} f(+1)(X))$$

(note: $f(1) \in K_0$). So from this we will get the orbits of ${}_0\mathcal{K}$ on ${}_0\mathcal{X}$ are the same as the orbits of K_0 on K/K_0.

Check: If $h(t) = f(e^{2\pi i t}) \exp(tX)$

$\cancel{\phi(g.f) = [g(-1)f(-1)\exp(\tfrac{1}{2}X)]^{-1}K_0 = \exp(-\tfrac{1}{2}X)K_0}$

then $(g.h)(t) = g(e^{2\pi i t})\, h(t)\, g(1)^{-1}$ so

$$\phi(g\cdot h) = \left[g(-1)f(-1)\exp(\tfrac{1}{2}X)g(1)^{-1}\right]^{-1}K_o$$
$$= g(1)\exp(-\tfrac{1}{2}X)K_o = g(1)\,\phi(h)$$

$$\cancel{\exp(-\tfrac{1}{2}\mathrm{Ad}\,g(1)X)K_o}$$

<u>Example:</u> Suppose we consider a compact group U considered as a symmetric space

$$\begin{array}{ccc} U & \overset{\Delta}{\hookrightarrow} & U\times U \\ \shortparallel & & \shortparallel \\ K_o & & K \end{array} \qquad \overline{(x,y)} = (y,x).$$

An element of ${}_o\mathcal{K}$ is an alg. map $S^1 \longrightarrow U\times U$, $z \longmapsto (f(z), g(z))$ such that

$$(f(\bar{z}), g(\bar{z})) = (g(z), f(z)) \qquad \text{i.e. } g(z) = f(\bar{z}).$$

Thus $\mathcal{U} \xrightarrow{\sim} {}_o\mathcal{K}$, $f \mapsto (f(z), f(\bar{z}))$. ${}_o\mathcal{X}$ consists of paths

$$\left(f(e^{2\pi i t}), f(e^{-2\pi i t})\right) e^{t(X,-X)}$$

$$= \left(h(t),\ h(-t)\right)$$

where $h(t) = f(e^{2\pi i t})e^{tX}$ is in $\mathcal{X}_o(U)$. So ${}_o\mathcal{X} \simeq \mathcal{X}_o(U)$. The endpoint map is:

$$\left(f(e^{2\pi i t})e^{tX}, f(e^{-2\pi i t})e^{-tX}\right) \mapsto \left(f(-1)e^{\frac{1}{2}X}, f(-1)e^{-\frac{1}{2}X}\right)^{-1}\Delta U$$

so if we identify $U\times U/\Delta U \xrightarrow{\sim} U$ via $(x,y)\mapsto xy^{-1}$, this goes to $e^{-\frac{1}{2}X} f(-1)^{-1} f(-1) e^{-\frac{1}{2}X} = e^{-X}$. Thus we get the same fibration over U.

<u>Question</u>: Suppose h is a special path in K, $h(t) = f(e^{2\pi i t})\, e^{tX}$. If h is in ${}_0\mathcal{X}$, then $\overline{h(t)} = h(-t)$. Conversely does an h in $\mathcal{X}$ such that $\overline{h(t)} = h(-t)$ belong to ${}_0\mathcal{X}$?

Consider such an h. Then $\overline{h(1)} = h(-1) = h(1)^{-1}$. If we can write $h(1) = e^{Y}$ with $\overline{Y} = -Y$, then $h(t) = g(e^{2\pi i t})\, e^{tY}$ and

$$\overline{h(t)} = \overline{g(e^{2\pi i t})}\, e^{-tY} = g(e^{-2\pi i t})\, e^{-tY}$$

implies $g \in {}_0K'$; so $h \in {}_0\mathcal{X}$. Thus we reach

<u>Question</u>: Given $x \in K$ such that $\overline{x} = x^{-1}$, is $x = e^{X}$ where $\overline{X} = -X$?

<u>Obvious counterexample</u>. Suppose the involution is trivial and x is an element of K_0 of order 2. In general if this question has an affirmative answer then every element of order 2 in K_0 must be of form e^{X} with $\overline{X} = -X$.

Case of trivial involution in K. Let $f: S^1 \to K$ be algebraic such that $f(z) = f(\bar{z})$. Then we get a map of $\mathbb{C}$-varieties $f: \mathbb{G}_m \longrightarrow G$ such that $f(z) = f(\frac{1}{z})$. Quotient of $\mathbb{G}_m$ by the action of $\mathbb{Z}_2 : z \longmapsto z^{-1}$ is $\mathbb{G}_a$, the map being $x = (z+z^{-1})$. (Clearly x generates the algebra of invariant functions).

So we get $f(z) = g(z+z^{-1})$ where $g: \mathbb{G}_a \longrightarrow G$ is algebraic, and $g(x) \in K$ for x in $\mathbb{R}$. If $u \in A(K)$, then $ug(x)$ is a polynomial in x which remains bounded for $x \in \mathbb{R}$. This is possible only if $ug(x)$ is constant. Thus g, hence f has to be constant.

So we see that in the case of the trivial involution case there are no non-trivial special paths in the symmetric space : $\mathcal{X} = K$.

~~Now go back to the first question on page 9. Suppose $h(t) = f(e^{2\pi i t})\, e^{tX}$ is a special path in K such that $\overline{h(t)} = h(-t)$. This means~~

$$\overline{f(e^{2\pi i t})}\, e^{t\bar{X}} = f(e^{-2\pi i t})\, e^{-tX}$$

~~or~~

$$e^{t\bar{X}} e^{+tX} = \left[\overline{f(e^{2\pi i t})}\right]^{-1} f(e^{-2\pi i t}) \in \mathcal{K}'$$

~~is nothing more than~~ $e^{\bar{X}} e^{X} = 1$).

Also in the trivial involution case, suppose given a special path $h(t) = f(e^{2\pi i t})e^{tX}$ in K such that $h(t) = h(-t)$. Then

$$f(e^{2\pi i t})\, e^{tX} = f(e^{-2\pi i t})\, e^{-tX}$$

So setting $t = \frac{1}{2}$ we get

$$f(-1)\, e^{\frac{1}{2}X} = f(-1)\, e^{-\frac{1}{2}X}$$

or $e^X = 1$ whence $h \in \mathcal{K}$ has to be constant by the preceding.

<u>Proposition</u>: Let $h(t) = f(e^{2\pi i t})\, e^{tX}$ be in $\mathcal{X}$ and satisfy $\overline{h(t)} = h(-t)$. Then h is in ${}_0\mathcal{X}$.

<u>Proof</u>: $\overline{f(-1)\, e^{\frac{1}{2}X}} = \overline{h(\tfrac{1}{2})} = h(-\tfrac{1}{2}) = f(-1)\, e^{-\frac{1}{2}X}$

hence if $y = h(\frac{1}{2}) = f(-1)e^{\frac{1}{2}X}$ then we have

$$\bar{y} = y e^{-X} \quad \text{or} \quad e^X = \bar{y}^{-1} y.$$

Now I know that every element of K/K_0 is of the form $e^{-\frac{1}{2}Y}K_0$ where $\bar{Y} = -Y$. Hence

$$y^{-1}K_0 = e^{-\frac{1}{2}Y}K_0$$

or $y^{-1} = e^{-\frac{1}{2}Y} k_0$ so

$$e^X = e^{\frac{1}{2}Y} k_0 k_0^{-1} e^{\frac{1}{2}Y} = e^{Y}$$

This means h can be put in the form $g(e^{2\pi i t})\, e^{tY}$

whence $\overline{g(z)} = g(\bar{z})$. QED.

~~The general picture: We have identified X with the building~~

Consider the inclusion map $K_0' \subset K'$ which gives us a map $\Omega(K/K_0) \longrightarrow \Omega(K)$. It takes a path $\lambda : [0,1] \longrightarrow K$ starting at 1 ending in K_0 and assoc. to it the loop in K given by

$$f(e^{2\pi i t}) = \begin{cases} \lambda(2t) & 0 \leq t \leq \frac{1}{2} \\ \overline{\lambda(-2t)} & -\frac{1}{2} \leq t \leq 0 \end{cases}$$

What is the composition with the map $\Omega(K) \to \Omega(K/K_0)$?
We get the look

$$\begin{cases} \lambda(2t)K_0 & 0 \leq t \leq \frac{1}{2} \\ \overline{\lambda(-2t)}K_0 & -\frac{1}{2} \leq t \leq 0 \end{cases}$$

in K/K_0. This is the ~~sum~~ difference of the loops $\lambda(t)K_0$ and $\overline{\lambda(t)}\,K_0$.

General picture: We have identified X with the building associated to G over the local field $F = \mathbb{C}[[z^{-1}]]\,[z$[redacted]$]$. Now the involution on K extends to a $\mathbb{C}$-anti-linear involution on G, which defines a real semi-simple group G_0 whose maximal compact subgroup is K_0:

$$\begin{array}{cc} G_0 & G \\ & \\ K_0 & K \end{array}$$

It should be the case that the involution $h \mapsto (t \mapsto \overline{h(-t)})$ on [redacted] X corresponds to the natural involution [redacted] on $G(F)$ with fixpt set $G_0(F_0)$, $F_0 = \mathbb{R}[[z^{-1}]]\,[z]$. Thus it should be possible to identify X with the building of G_0 over F_0

It is necessary to understand root theory for symmetric spaces. Start with standard situation

$$\begin{array}{ccc} G_0 & \subset & G \\ \cup & & \cup \\ K_0 & \subset & K \end{array}$$

where K is a simply-connected compact group, G its complexification, G_0 is a semi-simple real algebraic group with complex. G and K_0 its maximal compact. One has Cartan involutions θ, θ_0 of G_0 wrt K_0 resp. G wrt K, and the conjugation involution $\sigma x = \bar{x}$ of (G, K) with fixpts (G_0, K_0).

Lie algebra decomposition

$$\mathfrak{g} = \mathfrak{g}_0 \oplus i\mathfrak{g}_0$$
$$\mathfrak{g} = \mathfrak{k} \oplus i\mathfrak{k}$$

$$\mathfrak{k}_0 = \mathfrak{k} \cap \mathfrak{g}_0 \qquad \mathfrak{p}_0 = \mathfrak{g}_0 \cap i\mathfrak{k}$$

$$\mathfrak{g}_0 = \mathfrak{k}_0 \oplus \mathfrak{p}_0$$
$$\mathfrak{k} = \mathfrak{k}_0 \oplus i\mathfrak{p}_0$$

Next one lets $\mathfrak{a}_0$ be a maximal abelian subspace of $\mathfrak{p}_0$. It corresponds to a maximal split torus S_0 of G_0 having identity component A. $\mathfrak{a}_0$ extends to a Cartan subalg $\mathfrak{h}_0$ of $\mathfrak{g}_0$. Diagram of Cart. subalg. + tori

$$\begin{array}{ll} S_0 & \\ \cap & \\ T_0 A = H_0 & \qquad\qquad H \\ & \\ T_0 & \qquad\qquad T = T_0 T_- \end{array}$$

$\mathrm{Lie}(T_-) = i\mathfrak{a}$. Thus T_- is a maximal torus of K on which σ acts as -1. Now we have root decompositions

$$\mathfrak{g} = \mathfrak{h} + \sum_{\alpha \in \Phi}{}' \mathbb{C} X_\alpha$$

$$\mathfrak{k} = iE \oplus \sum_{\alpha \in \Phi_+} \mathbb{R}\{X_\alpha - X_{-\alpha},\ iX_\alpha + iX_{-\alpha}\}.$$

where E is spanned by the $H_\alpha = [X_\alpha, X_{-\alpha}]$.

Better: One starts with $\mathfrak{g}_0$ and shows there exists a compact form $\mathfrak{k}$ in $\mathfrak{g} = \mathfrak{g}_0 \otimes \mathbb{C}$ invariant under θ_0

Involutions are as follows

$\mathfrak{g} = \mathfrak{g}_0 + i\mathfrak{g}_0$ involution (anti-linear) σ

$\mathfrak{g} = \mathfrak{k} + i\mathfrak{k}$ ———— (————) θ

$\mathfrak{g} = \mathfrak{k}_0 + \mathfrak{p}_0$ involution (linear) $\theta_0 = \theta\sigma = \sigma\theta$

$$\mathfrak{g}_0 = \mathfrak{k}_0 + \mathfrak{p}_0 \qquad \text{involution } \theta = \theta_0 \text{ on } \mathfrak{g}_0.$$
$$\mathfrak{k} = \mathfrak{k}_0 + i\mathfrak{p}_0 \qquad \text{involution } \sigma = \theta_0 \text{ on } \mathfrak{k}.$$

Next let $\mathfrak{a}_0$ be a maximal abelian subspace of $\mathfrak{p}_0$ and $\mathfrak{h}_0$ any maximal abelian subalgebra of $\mathfrak{g}_0$ containing $\mathfrak{a}_0$. Then for $X \in \mathfrak{h}_0$, $Y \in \mathfrak{a}_0$

$$[X - \theta X, Y] = [X,Y] - \cancel{\theta}[X, \theta Y] = 0$$

and as $X - \theta X \in \mathfrak{p}_0$, maximality of $\mathfrak{a}_0$ implies $X - \theta X \in \mathfrak{a}$; thus $\theta \mathfrak{h}_0 \subset \mathfrak{h}_0$, so

$$\mathfrak{h}_0 = \mathfrak{t}_0 + \mathfrak{a}_0$$

Similarly $\quad \mathfrak{t} = \mathfrak{t}_0 + i\mathfrak{a}_0$

Next I want to look at the roots of $\mathfrak{g}$ with respect to $\mathfrak{h}$.

$$\mathfrak{g} = \mathfrak{h} + \sum_{\alpha \in \Phi} \mathbb{C} X_\alpha$$

Now θ_0 which is $\mathbb{C}$-linear moves this around. It preserves $\mathfrak{h} = \mathfrak{h}_0 \otimes \mathbb{C}$, hence we get

$$\mathfrak{h} = \underset{\theta_0 = 1}{\mathfrak{t}_0 \otimes \mathbb{C}} \oplus \underset{\theta_0 = -1}{\mathfrak{a}_0 \otimes \mathbb{C}}$$

So actually it is better to ask about the h_0-action on $\mathfrak{g}$.

Improvement: Suppose we start with the involution σ on K simply-connected and compact. Choose a torus T_- in K which is maximal ~~with~~ such that $\sigma = -1$ on T_-, and extend to a maximal torus T of K. Claim T is stable under σ. In effect if $X \in \mathrm{Lie}(T)$, then for Y in $\mathrm{Lie}(T_-)$ we have

$$[X-\bar{X}, Y] = [X,Y] - \overline{[X,\bar{Y}]} = [X,Y] + \overline{[X,Y]} = 0$$

as T is abelian. Thus $X-\bar{X}$ generates a 1-par. subgroup centralizing T_- and reversed by σ. $\therefore$ $X-\bar{X} \in \mathrm{Lie}(T_-)$, so $\sigma\,\mathrm{Lie}(T) \subset \mathrm{Lie}(T)$.

Possible notation: $\mathrm{Lie}(T_-) = 2\pi i\,\mathfrak{a}_0$ i.e. $\mathfrak{a}$ plays the role of E before.

$\mathfrak{a}_0$ is the Lie algebra of a maximal split torus S_0 of G_0. We ~~[illegible]~~ know that S_0 acting on $\mathfrak{g}_0$ splits into a sum of characters. Let $\Phi_0 \subset \mathfrak{a}_0^*$ be the set of ~~[illegible]~~ these characters $\neq 0$; these are called the roots of G_0 with respect to S_0. We have a surjection $\Phi \twoheadrightarrow \Phi_0 \cup \{0\}$

Z_0 is the centralizer of S_0 in G_0. It is the reductive group containing H_0 having the roots X_α where α vanishes on $\mathfrak{a}$. Φ_0 consists of the roots α such that $\alpha|\mathfrak{a} \neq 0$.

θ_0 is an involution (linear) of $\mathfrak{g}$ preserving $\mathfrak{h}$, hence

$$[H, X_\alpha] = \alpha(H) X_\alpha$$

$$[\theta_0 H, \theta_0 X_\alpha] = \alpha(H)\, \theta_0 X_\alpha$$

$$[H, \theta_0 X_\alpha] = \alpha(\theta_0 H)\, \theta_0 X_\alpha$$

Thus $\theta_0 \mathfrak{g}^\alpha = \mathfrak{g}^{\alpha\theta_0}$. There are two cases depending on whether $\alpha\theta_0 = \alpha$ or $\alpha\theta_0 \neq \theta_0$.

If $\alpha\theta_0 = \alpha$, then for $H \in \mathfrak{a}$ ~~we~~ have

$$\alpha(H) = \alpha(\theta_0 H) = -\alpha(H)$$

so $\alpha(H) = 0$. Thus $\alpha\theta_0 = \alpha \implies \alpha|\mathfrak{a} = 0$, and so α appears in Z_0. Note that in this case $\theta_0(\mathfrak{g}^\alpha) = \mathfrak{g}^\alpha$, so $\theta_0 X_\alpha = \pm X_\alpha$.

Recall that $\theta X_\alpha = -X_{-\alpha}$,

hence $\theta \mathfrak{g}^\alpha = \mathfrak{g}^{-\alpha}$. Thus $\mathfrak{g}^\alpha + \mathfrak{g}^{-\alpha}$ is fixed

If $\alpha|\mathfrak{a} = 0$, then for any $H \in \mathfrak{h}$ we have $H - \theta_0 H \in \mathfrak{a}$ so $\alpha(H) = \alpha(\theta_0 H)$. Conversely if $\alpha\theta_0 = \alpha$, then $H \in \mathfrak{a}$ implies

$$\alpha(H) = \alpha(\theta_0 H) = -\alpha(H)$$

so $\alpha(H) = 0$; hence $\alpha|\mathfrak{a} = 0$. In this case $\theta_0 \mathfrak{g}^{\alpha} = \mathfrak{g}^{\alpha\theta_0} = \mathfrak{g}^{\alpha}$ so $\theta_0 X_\alpha = \pm X_\alpha$. If $\theta_0 X_\alpha = -X_\alpha$ then $X_\alpha \in \mathfrak{p} = \mathfrak{p}_0 \otimes \mathbb{C}$, and if $H \in \mathfrak{a}$

$$[H, X_\alpha] = \alpha(H) X_\alpha = 0$$

which contradicts $\mathfrak{a}$ being a maximal abelian subspace of $\mathfrak{p}$. Thus $\theta_0 X_\alpha = X_\alpha$ so $X_\alpha \in \mathfrak{k}_0 \otimes \mathbb{C}$.

Recall $Z \subset G$ is the centralizer of the maximal split torus S. I want to select a Borel subgroup B in G such that ZB is a subgroup. Recall E is the real vector space generated by the lattice $\mathrm{Hom}(\mathbb{G}_m, H)$, and $\mathrm{Hom}(\mathbb{G}_m, S)$ generates the subspace $\mathfrak{a}_0$. Now I choose B so that $\mathfrak{a}_0 \cap C$ has a non-empty interior point of $\mathfrak{a}_0$, where C is the chambre in E

determined by B. In other words I take the roots of G wrt H and restrict them to $\mathfrak{a}_0$ thereby dividing $\mathfrak{a}_0$ into cones. I then take an open cone in $\mathfrak{a}_0$ and an open cone C in E of which the former cone is a face. Then I get a set of positive roots $\Phi_0^+ \subset \Phi_0$ and $\Phi^+ \subset \Phi$, such that if $\alpha \in \Phi$ is such that $\alpha|\mathfrak{a}_0 \subset \Phi_0^+$, then $\alpha \in \Phi^+$.

Now note that if $\alpha\theta_0 \neq \alpha$, i.e. $\alpha|\mathfrak{a} \neq 0$ then $\alpha, \alpha\theta_0$ has opposite signs on $\mathfrak{a}$, so $\alpha \in \Phi^+ \Rightarrow \alpha\theta_0 \in \Phi^-$. The parabolic group ZB we get has the roots $\alpha \in \Phi$ such that $\alpha|\mathfrak{a}$ is either 0 or in Φ^+.

Note that the unipotent radical of ZB has Lie algebra $\mathfrak{n}^+ = \bigoplus_{\alpha|\mathfrak{a} \in \Phi_0^+} \mathfrak{g}^\alpha$, that both θ_0 and θ carry this into $\mathfrak{n}^-$, hence $\mathfrak{n}^+$ is stable under $\sigma = \theta_0\theta$. Thus $\mathfrak{n}^+$ gives us a nilpotent group N_0^+ in G_0 with Lie algebra $\mathfrak{n}_0^+ = \mathfrak{n}^+ \cap \mathfrak{g}_0$. Moreover the roots of $\mathfrak{n}_0^+$ with respect to $\mathfrak{a}_0$ are the restrictions $\alpha|\mathfrak{a}_0$ which are $\neq 0$ with $\alpha \in \Phi^+$. So the weight space decomposition of $\mathfrak{g}_0$ looks like:

$$\mathfrak{g}_0 = \mathfrak{z}_0 + \sum_{\beta \in \Phi_0} \mathfrak{g}_0^\beta . \qquad \mathfrak{n}_0^+ = \sum_{\beta \in \Phi_0^+} \mathfrak{g}_0^\beta$$