February 14, 1975

Suppose to simplify that G is a finite group. Let X be a G-space, say X is a simplicial complex and $Y = X/G$ is also a complex. Consider the function attaching to each open simplex $y \in Y$, the set $f^{-1}(y)$ of open simplices over it. This is a functor from the poset $\mathrm{Simp}(Y)$ to the category of transitive G-sets which is contravariant.

So I should consider the category $\mathcal{C}$ of transitive G-sets. It is equivalent to the category whose objects are subgroups of G with

$$\mathrm{Hom}_{\mathcal{C}}(H, K) = \mathrm{Hom}_{G\text{-sets}}(G/H, G/K)$$

$$= (G/K)^H = \{xK \mid x^{-1}Hx \subset K\}.$$

There is an interesting quotient category $\overline{\mathcal{C}}$ where

$$\mathrm{Hom}_{\overline{\mathcal{C}}}(H, K) = \{x\, K\cdot\mathrm{Cent}(K) \mid x^{-1}Hx \subset K\}.$$

(Note $K\cdot\mathrm{Cent}(K)$ is the subgroup of G normalizing K and inducing an inner automorphism on K.)

Example: Suppose we take the category of trans. G-sets consisting of $X = G$ and $Y = G/H$.

$$\mathrm{Hom}(X,X) \xleftarrow{\sim} G \qquad g \longmapsto \text{right mult by } g^{-1}$$

$$\mathrm{Hom}(Y,Y) \xleftarrow{\sim} N(H)/H \qquad n \longmapsto (gH \longmapsto gn^{-1}H)$$

$$\mathrm{Hom}(X,Y) \xleftarrow{\sim} H\backslash G \qquad Hg \longmapsto (x \longmapsto xg^{-1}H)$$

Here G, $N(H)/H$ act to the right + left on $H\backslash G$ in the obvious way.

Recall that when we divide a category up into an "open" subcat $\{X\}$ and complementary "closed" subcat. $\{Y\}$, its homotopy type is a push-out

$$\begin{tikzcd}
 & B\{X \to Y\} \arrow[dl] \arrow[dr] & \\
B\{X\} \arrow[dr] & & B\{Y\} \arrow[dl] \\
 & B\mathcal{C} &
\end{tikzcd}$$

Now $\quad B\{X\} = B(G)$, $\quad B\{Y\} = B(N(H)/H)$,

$\{X \to Y\}$ = category assoc. to $H\backslash G$ with $(N(H)/H) \times G$ acting.

Now $(N(H)/H) \times G$ acts transitively on $H \backslash G$, and the stabilizer of He is $N(H)$. Proof: $(nH)Hg^{-1} = H$ $\Rightarrow Hng^{-1} = H$ $\Rightarrow$ $ng^{-1} \in H \Rightarrow$ $g \in N(H)$ and $nH = gH$. Hence

$$B\{X \longrightarrow Y\} = BN(H)$$

and so the category $\mathcal{C}$ fits into a h-pushout square

$$\begin{tikzcd} BN(H) \arrow[r] \arrow[d] & BG \arrow[d] \\ B(N(H)/H) \arrow[r] & B\mathcal{C}. \end{tikzcd}$$

To understand the above a bit better, suppose I try to understand a G-space X having just the orbit types G and G/H.

Let U be the open set where G acts freely and $Z = X - U$ the set of points whose stabilizers are conjugate to H. Then

$$G \times^N X^H \xrightarrow{\sim} Z \qquad N = N_G(H)$$

for if $g_1 x_1 = g_2 x_2$ with $x_i \in H$, then $g_2^{-1} g_1 x_1 = x_2$ so $g_2^{-1} g_1 H g_1^{-1} g_2 = H \Rightarrow g_2^{-1} g_1 \in N \Rightarrow g_1 = g_2 n$, $n x_1 = x_2$.

Supposing X is a manifold, so is X^H, and H acts freely on the normal sphere bundle of X^H in X.

Note that Z maps to G/N with fibre X^H, hence $Z = \coprod_{gN \in G/N} gX^H$. Thus Z is obtained from the free N/H-space X^H by lifting to N and inducing up to G.

Let V be a normal tube to Z, i.e. points of distance $\leq \varepsilon$ to Z for some invariant metric. (Riemannian metric – X is a manifold.) $V = G \times^N V_0$ where V_0 is a normal tube around X^H. Since G acts freely on ∂V, N acts freely on ∂V_0.

So we get the following situation

U free G-space

$Z_0 = X^H$ free N/H-space

∂V_0 free N-space

$\partial V_0 \xrightarrow{\alpha} U$ N-equivariant map

$\partial V_0 \xrightarrow{\beta} Z_0$ N-equivariant map

X is essentially the pushout of the arrows

$$U \longleftarrow G \times^N \partial V_0 \longrightarrow G \times^N Z_0$$

induced by α and β respectively.

Question: Does there exist a universal G-space P with orbit types G, G/H? Here universal means that there exists a G-map f such that the square

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d] & P \arrow[d] \\ X/G \arrow[r] & P/G \end{tikzcd}$$

is cartesian (+ some sort of homotopy uniqueness of f).

~~[illegible]~~

Put P_1 for the open set of P where G acts freely, and P_2 for the boundary of the normal tube around P^H. If P is universal, then by considering free G-spaces X, we see that $P_1 \to P_1/G$ would be a universal principal G-bundle, i.e. P_1 is contractible.

~~[illegible]~~

Next consider X having only the orbit type G/H. Then I have seen that $X = G \times^N X^H$, $G\backslash X = N\backslash X^H$, so this category of G-spaces is equivalent to free N/H-spaces. Thus $X \to P$ would be obtained from $X^H \to P^H$. Thus P^H is contractible.

Since P is essentially the cylinder of

$$PG = P_1 \longleftarrow G \times^N P_2 \longrightarrow G \times^N P^H = G \times^N P(N/H).$$

one would guess P_2 would be contractible in the universal case. Certainly this should be sufficient:

$$\begin{tikzcd} \mathcal{U} \arrow[d] & \partial V_0 \arrow[l] \arrow[r] \arrow[d] & Z_0 \arrow[d] \\ PG & PN \arrow[l] \arrow[r] & P(N/H). \end{tikzcd}$$

Conversely if you know P is universal one takes X to be the cylinder ~~of~~ a map from a free N-space to a free N/H space induced up to G. Then you would get a map of the free N-space to P_2 showing P_2 would be contractible.

February 21, 1975 Stability for Σ_n

Note that the normalizer of Σ_{n-1} in Σ_n is Σ_{n-1}. Thus I know that the universal Σ_n-space with orbit types Σ_n, Σ_n/Σ_{n-1} has the homotopy type of $B\Sigma_n/B\Sigma_{n-1}$.

Let X be such a Σ_n space, $\mathcal{U}$ the open set where Σ_n acts freely, and $Z = X - \mathcal{U}$. Then

$$Z = \Sigma_n \times^{\Sigma_{n-1}} X^{\Sigma_{n-1}}$$

Let V_0 be the link of $X^{\Sigma_{n-1}}$ in X, whence the link of Z is

$$V = \Sigma_n \times^{\Sigma_{n-1}} V_0$$ Σ_{n-1} acts freely on V_0.

Put $$Y = \Sigma_n \backslash X = \Sigma_n \backslash \mathcal{U} \underset{\Sigma_n \backslash V}{\cup} \Sigma_n \backslash Z$$

$$= \Sigma_n \backslash \mathcal{U} \underset{\Sigma_{n-1} \backslash V}{\cup} X^{\Sigma_{n-1}}$$

In the universal case $\mathcal{U} \sim P\Sigma_n$, $V \sim P\Sigma_{n-1}$, $X^{\Sigma_{n-1}} = pt$

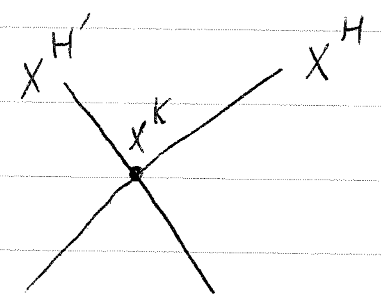
The picture: to give a Σ_n-space over Y with these orbit types, I give $Y = A \cup B$ and a n-sheeted covering of B with a reduction to an $(n-1)$-sheeted covering over $A \cap B$.

Suppose we look at G-spaces X with three orbit types G, G/H, G/K where $H \subset K$. So we have three strata: $X = X_0 \supset X_1 \supset X_2$. To simplify at first suppose $X = X_1$ and put $U = X_1 - X_2$ so that on U we have orbit type G/H, and on X_2 we have the orbit type G/K. Thus

$$U = G \times^{N(H)} U^H$$

$$X_2 = G \times^{N(K)} X^K$$

where $N(H)/H$ acts freely on U^H, $N(K)/K$ acts freely on X^K.



Points near X^K have for isotropy groups the subgroups $H' \subset K$ which are conjugate in G to H. Let T be the boundary of a tubular nbd. of X^K. Then

$$T = \coprod_{H'} T^{H'}$$

and $N(K)$ permutes these components around. Now

$$X \sim \left(G \times^{N(H)} U^H\right) \underset{\left(G \times^{N(K)} T\right)}{\cup} \left(G \times^{N(K)} X^K\right)$$

so

$$G \backslash X \sim \left(N(H)/H \backslash U^H\right) \underset{\left(N(K) \backslash T\right)}{\cup} \left(N(K)/K \backslash X^K\right)$$

Now let H_i run over ~~the~~ representatives for the $N(K)$-conjugacy classes of $H' \subset K$ with H' conjugate to H. Then $N(H_i)/H_i$ acts freely on $T^{H_i} \subset U^{H_i}$ so

$$N(K)\backslash T = \coprod_i \left(N(H_i)/H_i \backslash T^{H_i}\right)$$

For the universal G-space of these orbit types, one would expect U^H, T^{H_i}, X^K to be contractible, so

$$G\backslash X \sim B(N(H)/H) \underset{\coprod_i B(N(H_i)/H_i)}{\cup} B(N(K)/K)$$

seems to be ~~[illegible]~~ the same as the classifying space of the category consisting of the G-sets G/H, and G/K.

February 22, 1975

Let $\mathcal{C}_n$ be the category of non-empty finite sets of card $\leq n$ and injective maps between them.

Lemma 1: The inclusion $\mathcal{C}_{n-1} \xrightarrow{i} \mathcal{C}_n$ is null-homotopic

Proof:
$$\begin{tikzcd} & S \arrow[d, Rightarrow] \\ S \arrow[ur, mapsto, "i"] \arrow[r, mapsto] \arrow[dr, mapsto] & S \amalg pt \\ & pt \arrow[u, Rightarrow] \end{tikzcd}$$
gives a contraction of i to a constant functor.

Consider the spectral sequence associated to i

$$E^2_{pq} = H_p(\mathcal{C}_n, Y \longmapsto H_q(i/Y, \mathbb{Z})) \Longrightarrow H_{p+q}(\mathcal{C}_{n-1}, \mathbb{Z})$$

i/Y has a final object $Y \xrightarrow{id} Y$ if card $Y < n$. If card $Y = n$, then i/Y consists of $(X, X \hookrightarrow Y)$ with maps induced by ~~embeddings~~ injections of X. i/Y equivalent to the poset of proper subsets of Y, when card $Y = n$. Thus

$$H_q(i/\{1,\ldots,n\}, \mathbb{Z}) = \begin{cases} \mathbb{Z} & q = 0 \\ 0 & q \neq 0, n-2 \\ I_n & q = n-2 \end{cases}$$

where $I_n = \mathbb{Z}$ with Σ_n acting as the sign. Here $n \geq 3$.
If $n=2$, then we get only $H_0 = \mathbb{Z}[\Sigma_2]$.

In general we could write

$$0 \to \alpha(I_n[n-2]) \longrightarrow \mathbb{L}i_!(\mathbb{Z}) \longrightarrow \mathbb{Z}[0] \longrightarrow 0 \qquad (*)$$

where if M is a Σ_n-module then $\alpha(M) : \mathcal{C}_n \longrightarrow \mathcal{A}b$ is the functor sending X to 0 if $\mathrm{card}(X) < n$, and $\{1, \ldots, n\}$ into M.

$$\Sigma_n \overset{\varepsilon}{\hookrightarrow} \mathcal{C}_n$$

$$(\varepsilon_! M)(Y) = \varinjlim_{\varepsilon X \to Y} M(X)$$

$$\therefore \quad \alpha(M) = \varepsilon_! M$$

so $\varepsilon_!$ being exact we know

$$H_*(\mathcal{C}_n, \varepsilon_! M) = H_*(\Sigma_n, M).$$

So from $(*)$, we get a long exact sequence

$$\cdots \longrightarrow H_q(\mathcal{C}_{n-1}) \longrightarrow H_q(\mathcal{C}_n) \longrightarrow H_q(\Sigma_n, I_n[n-1]) \longrightarrow \cdots$$

$$H_q(\Sigma_n, I_n[n-1]) = H_{q-n+1}(\Sigma_n, I_n)$$

By lemma 1, $\tilde{H}_q(\mathcal{C}_{n-1}) \to \tilde{H}_q(\mathcal{C}_n)$ is the zero map, hence we get

$$\tilde{H}_q(\mathcal{C}_n) = 0 \qquad q < n-1$$

$$0 \to \tilde{H}_{n-1}(\mathcal{C}_n) \longrightarrow H_0(\Sigma_n, I_n) \longrightarrow \tilde{H}_{n-2}(\mathcal{C}_{n-1}) \to 0$$

$$0 \to H_q(\mathcal{C}_n) \longrightarrow H_{q-(n-1)}(\Sigma_n, I_n) \to H_{q-1}(\mathcal{C}_{n-1}) \to 0.$$

One gets a long exact sequence

$$\to H_p(\Sigma_n, I_n) \to H_{p-1}(\Sigma_{n-1}, I_{n-1}) \to \cdots$$

Note $C_{n-1} \to C_n$ is a equivalence in low degrees because the fibres are highly connected, hence as this functor is null-homotopic C_n must be highly connected.

Variant: Let $\mathcal{A}(n)$ denote the category consisting of non-empty subsets of $\{1,\ldots,n\}$, in which a map $\sigma \to \sigma'$ is a $g \in \Sigma_n$ such that $g\sigma \subset \sigma'$. This category has up to isomorphism one object of each cardinality, namely $\{1,\ldots,p\} \subset \{1,\ldots,n\}$

whose auto group is $\Sigma_p \times \Sigma_{n-p}$. Let $F_p\mathcal{A}(n)$ be the full subcategory consisting of subsets σ of cardinality $\leq p$. If

$$i : F_{p-1}\mathcal{A}(n) \longrightarrow F_p\mathcal{A}(n)$$

is the inclusion functor, then $i/\{1,\dots,p\}$ consists of $(\sigma,g) \ni g\sigma < \{1,\dots,p\}$ in which $(\sigma,g) \longrightarrow (\sigma',g')$ is $h \ni h\sigma \subset \sigma'$ and $g = g'h$. So we get equivalent category consisting of (σ,e) with $\sigma < \{1,\dots,p\}$. Thus $i/\{1,\dots,p\}$ is equivalent to the poset of proper subsets of $\{1,\dots,p\}$, so we will get

$$H_q\big(F_p\mathcal{A}(n), F_{p-1}\mathcal{A}(n)\big) = H_{q-(p-1)}\big(\Sigma_p \times \Sigma_{n-p}, I_p\big)$$

and the filtration of $\mathcal{A}_{\bullet}(n)$ will gives me the analysis I need for stability.

For example suppose I want to know that $H_q(F_p\mathcal{A}(n))$ stabilizes in n. Then working by induction on q I see that

$$H_q\big(F_p\mathcal{A}(n), F_{p-1}\mathcal{A}(n)\big)$$

stabilizes for $p-1 \geq 1$ or $p \geq 2$. What is $F_1\mathcal{A}(n)$; it is Σ_n acting on $\{1,\dots,n\}$, hence $F_1\mathcal{A}(n) \sim \Sigma_{n-1}$. Thus I conclude that $H_q(F_p\mathcal{A}(n), F_1\mathcal{A}(n))$ stabilizes, hence

$H_q(B\Sigma_n, B\Sigma_{n-1})$ stabilizes.

<u>Review Volodin–Wagner construction:</u> Let J be a poset and $j \mapsto H_j$ a functor from J to [redacted] subgroups of G. Then over J I can form the cofibred category associated to the [redacted] functor

$$j \longmapsto G/H_j$$

An object is a pair (j, gH_j), and a map $(j, gH_j) \longrightarrow (j', g'H_{j'})$ is a map $j \le j'$ in J such that $gH_j \mapsto g'H_{j'}$ i.e. $g'H_{j'} = gH_{j'}$. Observe that G acts to the left on this category over J.

<u>Example:</u> Take a Tits system $(G, B,$ [redacted] $N, S)$ and [redacted] recall that [redacted] subgroup $P \neq G$ such that $P \supset B$ are in 1-1 correspondence with nonempty subsets σ of S. Take $\{J = \sigma \subset S,\ \sigma \neq \phi\}$ and $H_\sigma = P_\sigma$. Because P_σ is its own normalizer, I can identify G/P_σ with the conjugates of P_σ

$$gP_\sigma \longmapsto gP_\sigma g^{-1}$$

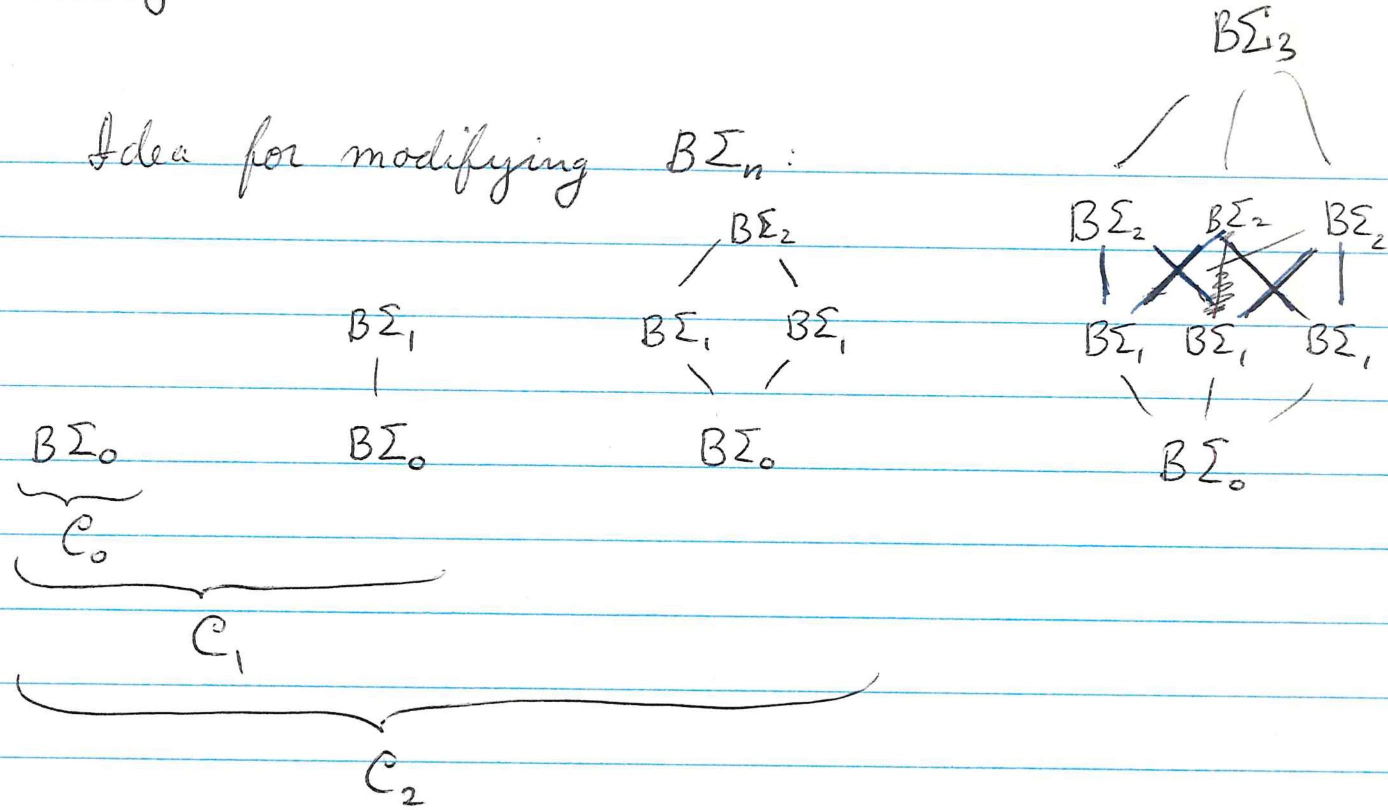
If $\quad gP_\sigma g^{-1} \subset g'P_\tau g'^{-1} \Rightarrow P_\sigma \subset g^{-1}g' P_\tau g'^{-1} g$

So $\quad g^{-1}g' \in P_\tau$ and $g'P_\tau = gP_\tau \supset gP_\sigma$. Thus the cofibred category is the building in this case.

Let I denote the cofibred category over J with fibres G/H_j. I is clearly a poset on which G acts and $G\backslash I \xrightarrow{\sim} J$ at least set-theoretically. But the strange feature here is that one has a cocart. section $J \longrightarrow I$ sending j to the coset H_j.

Conversely suppose G acts on a simplicial complex X, and let X be triangulated enough so that $Y = G\backslash X$ is a simplicial complex whose simplices are the orbits of simplices of X. Then $\mathrm{Simp}(X) \longrightarrow \mathrm{Simp}(Y)$ is fibred and the fibres are transitive G-sets. If $\exists$ a section $Y \xrightarrow{s} X$, then for each $\sigma \in Y$ the stabilizer of $s(\sigma)$ is a subgroup H_σ, and $\sigma \subset \tau \Rightarrow H_\tau \subset H_\sigma$, so we get a contravariant functor from $\mathrm{Simp}(Y)$ to subgroups of G, such that $X = G \times Y / \bigcup_\sigma H_\sigma \times \sigma$.

Idea for modifying $B\Sigma_n$:



Original motivation was that we have a map $B\Sigma_p \hookrightarrow B\Sigma_n$ for each subset σ of $\underline{n}$ with card $\sigma = p$. (Here we use the natural ordering of σ). Hence in our modified $B\Sigma_n$ we want to glue:

$$\begin{tikzcd} \coprod_\sigma B\Sigma_p \arrow[r] \arrow[d, "\text{fold}"] & B\Sigma_n \\ B\Sigma_p & \end{tikzcd}$$

This gives us an inductive construction of $B\Sigma_n$-modified as drawn above.

First try to understand the diagram attached with $B\Sigma_n$. I know it comes from a VW setup. Given

$\sigma \subset \underline{n} = \{1, \dots, n\}$ let H_σ be the subgroup of Σ_n fixing each element of σ. $\sigma \subset \tau \Rightarrow H_\sigma \supset H_\tau$ $\Rightarrow$ we have a map $G/H_\tau \longrightarrow G/H_\sigma$. G/H_σ can be identified with the set of embeddings $u : \sigma \hookrightarrow \underline{n}$, and the map $G/H_\tau \longrightarrow G/H_\sigma$ assoc. to $\sigma \subset \tau$ takes an embedding of τ and restricts it to σ. The link for attaching $B\Sigma_n$ to $\widetilde{B\Sigma_{n-1}}$ is therefore the poset consisting of pairs (σ, u), $\phi \neq \sigma \subset \{1, \dots, n\}$ and where $u : \sigma \hookrightarrow \underline{n}$. This is my old friend of simplices in $\Delta(n-1) \times \Delta(n-1)$ projecting non-degenerately in both directions.

So it would seem that the category I get from this model for $\widetilde{B\Sigma_n}$ consists of the sets $\underline{p}$ $0 \leq p \leq n$ in which a map $\underline{n-p} \longrightarrow \underline{n}$ consists of a subset σ of $\underline{n}$ of card p (i.e. an order-preserving embedding $\underline{n-p} \longrightarrow \underline{n}$) together with an embedding $\sigma \hookrightarrow \underline{n}$ which may be identified with an arbitrary embedding $\underline{p} \hookrightarrow \underline{n}$.

February 26, 1975. Example of U_n-manifolds.

I consider manifolds X with smooth U_n-action with only the orbit types U_n/U_{n-i}, $0 \le i \le k$.

Example: Let U_n act on $\mathrm{Hom}(\mathbb{C}^k, \mathbb{C}^n)$ by left multiplication. The stabilizer of $\theta : \mathbb{C}^k \to \mathbb{C}^n$ is the set of $g \in U_n$ such that $g\theta = \theta$, i.e. $g(\theta v) = \theta v$ for all $v \in \mathbb{C}^k$; i.e. g centralizes $\mathrm{Im}\,\theta$. This stabilizer is conjugate to U_{n-i} where $i = \dim(\mathrm{Im}\,\theta) \le k$.

The orbit space may be identified

$$H(k) = {}_{U_n}\backslash \mathrm{Hom}(\mathbb{C}^k, \mathbb{C}^n)$$

with the space $H(k)$ of hermitian forms $P \ge 0$ on $\mathbb{C}^k$. In effect given $\theta : \mathbb{C}^k \to \mathbb{C}^n$, the form $(\theta v, \theta v)$ on $\mathbb{C}^k$ is an invariant of the orbit $U_n\theta$. Moreover if $(\theta v, \theta v) = (\theta' v, \theta' v)$, then $\exists$ $\alpha : \mathrm{Im}\,\theta \xrightarrow{\sim} \mathrm{Im}\,\theta'$ such that $\alpha\theta = \theta'$; α is unitary, hence it extends to a $g \in U_n$ such that $g\theta = \theta'$.

Recall the orbit structure of U_k acting on $H(k)$. Any form $P \ge 0$ on $\mathbb{C}^k$ has a sequence of eigenvalues $0 \le \lambda_1 \le \dots \le \lambda_k < \infty$ and an eigenspace decomposition of $V = \mathbb{C}^k$. First thing to examine is the number of 0-eigenvalues.

Introduce the open set U_i where $\lambda_i < \lambda_{i+1}$.
Then U_i deforms down to the ~~sp~~ subspace where there are i eigenvalues equal to zero

$$U_i \sim Y_i(V)$$

$$U_i \cap U_j \sim Y_{i,j}(V) \qquad i<j.$$

Hence for $0 \le i_0 \le k$

$$U_{i_0} \cap \cdots \cap U_{i_j} \times^{U_k} PU_k \sim BU_{i_0} \times BU_{i_1 - i_0} \times \cdots \times BU_{i_j - i_{j-1}}$$

So it would appear that $H(k) \times^{U_k} PU_k$ is essentially the realization of the top category with object space

$$\coprod_{0 \le i \le k} BU_i$$

in which the space of maps from pts of BU_i to pts. of BU_j is $BU_i \times BU_{j-i}$.

To understand U_n-spaces with orbit types U_n/U_{n-i}, $0 \le i \le k$, one looks at the top. category whose objects are these U_n spaces and maps between them. U_n/U_{n-i} is the space of orthogonal i-frames in $\mathbb{C}^n$. $\therefore$

$$\mathrm{Hom}(U_n/U_{n-j},\, U_n/U_{n-i}) = \{(v_1,\dots,v_i) \text{ orth in } \mathbb{C}^n\}^{U_{n-j}}$$

$$= \{(v_1,\dots,v_i) \text{ orth in } \mathbb{C}^j\}$$

$$= U_j/U_{j-i}$$

In particular, the object U_n/U_{n-i} has the endo. group U_i. Finally the link between the ith stratum U_n/U_{n-i} and the j-th stratum U_n/U_{n-j} for $i<j$ is

$$U_j/U_{j-i} = \mathrm{Orth\,Emb}(\mathbb{C}^i, \mathbb{C}^j)$$

with U_j acting on the left, and U_i on the right. Thus the classifying space of this top. cat. is the same as the one with objects

$$BU_i \qquad 0 \le i \le k$$

and in which the arrows over $i<j$ is the space

$$PU_j \times^{U_j} U_j/U_{j-i} \times^{U_i} PU_i = PU_j \times^{U_j} U_j/U_i \times U_{j-i}$$

$$= BU_i \times BU_{j-i}$$

Hence you seem to be getting the category consisting of spaces of dim $\le k$ and their embeddings.

March 1, 1975

Review how I stratified the Grassmannian. Consider $Y = Y_p(V)$ and choose a fixed subspace W of codim p. Then I get strata

$$Y(k) = \{ A \in Y \mid \dim(A \cap W) = k \}$$

These are the orbits of the group of autos of V normalizing W. The map

$$Y(k) \longrightarrow Y_k(W) \times Y_{p-k}(V/W)$$
$$A \longmapsto (A \cap W \, , \; A/A \cap W)$$

is a homotopy equivalence. In the infinite case ($\dim W = \dim V/W = \infty$) the strata has the homotopy type of $BU_k \times BU_k$.

Generalize to the "gen. Grassmannian" of lattices. $\mathcal{L} = \mathbb{C}[z^{-1}]$-lattices in $\mathbb{C}[z, z^{-1}]^n = V$ and $W = \mathbb{C}[z]^n$. Here the analogue of the group of autos of V normalizing W is the group $\Gamma = GL_n(\mathbb{C}[z])$. Since W is given, a lattice L can be interpreted as a vector bundle over $\mathbb{P}^1$, and $\Gamma \backslash \mathcal{L}$ is the different iso classes:

$$\Gamma \backslash \mathcal{L} = \{ (p_1 \leq \cdots \leq p_n) \mid p_i \in \mathbb{Z} \}.$$

So I recall that a vector bundle E over $\mathbb{P}^1$ splits: $\mathcal{O}(p_1) \oplus \cdots \oplus \mathcal{O}(p_n)$. In terms of the lattice L I consider the least n such that $H^0(E(n)) \neq 0$, ie. $z^n L \cap W \neq 0$. The subspace $z^n L \cap V$ is unimodular in L. Now one one divides out by this unimodular subspace and continues. It more or less clear that the homotopy type of the stratum with integers

$$p_1 = \cdots = p_{a_1} < p_{a_1+1} = \cdots = p_{a_2} < \cdots \cdots \quad p_n$$

is the space $Y_{a_1 a_2 \cdots a_r}(\mathbb{C}^n)$. From another point of view the orbit with these integers is

$$GL_n(\mathbb{C}[z]) \Big/ \begin{pmatrix} GL_{a_1} & & \deg \leq p_{a_j} - p_{a_i} \\ & GL_{a_2 - a_1} & \\ & & \ddots \\ & & & GL_{a_r - a_{r-1}} \end{pmatrix}$$

which is heq to $\quad GL_n / GL_{a_1} \times \cdots \times GL_{a_r - a_{r-1}} = Y_{a_1 \cdots a_r}$

Next I want the normal bundle to the strata. For $Y_p(V)$ the normal bundle I know to be

$$\mathrm{Hom}(A \cap W, V/A+W)$$

since the tangent space at A is $\mathrm{Hom}(A, V/A)$, and the Lie alg. of Γ is $\theta \in \mathrm{End}(V) \ni \theta W \subset W$.

In the lattice case one has for the bundle E defined by L, the exact seq.

$$0 \longrightarrow H^0(E) \longrightarrow L \oplus W \longrightarrow V \longrightarrow H^1(E) \longrightarrow 0$$

since $L = \Gamma(E, \mathbb{P}^1 - 0)$, $W = \Gamma(E, \mathbb{P}^1 - \infty)$, etc. Thus one suspects the normal space to the stratum thru L is (NO see below).

$$\mathrm{Hom}(L \cap W, V/L+W) = \mathrm{Hom}(H^0(E), H^1(E)).$$

Tangent space to a lattice L should be

$$\mathrm{Hom}_{\mathbb{C}[z^{-1}]}(L, V/L).$$

Lie algebra of Γ is $\mathrm{End}_{\mathbb{C}[z]}(W)$.

$$\begin{tikzcd}
 & \mathfrak{g} \arrow[d, phantom, "\parallel"] & & \\
0 \arrow[r] & \mathrm{Hom}_{\mathbb{C}[z]}(W,W) \arrow[r] & \mathrm{Hom}_{\mathbb{C}[z]}(W,V) \arrow[d, phantom, "\parallel"] & \\
 & & \mathrm{Hom}_{\mathbb{C}[z,z^{-1}]}(V,V) \arrow[d, phantom, "\parallel"] & \\
0 \arrow[r] & \mathrm{Hom}_{\mathbb{C}[z^{-1}]}(L,L) \arrow[r] & \mathrm{Hom}_{\mathbb{C}[z^{-1}]}(L,V) \arrow[r] & \mathrm{Hom}_{\mathbb{C}[z^{-1}]}(L,V/L) \arrow[r] \arrow[d, phantom, "\parallel"] & 0 \\
 & & & \text{Tangent space} &
\end{tikzcd}$$

So it seems that the normal space to the Γ orbit through L is

$$\operatorname{Coker}\left\{ \operatorname{Hom}_{\mathbb{C}[z]}(W,W) \oplus \operatorname{Hom}_{\mathbb{C}[z^{-1}]}(L,L) \longrightarrow \operatorname{Hom}_{\mathbb{C}[z,z^{-1}]}(V,V) \right\}$$

which is

$$H^1\left(\underline{\operatorname{Hom}}(E,E)\right)$$

E being the vector bundle determined by (L,W). Now there is a map

$$H^1\left(\underline{\operatorname{Hom}}(E,E)\right) \longrightarrow \operatorname{Hom}(H^0E, H^1E)$$

and if $E = \bigoplus_i L_i$ this map is the direct sum of the maps

$$\bigoplus_{i,j} H^1(L_i^\vee \otimes L_j) \longrightarrow \bigoplus_{i,j} \operatorname{Hom}(H^0L_i, H^1L_j)$$

so it is far from being either injective or surjective.

So we can interpret strata as being vector bundles up to isos. and a normal vector as being an infinitesimal ~~[illegible]~~ deformation of a vector bundle.

March 14, 1975 SU

Conjugacy classes in SU_n form an $(n-1)$-simplex:

Given $\theta \in SU_n$ let its eigenvalues be

(*) $$e^{2\pi i u_1}, \dots, e^{2\pi i u_n}$$

where $0 \le u_1 \le \cdots \le u_n < 1$. Since $\det(\theta) = 1$, $u_1 + \cdots + u_n$ is an integer k, $0 \le k < n$. Replacing $u_1 \cdots u_n$ by

$$u_{n-k+1} - 1 \le \cdots \le u_n - 1 \le u_1 \le \cdots \le u_k$$

we see the eigenvalues of θ can be put in the form ~~(*), where~~ $e^{2\pi i t_1}, \dots, e^{2\pi i t_n}$ where

(**) $$\begin{cases} t_1 \le \cdots \le t_n \le t_1 + 1 \\ \text{and} \quad t_1 + \cdots + t_n = 0. \end{cases}$$

Suppose I have another representation (*)(**) for the eigenvalues of θ, say $t_1', \dots, t_n'$. If k is the number of $t_i' < 0$, then $0 \le k < n$ and the sequence

$$0 \le t_{k+1}' \le \cdots \le t_n' \le t_1' + 1 \le \cdots \le t_k' < 1$$

must coincide with the u-sequence; hence $t_i' = t_i$. So we have proved.

Prop. Given $\theta \in SU_n$ its eigenvalues can be uniquely represented $e^{2\pi i t_1}, \dots, e^{2\pi i t_n}$ where

$$(**)\qquad \begin{cases} t_1 \le \cdots \le t_n \le t_1 + 1 \\ \sum t_i = 0. \end{cases}$$

Next observe that the sequences $(t_1, \ldots, t_n)$ satisfying $(**)$ can be identified with points in the $(n-1)$-simplex Δ_{n-1} via the formulas

$$(t_1 \le \cdots \le t_n) \longmapsto 0 \le t_2 - t_1 \le \cdots \le t_n - t_1 \le 1$$

$$(-\mu, -\mu + x_1 \le \cdots \le -\mu + x_{n-1}) \longleftarrow 0 \le x_1 \le \cdots \le x_{n-1} \le 1$$

$$\mu = \tfrac{1}{n-1} \sum x_i$$

In the preceding fashion we can identify the conjugacy classes of SU_n with points in Δ_{n-1}. In fact one gets a map

$$SU_n \longrightarrow \Delta_{n-1}$$

whose fibres are the conjugacy classes

Next examine stabilizing: $SU_n \hookrightarrow SU_{n+1}$. More suitable for our purposes to represent eigenvalues of θ in the form

$$e^{2\pi i t_1} \qquad e^{2\pi i t_n}$$

where $\quad t_1 \ge t_2 \ge \cdots \ge t_n \ge t_1 - 1, \quad \sum t_i = 0.$

In this case stabilization consists of putting in zeroes in the sequence of t's.

An element of SU will have eigenvalues represented by a divisor in $[-1, 1]$ of amplitude 1, 0 of infinite multiplicity, and the sum of the points of divisor in $\mathbb{R}$ is zero.

I want to stratify SU_n by looking at the multiplicity of the largest of the t's. Thus θ belongs to the k-th stratum if in the sequence $t_1 \geq \cdots \geq t_n$ one has $t_1 = \cdots = t_k > t_{k+1}$. Notation $SU_n(k)$ for kth stratum.

~~Claim $SU_n(k) \longrightarrow Y_k(\mathbb{C}^n)$~~

~~$\theta \longmapsto$ eigenspace for $t_1 = \cdots = t_k$.~~

~~is a heq.~~

~~NO — eigenspace for $t_1 = \cdots = t_k$ has no meaning, because t and $t-1$ give the same eigenvalue.~~

Claim $SU_n(k)$ is contractible. Because if you push all the $t_{k+1}, \ldots$ in the negative direction, you end up with the sequence:

$$\underbrace{\lambda,\dots,\lambda}_{k\text{-times}},\ \underbrace{\lambda-1,\dots,\lambda-1}_{(n-k)\text{ times}}$$

where $k\lambda + (n-k)(\lambda-1) = 0$ or

$$n\lambda - n + k = 0$$

$$\text{or } \lambda = \frac{n-k}{n}$$

The corresponding matrix is $e^{2\pi i \frac{n-k}{n}}$ which is an n-th root of unity.

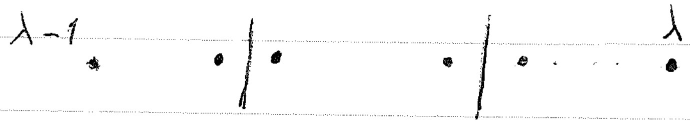
Next we want to understand how the k-th stratum is linked to the l-th stratum. Around the k-th stratum one puts an open set V_k consisting of those θ whose sequences are such that $t_k > t_{k+1}$. $k \geq 1$. V_0 is where maximum $t_1 <$ (minimum t_n)$+1$.

Better approach. Put $\lambda(\theta) =$ minimum of the t's $+1$. Then one can count the t's, $\lambda(\theta) \geq t_1 \geq t_2 \geq \cdots\cdots$ and define V_k to be those $\ni$ $t_k > t_{k+1}$ where $t_0 = \lambda(\theta)$. and we can deform by pushing $t_1, \dots, t_k$ up to $\lambda(\theta)$ and the rest down to $\lambda(\theta) - 1$ (n finite).

$$0 < \lambda(\theta) \leq 1$$

If $k<\ell$,

$V_k \cap V_\ell$ consists of θ for which $t_k > t_{k+1}$ and $t_\ell > t_{\ell+1}$.



so the homotopy type of $V_k \cap V_\ell$ is the Grassmannian of $\ell-k$ planes in V.

$$V_k \cap V_\ell \xrightarrow{\text{heq}} \mathcal{G}_{\ell-k}(V)$$

So it's clear now what the homotopy type of the resulting simplicial space is:

$$\coprod_{k\geq 0} V_k \sim \coprod_{k\geq 0} pt$$

$$\coprod_{k<\ell} V_k \cap V_\ell \sim \coprod_{0\leq k<\ell} BU_{\ell-k}$$

$$\coprod_{k<\ell<m} V_k \cap V_\ell \cap V_m \simeq \coprod_{0\leq k<\ell<m} BU_{\ell-k} \times BU_{m-\ell}$$

Thus what I am getting is the nerve of the monoid $\coprod_{n\geq 0} BU_n$ acting on $\coprod_{n\geq 0} pt$

<u>March 15, 1975</u>

Let A a d.v.r, F, k as usual, and let C be a smooth projective curve over A. Denote by C_η and C_0 the generic and special fibres. Let E be a vector bundle over C, let W be a sub-bundle of E_η. The map $j: C_\eta \to C$ being affine, I can identify quasi-coherent sheaves on C_η with quasi-coh. sheaves on C which are F modules. So I can form $W \cap E =$ subsheaf of E whose sections on C_η are in W. I have a map of exact sequence of quasi-coh. sheaves on C

$$\begin{array}{ccccccccc} 0 & \longrightarrow & j_* W & \longrightarrow & j_* E_\eta & \longrightarrow & j_*(E_\eta/W) & \longrightarrow & 0 \\ & & \uparrow & & \uparrow & & \uparrow & & \\ 0 & \longrightarrow & E \cap W & \longrightarrow & E & \longrightarrow & E/E\cap W & \longrightarrow & 0 \end{array}$$

which shows E, $E/E \cap W$ are A-torsion-free, hence flat over A. Thus we get

$$0 \longrightarrow (E \cap W)_0 \longrightarrow (E)_0 \longrightarrow (E/E\cap W)_0 \longrightarrow 0$$

exact on C_0. This shows $(E \cap W)_0$ is a vector bundle. Since

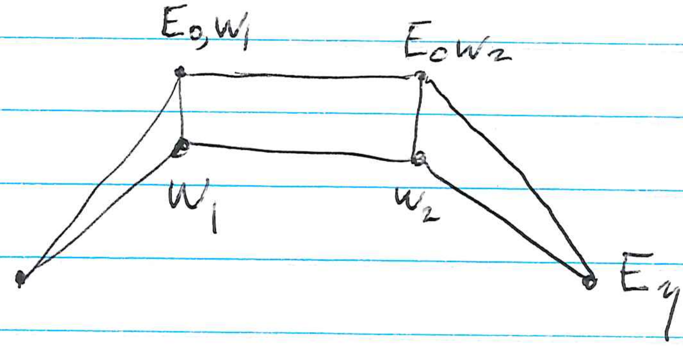
$$\chi((E\cap W)_0) = \chi(W) = \deg W + (\operatorname{rg} W)(1-g)$$

the degree of $(E \cap W)_0$ is the same as that of W.

So if $(E \cap W)_0$ is enlarged to a subbundle $E_{0,W}$ of E_0 then

$$\deg E_{0,W} \geq \deg W.$$

This proves that the canonical filtration of E_η will induce a filtration of E_0 with bigger degrees.



hence

$\boxed{\text{slope polygon of } E_\eta \subset \text{slope polygon of } E_0}$

Specializing a bundle makes the slope polygon increase.

March 16, 1975:

Recall the following idea for the K-theory of a curve. I wanted a space β_n of bundles of rank n, and more generally a space $\beta_{a_1,\dots,a_n}$ of filtered bundles with ranks $a_1, a_2, \dots, a_n$. Then I could form the simplicial Q-category:

$$F_2Q: \qquad \begin{tikzcd} \beta_{1,1} \arrow[r, shift left] \arrow[r, shift right] \arrow[ur] & \beta_1 & \beta_2 \end{tikzcd} \rightrightarrows pt.$$

Idea for $\beta_{1,1}$. Any exact sequence $0 \to L_1 \to E \to L_2 \to 0$ can be identified over $\mathbb{A}^1$ with $k[t]e_1 \rightarrowtail k[t]^2 \twoheadrightarrow k[t]e_2$ unique up to an element of $B_2(k[t])$. Thus the groupoid of such exact sequences can be obtained by letting $B_2(k[t])$ act on the set $\mathcal{L}_2$ of lattices at ∞ in $k[t]^2$. So my idea for the space $\beta_{1,1}$ was to take $B_2(k[t])$ acting on the space $\mathcal{L}_2$. This would give the homotopy type:

$$\begin{tikzcd} \Omega U_2 \arrow[r] \arrow[d, equal] & \beta_{1,1} \arrow[r] \arrow[d] & B(B_2 k) = BT_2 \arrow[d] \\ \Omega U_2 \arrow[r] & \mathrm{Map}(S_2, BU_2) \arrow[r] & BGL_2 k = BU_2 \end{tikzcd}$$

which is wrong, for there are no maps to β_1.

The right $\beta_{1,1}$ will have the homotopy type $\beta_1 \times \beta_1$.

Suppose then we have a bundle E on $\mathbb{P}^1$ extending $k[z]^2$. Thus E is given by a $R = k[[z^{-1}]]$ [redacted] lattice L in F^2, $F = k[[z^{-1}]][z]$. Form

$$L \cap Fe_1 = k[[z^{-1}]]\, z^p e_1$$

Hence $E \cap Fe_1$ is the line bundle given by $k[z]e_1$ and the lattice $k[[z^{-1}]]z^p e_1$, which is $\mathcal{O}(p)$.

$$L / Fe_1 \cap L = k[[z^{-1}]]\, z^q e_2 .$$

[redacted] Then L has a unique $k[[z^{-1}]]$-basis of the form

$$z^p e_1$$

$$f e_1 + z^q e_2$$

where $f \in F$ is unique modulo $k[[z^{-1}]]z^p$. In other words f may be uniquely chosen to be a Laurent polynomial

$$f = a_{p+1} z^{p+1} + a_{p+2} z^{p+2} + \cdots + a_m z^m$$ finite sum

Consider the orbit of L under $B_2(k[t])$, i.e. we are allowed to replace e_1 by ae_1, e_2 by [redacted] $ge_1 + de_2$ where $g \in k[t]$, $a, d \in k^*$. This means that I can modify f by scalars and by [redacted] adding $z^q k[z]$; thus I can ~~suppose that~~ modify f so that it has no terms of degree $\geq q$. Thus

q [redacted] $\leq p+1 \implies$ all L assoc. to (p,q) are conjugate under $B_2(k[z])$.

$q > p+1 \implies$ conjugacy classes of L assoc. to (p,q) under $B_2(k[z])$ form an affine space of $\dim = q-p-1$. (actually a proj. space union a point).

This should check with the fact that any sequence

$$(*) \qquad 0 \longrightarrow \mathcal{O}(p) \longrightarrow E \longrightarrow \mathcal{O}(q) \longrightarrow 0$$

[redacted] is classified up to isom. by an elt of

$$H^1(\underline{\mathrm{Hom}}(\mathcal{O}(q), \mathcal{O}(p))) = H^1(\mathcal{O}(p-q)) = \text{[redacted]}\ 0 \quad \text{if } p-q \geq -1 \text{ or } p+1 \geq q$$

and

$$\dim H^1(\mathcal{O}(p-q)) = \dim H^0(\mathcal{O}(-2+q-p))$$
$$= q-p-1 \quad \text{if} \quad q-p-1 \geq 0$$

Compute inf. defmns. of $(*)$ for $p < q$

$$0 \longrightarrow \mathrm{Hom}(\mathcal{O}(q), \mathcal{O}(p)) \longrightarrow \underline{\mathrm{End}}(E) \longrightarrow \mathcal{O} \times \mathcal{O} \longrightarrow 0$$

$$0 \longrightarrow H^0(\underline{\mathrm{End}}(E)) \longrightarrow k \times k \longrightarrow H^1(\mathcal{O}(p-q)) \longrightarrow H^1(\underline{\mathrm{End}}(E)) \longrightarrow 0$$

If $(*)$ splits, then $H^0(\underline{End}(E)) \simeq k \times k$ and so

$$H^1(\mathcal{O}(p-q)) \xrightarrow{\sim} H^1(\underline{End}(E))$$

which means, I guess, that the infinitesimal deformations fill out the iso classes of extensions. But if $(*)$ doesn't split, then we get

$$(+) \qquad 0 \longrightarrow k \longrightarrow H^1(\mathcal{O}(p-q)) \longrightarrow H^1(\underline{End}(E)) \longrightarrow 0.$$

I am now examining the moduli question for exact sequences of bundles over $\mathbb{P}^1$ of the form:

$$(*) \qquad 0 \longrightarrow L_1 \longrightarrow E \longrightarrow L_2 \longrightarrow 0.$$

The first invariant of such an exact sequence is $(p = \deg L_1,\ q = \deg L_2)$. This being fixed we have

$$(**) \qquad L_1 \simeq \mathcal{O}(p) \ , \quad L_2 \simeq \mathcal{O}(q)$$

these isos. unique up to elts. of k^*. Having fixed $(**)$, the sequence $(*)$ is classified by an element of

$$\mathrm{Ext}^1(\mathcal{O}(q), \mathcal{O}(p)) = H^1(\mathcal{O}(p-q)).$$

Thus the iso classes of exact sequences $(*)$ is

$$k^* \backslash H^1(\mathcal{O}(p-q))$$

which explains the exact sequence (+) above.

So when we form the space $\beta_{1,1}$ we want to topologize the set of lattices so that the map $L \longmapsto L \cap Fe_1$ is continuous. This means that we break the space $\mathcal{L}$ up into strata indexed by pairs (p,q), this stratum being lattices with basis

$$z^p e_1$$

$$(a_{p+1} z^{p+1} + \cdots) e_1 + z^q e_2$$

The stratum is contractible, so the space $\mathcal{L}$ now has the homotopy type $\mathbb{Z} \times \mathbb{Z}$. Now we let $B_2(k[z])$ act, and as $B_2(k[z]) \sim T_2 \subset U_2$, the space $\beta_{1,1}$ has the homotopy type

$$(\mathbb{Z} \times BU_1)^2 .$$

So the corresponding Steinberg homology will be the cone on the map

$$\mathrm{Map}(S^2, BU_1)^2 \longrightarrow \mathrm{Map}(S^2, BU_2)$$

Suppose we try to classify exact sequences of bundles over a curve C of the form

$$(1) \qquad 0 \longrightarrow L_1 \longrightarrow E \longrightarrow L_2 \longrightarrow 0$$

First invariant is $(cl(L_1), cl(L_2)) \in (\mathrm{Pic}\, C)^2$. The iso. classes of exact sequences (1) with L_1, L_2 fixed is

$$\mathrm{Ext}^1(L_2, L_1) = H^1(L_2^\vee \otimes L_1)$$

So the set of iso classes when we allow k^* to act on L_1, L_2 is

$$k^* \backslash H^1(L_2^\vee \otimes L_1)$$

(projective space union a point). As to deformations, we have

$$0 \longrightarrow L_2^\vee \otimes L_1 \longrightarrow \underline{\mathrm{End}}\, E \longrightarrow \mathcal{O} \times \mathcal{O} \longrightarrow 0$$

$$0 \longrightarrow H^0(L_2^\vee \otimes L_1) \longrightarrow H^0(\mathrm{End}\, E) \longrightarrow k \times k \longrightarrow$$

$$\longrightarrow H^1(L_2^\vee \otimes L_1) \longrightarrow H^1(\mathrm{End}\, E) \longrightarrow H^1(\mathcal{O})^2 \longrightarrow 0$$

has for image the line generated by class of (1).

represents deformation of L_1, L_2 in Pic

~~So what I see from this is~~

March 19, 1975

Recall resolution theorem. $\mathcal{M} = \mathrm{Mod}(A)$ A Dedekind $\mathcal{P} = \mathcal{P}(A)$. Then we factor $Q(\mathcal{P}) \to Q(\mathcal{M})$ by introducing the full subcat. $\mathcal{C}$ of $Q(\mathcal{M})$ consisting of objects of $\mathcal{P}$. Thus a map $P' \to P$ is $\mathcal{C}$ is an isom. $P' \simeq P_1/P_0$ where (P_0, P_1) is any layer of P (hence $P_0, P_1 \in \mathcal{P}_A$ also P_1/P_0, but not nec. P/P_1). Filter $\mathcal{C}$ by rank, so $F_n\mathcal{C}$ ~~consists~~ of P with rang $P \leq n$.

Given P of rank n, consider $F_{n-1}\mathcal{C}/P$. This is equivalent to the [illegible] poset of layers (P_0, P_1) in P $\ni$ $\mathrm{rank}(P_1/P_0) < n$. Call this poset $J(P)$. Put

$$J_+ = \text{subposet consisting of } (P_0, P_1) \text{ with } P_0 > 0$$
$$J_- = \text{———————— } \mathrm{rk}(P_1) < n$$

Then J_+ heq to poset of submodules P_0 of P with $P_0 > 0$ hence J_+ contractible; J_- is heq to poset of submodules P_1 of P with $\mathrm{rk}(P_1) < n$, hence J_- is contractible. Clearly $J = J_+ \cup J_-$ and J_+, J_- are closed, hence

$$J \sim \text{Susp of } J_+ \cap J_-$$

But $J_+ \cap J_-$ is heq to poset of submodules P' of P such that $0 < \mathrm{rank}\, P' < n$. Call this last poset Γ. Then we have a functor

$$\Gamma \xrightarrow{\lambda} T = \text{poset of proper subspaces of } F\otimes_A P$$

sending P' to $F \otimes_A P'$. Since
$$\mathrm{Hom}_\Gamma(P', W \cap P) = \mathrm{Hom}_T(F \otimes P', W)$$
(i.e. $P' \subset W \cap P \iff F \otimes P' \subset W$) it follows that λ has an adjoint $W \mapsto W \cap P$, hence λ is an h.eq. Thus we conclude
$$F_{n-1}\mathcal{C}/P \simeq \mathrm{Susp}\, T(F \otimes_A P)$$

Assuming A is a PID, the category $F_n\mathcal{C} - F_{n-1}\mathcal{C}$ consisting of modules P of rank n is equivalent to the monoid of embeddings $A^n \to A^n$

Mistake: [illegible] J consist of layers (P_0, P_1) in P with $P_1/P_0 \in \mathcal{P}$.

Deformations of a rang 2 bundle.

Let C be a complete non-sing curve over k. Assume that S is the spectrum of a d.v.r. over k and that I am given a rank 2 bundle E over $S \times C$. Assume over the generic point η of S that E_η is instable, whence we get an exact sequence of bundles

$$0 \longrightarrow W \longrightarrow E_\eta \longrightarrow E_\eta/W \longrightarrow 0$$

over C_η. Then we can extend this sequence to all of $S \times C$ to a sequence of sheaves flat over S

$$0 \longrightarrow E \cap W \longrightarrow E \longrightarrow E/E\cap W \longrightarrow 0.$$

Now taking special fibres

$$0 \longrightarrow (E\cap W)_0 \longrightarrow E_0 \longrightarrow (E/E\cap W)_0 \longrightarrow 0$$

one sees that $(E \cap W)_0$ is ~~locally~~ a line bundle, so it's more or less clear that $E \cap W$ is a line bundle on $S \times C$, which I will call L_1. Put

$$L_2 = \Lambda^2 E \otimes L_1^\vee$$

whence I will get from the pairing

$$L_1 \otimes E/L_1 \hookrightarrow \Lambda^2 E$$

a map $\quad E/L_1 \hookrightarrow L_2$

and hence an exact sequence

$$0 \longrightarrow L_1 \longrightarrow E \longrightarrow L_2 \longrightarrow Q \longrightarrow 0$$

where Q has 0-dimensional support.

Do the same analysis with $S = k[t]$. Here I know that $\mathrm{Pic}(C[t]) = \mathrm{Pic}\, C$, hence L_1 and L_2 are of the form

$$L_1'[t] \quad , \quad L_2'[t]$$

canonically for two line bundles L_1', L_2' of C. If we stay away from the set Z of points of $k[t]$ over which $Q \neq 0$, we have an exact sequence classified by an element of

$$\begin{aligned} \mathrm{Ext}^1(U \times C; L_2, L_1) &= H^1(U \times C, \underline{\mathrm{Hom}}(L_2, L_1)) \\ &= H^1(U \times C, pr_2^* \underline{\mathrm{Hom}}(L_2', L_1')) \\ &= H^1(C, \underline{\mathrm{Hom}}(L_2', L_1')) \otimes \Gamma(U, \mathcal{O}). \end{aligned}$$

This doesn't yield much information.

Instead let us examine Q at a point x of $S \times C$ where it is $\neq 0$. Then the exact sequence must be a Koszul type sequence. Thus if A is the reg. local ring of dim. 2 of $S \times C$ at x one has $A \simeq L_{1,x}, L_{2,x}$ and $A^2 \simeq E_x$, hence $Q_x \cong A/(a, b)$

where a, b is a regular sequence in A.

Conversely suppose we start with line bundles L_1, L_2 on $S \times C$ and a 0-dimensional quotient Q of L_2. Then put $K = \operatorname{Ker}(L_2 \twoheadrightarrow Q)$ and suppose E is an extension of K by L_1.

$$\begin{array}{ccccccc} \operatorname{Ext}^1(L_2, L_1) \to \operatorname{Ext}^1(K, L_1) & \longrightarrow & \operatorname{Ext}^2(Q, L_1) & \longrightarrow & \operatorname{Ext}^2(L_2, L_1) & & \\ & & \parallel & & \parallel & & \\ & & H^0(\mathcal{E}xt^2(Q, L_1)) & & 0 & \text{if } S \text{ affine} & \end{array}$$

So we will have to ~~know~~ know when E is locally free.

So suppose A is reg. local of dim 2, $Q = A/I$ where I is generated by a regular sequence, and

$$0 \longrightarrow A \longrightarrow E \longrightarrow I \longrightarrow 0$$

an extension representing an element of

$$\operatorname{Ext}^1(I, A) \simeq \operatorname{Ext}^2(Q, A)$$

For E to be isomorphic to A^2 it's more or less clear that under the canonical isomorphism

$$\operatorname{Ext}^2(A/I, A) \simeq A/I$$

given by the Koszul complex, the class of E must be a unit.

So therefore if we take two line bundles L_1, L_2 on $S \times C$ and a quotient $L_2 \twoheadrightarrow Q$ with Q zero-dimensional and the Kernel of $L_2 \twoheadrightarrow Q$ K locally generated by 2 elements, then if we take an ~~[illegible]~~ extension $E \in \operatorname{Ext}^1(K, L_1)$ mapping onto an element of $H^0(\mathcal{E}xt^2(Q, L_1))$

$$\begin{array}{c} H^0(\mathcal{E}xt^2(Q,L_1)) \\ \wr\wr \\ \prod_i A_i/I_i \end{array} \qquad \text{if } Q_{p_i} \simeq A_i/I_i$$

which maps onto a unit, then we get ~~[illegible]~~ a vector bundle E over $S \times C$.

Suppose for example $C = \mathbb{P}^1$ and I take $L_1, L_2 = \mathcal{O}(p), \mathcal{O}(q)$ with $p > q$. Then $\operatorname{Ext}^1(L_1, L_2) = H^1(\mathcal{O}(p-q)) = 0$. Messy.

March 23, 1975

Back to SU. Recall that the eigenvalues of an elt θ of SU_n may be uniquely represented in the form $e^{2\pi i t_1}, \ldots, e^{2\pi i t_n}$ where

$$t_1 \geq \cdots \geq t_n \geq t_1 - 1 \qquad \sum t_i = 0.$$

What is interesting to me is the integers k, l such that

$$t_n + 1 = t_1 = \cdots = t_k > t_{k+1} \qquad t_{n-l} < t_{n-l+1} = \cdots = t_n$$

Here $k \geq 0$, $l > 0$. In other words I put $\sigma(\theta) = t_n$ = minimum t, and then l = no. of t_i equal ~~to~~ $\sigma(\theta)$, k = no. of $t_i = \sigma(\theta) + 1$.

Given θ in the (k, l)-stratum, we can push all eigenvalues not equal to either $\sigma(\theta)+1$ or $\sigma(\theta)$ to zero, whence we end up with a matrix having the sequence

$$\begin{array}{ccc} \lambda - 1 & 0 & \lambda \\ \bullet & \bullet & \bullet \\ l & n-k-l & k \end{array}$$

$$k\lambda + l(\lambda - 1) = 0 \qquad (k+l)\lambda = l \qquad \lambda = \frac{l}{k+l}$$

so we get a single eigenvalue $e^{2\pi i \lambda}$ not 1. So the stratum is homotopy equivalent to $Y_{k+l}(\mathbb{C}^n)$, BU_{k+l} as $n \longrightarrow \infty$.

In order to pass from the (k', l') stratum to the (k,l) stratum $(k-k')$ of the t_i must head up and $l-l'$ of the t_i must go down. So the normal bundle should be

$$BU_{k-k'} \times BU_{k'+l'} \times BU_{l-l'} \longrightarrow BU_{k+l}$$

~~projection~~

Thus it's clear that I ought to get the category whose objects are pairs (k, P) where P is a projective and $0 \leq k < \mathrm{rank}(P)$, & in which $(k', P') \longrightarrow (k, P)$ is a Q-map $P' \simeq P_1/P_0 \dashrightarrow P$ Such that $k' + \mathrm{rank}(P_0) = k$.

$$n = \mathrm{rank}\ P = k + \overset{l}{\overset{\shortparallel}{(n-k)}}$$

$$= \underbrace{rg(P_0) + rg(P')}_{k'+l'} + rg(P/P_1)$$

Compute next the poset of $(k', P') \to (k, P)$. i.e. all layers (P_0, P_1) in P such that

$$rg(P_0) \leq k.$$

~~So we have the attraction $(P_0', P_1') \leq (P_0, P)$, so~~

Fix (k, l) $k>0$, $l>0$, and consider preceding strata: (k', l'), $k' \leq k$ $l' \leq l$ and not both equal. So I am interested in the poset of layers (P_0, P_1) in P such that

$rg(P_0) \leq k-1$

$rg(P_1) \geq k+1$ $\Leftarrow$ $\left(l' = l - rg(P/P_1) > 0 \right.$

$\left. n-k-n+rg\,P_1 > 0 \right)$

and such that $(P_0, P_1) < (0, P)$.

Still don't have the $k=0$ strata straight. Thing to do is this. Let k = number of maximum positive t_i and l = number of minimum negative t_i. So (k, l) is $(0,0)$ or both are positive. So in my category I also have the object $(0,0)$.

Objects are (P, k) with $0<k<rg(P)$ and $(0,0)$. A map $(P', k') \longrightarrow (P, k)$ is a Q-map $P' \simeq P_1/P_0$ such that $k' + rg\,P_0 = k$. Hence a map $(0,0) \longrightarrow (P, k)$ is a submodule (admiss) of rank k.

Now fix (P, k), $rg\,P > k > 0$, and consider preceding ~~strata~~ $(P', k') \longrightarrow (P, k)$, i.e layer (P_0, P_1) in P such that

$$k' + rg\, P_0 = k \implies rg\, P_0 < k$$
$$\text{or } rg\, P_0 = k \ \& \ P_0 = P_1.$$

also want

$$rg\,(P_1/P_0) > k' = k - rg\, P_0$$

$$\implies rg(P_1) > k \qquad \text{except if } rg(P_0) = rg(P_1) = k.$$

Thus the link of (P, k) appears to be the poset of layers (P_0, P_1) such that

$$rg\, P_0 < k < rg\, P_1$$
$$\text{or} \quad rg\, P_0 = k = rg\, P_1.$$

Also want $(P_0, P_1) \neq (0, P)$

Call this poset $J = J(P, k)$.
For each $Q \subset P$ of rank k one gets a minimal element of J.

$$\{Q \mid Q \in Y_k(P)\} \subset J \supset \left\{(P_0, P_1) \;\middle|\; \begin{matrix} rg\, P_0 < k \\ rg\, P_1 > k \end{matrix}\right\}.$$

Given Q compute the link of Q in J, i.e. the poset of (P_0, P_1) where $P_0 < Q$, $P_1 > Q$ and not both $P_0 = 0$, $P_1 = P$.

~~[illegible]~~ March 31, 1975. More on Schubert cells.

In what sense does the Schubert cell decomp of the Grassmannian constitute a CW decomp? Produce attaching maps.

Take $Y = \mathbb{P}_1(\mathbb{C}^n)$. $V_p = \mathbb{C}e_1 + \cdots + \mathbb{C}e_p$. Then

$$e_p = \mathbb{P}V_{p+1} - \mathbb{P}V_p$$

consists of ~~[illegible]~~ row forms $(* \cdots * \ 1 \ 0 \cdots 0)$. If I remove the center from e_p, there is the radial deformation

$$\mathbb{C}(a_1, \ldots a_p, 1) \longmapsto \mathbb{C}(a_1, \ldots, a_p, t) \qquad 1 \geq t \geq 0$$

of ~~[illegible]~~ e_p-center into $\mathbb{P}V_p$. This would gives the attaching map if I wish to work it out

~~[illegible]~~ Consider now $Y_2(V)$ and the cell given by (i_1, i_2) $\quad i_1 < i_2$ consisting of the row forms

$$\begin{array}{c} \begin{pmatrix} * \cdots\cdots * & 1 & & & \\ * \cdots\cdots * & 0 & * \cdots * & 1 & \end{pmatrix} \\ \qquad\quad\ \ \uparrow \qquad\qquad\ \uparrow \\ \qquad\quad\ \ i_1 \qquad\qquad\ i_2 \end{array}$$

i.e. $\quad C_{i_1 i_2} = \{ A \mid A \cap V_p \text{ jumps at } i_1, i_2 \}$.

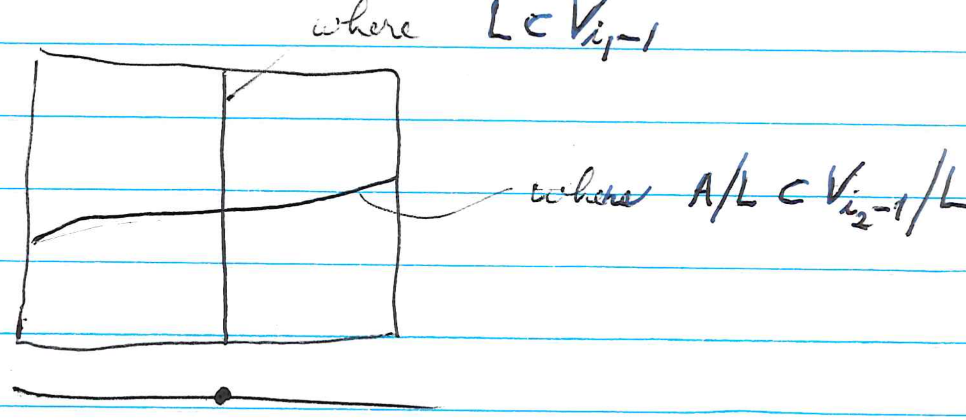
Can desingularize: $\tilde{C}_{i_1 i_2} = \left\{ (L \subset A) \,\middle|\, \begin{matrix} L \subset V_{i_1} \\ A \subset V_{i_2} \end{matrix} \right\}$

Then $\tilde{C}_{i_1 i_2}$ is a fibre bundle over $\mathbb{P}V_{i_1}$ with fibre $\mathbb{P}(V_{i_2}/L)$ over $L \in \mathbb{P}V_1$. The bad set, i.e. the complement of $C_{i_1 i_2}$, is the union of two sets:

$$L \in \mathbb{P}V_{i_1 - 1}$$

$$L \in \mathbb{P}V_{i_1} - \mathbb{P}V_{i_1 - 1} \quad \text{but} \quad A \subset V_{i_2 - 1}$$

Notice that since $i_1 < i_2$, we have $L \subset V_{i_1} \subset V_{i_2 - 1}$, hence this bad set is the union of two divisors crossing normally



Does same picture work for $C_{i_1 \ldots i_r}$ in $Y_r(V)$?

$$\tilde{C}_{i_1 \ldots i_r} = \{ 0 \subset A_1 \subset \cdots \subset A_r \mid A_j \subset V_{i_j} \}$$

Clearly works.

What I want to prove is that the inclusion map $C_{i_1 i_2} \subset Y_2(V)$ can be factored:

$$C_{i_1 i_2} \overset{\alpha}{\cong} \mathrm{Int}(D) \subset D \longrightarrow Y_2(V)$$

where D is the closed disk of same real dimension as $C_{i_1 i_2}$. (Previous reasoning shows α is not the obvious coordinatization.)

First step

$$\begin{tikzcd} C_{i_1 i_2} \arrow[r, phantom, "\subset"] \arrow[d] & \widetilde{C}_{i_1 i_2} \arrow[r] \arrow[d] & Y_2(V) \\ \mathbb{P}V_{i_1} - \mathbb{P}V_{i_1-1} \arrow[r, phantom, "\subset"] & \mathbb{P}V_{i_1} & \end{tikzcd}$$

By the case of $\mathbb{P}V$ the bottom inclusion factors

(*) $$\mathbb{P}V_{i_1} - \mathbb{P}V_{i_1-1} \subset D^{i_1-1} \xrightarrow{\gamma} \mathbb{P}V_{i_1}$$

$j = i_1 - 1$

$$\mathbb{C}(a_1, \ldots, a_j, 1) \longmapsto \frac{(a_1, \ldots, a_j)}{\sqrt{1+|a_1|^2+\cdots+|a_j|^2}}$$

$$(x_1, \ldots, x_j) \longmapsto (x_1, \ldots, x_j, \sqrt{1-|x|^2})$$

composite $$(a, 1) \longmapsto \frac{a}{\sqrt{1+|a|^2}} \longmapsto \left(\frac{a}{\sqrt{1+|a|^2}}, \sqrt{1-\frac{|a|^2}{1+|a|^2}}\right)$$

$$\sqrt{1-\frac{|a|^2}{1+|a|^2}} = \sqrt{\frac{1}{1+|a|^2}}, \qquad \sim (a, 1).$$

So we can pull $\tilde{C}_{i_1 i_2}$ back via the map γ. Because D^{i_1-1} is contractible the bundle $\gamma^*(\tilde{C}_{i_1 i_2})$ over D^{i_1-1} is trivial

$$\gamma^*(\tilde{C}_{i_1 i_2}) = D^{i_1-1} \times \mathbb{P}^{i_2-2}$$

and $C_{i_1 i_2}$ will be the complement of $D^{i_1-1} \times \mathbb{P}^{i_2-2-1}$. So we can use the product factorization

$$\begin{array}{ccc} C_{i_1 i_2} & = & D^{i_1-1} \times (\mathbb{P}^{i_2-2} - \mathbb{P}^{i_2-3}) \\ & & \cap \\ & & D^{i_1-1} \times D^{i_2-2} \\ & & \downarrow \\ & & D^{i_1-1} \times \mathbb{P}^{i_2-2} = \gamma^*(\tilde{C}_{i_1 i_2}) \end{array}$$

On to lattices: $R = \mathbb{C}[[\pi]]$. Consider the space X of lattices L with $\pi^{n+1}R^2 \subset L \subset R^2$ such that $\dim(R^2/L) = n+1$. Let $\tilde{X}$ be the set of flags $0 \subset A_1 \subset \cdots \subset A_{n+1}$ in $R^2/\pi^{n+1}R^2$ such that $\pi A_i \subset A_{i-1}$. Then we have a map

$$\tilde{X} \longrightarrow X$$

$$(A_1, \ldots, A_{n+1}) \longmapsto A_{n+1}.$$

Claim $\tilde{X}$ is non-singular.

Change notation: $R^2 \subset L \subset \pi^{-N}R^2$ $\dim(L/R^2) = n$.

X_n to consist of $R^2 = A_0 \subset A_1 \subset \cdots \subset A_n \ni \pi A_j \subset A_{j-1}$. Clearly $X_n \longrightarrow X_{n-1}$ is a $\mathbb{P}^1$-bundle, hence each X_n is non-singular. Generically A_n projects non-trivially in $\pi^{-n}R^2/\pi^{-n+1}R^2$; for A_n in this open set one has $A_i = \pi^{n-i}A_n + R^2$, so X_n is isomorphic to X on this open set.

Next consider the set of $A_n = L$ such that L projects non-trivially in $\underbrace{\pi^{-n}R^2}_{V_{2n}} / \underbrace{R\pi^{-n}e_1 + R\pi^{-n+1}e_2}_{V_{2n-1}}$ and $\dim L/R^2 = n$. This is a cell such that each L in it has distinguished basis

$$x_1 = e_1$$
$$x_2 = \pi^{-n}(e_2 + (a_0 + a_1\pi + \cdots + a_{n-1}\pi^{n-1})e_2)$$

Now I want to see this open cell in X inside of the resolution $\tilde{X} = X_n$. First note

$$A_i = \pi^{n-i}L + R^2$$

has the distinguished basis

$$e_1$$
$$\pi^{-i}(e_2 + (a_0 + \cdots + a_{i-1}\pi^{i-1})e_1).$$

e.g. A_1 has basis e_1, $\pi^{-1}(e_2 + a_0 e_1)$ so it is any line in $\pi^{-1}R^2/R^2$ not $= k\pi^{-1}e_1$. Having chosen A_{i-1} to stay within the open cell A_i must be chosen to avoid one point.

Note that X_n is in fact a product of projective lines.

$$X_n = (\mathbb{P}^1)^n$$

because I can use the scattering matrix to pull A_{n-1} back to R^2 whence choices for A_n will correspond to lines in $\pi^{-1}R^2/R^2$. Now the bad set inside of X_n is where two consecutive lines coincide, so if I remove this diagonal union I will get the space of things at distance n. And if I require A_1 to go in the right direction, i.e. $A_1/R^2 \neq ke_1$, then I get the big cell. Seems like Bott's geodesics.

Generalize to rank r. Recall that a lattice L is classified as to its B-orbit by the set of integers p at which the filtration $L \cap V_p$ jumps. If $S(L) = \{p \mid L \cap V_p > L \cap V_{p-1}\}$, then $p \in S(L) \Rightarrow p-r \in S(L)$; for $p \in S(L) \Rightarrow \exists\, x \in L \cap V_p,\ x \notin V_{p-1} \Rightarrow \pi x \in L \cap V_{p-r};\ \pi x \notin V_{p-r-1}$. Thus

$$S(L) = \coprod_{i=1}^{r} \{ p_i - nr \mid n \geq 0 \}$$

where $p_i \equiv i \pmod{r}$.

Given such a subset $S \subset \mathbb{Z}$, the set of L with $s(L) = S$ is the Schubert cell in the set of lattices assoc. to S. ~~Now suppose that~~ What I want to do is to construct this cell inductively, in such a way that I can see its closure, in fact, a desingularization of its closure.

The subset $S \subset \mathbb{Z}$ being given I will consider chains of lattices L_p $p \in S$ such that $L_p \subset V_p$ & if $p' < p$ are successive elements of S then $L_{p'} \subset L_p$ is of codim 1, and such that $L_p = V_p$ for $p \ll 0$. The set of such chains I will denote $\tilde{C}_S$. Set $h(L)$ = index of L with respect to the lattice $V_0 = \pi R^r$, and put $L^h = L_p$ where $h = h(L_p)$. In this way I have relabelled the chain so that L^h is defined for $h \leq h_0$ and L^h is of index h.

I want to show that $\tilde{C}_S$ is an iterated projective space bundle. Thus I have to see what the choice for L_p is given all the previous ones. Let p' be the predecessor of p in S, whence $p-r \leq p' < p$. Then L_p is to be chosen

$$L_{p'} \underset{\dim 1}{\subset} L_p \subset \pi^{-1} L_{p'} \cap V_p$$

so we get a projective spaces. ~~Notice that $L_p / L_{p'}$ dim 1 $\Rightarrow$ $L_p \subset \pi^{-1} L_{p'}$ so this condition on~~ ?