May 6, 1973. Stable bundles on a Riemann surface

Sheshadri – Narashiman thm: Let $\tilde{C} \to C$ be the universal covering + Γ the Galois group of a curve C over $\mathbb{C}$. Then if V is a unitary representation of Γ over $\mathbb{C}$, the bundle

$$\tilde{C} \times^{\Gamma} V \longrightarrow C$$

is a direct sum of stable bundles of degree 0. Conversely a vector bundle which is a direct sum of stable bundles of degree zero is obtained in this way.

Weil thm. A bundle E on C is obtained from a finite dimensional representation of Γ $\Longleftrightarrow$ each of its indecomposable components has degree zero.

Note that E is obtained from a representation of Γ $\Longleftrightarrow$ it has a integrable connection. Since we are over a curve integrability is immediate, so we need only a connection, i.e. a splitting of the sequence

$$(*) \qquad 0 \longrightarrow E \otimes \Omega \longrightarrow J_1(E) \longrightarrow E \longrightarrow 0 .$$

Now

$$\mathrm{Ext}^1(E, E\otimes\Omega) = H^1(\underline{\mathrm{Hom}}(E,E)\otimes\Omega)$$

is dual to

$$H^0(\underline{\mathrm{Hom}}(E,E)^{\vee}) =$$

$$= H^0(\underline{\mathrm{Hom}}(E,E)) \qquad \text{by (trace } AB)$$

$$= \mathrm{End}(E,E)$$

The question now is just what the linear function on

End(E) belonging to (*) is. We want to split

(1) $$0 \longrightarrow \underline{\mathrm{Hom}}(E,E)\otimes\Omega \longrightarrow \mathrm{Hom}_{/id_E}(E, J_1(E)) \longrightarrow \mathcal{O} \longrightarrow 0$$

which is classified by a class in

$$H^1(\underline{\mathrm{Hom}}(E,E)\otimes\Omega) \quad \text{dual to} \quad \mathrm{End}(E).$$

If E is simple, i.e. $\mathrm{End}(E) = k$, then

$$H^1(\underline{\mathrm{Hom}}(E,E)\otimes\Omega) \xrightarrow{\text{trace}} \underset{\overset{\|}{k}}{H^1(\Omega)} \longrightarrow 0$$

both are 1-dimensional and the image of the canonical class is the Atiyah first Chern class, hence it is the degree of E. Conclude:

<u>Prop</u>: If $\mathrm{End}(E) = k$, then E has a connection (necessarily integrable) iff $\deg(E) = 0$.

But in general we have the exact sequence

$$0 \longrightarrow H^0(sl(E)\otimes\Omega) \longrightarrow H^0(gl(E)\otimes\Omega) \longrightarrow H^0(\Omega) \supset$$
$$\hookrightarrow H^1(sl(E)\otimes\Omega) \longrightarrow H^1(gl(E)\otimes\Omega) \longrightarrow \underset{\overset{\|}{k}}{H^1(\Omega)} \longrightarrow 0$$

where $gl(E) = \underline{\mathrm{End}}(E)$, and $sl(E)$ is the subsheaf of endos. of trace zero. This exact sequence splits (char. 0). If Weil's theorem is true one has to be able to see why the class of (1) dies if its ?

The point is that given E there is a canonical map End(E) ⟶ k, defined by

$$H^0(\underline{\mathrm{End}}(E)) \times H^1(\underline{\mathrm{End}}(E) \overset{\chi}{\wedge} \otimes \Omega) \longrightarrow H^1(\Omega)$$

$$\alpha \qquad\qquad \beta\otimes\omega \longmapsto \int \mathrm{tr}(\alpha * \beta)\,\omega$$

and I want a formula for this map. The conjecture would be that if

$$E = \oplus E_i$$

is a decomposition into indecomposables, then the function is

(2) $$\alpha \longmapsto \sum \mathrm{tr}(\alpha | E_i) \cdot \deg(E_i)$$

Possibly a way to proceed is to use the fact that E has a reduction to the Borel subgroup:

Given $\alpha \in \mathrm{End}(E)$ we can find a full flag in E which is invariant under α (do so at the generic point). Then the bundle is reduced to the Borel subgroup and so we have a map of Atiyah extensions

$$\begin{tikzcd}
0 \arrow[r] & \mathfrak{bor}(E) \arrow[r] \arrow[d] & {} \arrow[r] & \mathcal{O} \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & \mathfrak{gl}(E) \arrow[r] & {} \arrow[r] & \mathcal{O} \arrow[r] & 0
\end{tikzcd}$$

On the other hand $\alpha \in H^0(\mathfrak{bor}(E)) \subset H^0(\mathfrak{gl}(E))$, so under the duality pairing $(A,B)\mapsto tr(AB)$, $\mathfrak{gl}E \times \mathfrak{gl}E \to \mathcal{O}$ we just restrict to the Borel subgroup. What this clearly gives is the formula

$$\alpha \longmapsto \sum tr(\alpha | L_i)\cdot deg(L_i)$$

(the last factor results from the line bundle situation).

Now if E is indecomposable the eigenvalues $tr(\alpha|L_i)$ must all be the same,* so we get

$$\alpha \longmapsto tr(\alpha|E)\ deg E$$

* construct idempotents from min. poly of α

for an indecomposable bundle, proving (2). Thus we have a proof of Weil's theorem.

Weil's thm: A vector bundle E on a curve C has a connection (necessarily integrable) $\Longleftrightarrow$ each indecomposable factor has degree zero.

Mumford's method for constructing the moduli space.

Let $\mathcal{F}$ be a limited family of vector bundles with the same rank and degree, hence the same Hilbert polynomial. Twisting with $\mathcal{O}(m)$ for m sufficiently large, one can make $\dim H^0(E(m))$ independent of E and such that it generates $E(m)$. (More precisely, one arranges that $H^1(E(m-1))=0$ and this implies $H^1(E(m))=0$ and $H^0(E(m))$ generates $E(m)$.) Thus upon choosing a basis for $H^0(E(m))$ we get a point in a Hilbert scheme of a certain type. Thus by rigidifying E to $(E + V \xrightarrow{\sim} H^0(E(m)))$ we can parameterize by a variety of same sort. Now we have to form a quotient by the action of $\mathrm{Aut}(V)$ to parameterize the iso. classes of $\mathcal{F}$.

~~Now Mumford's idea is to forget the Hilbert scheme~~

In order for Mumford to apply his theory he needs to determine the stable points of the Hilbert scheme. For this he chooses a large number of points on the surface and associates to a vector bundle quotient of $\mathcal{O}\otimes V$ the fibres at each of the points, thus getting a sequence of quotients $V \twoheadrightarrow W_1, V \twoheadrightarrow W_2, \ldots, V \twoheadrightarrow W_N$ of V. The idea now is simply to show that if the points are properly chosen than a quotient of $\mathrm{Aut}(V)$ is stable iff the sequence of points in the Grassmannian of V is.

Mumford-Sheshadri example of an exact category.

N fixed integer ≥ 0. Objects consist of $(V, W_1, \ldots, W_N)$ where V is a vector space and W_i are a sequence of subspaces of the same dimension. An exact sequence

$$0 \longrightarrow (V', W_i') \longrightarrow (V, W_i) \longrightarrow (V'', W_i'') \longrightarrow 0$$

is defined to be an exact sequence

$$0 \to V' \to V \longrightarrow V'' \longrightarrow 0$$

such that for each i we have a map of exact seq.

$$\begin{tikzcd}
0 \arrow[r] & W_i' \arrow[r] \arrow[d, phantom, "\cap"] & W_i \arrow[r] \arrow[d, phantom, "\cap"] & W_i'' \arrow[r] \arrow[d, phantom, "\cap"] & 0 \\
0 \arrow[r] & V' \arrow[r] & V \arrow[r] & V'' \arrow[r] & 0
\end{tikzcd}$$

Call this exact category $\mathcal{G}^N$. Here's how it arises:

Let X be a projective variety over k (alg. closed say) and $\mathcal{O}(1)$ a very ample line bundle. Consider the exact category of vector bundles over X such that E is generated by $H^0(E)$ and $H^1(E) = 0$. Let $x_1, \ldots, x_N$ be a sequence of closed points. Then

$$E \longmapsto \left(H^0(E), \operatorname{Ker}\{ev_{x_i} : H^0(E) \longrightarrow E(x_i)\}\right)$$

is exact from such vector bundles to $\mathcal{G}^N$.

I want to determine the K-theory of $\mathcal{G}^N$. Let $\mathcal{G}^{N\prime} \subset \mathcal{G}^N$ be the full subcategory consisting of (V, W_i) such that the W_i are independent, i.e. $\bigoplus W_i \hookrightarrow V$.

This is closed under extensions since if we have an exact sequence

$$0 \longrightarrow (V', W_i') \longrightarrow (V, W_i) \longrightarrow (V'', W_i'') \longrightarrow 0$$

then we have a map of exact sequences

$$\begin{tikzcd}
0 \arrow[r] & \oplus W_i' \arrow[r] \arrow[d] & \oplus W_i \arrow[r] \arrow[d] & \oplus W_i'' \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & V' \arrow[r] & V \arrow[r] & V'' \arrow[r] & 0
\end{tikzcd}$$

so by the serpent lemma $(V, W_i) \in \mathcal{G}^{N'} \Longrightarrow (V', W_i') \in \mathcal{G}^{N'}$ and $\mathcal{G}^{N'}$ is closed under extensions. On the other hand we know that given $(V, W_i) \in \mathcal{G}^N$, then set

$$\begin{array}{ccl}
\bar{V} & = & \bigoplus_{j=1}^{N} V \\
\cup & & \quad\cup \\
\bar{W}_i & = & \bigoplus_{j=1}^{N} \begin{cases} 0 & j \neq i \\ W_i & j = i \end{cases}
\end{array}$$

so that we have a map

$$\bar{V} \twoheadrightarrow V$$

carrying $\bar{W}_i \xrightarrow{\sim} W_i$. Thus we have an exact sequence

$$0 \to (\mathrm{Ker}(\bar{V} \to V), 0_i) \longrightarrow (\bar{V}, \bar{W}_i) \longrightarrow (V, W_i) \longrightarrow 0$$

showing that any member of $\mathcal{G}^N$ is a quotient of some member of $\mathcal{G}^{N'}$. So the resolution thm. implies $\mathcal{G}^{N'}$

and $\mathcal{G}^N$ have the same K-theory.

But now it is clear that $\mathcal{G}^{N'}$ is the exact category of exact sequences

$$0 \to \oplus W_i \to V \to V'' \to 0$$

so there are obvious characteristic filtrations which show that the K-theory obtained is the direct sums of copies of $K_*(k)$, $N+1$ copies.

Nothing new is obtained in this way about the K-theory of the curve C. Thus we get from sending

$$E \longmapsto (H^0(E), \operatorname{Ker}\{ev_x : H^0(E) \to E_x\})$$

the two ingredients of $K_*(C)$ which we knew about before:

$$\begin{tikzcd} K_*(C) \arrow[r, "\text{rest.}", "\text{to } x"'] \arrow[r, bend right=40, "\int_C = H^0 - H^1"'] & K_*(k) \end{tikzcd}$$

<u>Question:</u> Describe the exact functors from suff. pos. vector bundles on X to k-modules?

Vector bundles on elliptic curves. (Atiyah's paper)

First let E be a rank n vector bundle over a curve C and let $0 \subset E_1 \subset \cdots \subset E_n = E$ be a maximal flag with quotients L_i. Then the first estimate is an upper bound for $\deg(L_i) - \deg(L_{i-1})$. To derive it we can suppose E two dimensional

$$0 \longrightarrow L_1 \longrightarrow E \longrightarrow L_2 \longrightarrow 0$$

and that

$$0 < \deg(E) + 2(1-g) \leq 2$$

since we can ⊗ with a line bundle. Then $RR \Rightarrow H^0(E) \neq 0$ $\Rightarrow \deg(L_1) \geq 0$ so

$$\deg(L_2) - \deg(L_1) \leq \deg(L_2) + \deg(L_1) = \deg E \leq 2g$$

Thus for a maximal flag

$$\boxed{\deg(L_i) - \deg(L_{i-1}) \leq 2g}$$

The next estimate concerns the case where E is indecomposable whence for $0 < i < n$

$$0 \neq \mathrm{Ext}^1(E_n/E_i, E_i) = H^1(\underline{\mathrm{Hom}}(E_n/E_i, E_i))$$

dual to $H^0(\underline{\mathrm{Hom}}(E_i, E_n/E_i) \otimes \Omega) \neq 0$. Then fixing i we have

$$f_i : E_i \xrightarrow{\neq 0} E_n/E_i \otimes \Omega$$

Then let j be minimal $\ni$ $f_i | E_j$ is $\neq 0$, whence $f_i : L_j \hookrightarrow E_n/E_i \otimes \Omega$

and so $\deg(L_j) - (2g-2) \leq \deg(L_{i+1})$

for some $1 \leq j \leq i$. Thus want

$$\boxed{\deg(L_i) \geq \deg(L_1) - (i-1)(2g-2)}$$

E ind.

and we can get this by induction

$$\begin{aligned} \deg(L_{i+1}) &\geq \deg(L_j) - (2g-2) \\ &\geq \deg(L_1) - (j-1)(2g-2) - (2g-2) \\ &\geq \deg(L_1) - (i)(2g-2) \qquad \text{since } j \leq i. \end{aligned}$$

But now suppose we have an <u>elliptic curve</u> whence $\Omega = \mathcal{O}$. Here to show E_i can be chosen so that $L_1 \subset L_i$ for all i. Now again using induction we have

$$f_i : E_i \xrightarrow{\not\equiv 0} E_n / E_i$$

hence $\exists j \leq i$

$$f_i : L_j \hookrightarrow E_n / E_i$$

~~so we can suppose L_{i+1} chosen~~ If the maximal degree $= \deg(L_{i+1})$ is the same as $\deg(L_j)$ we can suppose L_{i+1} chosen to be $f_i(L_j)$. Otherwise

$$\deg(L_j) < \deg(L_{i+1})$$

and we know $\exists$ $L_1 \hookrightarrow L_{i+1}$. In either case L_{i+1} can be assumed to contain L_1

Now suppose E is indecomposable of rank n and

$\deg(E) = d$ where $0 < d \leq n$. [redacted] We know that [redacted] $H^0(E) \neq 0$ by RR, hence $\deg(L_1) \geq 0$.

If $\deg(L_1) = 1$, then [redacted] $L_1 \subset L_i$ + $\sum \deg L_i = d \leq n$ $\Rightarrow$ $d = n$ and all $L_i = L$, so $E = L_1 \otimes F_n$ where F_n is the unique indecomposable bundle which is a successive extension of $\mathcal{O}$'s of rank n. Thus E is semi-stable of slope 1.

On the other hand suppose $\deg(L_1) = 0$. [redacted] I claim that the map

$$\mathcal{O} \otimes H^0(E) \longrightarrow E$$

is a vector bundle injection. To see this it is enough to show that [redacted] $s(x) \neq 0$ $\forall x \in X, \forall s \in H^0(E)$, $s \neq 0$. But otherwise the subline bundle L of E generated by s would have $\deg > 0$ which is impossible.

~~Thus we have an exact sequence of vector bundles~~

~~$0 \to \mathcal{O} \otimes H^0(E) \longrightarrow E \longrightarrow E' \longrightarrow 0$~~

~~hence~~

~~$0 \to H^0(E) \xrightarrow{\sim} H^0(E) \longrightarrow H^0(E') \xrightarrow{\delta} H^0(E) \longrightarrow H^1(E) \to H^1(E') \to 0$~~

~~hence~~ ~~$H^0(E') \xhookrightarrow{\delta} H^1$~~

But $h^0(E) \geq \deg(E)$, so that the first thing to note is that $d \geq n$ is impossible in this case since otherwise E would be trivial.

If $d = n-1$, then must have $h^0(E) = n-1$, so

E is an extension of $\Lambda^n E$ by $\mathcal{O}^{n-1}$. In general define E' by

$$0 \longrightarrow \mathcal{O}\otimes H^0(E) \longrightarrow E \longrightarrow E' \longrightarrow 0$$

so that $E' = \Lambda^n E$ if $d = n-1$. Since $h^0(E) - h^1(E) = n-1$ and $h^0(E) = n-1$ we have $h^1(E) = 0$ and therefore the above should be the canonical extension ?

Better: Let L have degree k. Then

$$\mathrm{Ext}^1(L, \mathcal{O}\otimes_k V)$$
$$= H^1(L)\otimes_k V \qquad = \mathrm{Hom}(H^0(L), V)$$

so there is a canonical extension

$$0 \longrightarrow \mathcal{O}\otimes H^0(L) \longrightarrow \, ? \longrightarrow L \longrightarrow 0$$

and from the coh. exact sequence we get

$$0 \longrightarrow H^0(L) \longrightarrow H^1(\mathcal{O}\otimes H^0(L)) \longrightarrow H^1(?) \longrightarrow 0$$

hence $H^1(?) = 0$. It is clear that ? is indecomposable since any endo. of ? induces one of the exact sequences.

Suppose $d = n-2$. $h^0(E) - h^1(E) = d$. There are two possibilities: $h^0(E) = n-1$ so E' is a line bundle L of degree $n-2$

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{O}^{n-1} \arrow[r] & E \arrow[r] & L \arrow[r] & 0 \\
0 \arrow[r] & \mathcal{O}\otimes H^0(L) \arrow[r] \arrow[u] & ? \arrow[r] \arrow[u] & L \arrow[r] \arrow[u, equal] & 0
\end{tikzcd}$$

But if $H^0(L)$ is of dim $n-2$, then the image of $\mathcal{O}\otimes H^0(L)$ sits is a proper ~~[illegible]~~ bundle of $\mathcal{O}^{n-1}$ and so it is clear that $E = ? \oplus \mathcal{O}$. This case doesn't happen.

Thus $h^0(E) = n-2$, $h^1(E) = 0$ and E' is of rank two and E is the canonical extension

$$0 \longrightarrow \mathcal{O}\otimes H^0(E') \longrightarrow ? \longrightarrow E' \longrightarrow 0$$

It follows that E' is indecomposable of rank 2 and degree $n-2$.

In general let E be indecomposable with degree $d>0$ and $\deg(L_1) = 0$ ~~so~~ that we have

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{O}\otimes H^0(E) \arrow[r] & E \arrow[r] & E' \arrow[r] & 0 \\
0 \arrow[r] & \mathcal{O}\otimes H^0(E') \arrow[r] \arrow[u, "\alpha"] & ? \arrow[r] \arrow[u] & E' \arrow[r] \arrow[u, equal] & 0
\end{tikzcd}$$

where the bottom is the canonical extension. Since $H^0(E') \xhookrightarrow{\delta} H^1(\mathcal{O}\otimes H^0(E)) \cong H^0(E)$, the map α if it is onto then it is an isom. But if α isn't onto then we get

$$\begin{tikzcd}
0 \arrow[r] & ?/\mathcal{O}\otimes \operatorname{Ker}\alpha \arrow[r] & E \arrow[r] & \mathcal{O}\otimes \operatorname{Coker}\alpha \arrow[r] \arrow[l, bend right] & 0
\end{tikzcd}$$

so α has to be an isomorphism. Conclude that an indecomposable with $\deg \overset{=d}{>} 0$ and $\deg(L_1) = 0$ is the canonical extension of an E'. Clearly E' is indecomposable of the same degree as E. ~~But more~~.

~~$h^0(E) = h^1(E)$~~

<u>Lemma</u>: Let E a vector bundle with $\deg(L_1) = 0$. Then the map

$$\mathcal{O} \otimes H^0(E) \longrightarrow E$$

is a subbundle inclusion.

This holds for any curve.

For an elliptic curve with E decomposable we have putting $E' = E / \mathcal{O} \otimes H^0(E)$

$$\begin{aligned} \operatorname{Ext}^1(E', \mathcal{O} \otimes_k V) &= H^1(E'^*) \otimes V \\ &= H^0(E' \otimes \Omega)^* \otimes V \\ &= \operatorname{Hom}(H^0(E' \otimes \Omega), V) \end{aligned}$$

so we have a canonical extension

$$\begin{tikzcd} 0 \arrow[r] & \mathcal{O} \otimes H^0(E' \otimes \Omega) \arrow[r] \arrow[d, "\alpha"] & \tilde{E}'_{can} \arrow[r] \arrow[d] & E' \arrow[r] \arrow[d, equal] & 0 \\ 0 \arrow[r] & \mathcal{O} \otimes H^0(E) \arrow[r] & E \arrow[r] & E' \arrow[r] & 0 \end{tikzcd}$$

← hence a map

and when E is indecomposable it follows that α is onto. But $\delta : H^0(E') \hookrightarrow H^1(\mathcal{O} \otimes H^0(E)) \simeq H^0(E)$, hence α has to be an isomorphism. Clearly E' is indecomposable.

Conversely suppose E' indecomposable and consider the canonical extension:

$$0 \longrightarrow \mathcal{O} \otimes H^0(E' \otimes \Omega) \longrightarrow \widetilde{E}'_{can} \longrightarrow E' \longrightarrow 0$$

Its pretty clear that $\delta : H^0(E') \xrightarrow{\sim} H^1(\mathcal{O}) \otimes H^0(E' \otimes \Omega)$ so that $H^0(E' \otimes \Omega) = H^0(\widetilde{E}'_{can})$. In particular any endo. of E'_{can} induces one of the whole exact sequence, and so $\widetilde{E}'_{can}$ has to be indecomposable when E' is.

So now start with an indecomposable E_n of deg > 0 such that $\deg(L_1) = 0$ and form E' and note that $H^0(E) = H^0(E')$ and $H^1(E) = H^1(E')$. Now look at $\deg(L_1')$ for E'. If >0, then $H^1(E') = 0$ since $L_1' \subset L_i'$ for all i. So by induction we can establish that $H^1(E) = 0$, ~~so~~ so

~~$\mathrm{rank}(E') \leq \mathrm{rank}(E) \leq h^0(E)$~~ $h^0(E) = d$.

Now note that if we start with ~~deg~~ $0 < d < n$ then $\deg(L_1) = 0$, and so

$$\mathrm{rank}(E') = \mathrm{rank}(E) - d$$
$$\deg(E') = d.$$

On the other hand, ~~deg~~ once we arrange $0 < \deg E \leq$ ~~deg~~ $rg(E)$ then $\deg(L_1) = 0$ iff $\deg(E) < rg(E)$. (Recall: $\deg(L_1) > 0 \Rightarrow$ (as $L_1 \subset L_i$) all $L_i = L_1$ so $\deg(E) = rg(E)$. But also if $\deg(L_1) = 0$, then $\mathcal{O} \otimes H^0(E) \hookrightarrow E$ + $h^0(E) \geq rg(E)$ impossible)

Thus for $n > d$ we have

$$\mathcal{E}(n,d) \xrightarrow{\sim} \mathcal{E}(n-d, d)$$
$$E \longmapsto E'$$

where $\mathcal{E}(n,d)$ denotes the set of iso. classes of indecomposable bundles of rank n and degree d.

Thm: All $\mathcal{E}(n,d)$ are in 1-1 correspondence with the elliptic curve C.

Proof: (Use induction on n) By tensoring with a line bundle of deg k get

$$\mathcal{E}(n,d) \simeq \mathcal{E}(n, d+kn)$$

so can suppose $0<d \leq n$. If $d=n$ we know that we have an isom with C: $L \mapsto L \otimes F_n$. If $d<n$, then we have (as above) an isomorphism

$$\mathcal{E}(n,d) \xrightarrow{\sim} \mathcal{E}(n-d, d)$$

so the induction marches.

Prop: E indecomposable of degree $>0 \Rightarrow H^1(E)=0$.

Pf: If $\deg(L_1) > 0$, then because $L_1 \subset L_i$ E is a successive extension of line bundles of deg >0, hence $H^1(E)=0$. On the other hand if $\deg(L_1)=0$, then we have $H^1(E) = H^1(E')$ where E' is indecomposable, + of the same degree, but with smaller rank.

~~When rank E≥2,~~

Prop: An indecomposable E such that $\frac{\deg E}{\operatorname{rank} E} \notin \mathbb{Z}$ is stable and conversely.

Proof: Any stable bundle is indecomposable (in fact $\operatorname{End}(E)=k$), and we have seen that if rank E divides deg E, Proposition: Let $0 \to E' \to E \to E'' \to 0$ be an exact sequence of vector bundles over a curve C of genus g. If the sequence doesn't split, then

$$\mu_{\min}(E') - \mu_{\max}(E'') \leq 2g-2 .$$

Proof: $\mathrm{Ext}^1(E'',E') = H^1(\underline{\mathrm{Hom}}(E'',E'))$ is dual to $H^0(\underline{\mathrm{Hom}}(E',E'')\otimes\Omega) = \mathrm{Hom}(E',E''\otimes\Omega)$, so if the sequence doesn't split $\exists$

$$f: E' \longrightarrow E''\otimes\Omega \quad \neq 0.$$

Then $\mathrm{Coim}(f) = E'/\mathrm{Ker} f$ is a $\neq 0$ quotient bundle of E' so

$$\mu_{\min}(E') \leq \mu(\mathrm{Coim} f)$$

and $\mathrm{Im}(f)$ is a $\neq 0$ subbundle of $E''\otimes\Omega$ so

$$\mu(\mathrm{Im}(f)) \leq \mu_{\max}(E''\otimes\Omega) = \mu_{\max}(E'') + 2g-2$$

and since $\mathrm{Coim} f \hookrightarrow \mathrm{Im} f$ have the same rank

$$\mu(\mathrm{Coim} f) \leq \mu(\mathrm{Im} f).$$

The proposition follows.

Corollary: On an elliptic curve an indecomposable bundle is semi-stable.

In effect there can be only one slope

it is indecomposable and

Prop. On an elliptic curve a bundle is stable iff ^ its rank and degree are relatively prime $(\mathbb{Z}d+\mathbb{Z}r=\mathbb{Z})$.

Quasi-Proof: E indecomposable $\Rightarrow$ E semi-stable. If $(d,r)=1$, then have for any $E'\subset E$ that $\frac{d'}{r'}\leq\frac{d}{r}$; must be $<$ ~~[illegible]~~ because by assumption $\frac{d}{r}$ is in lowest terms.

For the converse suppose E is stable of degree d rank r with $(d,r)=1$, and consider the bundle $E\otimes F_n$ with invariants (dn, rn). Then $E\otimes F_n$ is semi-stable with quotients ~~[illegible]~~ isom to E.

$$0\subset E\otimes\mathcal{O}\subset E\otimes F_2\subset\cdots\subset E\otimes F_n$$

Claim this filtration is characteristic; by induction one should be able to see that

$$\mathrm{Hom}(E, E\otimes F_n) = k$$
$$\mathrm{Ext}^1(E, E\otimes F_n) = k$$

so that any idempotent operator $\neq 0$ has to preserve the filtration and so [illegible] $=1$. Thus $E\otimes F_n$ has to be indecomposable.

So now more or less it is clear that $E\mapsto E\otimes F_n$ will be a map from $\mathcal{E}(d,r)\longrightarrow\mathcal{E}(nd,nr)$ which has to be onto by isogeny rules. This point should be in Atiyah's paper.

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no good

May 10, 1973

Possible proof of $K_q(A[t]) = K_q(A) \oplus Nil_{q-1}(A)$:

Try to lift $A[t]$-modules to graded $A[T_0, T_1]$-modules. ~~Recall that~~ Let $\mathcal{P}gr(A[T_0,T_1])$ be the cat. of f.g. proj graded $A[T_0,T_1]$-modules $P = \bigoplus_{n \geq 0} P_n$. Recall that non-canonically

$$(1) \qquad P \cong \bigoplus_{k=0}^{N} A[T_0,T_1](-k) \otimes_A L_k \qquad L_k \in \mathcal{P}(A)$$

and that

$$(2) \qquad K_q(\mathcal{P}gr(A[T_0,T_1])) = K_q(A)[z]$$

$$\left(A[T_0,T_1](-k)\otimes_A ?\right)_* x \longleftarrow\!\shortmid \; x \cdot z^k \; .$$

Now we have the localization map

$$\mathcal{P}gr(A[T_0,T_1)) \longrightarrow \mathcal{P}(A[t])$$

$$P \longmapsto P/(T_0-1)P \qquad t = \text{image of } T_1.$$

and it is clear from (1) that the image of this consists of objects of the form $A[t]\otimes_A L$, $L \in \mathcal{P}(A)$. Call this full subcategory $\mathcal{P}' \subset \mathcal{P}(A[t])$.

Now given $V \in \mathcal{P}'$ we have the building consisting of those P such that

$$P/(T_0-1)P \xrightarrow{\sim} V.$$

Better: Form the category $\mathcal{C}$ whose objects are $P \in \mathcal{P}gr(A[T_0,T_1])$ and in which a morphism $P \to P'$ is an inclusion on whose kernel P'/P, T_0 is nilpotent and such that $P'/P \in \mathcal{P}gr(A[T_1])$. Then have

$$(3) \qquad \lambda : \mathcal{C} \longrightarrow \mathcal{I}so(\mathcal{P}')$$
$$P \longmapsto P/(T_0-1)\cancel{P} = \varinjlim \{ P_0 \xrightarrow{T_0} P_1 \xrightarrow{T_0} P_2 \to \cdot \}$$

which we want to show is a homotopy equivalence. So we consider λ/V. Observe that

$$P_0 \hookrightarrow P_1 \hookrightarrow P_2 \hookrightarrow \cdots \varinjlim P_n = P/(T_0-1)P$$

so that given $P/(T_0-1)P \xrightarrow{\sim} V$ in λ/V we can identify P_n with an A-submodule of V. In fact to give an $A[T_0,T_1]$-module M which is T_0-torsion-free and such that $\lambda M \simeq V$ is the same thing as giving a sequence of A-submodules $M_n \subset V$, $n \geq 0$ such that $M_n \subset M_{n+1}$ and $tM_n \subset M_{n+1}$ and $\bigcup M_n = V$. For M to be in $\mathcal{P}gr(A[T_0,T_1])$ probably means that

$$P_n/P_{n-1} + tP_{n-1} \in \mathcal{P}(A) \qquad (=0 \quad n>>0)$$

for each n, and that $T_1 : P_n/P_{n-1} \longrightarrow P_{n+1}/P_n$ is injective $\forall n$. Thus these conditions imply

$$P/(T_0,T_1)P \in \mathcal{P}(A)$$

$$\mathrm{Tor}_1^{A[T_0,T_1]}(A, P) = 0$$

so by the lemma in the paper, $P \in \mathcal{P}gr(A)$.

So it is clear that λ/V is equivalent to the ordered set of such families $\{P_n \subset V\}$, where $\{P_n\} \leq \{P_n'\} \iff P_n \subset P_n'$ for all n and

$$\bigoplus_n P_n'/P_n \in \mathcal{P}gr(A[T_1])$$

$$'' \quad \text{is killed by } T_0^N$$

I want to show this ordered set is directed (below). Thus given P, P' I want to find $P'' \leq P, P'$. Since P is fin. gen. $T_0^N P \subset P'$ for some N, and since $T_0^N P \leq P$ we can suppose $P \subset P'$. Now $\exists\, n \ni T_0^n P' \subset P$. Then have

$$0 \longrightarrow P/T_0^n P' \longrightarrow P'/T_0^n P' \longrightarrow P'/P \longrightarrow 0 .$$

Now because P, P' are projective over $A[T_1]$, we know P'/P has projective dim 1 over $A[T_1]$, hence $P/T_0^n P'$ is projective over $A[T_1]$. ~~But if m is large enough so that $T_0^m P \subset T_0^n P'$, then~~ But $P/T_0^n P'$ is a quotient of $P/T_0^n P$, hence $P/T_0^n P$ is finitely gen. over $A[T_1]$. Thus $T_0^n P' \leq P, P'$ as desired.

Let $\bar{\mathcal{C}}$ be the arrow cat. of $\mathcal{C}$, and let $\mathcal{H}$ denote the category of those gr $A[T_0, T_1]$-modules $M = M_0 \oplus \cdots$ such that (i) T_0 nilp. on M, (ii) $M \in P(A[T_1])$. Then we have a functor

$$(4) \qquad \bar{\mathcal{C}} \longrightarrow Q(\mathcal{H}) \qquad (P \hookrightarrow P') \longmapsto P'/P .$$

Claim this functor is fibred and the fibre over $M \in \mathcal{H}$ is the groupoid of $P \twoheadrightarrow M$ with $P \in \mathcal{P}gr(A[T_0, T_1])$. Clearly this is the fibre, so all we have to do is to check ~~to prove this~~ that if we have $P \twoheadrightarrow M$, then the Kernel is in $\mathcal{P}gr(A[T_0, T_1])$. But ~~hypothesis~~ we have the char. sequence

$$0 \longrightarrow A[T_0, T_1] \otimes_{A[T_1]} M \longrightarrow A[T_0, T_1] \otimes_{A[T_1]} M \longrightarrow M \longrightarrow 0$$

showing $M \in \mathcal{P}_1 gr(A[T_0, T_1])$, so this is clear.

May 11

(What I am trying to do is this: Identify $\mathcal{P}(A[t])$ modules with graded $A[T_0, T_1, T_0^{-1}]$-modules and construct the localization theorem in this graded setting)

We now have that $\mathcal{C} \xrightarrow{\ker} \mathrm{Iso}(\mathcal{P}')$ and that $\overline{\mathcal{C}}$ (= arrow category of $\mathcal{C}$) fibres over $Q(\mathcal{H})$. Thus we get a spectral sequence

$$E^2_{pq} = H_p(Q(\mathcal{H}), M \longmapsto H_q(\{P \twoheadrightarrow M\})) \Longrightarrow H_q(\mathrm{Is}\,\mathcal{P}')$$

which I want to localize.

So the idea is to let $I = \mathrm{Is}(\mathcal{P}gr(A[T_0, T_1]))$ act on the category $\overline{\mathcal{C}}$ and $\mathrm{Is}(\mathcal{P}')$ and we get a fibration

$$\mathrm{I}_I \longrightarrow \mathrm{Is}\,\mathcal{P}'_I \longrightarrow Q(\mathcal{H})$$

and hence a long exact homotopy sequence

$$\begin{tikzcd}
K_q \mathcal{H} \arrow[r, "\alpha"] & K_q(P_{gr} A[T_0, T_1]) \arrow[r] \arrow[d, phantom, "\parallel"] & K_q A[t] \arrow[r] & K_{q-1} \mathcal{H} \arrow[r] & \cdots \\
 & K_q A[z] & & &
\end{tikzcd}$$

Next we have to compute the map α. ?

$A = k$ field, $B = k[T_0, T_1]$. Take $V = k[t]^2$.
Now consider those $P = P_0 \oplus \cdots \oplus P_n \oplus \cdots\cdots$ free over B of rank 2 such that $V = P/(T_0 - 1)P$. Such a P is isomorphic to $B(-p) \oplus B(-q)$ where $0 \leq p \leq q$.

June 6, 1973: $p^nA=0 \Rightarrow K_nA[p^{-1}] \xrightarrow{\sim} K_n(A[t])[p^{-1}]$.

We can prove the fundamental thm. as follows

$$\begin{tikzcd}\mathbb{P}^1_A & Sp(A[t^{-1}]) \arrow[l, hook', "j"']\end{tikzcd}$$

$$\xrightarrow{\partial} K_q(Nil\,A) \longrightarrow K_q(\mathbb{P}^1_A) \xrightarrow{j^*} K_q(A[t^{-1}]) \xrightarrow{\partial} \cdots$$

$Nil(A) =$ $A[t]$-mods. $M \ni$ $M \in \mathcal{P}(A)$, $t^nM=0$ some n

Think of these as sheaves on $\mathbb{P}^1_A$ with support at $t=0$. which are finite loc. free over A.

I want now to consider the effect of a homom $A[t] \longrightarrow A[x]$ sending $t \longmapsto x^n$.

$$A[T_0,T_1] \longrightarrow A[X_0,X_1] \qquad T_i \longmapsto X_i^n$$

$$t = \frac{T_1}{T_0} \longmapsto \left(\frac{X_1}{X_0}\right)^n = x^n$$

certainly finite locally free map: $\mathbb{P}^1_A \xrightarrow{f} \mathbb{P}^1_A$

$x^n \longleftarrow\!\shortmid t$

So it ought to induce a map of localization sequences

$$\begin{tikzcd}
Nil(A) \arrow[d, "f_*"'] & \mathcal{P}(\mathbb{P}^1_A) \arrow[r, "j^*"] \arrow[d, "f_*"] & \mathcal{P}(A[x^{-1}]) \arrow[d, "f_*"] \\
Nil\,A & \mathcal{P}(\mathbb{P}^1_A) \arrow[r, "j^*"] & \mathcal{P}(A[t^{-1}])
\end{tikzcd}$$

f_* of M with x acting as $\nu: M \longrightarrow M$ is M with t acting as x^n.

Can also check this using the map

$$\begin{tikzcd}
\mathcal{N}il(A) \arrow[r] \arrow[d] & \mathcal{P}_1(A[x]) \arrow[r] \arrow[d, "f_*"] & \mathcal{P}_1(A[x,x^{-1}]) \arrow[d, "f_*"] \\
\mathcal{N}il(A) \arrow[r] & \mathcal{P}_1(A[t]) \arrow[r] & \mathcal{P}_1(A[t,t^{-1}])
\end{tikzcd}$$

So we seem to be able to do the following:

~~Let us form the inductive system~~

~~$n \mapsto K_0(A[t^{-n}])$~~

~~where~~

Suppose A is of characteristic p. Then I can consider the operation

$$\begin{tikzcd}
K_0(A[t]) \arrow[r, "f_*"] & K_0(A[t^p]) \arrow[r, "f^*"] & K_0(A[t]) \\
K_0 A \arrow[u, "i^*"] \arrow[r, "p"'] & K_0 A \arrow[u, "i^*"'] &
\end{tikzcd}$$

$f_* i^* = p \cdot i^*$ because

$$f_* i^*(E) = A[t] \otimes_A E = \coprod_{i=0}^{p-1} A[t^p] t^i \otimes_A E$$

Now

$$f^* f_* V = \left(A[t] \otimes_{A[t^p]} A[t] \right) \otimes_{A[t]} V$$

and because we are in characteristic p the aug. ideal in $A[t] \otimes_{A[t^p]} A[t]$ is nilpotent

$$(t \otimes 1 - 1 \otimes t)^p = 0$$

etc., so it should be the case that

$$f^* f_* = \text{mult. by } p.$$

and so iterating

$$(f^2)^* (f^2)_* = f^* f^* f_* f_* = p f^* f_* = p^2$$

etc. But now have

Lemma: If $Nil^n(A) = A[t]/t^n$ mods which are in $P(A)$, then

$$K_* Nil(A) = \varinjlim_n K_* Nil^n(A)$$

We have computed that

$$\begin{tikzcd} K_q(A[t]) \arrow[r, "f_*"] \arrow[d, "d"] & K_q(A[t^p]) \arrow[d, "d"] \\ Nil_{q-1}(A) \arrow[r, "g"] & Nil_{q-1}(A) \end{tikzcd}$$

commutes where g sends (P,ν) to (P,ν^p). Thus any element of $Nil_{q-1}(A)$ gets killed by $(f^N)_*$, so we get

Prop: If A of char. p, then $Nil_q(A)$ is p-torsion for $q \geq 0$, hence

$$K_{q+1}(A)[p^{-1}] \xrightarrow{\sim} K_{q+1}(A[t])[p^{-1}] \quad \text{for } q \geq 0.$$

But we also know that $K_0 A$ is canonically a direct summand of $K_1(A[x,x^{-1}])$. Thus from the isom

$$K_1(A[x,x^{-1}])[p^{-1}] \xrightarrow{\sim} K_1(\underset{\substack{\| \\ A[t][x,x^{-1}]}}{A[x,x^{-1}][t]})[p^{-1}]$$

one sees the above prop. holds for K_0 also.

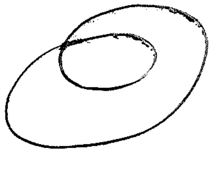
Question: Is ~~Conjecture:~~ $K_{-n}(A)[\frac{1}{p}] = 0$, $-n < 0$?

This amounts to showing that $K_{-1}(A)[\frac{1}{p}] = 0$

Question: If A is noetherian of char. p, is

$$K_n A[\tfrac{1}{p}] \xrightarrow{\sim} K_n'(A)[\tfrac{1}{p}] \ ?$$

$\boxed{\text{NO}}$, take A to be affine singular curve, whose ~~Picard group~~ Picard group has a $\mathbb{G}_m$ part.
Precisely take a ~~[illegible]~~ plane cubic X with ordinary



double point; its normalization $\bar{X}$ is $\mathbb{P}_1$ (project thru the double point). One has

$$K_0(\bar{X}) = \mathbb{Z}\oplus \mathrm{Pic}(\bar{X}) = \mathbb{Z}\oplus\mathbb{Z}$$

$$K_0(X) = \mathbb{Z}\oplus \mathrm{Pic}(X)$$

$$\begin{tikzcd} 0 \arrow[r] & k^* \arrow[r] & \mathrm{Pic}(X) \arrow[r] & \mathrm{Pic}(\bar{X}) \arrow[r] \arrow[d, phantom, "\parallel"] & 0 \\ & & & \mathbb{Z} & \end{tikzcd}$$

e.g. a line bundle on X is same as one on $\bar{X}$ with a way of identifying the fibres at P and Q. Thus

$$K_0X = \mathbb{Z}\oplus(\mathbb{Z}\oplus k^*)$$

$$\begin{tikzcd} K_1F \arrow[r] & \coprod_{x\in X}\mathbb{Z} \arrow[r] & K_0'X \arrow[r] & K_0'F \arrow[r] & 0 \\ K_1F \arrow[r] \arrow[u, "\wr"] & \coprod_{x\in\bar{X}}\mathbb{Z} \arrow[r] \arrow[u] & K_0'\bar{X} \arrow[r] \arrow[u] & K_0'F \arrow[r] \arrow[u, "\wr"] & 0 \end{tikzcd}$$

yields

$$\begin{tikzcd} & \mathbb{Z}\oplus\mathbb{Z} \arrow[d, phantom, "\parallel"] \arrow[r, phantom, "(rg, deg)"] & & \\ \mathbb{Z} \arrow[r] & K_0'\bar{X} \arrow[r] & K_0'X \arrow[r] & 0 \\ 1 \arrow[r, mapsto] & P-Q & & \end{tikzcd}$$

so

$$K_0'X = \mathbb{Z}\oplus\mathbb{Z}$$

and hence the map $K_0X \longrightarrow K_0'X$ has k^* for its kernel. This will persists upon removing a simple point from X, so we can make this example affine if we wish!

Direct proof that $\mathrm{Nil}_*(A)$ is p-torsion when $p \cdot A = 0$.

Consider $(M, \nu) \in \mathrm{Nil}^{p^r}(A)$ so that $\nu^{p^r} = 0$. Then we will show that

$$A[t] \otimes_{A[t^{p^r}]} M \qquad t^{p^r} \text{ acts as } \nu^{p^r}$$

has a canonical filtration with p^r quotients all isomorphic to M.

$$A[t] \otimes_{A[t^{p^r}]} A[t] = A \otimes_{\mathbb{F}_p} \left(\mathbb{F}_p[t] \otimes_{\mathbb{F}_p[t^{p^r}]} \mathbb{F}_p[t] \right)$$

$$I = (t \otimes 1 - 1 \otimes t) \text{ in } \mathbb{F}_p[t] \otimes_{\mathbb{F}_p[t^{p^r}]} \mathbb{F}_p[t] = \mathbb{F}_p[t', t''] / (t'^{p^r} - t''^{p^r})$$

Thus filtering by powers of I we get

$$\operatorname{gr} \mathbb{F}_p[t', t''] / \underset{\substack{\| \\ (t' - t'')^{p^r}}}{(t'^{p^r} - t''^{p^r})} = \mathbb{F}_p[t]\,[t' - t''] / (t' - t'')^{p^r}$$

Thus we filter: $I^j \left(A[t] \otimes_{A[t^{p^r}]} M \right)$ $\quad 0 \leq j \leq p^r$ and the assoc. graded $A[t]$-module is just M with t acting as ν.

Thus conclude that

$$\begin{tikzcd}
K_* \mathrm{Nil}^{p^r}(A) \arrow[r, "\text{inc.}"] \arrow[d, "\text{forg.}"'] \arrow[dr, "M \mapsto A[t] \otimes_{A[t^{p^r}]} M"] & K_* \mathrm{Nil}(A) \arrow[d, "p^r"] \\
K_* A \arrow[r, "p^r (\text{put } t = 0)"'] & K_* \mathrm{Nil}(A)
\end{tikzcd}$$

commutes, so done.

Question: If A is commutative, then $A^{\bullet}$ acts on $K_*(A[t])$ by $t \mapsto \lambda t$, $\lambda \in A^{\bullet}$. Is this action non-trivial?

In fact this action extends to the multiplicative monoid of A, but here it is non-trivial in general, for $\lambda = 0$ gives the projection of $K_*(A[t])$ onto $K_*(A)$ given by $t \mapsto 0$.

Origin of question: To prove $Nil_*(A)$ is torsion one might try restricting from $A[t]$ to $A[t^n]$ and then extending.

$$A[t] \otimes_{A[t^n]} A[t] = A[x,y]/(x^n - y^n) \qquad \begin{matrix} x = t\otimes 1 \\ y = 1 \otimes t \end{matrix}$$

If $T^n - 1$ splits into linear factors over A, then

$$x^n - y^n = \prod_{\zeta^n = 1} (x - \zeta y)$$

and so

$$gr \quad A[t] \otimes_{A[t^n]} A[t] = \prod_{\zeta} A[t\otimes 1, 1\otimes t] / t\otimes 1 - \zeta 1 \otimes t$$

and the effect on Nil is to replace (M, ν) by sum of $(M, \zeta\nu)$ as ζ runs over roots of $T^n - 1$.

Question: Is it possible that for any $\mathbb{Q}$-algebra Nil is zero?

NO: One has:

$$K_1(A[\varepsilon]) = K_1A \oplus A$$

where the A factors is picked up by

$$K_1(A[\varepsilon]) \xrightarrow{\det} A[\varepsilon]^\circ = A^\circ \oplus A.$$

Thus

$$Nil_0(A[\varepsilon]) = K_1(A[\varepsilon][t])/K_1(A[\varepsilon])$$

$$= Nil_0(A) \oplus A[t]/A$$

The point is that nilpotent elts ν in A contribute to units $1+t\nu$ in $A[t]$ which do not come from A°. This examples suggests:

<u>Question</u>: If l is a prime number invertible in A, is $Nil_*(A)$ uniquely l-divisible?

June 11, 1973

Let S be a mult. system in the center of A consisting of non-zero divisors. Let $P \in \mathcal{P}(S^{-1}A)$ and write

$$P \oplus Q = S^{-1}L$$

where $L \in \mathcal{P}(A)$. Denote $\pi' : S^{-1}L \to P$, $\pi'' : S^{-1}L \to Q$ the projections and put

$$L' = \pi'(L) \subset P, \quad L'' = \pi''(L) \subset Q$$

Then L', L'' are finite type over A, so $\exists\ s \in S$ such that

$$L \subset L' + L'' \subset s^{-1}L$$

Now suppose $A = B[t]$, $S = \{t^n\}$, where B is regular coherent. Then

$$t^{-n}L/L \in \mathcal{P}(A/t^nA)$$

is finitely presented over B, and $L'+L''/L$ being finitely generated over A and killed by t^n is fin. gen. over B. Since B is coherent, $L'+L''/L$ is finitely presented over B, and since B is regular we can find a $B[t]/t^n$ resolution

$$0 \to K \to (B[t]/t^n)^{r_m} \to \cdots\cdots \to (B[t]/t^n)^{r_0} \to L'+L''/L \to 0$$

where $K \in \mathrm{Nil}(B)$. As $\mathrm{Nil}(B) \subset \mathcal{P}_1(B[t])$, one has

$$L'+L''/L \in \mathcal{P}_\infty(B[t])$$

$$\Rightarrow \quad L'+L'' \in \mathcal{P}_\infty(A)$$

But the sum $L'+L''$ is direct, so we need

<u>Lemma:</u> $P_n(A)$ is Karoubian for each n.

<u>Proof:</u> ~~If $R \in Mod(A)$, then $R \in P(A) \Leftrightarrow M \mapsto Hom(R, M)$ exact and commutes with $\varinjlim$~~

True for $n=0$. Given $L'\oplus L'' \in P_n(A)$, then $L'\oplus L''$ fin gen. $\Rightarrow L'+L''$ fin. gen. $\Rightarrow \exists$

$$0 \longrightarrow \begin{matrix} R' \\ \oplus \\ R'' \end{matrix} \longrightarrow \begin{matrix} P' \\ \oplus \\ P'' \end{matrix} \longrightarrow \begin{matrix} L' \\ \oplus \\ L'' \end{matrix} \longrightarrow 0$$

with $P', P'' \in P(A)$. Then get $R'\oplus R'' \in P_{n-1}(A) \xRightarrow{\text{induction}}$ $R', R'' \in P_{n-1}(A) \Rightarrow L', L'' \in P_n(A)$. done

Thus we see that $L', L'' \in P_\infty(A)$ so we have proved

<u>Prop:</u> If B is regular coherent, then $K_0 B[t] \to K_0(B[t,t^{-1}])$ is onto, hence $K_0(B) \xrightarrow{\sim} K_0(B[t])$

Review the argument. Given A ring and $P \in \mathcal{P}(A[t,t^{-1}])$ write

$$P \oplus Q = S^{-1}L \qquad L = A[t]^m \qquad S = \{t^n\}$$

and let $\pi' : A[t^{-1}]^m \longrightarrow P$, $\pi'' : A[t,t^{-1}]^m \to Q$ be the two projections. Put

$$L' = \pi'(A[t]^m) \,, \quad L'' = \pi''(A[t]^m).$$

Then have

$$A[t]^m \subset L' \oplus L'' \subset t^{-n}\cdot A[t]^m$$

for some n since L', L'' are fin. gen. over $A[t]$. Now the point is that

$$J = t^{-n}A[t]^m / L' \oplus L''$$

is a finitely presented $A[t]$-module killed by t^n. Hence J is finitely presented over A, so if we assume A regular coherent (i.e. $\mathcal{P}_\infty(A) = Modfp(A)$) we have $J \in \mathcal{P}_\infty(A)$. This means that if we construct a resolution inductively

$$0 \to R_k \longrightarrow (A[t]/t^n)^{r_k} \longrightarrow \cdots\cdots\cdots \longrightarrow (A[t]/t^n)^{r_0} \longrightarrow J \to 0$$

we eventually find that R_k is in $\mathcal{P}(A)$. Then by char. seq.

$$0 \longrightarrow A[t] \otimes R_k \longrightarrow A[t] \otimes R_k \longrightarrow R_k \longrightarrow 0$$

we know $R_k \in \mathcal{P}_1(A[t])$, so $J \in \mathcal{P}_\infty(A[t]) \Longrightarrow L' \oplus L'' \in \mathcal{P}_\infty(A[t]) \Longrightarrow L' \in \mathcal{P}_\infty(A[t])$. Since $P = S^{-1}(L')$, it follows that $cl(P)$ in $K_0(A[t,t^{-1}])$ is in the image of $K_0(A[t])$.

This argument seems to show that if S is a multiplicative system of central non-zero-divisors in a ring B such that

$$\mathcal{H}_S(A) = \{M \in \mathrm{Modfp}(B) \mid S^{-1}M = 0\}$$

then any projective $S^{-1}B$-module P is of the form $S^{-1}L$ for some $L \in \mathcal{P}_\infty(B)$. In fact it seems that

$$\mathcal{P}_\infty(B) \longrightarrow \mathcal{P}_\infty(S^{-1}B)$$

is onto objects. For if this be true for anything in $\mathcal{P}_{n-1}(S^{-1}B)$, then given $X \in \mathcal{P}_n(S^{-1}B)$ choose $X = P/Y$

$$\begin{tikzcd}
0 \arrow[r] & Y \arrow[r] \arrow[d, equal] & P \arrow[r] \arrow[d, equal] & X \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & S^{-1}K \arrow[r] & S^{-1}L \arrow[r] & S^{-1}(L/K) \arrow[r] & 0
\end{tikzcd}$$

with $P \in \mathcal{P}(S^{-1}B)$, so that $Y \in \mathcal{P}_{n-1}(S^{-1}B)$. Thus $Y = S^{-1}K$ with $K \in \mathcal{P}_\infty(B)$, ~~so~~ $P = S^{-1}L$, $L \in \mathcal{P}_\infty(B)$ and we can assume $K \subset L$ as K is fin. gen. Thus $X = S^{-1}(L/K)$ where $L/K \in \mathcal{P}_\infty(B)$.

Probably also true that any K in $X \in \mathcal{P}_n(S^{-1}B)$ $\ni$ K fin. type and $S^{-1}K = X \Rightarrow K \in \mathcal{P}_\infty(B)$. Because we can sandwich $L \subset K \subset s^{-1}L$.

June 16, 1973

Let $\mathcal{S}$ be a cat. with product $+$ (ACU)

(H): $\mathcal{S}$ is a groupoid, $? + S$ faithful $\forall S$.

Suppose $\mathcal{S}$ acts on $\mathcal{X}$ $\ni$

i) every arrow in $\mathcal{X}$ is a mono.

ii) $?\#X : \mathcal{S} \to \mathcal{X}$ faithful $\forall X \in \mathcal{X}$.

Let

$$p : \langle \mathcal{S}, \overset{\mathcal{S}^{-1}\mathcal{X}}{\overset{\shortparallel}{\mathcal{S} \times \mathcal{X}}} \rangle \longrightarrow \langle \mathcal{S}, \mathcal{X} \rangle$$

be the functor induced by the projection $pr_2 : \mathcal{S} \times \mathcal{X} \longrightarrow \mathcal{X}$

Fix an object X_0 of $\langle \mathcal{S}, \mathcal{X} \rangle$ and consider the category $X_0 | p$. An object consists of an (S, X) in $\mathcal{S}^{-1}\mathcal{X}$ and an arrow $\xi : X_0 \to X$ in $\langle \mathcal{S}, \mathcal{X} \rangle$, say given by

$$T \# X_0 \xrightarrow{\alpha} X.$$

Thus an object is

$$\begin{tikzcd}
(S, X, T\#X_0 \xrightarrow{\alpha} X) \arrow[d] \\
(S', X', T'\#X_0 \xrightarrow{\alpha'} X')
\end{tikzcd}$$

and a morphism

is given by a $u \in \mathcal{S}$ &

$$u + S \simeq S', \quad u \# X \to X'$$

~~comp. with α, α'~~; carrying α to α'.

To be precise: an object of $X_0\backslash p$ consists of $((S,X), \xi : X_0 \to X)$ and a map

$$((S,X),\xi) \longrightarrow ((S',X'),\xi')$$

is a map $(S,X) \longrightarrow (S',X')$ carrying ξ to ξ'. Clearly this is cofibred over $\mathcal{S}^{-1}(pt)$ – composition $((S,X),\xi) \mapsto (S,X) \mapsto S$ of two cofibred functors. The fibre over S is the cof. cat over $\mathcal{X}$ assoc. to functor $X \mapsto \mathrm{Hom}_{\langle \mathcal{S},\mathcal{X}\rangle}(X_0, X)$. But we can identify this:

$$(X,\xi) \longmapsto (T_\xi,\ \alpha_\xi : T_\xi \# X_0 \longrightarrow X).$$

Better: Define

$$\mathcal{S}^{-1}\mathcal{S} \longrightarrow X_0\backslash p$$

$$(S,T) \longmapsto ((S, T\#X_0),\ X_0 \longrightarrow T\#X_0)$$

This should be a heq. In the case where $\mathcal{X}$ is a groupoid it is an equivalence because to give (S,X) and $\xi : X_0 \to X$ means up to unique iso we give the T_ξ such that ξ is

$$T_\xi \# X_0 \xrightarrow{\sim} X.$$

Idea: We can define a functor

(1) $\quad X_0\backslash p \longrightarrow \mathcal{S}^{-1}\mathcal{S}$

sending (S,X,ξ) to (S,T_ξ). Check functorality.

Given $(S, X, \xi) \longrightarrow (S', X', \xi')$ i.e. given

$T + S \xrightarrow{\sim} S'$

$T \# X \longrightarrow X'$

$$\begin{tikzcd}
{}^{\exists}\, T\#(T_\xi \# X_0) \arrow[r, "\alpha_\xi"] \arrow[d, equal] & T\#X \arrow[r] & X' \\
(T \perp T_\xi)\#X_0 \arrow[r, phantom, "\cong"] & T_{\xi'}\#X_0 \arrow[ur, "\alpha_{\xi'}"'] &
\end{tikzcd}$$

commutes

we certainly get a map $(S, T_\xi) \longrightarrow (S', T_{\xi'})$

On the other hand we have the functor

(2) $$\mathcal{S}^{-1}\mathcal{S} \longrightarrow X_0 \backslash p$$

$$(S, T). \longmapsto (S, T\#X_0, \ id : T\#X_0 \longrightarrow T\#X_0)$$

Clearly the functor (1)(2) is isomorphic to identity. But the composition (2)(1) is homotopic to id since we have

$$\begin{tikzcd}
(S, X, \xi) \arrow[r, mapsto] \arrow[drr, mapsto] & (S, T_\xi) \arrow[r, mapsto] & (S, T_\xi\#X_0, \ T_\xi\#X_0 \to T\#X_0) \arrow[d, "\alpha_\xi"] \\
 & & (S, X, \xi)
\end{tikzcd}$$

Finally if we have a map $X_1 \xrightarrow{\beta} X_0$ in $\langle \mathcal{S}, \mathcal{X} \rangle$ the following square:

$$\begin{tikzcd}
X_0 \backslash p \arrow[r, "(1)"] \arrow[d] & S^{-1}S \arrow[d, "\text{mult by } T_\beta"] \\
X_1 \backslash p \arrow[r, "(1)"] & S^{-1}S
\end{tikzcd}$$

commutes:

$$\begin{tikzcd}
\left(S, X, T_\xi \# X_0 \xrightarrow{\alpha_\xi} X\right) \arrow[r, mapsto] \arrow[d, mapsto] & (S, T_\xi) \\
\left(S, X, T_\xi \# (T_\beta \# X_1) \to T_\xi \# X_0 \xrightarrow{\alpha_\xi} X_1\right) \arrow[r, mapsto] & (S, T_\xi \# T_\beta)
\end{tikzcd}$$

and so we see that we can apply Thm. B to conclude $X_0 \backslash p$ is the h-fibre over X_0 of the functor p.

July 11, 1973

Terminology: Category with product = symmetric monoidal category in MacLane's terminology = $\otimes$-cat ACU in Saavedra's terminology.

Up to equivalence we can ~~change the~~ replace such a category by one in which the associativity and unity isos are identity maps. The commutativity iso

$$\tau_{S_1 S_2} : S_1 S_2 \simeq S_2 S_1$$

is then required to satisfy:

i) $$\tau_{S_1 S_2} = \tau_{S_2 S_1}^{-1}$$

ii)
$$\begin{tikzcd}
S_1 S_2 T \arrow[rr, "\sim"] \arrow[dr, "\wr"'] & & T S_1 S_2 \\
& S_1 T S_2 \arrow[ur, "\sim"'] &
\end{tikzcd}$$

iii)
$$\begin{tikzcd}
OS \arrow[rr, "\sim"] \arrow[dr, equal] & & SO \arrow[dl, equal] \\
& S &
\end{tikzcd}$$

Product-preserving functor = monoidal functor compatible with the commutativity isos.

Now suppose $f : \mathcal{S} \to \mathcal{X}$ is a product-preserving functor between categories with product. Then I make $\mathcal{S}$ act on $\mathcal{X}$ in the natural way:

$$\mathcal{S} \xrightarrow{f} \mathcal{X} \underset{\substack{\uparrow\text{-translation} \\ \text{left action}}}{\longrightarrow} \underline{\text{Hom}}(\mathcal{X}, \mathcal{X})$$

To show that $\langle \mathcal{S}, \mathcal{X} \rangle$ and $\mathcal{S}^{-1}\mathcal{X} = \langle \mathcal{S}, \mathcal{S} \times \mathcal{X} \rangle$ are categories with product. Enough to do former.

Recall that $\text{Ob}\,\langle \mathcal{S}, \mathcal{X} \rangle = \text{Ob}\,\mathcal{X}$ and that a map from X to X' in $\langle \mathcal{S}, \mathcal{X} \rangle$ is an eq. class of pairs $(S, SX \to X')$. Define a product functor

$$\langle \mathcal{S}, \mathcal{X} \rangle \times \langle \mathcal{S}, \mathcal{X} \rangle \longrightarrow \langle \mathcal{S}, \mathcal{X} \rangle$$

$$(X_1, X_2) \longmapsto X_1 X_2$$

$$\big(cl(S_1, S_1X_1 \to X_1'),\, cl(S_2, S_2X_2 \to X_2')\big) \longmapsto \text{class of}$$

$$\big(S_1S_2\,,\ (S_1S_2)(X_1X_2) \simeq S_1X_1\,S_2X_2 \longrightarrow X_1'X_2'\big)$$

Here to save writing we put SX for $f(S)\cdot X$ and the iso is

$$\begin{aligned} f(S_1S_2)(X_1X_2) &\simeq (fS_1 \cdot fS_2)(X_1X_2) && \text{product iso} \\ &= fS_1\big(fS_2\,(X_1X_2)\big) && \text{assoc} \\ &= fS_1\big((fS_2 \cdot X_1)X_2\big) && \text{assoc.} \end{aligned}$$

$$\simeq fS_1((X_1 \cdot fS_2)X_2) \qquad \text{comm.}$$

$$= fS_1(X_1(fS_2 \cdot X_2)) \qquad \text{assoc}$$

$$= (fS_1 \cdot X_1)(fS_2 \cdot X_2). \qquad \text{assoc.}$$

If $\mathcal{S}, \mathcal{X}, f$ are strict it is simply the map

$$f(S_1S_2)X_1X_2 = fS_1fS_2X_1X_2 \simeq fS_1 \cdot X_1 \cdot fS_2 \cdot X_2$$

induced by the commutativity isom of $\mathcal{X}$.

It is clear this procedure above does carry equivalence classes to equiv. classes. We must check it is compatible with composition. So suppose given

$$S_1X_1 \longrightarrow X_1' \qquad S_2X_2 \longrightarrow X_2'$$

$$S_1'X_1' \longrightarrow X_1'' \qquad S_2'X_2' \longrightarrow X_2''$$

Composing we get

$$S_1'S_1X_1 \longrightarrow S_1'X_1' \longrightarrow X_1'', \quad S_2'S_2X_2 \rightarrow S_2'X_2' \rightarrow X_2''$$

which goes to

$$S_1'S_1S_2'S_2X_1X_2 \simeq S_1'S_1X_1S_2'S_2X_2 \longrightarrow X_1''X_2''$$

On the other hand composing

$$S_1S_2X_1X_2 \simeq S_1X_1S_2X_2 \longrightarrow X_1'X_2'$$
$$S_1'S_2'X_1'X_2' \simeq S_1'X_1'S_2'X_2' \longrightarrow X_1''X_2''$$

gives

$$\begin{tikzcd}
S_1'S_2'S_1S_2X_1X_2 \arrow[r, "\simeq"] & S_1'S_2'S_1X_1S_2X_2 \arrow[r, dash] \arrow[d, phantom, "\cong"] & S_1'S_2'X_1'X_2' \arrow[d, phantom, "\cong"] \\
 & S_1'S_1X_1S_2'S_2X_2 \arrow[r] & S_1'X_1'S_2'X_2' \arrow[d] \\
 & & X_1''X_2''
\end{tikzcd}$$

So to see these two results are the same we must establish commutativity in the square)

$$\begin{tikzcd}
 & S_1'S_2'S_1S_2X_1X_2 \arrow[r, "S_2X_1", "\simeq"'] \arrow[dl, phantom, "\cong", "\tau_{S_2'S_1}"'] & S_1'S_2'S_1X_1S_2X_2 \arrow[dd, Rightarrow, no head] \arrow[d, phantom, "\cong"', "{S_2', S_1X_1}"] \\
S_1'S_1S_2'S_2X_1X_2 \arrow[rr, "\simeq", "{X_1, S_2'S_2}"'] \arrow[dr, phantom, "\cong"] & & S_1'S_1X_1S_2'S_2X_2 \arrow[dl, phantom, "\cong"] \\
 & S_1'S_1S_2'X_1S_2X_2 &
\end{tikzcd}$$

which is clear.

What matters is that the composition of

[illegible]

$$\begin{tikzcd}
(X_1, X_2) \arrow[dd] \arrow[dddd, bend right] \arrow[r, mapsto] & X_1 X_2 \\
{} \arrow[r, mapsto] & (S_1 S_2)(X_1, X_2) \simeq (S_1 X_1)(S_2 X_2) \\
(S_1 X_1, S_2 X_2) \arrow[dd] \arrow[r, mapsto] & (S_1 X_1)(S_2 X_2) \\
{} \arrow[r, mapsto] & (S_1' S_2')(S_1 X_1, S_2 X_2) \simeq S_1' S_1 X_1 S_2' S_2 X_2 \\
(S_1' S_1 X_1, S_2' S_2 X_2) \arrow[r, mapsto] & S_1' S_1 X_1 \, S_2' S_2 X_2 \\
{} \arrow[r, mapsto] & (S_1' S_1 \, S_2' S_2) X_1 X_2 \simeq S_1' S_1 X_1 S_2' S_2 X_2
\end{tikzcd}$$

Thus want we really have to show is that the outside square in

$$\begin{tikzcd}
S_1' S_2' S_1 S_2 X_1 X_2 \arrow[r, dash] \arrow[d, dash] & S_1' S_2' S_1 X_1 S_2 X_2 \arrow[d, dash] \arrow[ddr, dash] & {} \\
S_1' S_1 \, S_2' S_2 X_1 X_2 \arrow[r, dash] \arrow[drr, dash] & S_1' S_1 S_2' X_1 S_2 X_2 \arrow[dr, dash] & {} \\
{} & {} & S_1' S_1 X_1 S_2' S_2 X_2
\end{tikzcd}$$

commutes. Clear as the triangles commute by hyp.

June 30, 1973. monoidal categories.

Definition of monoidal category (terminology of Maclane; it is called ⊗-cat AU in Saavedra). A monoidal category ~~consists of the following data:~~ is a cat S equipped with the following extra structure

~~i) a category S~~

i) a functor $\perp : S \times S \longrightarrow S$, denoted $(S_1, S_2) \longmapsto S_1 \perp S_2$ or simply $(S_1, S_2) \longmapsto S_1 S_2$, and called the product functor of the monoidal category

ii) an isomorphism

$$a : (S_1 S_2) S_3 \simeq S_1 (S_2 S_3)$$

of functors from $S \times S \times S$ to S called the associativity isom of ~~the monoidal cat.~~ S

iii) an object O of S, called the unit of S and two isomorphisms

$$OS \simeq S \quad , \quad SO \simeq S$$

of functors from S to S, called the left and right unit iso's of S respectively, which coincide for $S = 0$.

These data are subject to the following conditions:

1) The associativity pentagon

$$\begin{tikzcd}
& ((S_1 S_2) S_3) S_4 \arrow[dl, "\sim"'] \arrow[dr, "\sim"] & \\
(S_1 S_2)(S_3 S_4) \arrow[dr, "\sim"'] & & (S_1 (S_2 S_3)) S_4 \arrow[d, "\wr"] \\
& S_1 (S_2 (S_3 S_4)) & S_1 ((S_2 S_3) S_4) \arrow[l, "\sim"']
\end{tikzcd}$$

should commute

~~Alternative language: A monoidal structure on a cat. $\mathcal{S}$ consists of the following data~~

Two possible $OO \simeq O$ coincide

2) The triangles

$$\begin{tikzcd}[column sep=tiny] (OS_1)S_2 \arrow[rr, "\simeq"] \arrow[dr, "\wr"'] & & O(S_1S_2) \arrow[dl, "\wr"] \\ & S_1S_2 & \end{tikzcd} \qquad \begin{tikzcd}[column sep=tiny] (S_1O)S_2 \arrow[rr, "\simeq"] \arrow[dr, "\wr"'] & & S_1(OS_2) \arrow[dl, "\wr"] \\ & S_1S_2 & \end{tikzcd} \qquad \begin{tikzcd}[column sep=tiny] (S_1S_2)O \arrow[rr, "\simeq"] \arrow[dr, "\wr"'] & & S_1(S_2O) \arrow[dl, "\wr"] \\ & S_1S_2 & \end{tikzcd}$$

should commute. (According to MacLane the middle ones + assoc. pentagon $\Longrightarrow$ other two).

~~Example: Let $\mathcal{S}$ be a monoidal object in Cat (= a 2-category with a unique object). The~~

~~A monoidal cat is a category equipped with a monoidal structure.~~

Example: Call a monoidal cat strict if the associativity and unit isos are identity maps. Clearly strict monoidal cats are the same thing as monoid-objects in Cat (which can also be identified with 2-categories with a unique object), for example Hom(X, X).

Definition of monoidal functor: Let $\mathcal{S}$, $\mathcal{T}$ be monoidal categories, ~~By a monoidal functor~~ ~~and let $F: \mathcal{S} \to \mathcal{T}$ be a functor.~~ By a monoidal functor $F: \mathcal{S} \longrightarrow \mathcal{T}$ we mean a functor $S \mapsto F(S)$ from $\mathcal{S}$ to $\mathcal{T}$ equipped with the following extra structure:

i) an isom $F(S_1S_2) \simeq F(S_1)F(S_2)$ of functors from $\mathcal{S} \times \mathcal{S}$ to $\mathcal{T}$ called the product isom of F.

ii) an isom $F(0) \simeq 0$ called the <u>unit</u> <u>isom.</u> of F.

These data are subject to the conditions:

1) (compatibility with associativity data):

$$\begin{tikzcd}
F((S_1S_2)S_3) \arrow[r, dash, "\sim"] \arrow[d, dash, "\wr"] & F(S_1S_2)F(S_3) \arrow[r, dash, "\sim"] & {[F(S_1)F(S_2)]F(S_3)} \arrow[d, dash, "\wr"] \\
F(S_1(S_2S_3)) \arrow[r, dash, "\sim"] & F(S_1)F(S_2S_3) \arrow[r, dash, "\sim"] & {F(S_1)[F(S_2)F(S_3)]}
\end{tikzcd}$$

2) (compatibility with unit data)

$$\begin{tikzcd}
F(0S) \arrow[r, dash, "\sim"] \arrow[d, dash, "\wr"] & F(0)F(S) \arrow[d, dash, "\wr"] \\
F(S) \arrow[r, dash, "\sim"] & 0F(S)
\end{tikzcd}
\qquad
\begin{tikzcd}
F(S0) \arrow[r, dash, "\sim"] \arrow[d, dash, "\wr"] & F(S)F(0) \arrow[d, dash, "\wr"] \\
F(S) \arrow[r, dash, "\sim"] & F(S)0
\end{tikzcd}$$

Definition of natural transf of monoidal functors:
Let $F, G : \mathcal{S} \longrightarrow \mathcal{T}$ be two monoidal functors. By a monoidal natural transf of monoidal functors $u : F \longrightarrow G$ we mean a natural transf of functors such that

$$\begin{tikzcd}
F(S_1S_2) \arrow[r] \arrow[d, dash, "\wr"] & G(S_1S_2) \arrow[d, dash, "\wr"] \\
F(S_1)F(S_2) \arrow[r] & G(S_1)G(S_2)
\end{tikzcd}
\qquad
\begin{tikzcd}
F(0) \arrow[rr] \arrow[dr, "\wr"] & & G(0) \arrow[dl, "\wr"] \\
& 0 &
\end{tikzcd}$$

Commutes.

Composition of monoidal functors. Given monoidal functors

$$F : \mathcal{S} \longrightarrow \mathcal{T} \qquad S \mapsto FS, \quad F(S_1S_2) \simeq F(S_1)F(S_2)$$

$$G : \mathcal{T} \longrightarrow \mathcal{U} \qquad T \longmapsto GT, \quad G(T_1T_2) \simeq G(T_1)G(T_2)$$

let GF be equipped with the product isom

$$(GF)(S_1S_2) = G(F(S_1S_2)) \simeq G(F(S_1)F(S_2)) \simeq GF(S_1)GF(S_2)$$

and the unit iso

$$(GF)(0) = G(F(0)) \simeq G(0) \simeq 0.$$

Claim then GF is a monoidal functor from $\mathcal{S}$ to $\mathcal{U}$.

$$\begin{tikzcd}
& G(F((S_1S_2)S_3)) \arrow[rr] & & G(F(S_1S_2)\cdot FS_3) \arrow[r, dash] \arrow[d, dash] & GF(S_1S_2)\cdot GFS_3 \arrow[d, dash] \\
G(F(S_1(S_2S_3))) \arrow[ur, equal] \arrow[d, dash] & \text{assoc-comp of } F & & G((FS_1\cdot FS_2)FS_3) \arrow[r, dash] & G(FS_1\cdot FS_2)\cdot GFS_3 \arrow[d, dash] \\
G(FS_1\cdot F(S_2S_3)) \arrow[rr, dash] \arrow[d, dash] & & G(FS_1\cdot(FS_2\cdot FS_3)) \arrow[ur, equal] \arrow[d, dash] & \text{assoc. comp of } G & (GFS_1\cdot GFS_2)GFS_3 \\
GFS_1\cdot GF(S_2S_3) & & GFS_1\, G(FS_2\cdot FS_3) \arrow[ll, dash] & & GFS_1(GFS_2\cdot GFS_3) \arrow[ll, dash] \arrow[u, equal]
\end{tikzcd}$$

The above comm diagram shows GF is compatible with associativity data

$$\begin{tikzcd}
GF(OS) \arrow[r] \arrow[dd] & G(FO.FS) \arrow[r] \arrow[d] & GFO.GFS \arrow[d] \\
\text{unit comp of } F & G(O.FS) \arrow[r] \arrow[dl] & GO.GFS \arrow[d] \\
GFS & \text{unit comp of } G & O.GFS \arrow[ll]
\end{tikzcd}$$

The above commutative diagram shows that GF is compatible with the unit data.

Put $\underline{Hom}^{\perp}(\mathcal{S},\mathcal{T})$ for the category of monoidal functors from $\mathcal{S}$ to $\mathcal{T}$ and monoidal natural transfs. Clearly the composition just defined gives rise to a functor

$$\begin{aligned}\underline{Hom}^{\perp}(\mathcal{S},\mathcal{T}) \times \underline{Hom}^{\perp}(\mathcal{T},\mathcal{U}) &\longrightarrow \underline{Hom}^{\perp}(\mathcal{S},\mathcal{U})\\ (F \quad , \quad G) &\longmapsto GF\end{aligned}$$

Evident transitivity requirements hold, so it is clear that we have established

Assertion: Monoidal categories, monoidal functors, and their monoidal natural transformations form a 2-category.

So we now ~~define an equiv~~ say that a monoidal functor $F: \mathcal{S} \to \mathcal{T}$ is an equivalence of monoidal categories if $\exists$ a monoidal functor $G: \mathcal{T} \to \mathcal{S}$ and monoidal isos.

$$GF \simeq id_{\mathcal{S}}, \quad FG \simeq id_{\mathcal{T}}.$$

Lemma: A monoidal functor $F: S \to T$ is an equivalence of monoidal categories iff it is an equivalence of categories.

Proof: If G ~~[illegible]~~ is a functor quasi-inverse to F, then ~~[illegible]~~

~~$S_1S_2 \simeq GF(S_1S_2) \simeq G(FS_1FS_2)$~~

~~$G(T_1T_2) \simeq G(FGT_1 FGT_2) \simeq GF(GT_1GT_2) \simeq GT_1GT_2$~~

to the isomorphism in T

$$F(GT_1 \cdot GT_2) \simeq FGT_1 \cdot FGT_2 \simeq T_1 T_2$$

corresponds an isomorphism

(1) $$GT_1 \cdot GT_2 \simeq G(T_1T_2)$$

in S. And similarly to

$$F(GO) \simeq O \simeq FO$$

corresponds (2): $GO \simeq O$. Better: (1) (2) are the unique ~~maps~~ such that

$$\begin{tikzcd} F(GT_1 GT_2) \arrow[r, "F((1))", "\simeq"'] \arrow[d, "\wr"', "\text{product iso of } F"] & FG(T_1T_2) \arrow[d, "\wr"] \\ FGT_1 \cdot FGT_2 \arrow[r, "\simeq"] & T_1 \cdot T_2 \end{tikzcd}
\qquad
\begin{tikzcd} F(GO) \arrow[rr, "F((2))", "\simeq"'] \arrow[dr, "\wr"'] & & FO \arrow[dl, "\wr", "\text{unit iso of } F"'] \\ & O & \end{tikzcd}$$

commute. ~~The proof~~ To show that (1) (2) constitutes a monoidal functor structure on G we must prove that certain diagrams constructed from these maps commute. But because F transports (1) (2) into the product iso of F and unit iso of F ~~and~~ the diagrams in question commute, since they already do so for F.

~~Better: We use the equivalence G to transport the~~

<u>Definition:</u> An (left) <u>action</u> of a monoidal category $\mathcal{S}$ on a category $\mathcal{X}$ consists of

i) a functor $\# : \mathcal{S} \times \mathcal{X} \longrightarrow \mathcal{X}$ denoted $(S,X) \mapsto S\#X$, or simply $(S,X) \longmapsto SX$.

ii) an isom

$$(S_1S_2)X \xrightarrow{\sim} S_1(S_2X)$$

of functors $\mathcal{S} \times \mathcal{S} \times \mathcal{X} \longrightarrow \mathcal{X}$, called the <u>assoc isom</u> of the action

iii) an isom

$$OX \simeq X$$

of functors $\mathcal{X} \longrightarrow \mathcal{X}$ called the <u>unit isom</u> of the action.

These data are supposed to satisfy

1) ~~[illegible]~~ .

$$\begin{tikzcd}
((S_1S_2)S_3)X \arrow[r, "\simeq"] \arrow[d, "\wr"] & (S_1S_2)(S_3X) \arrow[r, "\simeq"] & S_1(S_2(S_3X)) \\
(S_1(S_2S_3))X \arrow[r, "\simeq"] & S_1((S_2S_3)X) \arrow[ur, "\sim"'] &
\end{tikzcd}$$

Commutes

2)
$$\begin{tikzcd}
(SO)X \arrow[rr, dash, "\simeq"] \arrow[dr, dash, "\wr"'] & & S(OX) \arrow[dl, dash, "\wr"] \\
& SX &
\end{tikzcd}
\qquad
\begin{tikzcd}
(OS)X \arrow[rr, dash, "\simeq"] \arrow[dr, dash, "\wr"'] & & O(SX) \arrow[dl, dash, "\wr"] \\
& SX &
\end{tikzcd}$$

commute.

Example: $\exists$ canonical action of $\mathcal{S}$ on itself with ~~product~~ action functor $(S,X) \longmapsto SX$ the product functor of ~~[illegible]~~ $\mathcal{S}$, with assoc. iso. the assoc iso of $\mathcal{S}$, and with unit isom. the left unit isom $OX \simeq X$ of $\mathcal{S}$.

~~Alternative definition: Given an action of $\mathcal{S}$ on $\mathcal{X}$ and with a monoidal functor $\mathcal{S} \to \underline{\mathrm{Hom}}(\mathcal{X},\mathcal{X})$ as follows~~

Alternative definition: An action of $\mathcal{S}$ on $\mathcal{X}$ may be identified with a monoidal functor

$$\mathcal{S} \longrightarrow \underline{\mathrm{Hom}}(\mathcal{X},\mathcal{X}).$$

In effect the action functor $\mathcal{S} \times \mathcal{X} \longrightarrow \mathcal{X}$, $(S,X) \longmapsto SX$ may be identified with a functor $\mathcal{S} \to \underline{\mathrm{Hom}}(\mathcal{X},\mathcal{X})$, $S \mapsto (X \mapsto SX)$. The assoc. isom. for the ~~[illegible]~~ action ~~[illegible]~~ corresponds to product iso for the functor, and similarly for the unit iso. Conditions (1) and (2) of definition of monoidal functor and action coincide

Let $\mathcal{X}, \mathcal{Y}$ be cats with $\mathcal{S}$-action

<u>Definition of action-preserving functor</u>: An action-preserving functor $F: \mathcal{X} \to \mathcal{Y}$ is a functor equipped with an isomorphism

$$SF(X) \xrightarrow{\sim} F(SX)$$

action-preserving iso

of functors $\mathcal{S} \times \mathcal{X} \to \mathcal{Y}$ satisfying

1) (compat with assoc. $\overset{\text{iso}}{}$ of the action)

$$\begin{tikzcd}
(S_1S_2)F(X) \arrow[rr, dash, "\sim"] \arrow[d, equal] & & F((S_1S_2)X) \arrow[d, equal] \\
S_1(S_2F(X)) \arrow[r, dash, "\sim"] & S_1F(S_2X) \arrow[r, dash, "\sim"] & F(S_1(S_2X))
\end{tikzcd}$$

2) (compatible with unit iso. for the action)

$$\begin{tikzcd}
OF(X) \arrow[rr, dash, "\sim"] \arrow[dr, equal] & & F(OX) \arrow[dl, equal] \\
& F(X) &
\end{tikzcd}$$

Given action-preserving functors $F, G: \mathcal{X} \to \mathcal{Y}$ a morphism $F \to G$ of action-preserving functors is a natural transf compatible with the action-preserving isom:

$$\begin{tikzcd}
SF(X) \arrow[r, dash, "\sim"] \arrow[d] & F(SX) \arrow[d] \\
SG(X) \arrow[r, dash, "\sim"] & G(SX)
\end{tikzcd}$$

Category <u>$\mathrm{Hom}^{\mathcal{S}}(\mathcal{X}, \mathcal{Y})$</u> of action-preserving functors

Composition: Given action-preserving functors

$$\mathcal{X} \xrightarrow{F} \mathcal{Y} \xrightarrow{G} \mathcal{Z}$$

~~The~~ we equip GF with the isom

$*$ $\quad S(GF(X))$ ~~$SG(F(X))$~~ $\simeq G(SF(X)) \simeq S(GF(X))$.

Verify that in this way **GF** is an action-preserving functor:

$$\begin{tikzcd}
(S_1S_2)GFX \arrow[rr, dash, "\sim"] \arrow[d, equal] \arrow[dr, phantom, "\text{because } G \text{ is action-preserving}"] & & G(S_1S_2)FX \arrow[r, dash, "\sim"] \arrow[d, equal] & GF(S_1S_2)X \arrow[dd, equal] \\
S_1S_2GFX \arrow[r, phantom, "\simeq"] & S_1GS_2FX \arrow[r, dash, "\sim"] \arrow[d, "\wr"] \arrow[dr, phantom, "\text{triv.}"] & GS_1S_2FX \arrow[d, "\wr"] \arrow[r, phantom, "\text{because } F \text{ is action-preserv.}"] & \\
& S_1GFS_2X \arrow[r, dash, "\sim"] & GS_1FS_2X \arrow[r, dash, "\sim"] & GFS_1S_2X
\end{tikzcd}$$

The above diagram shows GF with $*$ is comp. with assoc iso.

$$\begin{tikzcd}
OGFX \arrow[r, dash, "\sim"] \arrow[dr, "\text{because } G \text{ is act.-pres.}"'] & GOFX \arrow[r, dash, "\sim"] \arrow[d, "\wr"] & GFOX \arrow[dl, "\text{because } F \text{ is action preserv.}"] \\
& GFX &
\end{tikzcd}$$

The above shows $(GF, *)$ is compat. with unit isos.

Clearly composition defines a functor

$$\underline{Hom}^{\mathcal{S}}(\mathcal{X},\mathcal{Y}) \times \underline{Hom}^{\mathcal{S}}(\mathcal{Y},\mathcal{Z}) \longrightarrow \underline{Hom}^{\mathcal{S}}(\mathcal{X},\mathcal{Z})$$

$$(F \quad , \quad G) \qquad \longmapsto \quad GF$$

which is associative with identities. Thus have verified

<u>Assertion</u>: Categories with $\mathcal{S}$-action, action-preserving functors, and morphisms of action-preserving functors form a 2-category.

In particular $\underline{Hom}^{\mathcal{S}}(\mathcal{X},\mathcal{X})$ is a strict monoidal category.

It should be clear what a right action of a monoidal cat. $\mathcal{S}$ on a cat. $\mathcal{X}$ is. It is a left action of $\mathcal{S}^*$ on $\mathcal{X}$, where $\mathcal{S}^*$ denotes the category $\mathcal{S}$ equipped with the opposite monoidal structure, i.e. in which the functor $\perp : \mathcal{S}^* \times \mathcal{S}^* \longrightarrow \mathcal{S}^*$ is $(S,T) \longmapsto TS$.

<u>Example</u>: $\mathcal{S}^*$ acts on $\mathcal{S}$ with product

$$\mathcal{S}^* \times \mathcal{S} \longrightarrow \mathcal{S} \;, \quad (S,X) \longmapsto XS$$

and assoc. + unity isos. obtained from those of $\mathcal{S}$ in the evident way.

~~[illegible]~~

We propose now to define a monoidal functor

(*) $$\lambda : \mathcal{S} \longrightarrow \underline{\mathrm{Hom}}^{\mathcal{S}^*}(\mathcal{S},\mathcal{S})$$

where $\mathcal{S}^*$ acts on $\mathcal{S}$ in the way just described. Given an object S one ~~associates~~ equips the functor $X \mapsto SX$ from $\mathcal{S}$ to itself together with the isomorphism

$$(SX)T \simeq S(XT)$$

of functors from $\mathcal{S}\times\mathcal{S}^*$ to $\mathcal{S}$ given by the assoc. isom of $\mathcal{S}$. In this way we obtain an $\mathcal{S}^*$-action-preserving functor from $\mathcal{S}$ to $\mathcal{S}$:

$$\lambda S = \Big(X \longmapsto SX \,,\ (SX)T \simeq S(XT) \Big).$$

(The two ~~[illegible]~~ axioms for ~~[illegible]~~ action-preserving result from the axioms for a monoidal cat). In this way we obtain a functor (*).

To make (*) a monoidal functor we equip (*) with the isom

(**) $$\lambda(S_1 S_2) \xrightarrow{\sim} \lambda S_1 \cdot \lambda S_2$$

given by the assoc. isom.

$$(S_1 S_2)X \xrightarrow{\sim} S_1(S_2 X)$$

It is necessary to check this is an isom of $\mathcal{S}^*$-action-preserving functors. Thus we must show commutativity of

$$\begin{tikzcd}
((S_1S_2)X)T \arrow[rr, phantom, "\simeq"] \arrow[d, "\wr"] & & (S_1S_2)(XT) \arrow[d, "\wr"] \\
(S_1(S_2X))T \arrow[r, phantom, "\simeq"] & S_1((S_2X)T) \arrow[r, phantom, "\simeq"] & S_1(S_2(XT))
\end{tikzcd}$$

(bottom row is the action iso for $\lambda S_1 \cdot \lambda S_2$), which results from the pentagon axiom.

We also must equip λ with

(**)' $$\lambda 0 \simeq id$$

given by $$0X \simeq X.$$

To see this is an isomorphism of $\mathcal{S}^*$-action-preserving functors, we need to check commutativity of

$$\begin{tikzcd}
(0X)T \arrow[r, phantom, "\simeq"] \arrow[d, "\wr"] & 0(XT) \arrow[d, "\wr"] \\
XT \arrow[r, phantom, "="] & XT
\end{tikzcd}$$

which is clear.

Now the verification that (**) and (**)' make λ into a monoidal functor requires checking the comm. of certain diagrams in $\underline{Hom}^{\mathcal{S}^*}(\mathcal{S},\mathcal{S})$ and this can be verified objectwise. So it's all clear now that λ is a monoidal functor.

Proposition: $\lambda : \mathcal{S} \longrightarrow \underline{Hom}^{\mathcal{S}^*}(\mathcal{S},\mathcal{S})$ is an equivalence of monoidal categories. In particular any monoidal category is equivalent (as monoidal cat.) to a strict monoidal category.

<u>Proof</u>. By the lemma it suffices to show λ is an equivalence of categories. Define a functor

$$\mu : \underline{\mathrm{Hom}}^{\mathcal{S}^*}(\mathcal{S}, \mathcal{S}) \longrightarrow \mathcal{S}$$

$$\mu\,(X \mapsto FX,\ (FX)T \simeq F(XT)) = FO.$$

Then

$$\mu\lambda(S) = \mu\,(X \mapsto SX,\ (SX)T \simeq S(XT))$$

$$= SO$$

is isomorphic to the id functor of $\mathcal{S}$. Also

$$\lambda\mu(F) = (X \mapsto (FO)X,\ (FO\cdot X)T \simeq \overset{FO\cdot(XT)}{\cancel{}})$$

Because F is $\mathcal{S}^*$ action-preserving we have a canon isom

$$(FO)X \simeq F(OX) \simeq FX$$

of functors from $\mathcal{S}$ to $\mathcal{S}$. To show this is a morphism of action-preserving functors we have to verify:

$$\begin{tikzcd}
((FO)X)T \arrow[rr, phantom, "\simeq"] \arrow[d, phantom, "\wr"] & \arrow[d, phantom, "\checkmark"] & (FO)(XT) \arrow[d, phantom, "\wr"] \\
F(OX)T \arrow[r, phantom, "\simeq"] \arrow[d, phantom, "\wr"] & F((OX)T) \arrow[r, phantom, "\simeq"] \arrow[dr, dash, "\wr"'] & F(O(XT)) \arrow[d, phantom, "\wr"] \\
(FX)T \arrow[rr, phantom, "\simeq"] \arrow[ur, phantom, "\checkmark"] & & F(XT) \arrow[u, phantom, "\checkmark"', shift right=4]
\end{tikzcd}$$

So done.

What must be true therefore is that

$$\begin{tikzcd}(OS)T \arrow[rr, phantom, "\simeq"] \arrow[dr, "l_S\cdot T"', "\wr"] & & O(ST) \arrow[dl, "l_{ST}", "\wr"'] \\ & ST & \end{tikzcd}
\qquad
\begin{tikzcd}(SO)T \arrow[rr, phantom, "\simeq"] \arrow[dr, "r_S\cdot T"', "\wr"] & & S(OT) \arrow[dl, "S\cdot l_T", "\wr"'] \\ & ST & \end{tikzcd}
\qquad
\begin{tikzcd}(ST)O \arrow[rr, phantom, "\simeq"] \arrow[dr, "r_{ST}"', "\wr"] & & S(TO) \arrow[dl, "S\cdot r_T", "\wr"'] \\ & ST & \end{tikzcd}$$

must imply $r_O = l_O : OO \longrightarrow O$. Here is a direct proof:

Start with

$$\begin{tikzcd}(OO)O \arrow[rrd, bend left, "l_O O"] \arrow[rd, "\wr"] \arrow[rdd, "r_O O"'] & & \\ & O(OO) \arrow[r, "l_{OO}", "\sim"'] \arrow[d, "Ol_O"'] & OO \arrow[d, "l_O"] \\ & OO \arrow[r, "l_O"'] & O\end{tikzcd}$$

Since l_O is an isom $\Longrightarrow r_O O = l_O O$. But because of

$$\begin{tikzcd}XO \arrow[r, "r_X", "\sim"'] \arrow[d, "fO"'] & X \arrow[d, "f"] \\ YO \arrow[r, "r_Y", "\sim"'] & Y\end{tikzcd}$$

hence $fO = gO \iff f = g$

we have $r_O O = l_O O \Longrightarrow r_O = l_O$.

~~Corollary (MacLane's) [illegible]~~

Above shows that any monoidal category is equivalent to a strict one. Given a word w in the free $\overset{\text{non-}}{\wedge}$associative ~~[illegible]~~ set with one generator e.g.

$$((x(xx))x)x$$

it determines a functor $\mathcal{S}^n \longrightarrow \mathcal{S}$, where $n =$ length of w. Using the associativity isom. one gets ~~[illegible]~~ the notion of an elementary transition between words, and composing these one gets various isomorphisms between the functors $\mathcal{S}^n \longrightarrow \mathcal{S}$ associated to the different words. The coherence theorem states that any two such isos between the functors assoc. to two given words coincide. To prove this one reduces immediately to ~~[illegible]~~ the case where $\mathcal{S}$ is strict, whence all words of length n lead to same functor $\mathcal{S}^n \longrightarrow \mathcal{S}$, and all $\overset{\text{transition}}{\wedge}$isos. between these are the identity.

Let $\mathcal{S}$ be a monoidal category, let

$$\bar{\mathcal{S}} = \underline{\mathrm{Hom}}^{\mathcal{S}^*}(\mathcal{S}, \mathcal{S})$$

so that we have the equivalence $\mathcal{S} \longrightarrow \bar{\mathcal{S}}$ as object. We define a new category $\tilde{\mathcal{S}}$ as follows. $Ob(\tilde{\mathcal{S}}) =$ free monoid generated by $Ob(\mathcal{S})$. Since $\bar{\mathcal{S}}$ is strict, the canon map $\mathcal{S} \longrightarrow \bar{\mathcal{S}}$ induces a map of monoids $Ob(\tilde{\mathcal{S}}) \longrightarrow Ob(\bar{\mathcal{S}})$.

In other words an element of $Ob(\tilde{\mathcal{S}})$ is a word $S_1 \cdots S_n$ of objects of $\mathcal{S}$ $(n \geq 0)$ and this word gets sent to the functor

$$X \longmapsto S_1(S_2 \cdots (S_n X) \cdots)$$

with its evident $\mathcal{S}^*$-action-preserving structure. Now we define morphisms in $\tilde{\mathcal{S}}$ so that

$$\tilde{\mathcal{S}} \longrightarrow \bar{\mathcal{S}}$$

$$S_1 \cdots S_n \longmapsto (X \mapsto S_1(S_2 \cdots (S_n X) \cdots))$$

is an equivalence of categories. Define product $\tilde{\mathcal{S}} \times \tilde{\mathcal{S}} \longrightarrow \tilde{\mathcal{S}}$ functor by sending

$$f : S_1 \cdots S_n \longrightarrow S_1' \cdots S_m' \quad \text{i.e.} \quad S_1 S_2 \cdots S_n X \longrightarrow S_1' \cdots S_m' X$$

$$g : T_1 \cdots T_p \longrightarrow T_1' \cdots T_q' \qquad T_1 \cdots T_p X \longrightarrow T_1' \cdots T_q' X$$

into
$$f \cdot g : S_1 \cdots S_n T_1 \cdots T_p \longrightarrow S_1' \cdots S_m' T_1' \cdots T_q'$$

the composition of the natural transformations. (Lemma here is that if M is a monoid, $\bar{\mathcal{S}}$ is a strict monoidal category, and if $M \longrightarrow Ob(\bar{\mathcal{S}})$ is given, then the category $\mathcal{M}$ having M for objects and over $\bar{\mathcal{S}}$ ∋ $\mathcal{M} \to \bar{\mathcal{S}}$ is fully faithful is a monoidal cat. In fact we have

$$\mathcal{M} = D(M) \times_{D(Ob\,\bar{\mathcal{S}})} \bar{\mathcal{S}}$$

where $D(M)$ is category with objects $= M$, arrows $= M \times M$, etc)

Thus we see $\tilde{\mathcal{S}}$ is as strict monoidal cat. and the functor $\mathcal{S} \to \tilde{\mathcal{S}}$ is an equivalence of monoidal categories. This is because

$$\begin{tikzcd}
\mathcal{S} \arrow[rr, "\text{equiv}"] \arrow[dr] & & \bar{\mathcal{S}} \\
& \tilde{\mathcal{S}} \arrow[ur, "\text{equiv.}"'] &
\end{tikzcd}$$

where equiv means an equivalence of monoidal categories.

Now let $f: \mathcal{S} \to \mathcal{T}$ be a monoidal functor. If $\mathcal{T}$ is strict, then we get a monoid map

$$f: Ob(\tilde{\mathcal{S}}) \longrightarrow Ob(\mathcal{T})$$

extending the given map $f: Ob(\mathcal{S}) \longrightarrow Ob(\mathcal{T})$. Now if $S_1 \cdots S_n$ is an object of $\tilde{\mathcal{S}}$, we know it is canon isomorphic to $S_1 + (S_2 + \cdots + S_n)$ of $\mathcal{S}$, hence $\exists$ canon isom

$$f(S_1 + (\cdots + S_n)) \simeq f(S_1) \cdots f(S_n) = f(S_1 \cdots S_n) \; ;$$

as f is monoidal. Thus since $\mathcal{S} \to \tilde{\mathcal{S}}$ is an equivalence, we can extend f to a functor $f: \tilde{\mathcal{S}} \to \mathcal{T}$ such that

$$\begin{tikzcd}
f(S_1 \cdots S_n) \arrow[rr, equal] \arrow[dr, dash, "\wr"'] & & f(S_1) \cdots f(S_n) \arrow[dl, dash, "\wr"] \\
& f(S_1 + (S_2 + \cdots S_n) \cdot\cdot)) &
\end{tikzcd}$$

Thus it is fairly clear that $f: \tilde{\mathcal{S}} \to \mathcal{T}$ has to be a strict monoidal functors. In fact, it seems we have adjoint ~~In fact, it seems that the map~~ ~~functor $\mathcal{S} \to \tilde{\mathcal{T}}$ is [illegible] universal map to a strict~~ ~~[illegible]~~ functors:

$$\begin{tikzcd}
\left(\begin{matrix}\text{monoidal cats} \\ \text{+ functors}\end{matrix}\right) \arrow[r, shift left, "\mathcal{S} \mapsto \tilde{\mathcal{S}}"] & \left(\begin{matrix}\text{strict monoidal cats.} \\ \text{+ strict — functors}\end{matrix}\right) \arrow[l, shift left, "\text{forget}"]
\end{tikzcd}$$

Improvement: Given $f: \mathcal{S} \to \mathcal{T}$ monoidal with $\mathcal{T}$ strict I wish to show $\exists$ unique $g: \tilde{\mathcal{S}} \to \mathcal{T}$ strict monoidal such that $g|\mathcal{S} = f$

Uniqueness of g: Because $\mathcal{S} \to \tilde{\mathcal{S}}$ is an equivalence, given g_1, g_2 there is a unique isom. $\theta: g_1 \xrightarrow{\sim} g_2$ inducing the identity on $\mathcal{S}$. To show θ is the identity. Given $S_1 \cdots S_n \in \tilde{\mathcal{S}}$

$$g_i(S_1 \cdots\cdots S_n) = g_i(S_1) \cdots\cdots g_i(S_n) \qquad i = 1, 2$$

$$= fS_1 \cdots fS_n$$

But in $\tilde{\mathcal{S}}$ there is an isom canonical

$$S_1 \cdots S_n \simeq S_1 + (S_2 + \cdots + S_n) \cdots)$$

product iso for $\mathcal{S} \to \tilde{\mathcal{S}}$

with + = product in $\mathcal{S}$. Thus

~~$g_1(S_1 \cdots S_n) \simeq f(S_1 + (\cdots + S_n))$~~

~~$\| \qquad fS_1 \cdots fS_n \quad \wr| \leftarrow$ product-pres. isom. for f~~

~~? ? ?~~

$$\begin{tikzcd}
g_1(S_1 \cdots S_n) \arrow[r, "\sim", "g_1(\text{product isom for } \mathcal{S} \to \tilde{\mathcal{S}})"'] \arrow[d, equal, "\text{product iso for } g"'] \arrow[rr, phantom, "", near start] & g_1(S_1 + (\cdots + S_n)) = f(S_1 + (\cdots S_n) \cdots) \arrow[d, phantom, "4"] \arrow[d, "\wr", "\text{product isom for } f"'] \\
g_1 S_1 \cdots g_1 S_n & f S_1\, f S_2 \cdots f S_n
\end{tikzcd}$$

To avoid confusion introduce $c : \mathcal{S} \to \tilde{\mathcal{S}}$. Then

$$c S_1 \cdots c S_n \simeq c(S_1 + (\cdots S_n)) \qquad \text{product iso for } c : \mathcal{S} \to \tilde{\mathcal{S}}$$

$$\begin{tikzcd}
g_1(c S_1 \cdots c S_n) \arrow[r, "\simeq"] \arrow[d, equal, "\text{product iso for } g"'] & \underset{f}{g_1 c}(S_1 + (\cdots S_n)) \arrow[d, "\wr", "\text{product iso for } f"'] \\
g_1 c S_1 \cdots g_1 c S_n \arrow[r, equal] & f S_1 \cdots f S_n
\end{tikzcd}$$

commutes because we are assuming $f = g_1 c$ as monoidal functors. Now if $\theta : g_1 \xrightarrow{\sim} g_2$ is such that $\theta \cdot c : g_1 c \xrightarrow{\sim} g_2 c$ is identity of f, it follows from the fact that f

$$\overset{\sim}{\overline{}} \quad f S_1 \cdots f S_n$$

$$\begin{tikzcd}
g_1(c S_1 \cdots c S_n) \arrow[r, "\simeq"] \arrow[d, "\theta"'] & g_1 c(S_1 + \cdots) = f(S_1 + \cdots) \arrow[d, "\theta"', shift right=8] \arrow[d, "id", shift left=8] \arrow[d, phantom, "\$"] \\
g_2(c S_1 \cdots c S_n) \arrow[r, "\simeq"] & g_2 c(S_1 + \cdots) = f(S_1 + \cdots)
\end{tikzcd}$$

Thus θ is the identity as claimed.

Existence of g. Define g on objects:

$$g(c(S_1)\cdots c(S_n)) = fS_1\cdots fS_n$$

Better we use the [illegible] isom.

$$c(S_1)\cdots c(S_n) \simeq c(S_1+\cdots S_n)$$

?

Commutativity:

direct product:

If $\mathcal{S}$, $\mathcal{T}$ are monoidal categories, then so is $\mathcal{S}\times\mathcal{T}$ with product functor

$$(S,T)(S',T') = (SS',TT')$$

with associativity and unity datas = the direct direct of those of $\mathcal{S}$ and $\mathcal{T}$. Then have strict monoidal functors

$$\mathcal{S} \xleftarrow{pr_1} \mathcal{S}\times\mathcal{T} \xrightarrow{pr_2} \mathcal{T}$$

such that

$$\underline{\mathrm{Hom}}^+(\mathcal{U},\mathcal{S}\times\mathcal{T}) \longrightarrow \underline{\mathrm{Hom}}^+(\mathcal{U},\mathcal{S})\times\underline{\mathrm{Hom}}^+(\mathcal{U},\mathcal{T})$$

is an isomorphism of categories.

point: The punctual category pt has a unique monoidal structure. Claim

$$\underline{\mathrm{Hom}}^+(pt,\mathcal{S})$$

is the category of pairs $(S, 0 \overset{\sim}{\to} S)$. It is thus a contractible groupoid. In effect a. monoidal functor $F: pt \longrightarrow \mathcal{S}$ consists of an object $F(\cdot)$ with product + unity isos

$$F(\cdot\;\cdot) \simeq F(\cdot)F(\cdot)$$

$$F(\cdot) \simeq 0$$

satisfying conditions of assoc. + unity. The unity condition says:

$$\begin{tikzcd} F(\cdot\,\cdot) \arrow[r, phantom, "\simeq"] \arrow[d, equal] & F(\cdot)F(\cdot) \arrow[d, phantom, "\wr"] \\ F(\cdot) \arrow[r, phantom, "\simeq"] & 0\,F(\cdot) \end{tikzcd}$$

commutes, [illegible] which implies $F(\cdot) \simeq 0$ determines the product isom. On the other hand, it is clear that given $S \simeq 0$ we [illegible] can make [illegible] S a new 0-object of $\mathcal{S}$, hence we get a ~~product functor~~ monoidal functor $F: pt \to \mathcal{S}$ with [illegible] $F(\cdot) = S$.

There is a distinguished monoidal functor

$$pt \xrightarrow{F} \mathcal{S}$$

sending $\cdot$ to 0 with $F(\cdot) \xrightarrow{\sim} 0$ = the identity arrow of 0. We will denote this monoidal functor $0_{\mathcal{S}} : pt \to \mathcal{S}$ or simply $0 : pt \to \mathcal{S}$.

Problem: To describe $\underline{Hom}^{+}(\mathcal{S}\times\mathcal{T}, \mathcal{U})$:

Definition: Let $F: \mathcal{S} \to \mathcal{U}$, $G: \mathcal{T} \to \mathcal{U}$ be monoidal functors. A *commutation-isomorphism* between F, G is an isom

$$F(S)G(T) \simeq G(T)F(S)$$

of functors $\mathcal{S}\times\mathcal{T} \to \mathcal{U}$ such that the following conditions hold:

compatible with product iso for F

$$\begin{tikzcd}
F(S_1 \cdot S_2)G(T) \arrow[r, "\simeq"] \arrow[d, "\wr"] & G(T)F(S_1 S_2) \arrow[d, "\wr"] \\
(FS_1\, FS_2)G(T) \arrow[d, equal] & G(T)(FS_1 \cdot FS_2) \arrow[d, equal] \\
FS_1(FS_2\, GT) \arrow[d, "\wr"] & (GT \cdot FS_1)FS_2 \arrow[d, "\wr"] \\
FS_1(GT \cdot FS_2) \arrow[r, equal] & (FS_1 \cdot GT)FS_2
\end{tikzcd}$$

compatible with unit iso for F:

$$\begin{tikzcd}
F(0)\,G(T) \arrow[rr, "\simeq"] \arrow[d, "\wr"] & & G(T)\,F(0) \arrow[d, "\wr"] \\
0\,G(T) \arrow[dr, equal] & & G(T)\,0 \arrow[dl, equal] \\
& G(T) &
\end{tikzcd}$$

Similarly the commutation isom. must be compatible with the product and unit isom. of G.

It is a well-known and useful fact (ascribed to Isbell in [May]) that every monoidal category $\mathcal{S}$ is equivalent to a strict monoidal category whose objects are elements of the free monoid generated by $Ob(\mathcal{S})$. A complete statement is as follows.

Proposition 2: (i) Given a monoidal category $\mathcal{S}$ there exists a strict monoidal category and an equivalence of monoidal categories $\xi : \mathcal{S} \to \tilde{\mathcal{S}}$ such that $Ob(\xi) : Ob(\mathcal{S}) \to Ob(\tilde{\mathcal{S}})$ induces an isomorphism of the monoid $Ob(\tilde{\mathcal{S}})$ with the free monoid generated by $Ob(\mathcal{S})$. These properties determine the pair $(\tilde{\mathcal{S}}, \xi)$ up to isom.

(ii) Any monoidal functor $F : \mathcal{S} \to \mathcal{T}$, where $\mathcal{T}$ is a strict monoidal category, factors uniquely

$$\mathcal{S} \xrightarrow{\xi} \tilde{\mathcal{S}} \xrightarrow{F'} \mathcal{T}$$

where F' is a strict monoidal functor. Thus the [illegible] inclusion functor

$$\left\{\begin{matrix}\text{strict monoidal categories}\\ \text{strict monoidal functors}\end{matrix}\right\} \subset \left\{\begin{matrix}\text{monoidal cats}\\ \text{monoidal functors}\end{matrix}\right\}$$

has the left adjoint: $\mathcal{S} \mapsto \tilde{\mathcal{S}}$.

One proof of this result is based on MacLane's coherence [illegible] theorem (compare [illegible] [May]). We will sketch another proof in § 3.

~~Let X be a set on which a monoid~~

Let S be a monoid, and let X be a set on which S acts. One can form a category whose objects are the elements of X and in which a morphism from x_1 to x_2 'is' an element s of S such that $sx_1 = x_2$. In the following chapter ~~we propose~~ ~~to generalize this construction to the situation~~ we present an analogue of this construction where S is replaced by a monoidal category $\mathcal{S}$, ~~that is a~~ [illegible] ~~associative unit~~ ~~and X becomes a category X on which S acts.~~ ~~a monoidal category~~ that is, a category with a product functor which is associative and unitary up to coherent isomorphisms (see [MacLane]), and where X becomes a category on which $\mathcal{S}$ acts.

monoidal categories and functors

Definition: A *monoidal category* is a category $\mathcal{S}$ equipped with

i) a functor $\mathcal{S}\times\mathcal{S} \longrightarrow \mathcal{S}$ which will be denoted $(S_1, S_2) \longmapsto S_1 S_2$ and called the *product* of $\mathcal{S}$

ii) an object of $\mathcal{S}$, which will be denoted $0_{\mathcal{S}}$, or simply 0, and called the *unit* of $\mathcal{S}$

iii) an isomorphism

$$(S_1 S_2) S_3 \simeq S_1 (S_2 S_3)$$

of functors from $\mathcal{S}\times\mathcal{S}\times\mathcal{S}$ to $\mathcal{S}$, called the *associativity isom.* of $\mathcal{S}$

iv) isomorphisms

$$0S \simeq S \quad , \quad S0 \simeq S$$

of functors from $\mathcal{S}$ to $\mathcal{S}$, called the left and right *unity isomorphisms*, respectively.

The associativity and unity isomorphisms are required to satisfy the conditions that the diagrams

(1)
$$\begin{tikzcd}
S_1(S_2(S_3 S_4)) \arrow[r, dash, "\simeq"] \arrow[d, dash, "\wr"] & S_1((S_2 S_3) S_4) \arrow[r, dash, "\simeq"] & (S_1(S_2 S_3)) S_4 \arrow[d, dash, "\wr"] \\
(S_1 S_2)(S_3 S_4) \arrow[rr, dash, "\simeq"] & & ((S_1 S_2) S_3) S_4
\end{tikzcd}$$

(2)
$$\begin{tikzcd}
(S_1 0) S_2 \arrow[rr, dash, "\simeq"] \arrow[dr, dash, "\wr"'] & & S_1 (0 S_2) \arrow[dl, dash, "\wr"] \\
& S_1 S_2 &
\end{tikzcd}$$

commutes for any objects S_i of $\mathcal{S}$.

Remarks: From the commutativity of the above diagrams one can deduce the commutativity of

$$\begin{tikzcd}(S_1S_2)0 \arrow[dr, "s"'] & \simeq & S_1(S_20) \arrow[dl, "s"] \\ & S_1S_2 & \end{tikzcd} \qquad \begin{tikzcd}(0S_1)S_2 \arrow[dr, "s"'] & \simeq & 0(S_1S_2) \arrow[dl, "s"] \\ & S_1S_2 & \end{tikzcd}$$

and the fact that the two isomorphisms: $00 \simeq 0$ obtained from the left and right unity isomorphisms are equal. (see [Kelly]). J. Alg 1 (1964) 397-402

A monoidal category will be called strict if its associativity and unity isomorphisms are identity maps. A strict monoidal category is the same thing as a monoid object in the category of categories, hence the terminology monoid category will also be used.

If $\mathfrak{X}$ is a category, the category $\underline{\mathrm{Hom}}(\mathfrak{X}, \mathfrak{X})$ of functors from $\mathfrak{X}$ to itself is a strict monoidal category with product given by composition of functors. More generally, the category of endomorphisms of an object in a 2-category is a strict monoidal category.

Definition: Let $\mathcal{S}, \mathcal{T}$ be monoidal categories. By a monoidal functor $F: \mathcal{S} \to \mathcal{T}$ we mean a functor $S \mapsto FS$ from $\mathcal{S}$ to $\mathcal{T}$ equipped with

i) an isomorphism of functors

$$F(S_1S_2) \simeq F(S_1)F(S_2)$$

from $\mathcal{S} \times \mathcal{S}$ to $\mathcal{T}$, called the product isom of F

ii) an isomorphism $F(0) \simeq 0$, called the unity isomorphism of F.

These are ~~supposed~~ required to satisfy the following compatibility ~~commutativity~~ conditions: expressed by the commutativity of the diagrams:

$$\begin{tikzcd}
F((S_1S_2)S_3) \arrow[r, phantom, "\simeq"] \arrow[d, phantom, "\wr"] & F(S_1S_2)F(S_3) \arrow[r, phantom, "\simeq"] & {[F(S_1)F(S_2)]F(S_3)} \arrow[d, phantom, "\wr"] \\
F(S_1(S_2S_3)) \arrow[r, phantom, "\simeq"] & F(S_1)F(S_2S_3) \arrow[r, phantom, "\simeq"] & F(S_1)[F(S_2)F(S_3)]
\end{tikzcd}$$

$$\begin{tikzcd}
F(0S) \arrow[r, phantom, "\simeq"] \arrow[d, phantom, "\wr"] & F(0)F(S) \arrow[d, phantom, "\wr"] \\
F(S) \arrow[r, phantom, "\simeq"] & 0F(S)
\end{tikzcd}
\qquad
\begin{tikzcd}
F(S0) \arrow[r, phantom, "\simeq"] \arrow[d, phantom, "\wr"] & F(S)F(0) \arrow[d, phantom, "\wr"] \\
F(S) \arrow[r, phantom, "\simeq"] & F(S)0
\end{tikzcd}$$

A monoidal functor will be called <u>strict</u> if its product and unity isomorphisms are identity maps. ~~All morphisms of monoid categories~~ A monoid functor between monoid categories is an example of a strict monoidal functor.

<u>Composition of monoidal functors:</u> If $F: \mathcal{S} \to \mathcal{T}$ and $G: \mathcal{T} \to \mathcal{U}$ are monoidal functors, ~~then $S \mapsto G(F(S))$ is a monoidal functor~~ let the functor $S \mapsto GF(S)$ be equipped with the product and unity isoms induced by those of F and G in the obvious way:

$$GF(S_1S_2) \simeq G(F(S_1)F(S_2)) \simeq GF(S_1)GF(S_2)$$

$$GF(0) \simeq G(0) \simeq 0$$

In this way one obtains a monoidal functor $GF: \mathcal{S} \to \mathcal{U}$. Further, with this composition monoidal categories and monoidal functors form a category ~~containing~~ of which the category of monoid categories + monoid functor is a subcategory.

Morphism of monoidal functors: If F and G are monoidal functors from $\mathcal{S}$ to $\mathcal{T}$, then by a morphism of monoidal functors $u: F \to G$, or monoidal natural transformation, we mean a ~~natural transformation of the underlying functors~~ morphism of functors which is compatible with the products and unity isomorphisms of F and G. We denote the category consisting of monoidal functors from $\mathcal{S}$ to $\mathcal{T}$ and morphisms between them by

$$\underline{\underline{\mathrm{MHom}}}(\mathcal{S},\mathcal{T}).$$ (MHom is to be script)

Composition of monoidal functors gives rise to functors

$$\underline{\mathrm{MHom}}(\mathcal{S},\mathcal{T}) \times \underline{\mathrm{MHom}}(\mathcal{T},\mathcal{U}) \to \underline{\mathrm{MHom}}(\mathcal{S},\mathcal{U})$$

of the sort that monoidal categories, monoidal functors, and monoidal natural transformations form a 2-category.

Equivalence of monoidal categories: A monoidal functor $F: \mathcal{S} \to \mathcal{T}$ will be called an equivalence of monoidal categories if there exists a monoidal functor $G: \mathcal{T} \to \mathcal{S}$ and isomorphisms of monoidal functors $GF \simeq id_{\mathcal{S}}$, $FG \simeq id_{\mathcal{T}}$ which are mutually compatible in the sense that

$$\begin{tikzcd} FGF \arrow[r, phantom, "\simeq"] \arrow[d, phantom, "\wr"] & id_{\mathcal{T}} F \arrow[d, equal] \\ F id_{\mathcal{S}} \arrow[r, equal] & F \end{tikzcd} \qquad \begin{tikzcd} GFG \arrow[r, phantom, "\simeq"] \arrow[d, phantom, "\wr"] & G id_{\mathcal{T}} \arrow[d, equal] \\ id_{\mathcal{S}} G \arrow[r, equal] & G \end{tikzcd}$$

commute.

<u>Proposition 1</u>: <u>A monoidal functor is an equivalence of monoidal categories if it is an equivalence of the underlying categories.</u>

<u>Proof</u>: Let $F : \mathcal{S} \to \mathcal{T}$ be a monoidal functor which is an equivalence of the underlying categories, and let $G : \mathcal{T} \to \mathcal{S}$ be a functor quasi-inverse to F so that there are compatible isomorphisms $GF \simeq id$, $FG \simeq id$. We equip G with the unique product isomorphism ~~$G(T_1T_2) \simeq G(T_1)G(T_2)$~~ and [illegible] unity isom [illegible] which are compatible with those of F in the sense that the diagrams

$$\begin{tikzcd}
F(G(T_1T_2)) \arrow[r, dash, "\sim"] \arrow[d, dash, "\wr"'] & F(G(T_1)G(T_2)) \arrow[d, dash, "\wr"] \\
T_1T_2 \arrow[r, dash, "\sim"] & FG(T_1)FG(T_2)
\end{tikzcd}$$

$$\begin{tikzcd}
F(G(0)) \arrow[rr, dash, "\sim"] \arrow[dr, dash, "\wr"'] & & F(0) \arrow[dl, dash, "\wr"] \\
& 0 &
\end{tikzcd}$$

commute. [illegible] It is routine to check that G is a monoidal functor, and that the isos. $FG \simeq id$, $GF \simeq id$ are compatible with the monoidal structure. Therefore F is an equivalence of monoidal categories.

Commuting monoidal functors.

~~Given monoids S, T, U, one knows that a monoid homomorphism $H: S \times T \to U$ is essentially the same thing~~

~~If S, T, U are monoids, one knows that to give a monoid homomorphisms $H: S \times T \to U$ is the same thing as giving a pair of monoid homomorphisms $F: S \to U$, $G: T \to U$ which commute: $F(s)G(t) = G(t)F(s)$.~~

Given monoids S, T, U, one knows that a monoid homomorphism $H: S \times T \to U$ is uniquely expressible in the form $H(s,t) = F(s)\,G(t)$ where $F: S \to U$, $G: T \to U$ are monoid homomorphisms which commute: $F(s)G(t) = G(t)F(s)$. We now discuss the generalization of this fact to ~~the~~ monoidal categories.

Definition: Let $F: \mathcal{S} \to \mathcal{U}$ and $G: \mathcal{T} \to \mathcal{U}$ be monoidal functors. By a commutation isomorphism between F and G we mean an isom

$$c: F(S)\,G(T) \simeq G(T)\,F(S)$$

of functors from $\mathcal{S} \times \mathcal{T}$ to $\mathcal{U}$ which is compatible with the product + unity isoms. of F and G: Compatibility with the product iso. of F means commutativity of

$$\begin{tikzcd}
F(S_1 S_2)\,G(T) \arrow[r, phantom, "\simeq"] \arrow[d, phantom, "\wr"] & (F(S_1)\,F(S_2))\,G(T) \\
G(T)\,F(S_1 S_2) &
\end{tikzcd}$$

$$\begin{tikzcd}
F(S_1S_2)\,G(T) \arrow[r, dash, "\simeq"] \arrow[d, dash, "\wr"'] & G(T)\,F(S_1S_2) \arrow[d, dash, "\wr"'] \\
(F(S_1)\,F(S_2))\,G(T) \arrow[d, dash, "\wr"'] & G(T)\,(F(S_1)\,F(S_2)) \arrow[d, dash, "\wr"'] \\
F(S_1)\,(F(S_2)\,G(T)) \arrow[d, dash, "\wr"'] & (G(T)\,F(S_1))\,F(S_2) \arrow[d, dash, "\wr"'] \\
F(S_1)\,(G(T)\,F(S_2)) \arrow[r, dash, "\simeq"] & (F(S_1)\,G(T))\,F(S_2)
\end{tikzcd}$$

while compatibility with the unity iso of F means that

$$\begin{tikzcd}
F(\mathbb{1})\,G(T) \arrow[rr, dash, "\simeq"] \arrow[d, dash, "\wr"'] & & G(T)\,F(\mathbb{1}) \arrow[d, dash, "\wr"'] \\
\mathbb{1}\,G(T) \arrow[r, phantom, "\simeq"] & G(T) \arrow[r, phantom, "\simeq"] & G(T)\cdot\mathbb{1}
\end{tikzcd}$$

commutes. Compatibility with the product + unity isos of G is defined similarly.

? By abuse of language we say two monoidal functors <u>commute</u> when there is given a commutation isomorphism between them.

Example: We consider the monoidal functor

$$in_1 = (id_{\mathcal{S}}, \mathbb{1}) : \mathcal{S} \to \mathcal{S} \times \mathcal{T} \quad , \quad S \mapsto (S, \mathbb{1})$$

where $\mathbb{1} : \mathcal{S} \longrightarrow \mathcal{T}$ denotes the monoidal functor given by the constant functor with value $\mathbb{1}_{\mathcal{T}}$, equipped with unity isom equal to the identity map of $\mathbb{1}_{\mathcal{T}}$, and with product iso furnished by the unity iso $\mathbb{1}_{\mathcal{T}} \mathbb{1}_{\mathcal{T}} \simeq \mathbb{1}_{\mathcal{T}}$ of $\mathcal{T}$. Similarly we have the monoidal functor

$$in_2 = (\mathbb{1}, id_{\mathcal{T}}) : \mathcal{T} \longrightarrow \mathcal{S} \times \mathcal{T}, \quad T \mapsto (\mathbb{1}, T).$$

We claim the composite

$$\begin{aligned} (S,\mathbb{1})(\mathbb{1},T) &= (S\mathbb{1}, \mathbb{1}T) \\ &\simeq (S, T) \\ &\simeq (\mathbb{1}S, T\mathbb{1}) = (\mathbb{1},T)(\mathbb{1},S) \end{aligned}$$

is a commutation isomorphism between in_1 and in_2. To show this we have to check the commutativity of certain diagrams constructed from the assoc. & unity isos. of $\mathcal{S}$ and $\mathcal{T}$. It is clear that we can, without loss of generality, replace $\mathcal{S}$ and $\mathcal{T}$ by equivalent mon. cats. Hence we can suppose $\mathcal{S}, \mathcal{T}$ are strict monoidal cats, in which case the relevant diagrams commute because all the arrows occurring in them are identity maps.

~~We are now going to sho~~

~~Given a commutation isom c between $F: \mathcal{S} \to \mathcal{U}$, $G: \mathcal{T} \to \mathcal{U}$, and a monoidal functor $E: \mathcal{U} \to \mathcal{U}'$,~~

~~Let $E: \mathcal{U} \to \mathcal{U}'$ be a~~

We are now going to show that the above example is in some sense the universal example of a ~~pair of monoidal functors~~ commutation isom between monoidal functors.

Let c be a commutation isom. between ~~functors~~ $F: \mathcal{S} \to \mathcal{U}'$, $G: \mathcal{T} \to \mathcal{U}'$, and let $H: \mathcal{U}' \to \mathcal{U}$ be a monoidal functor. One verifies easily that the composite isom

~~$E(F(S))E(G(T)) \simeq E(F(S)G(T)) \simeq E(G(T)F(S)) \simeq E(G(T))E(F(S))$ (Ec)~~

( )

$$\begin{tikzcd}
H(F(S))\,H(G(T)) \arrow[r, "\sim"] & H(F(S)G(T)) \arrow[d, "\wr"', "H(c)"] \\
& H(G(T)F(S)) \arrow[r, "\sim"] & H(G(T))\,H(F(S))
\end{tikzcd}$$

~~[illegible]~~ is a commutation isomorphism between $EF: \mathcal{S} \to \mathcal{U}$ and $EG: \mathcal{T} \to \mathcal{U}$. In particular taking $\mathcal{U}' = \mathcal{S} \times \mathcal{T}$ and F, G to be ~~the functors~~ in_1, in_2, respectively, we have associated to any monoidal functor $H: \mathcal{S} \times \mathcal{T} \to \mathcal{U}$ a pair of monoidal functors

$Hin_1: \mathcal{S} \to \mathcal{U}$, $S \longmapsto H(S, \mathbb{1})$

$Hin_2: \mathcal{T} \to \mathcal{U}$, $T \longmapsto H(\mathbb{1}, T)$

equipped with the commutation isomorphism:

$$\begin{aligned} H(S,\mathbb{1})\,H(\mathbb{1},T) &\simeq H(S\mathbb{1},\mathbb{1}T) \\ &\simeq H(S,T) \\ &\simeq H(\mathbb{1}S, T\mathbb{1}) \\ &\simeq H(\mathbb{1},T)\,H(S,\mathbb{1}). \end{aligned}$$

~~We wish to show that up to isom. a monoidal functor $H: \mathcal{S}\times\mathcal{T}\to\mathcal{U}$ may be identified with a pair of mon. functors $F: \mathcal{S}\to\mathcal{U}$, $G: \mathcal{T}\to\mathcal{U}$ and a commutation isom between them. To formulate this precisely we introduce the category $\mathcal{H}(\mathcal{S},\mathcal{T};\mathcal{U})$ defined as follows. An object is a triple (F,G,c) where F:~~

We wish to show that in this way a mon. functor $H: \mathcal{S}\times\mathcal{T}\to\mathcal{U}$ may be identified, up to isomorphism, with a triple (F,G,c) consisting of monoidal functors $F: \mathcal{S}\to\mathcal{U}$, $G: \mathcal{T}\to\mathcal{U}$ and a comm. isom c between them. To formulate this precisely, let $\mathcal{H}(\mathcal{S},\mathcal{T};\mathcal{U})$ denote the category of such triples in which a morphism $(F,G,c)\to(F',G',c')$ consists of two morphisms $F\to F'$, $G\to G'$ of monoidal functors which are compatible with c and c' in the evident sense. Then it is clear that by associating to $H: \mathcal{S}\times\mathcal{T}\to\mathcal{U}$ the pair $H\mathrm{in}_1, H\mathrm{in}_2$ with the commutation iso described above, we obtain a functor

$$(\ \ )\qquad \underline{\mathrm{MHom}}(\mathcal{S}\times\mathcal{T},\mathcal{U}) \longrightarrow \mathcal{H}(\mathcal{S};\mathcal{T};\mathcal{U}).$$

The desired result may now be formulated as follows:

<u>Proposition</u>: The functor ( ) is an equivalence of categories.

To prove this we construct a functor in the opposite direction. Let (F, G, c) be an object of $\mathcal{H}(\mathcal{S}, \mathcal{T}; \mathcal{U})$ and let H be the functor from $\mathcal{S} \times \mathcal{T}$ to $\mathcal{U}$ given by

$$H(S,T) = F(S)\,G(T).$$

Define a product isom for H to be the composition

$$\begin{aligned}
H((S_1,T_1)(S_2,T_2)) &= H(S_1S_2, T_1T_2) \\
&= F(S_1S_2)\,G(T_1T_2) \\
&\cong (F(S_1)F(S_2))(G(T_1)G(T_2)) \\
&\simeq F(S_1)\left[F(S_2)(G(T_1)G(T_2))\right] \\
&\simeq F(S_1)\left[(F(S_2)G(T_1))\,G(T_2)\right] \\
&\simeq F(S_1)\left[(G(T_1)F(S_2))\,G(T_2)\right] \\
&\simeq F(S_1)\left[G(T_1)(F(S_2)G(T_2))\right] \\
&\simeq (F(S_1)G(T_1))(F(S_2)G(T_2)) \\
&= H(S_1,T_1)\,H(S_2,T_2).
\end{aligned}$$

Define the unity isomorphism for H to be the composite

$$H(1,1) = F(1)\,G(1) \simeq 1 \cdot 1 \simeq 1$$

~~Lemma: The functor H equipped with the above product and unity isomorphisms is a monoidal functor from $\mathcal{S} \times \mathcal{T}$ to $\mathcal{U}$.~~

~~Proof. We first reduce to the case where $\mathcal{S}, \mathcal{T}, \mathcal{U}$ are strict monoidal categories and F, G are strict monoidal functors. Notice that to prove the lemma we have only to show~~

~~the commutativity certain diagrams in $\mathcal{U}$ constructed from the comm. isom and the isomorphisms belonging to the monoidal structure of $\mathcal{S}, \mathcal{T}, \mathcal{U}, F, G$. It is clear that if two triples (F, G, c) and (F', G', c') are isomorphic, then the lemma holds for one iff it does for the other. Further if the lemma holds for (F, G, c) then it also does for the triples obtained by pulling back with respect to monoidal functors $\mathcal{S}' \to \mathcal{S}$, $\mathcal{T} \to \mathcal{T}'$, and pushing forward with respect to $\mathcal{U} \to \mathcal{U}'$. Thus it is clear that without loss of generality we can replace $\mathcal{S}, \mathcal{T}, \mathcal{U}, F, G$ by $\tilde{\mathcal{S}}, \tilde{\mathcal{T}}, \tilde{\mathcal{U}}, \tilde{F}, \tilde{G}$ and so reduce to the strict case.~~

Lemma 1: The functor H equipped with the above prod + unity isos is a monoidal functor from $\mathcal{S} \times \mathcal{T}$ to $\mathcal{U}$.

Proof: ~~To prove this lemma~~ we must show ~~that certain~~ the commutativity of certain diagrams in $\mathcal{U}$ constructed from the comm. iso c and the isos. given by the monoidal structure of $\mathcal{S}, \mathcal{T}, \mathcal{U}, F, G$. Since the canonical functor $\gamma : \mathcal{U} \longrightarrow \tilde{\mathcal{U}}$ is an equivalence of monoidal categories, it evidently suffices to prove the lemma for the ~~$\gamma F : \mathcal{S} \to \tilde{\mathcal{U}}$, $\gamma G : \mathcal{T} \to \tilde{\mathcal{U}}$ and the comm~~ triple consisting of $\gamma F : \mathcal{S} \longrightarrow \tilde{\mathcal{U}}$, $\gamma G : \mathcal{T} \longrightarrow \tilde{\mathcal{U}}$ and the commutation isom between them induced from c. Since $\mathcal{S} \to \tilde{\mathcal{S}}$, $\mathcal{T} \to \tilde{\mathcal{T}}$ are equivalences of mon cats, one sees easily that this induced comm. iso extends to a commutation isom $\tilde{c}$ between $\tilde{F} : \tilde{\mathcal{S}} \longrightarrow \tilde{\mathcal{U}}$, $\tilde{G} : \tilde{\mathcal{T}} \longrightarrow \tilde{\mathcal{U}}$. It therefore suffices to prove the lemma for $(\tilde{F}, \tilde{G}, \tilde{c})$. Thus without loss of generality, we can suppose $\mathcal{S}, \mathcal{T}, \mathcal{U}, F, G$ are strict monoidals cats. + functors.

In this case the product isom for $H(S,T) = F(S)\,G(T)$ becomes

$$H((S_1,T_1)(S_2,T_2)) = FS_1\,FS_2\,GT_1\,GT_2$$
$$\simeq FS_1\,GT_1\,FS_2\,GT_2 = H(S_1,T_1)\,H(S_2,T_2)$$

and the condition expressing compatible of the product isom with associativity isomorphisms is for

$$\begin{tikzcd}
H((S_1,T_1)(S_2,T_2)(S_3,T_3)) \arrow[r, dash, "\simeq"] \arrow[d, dash, "\wr"] & H((S_1,T_1)(S_2,T_2))\, H(S_3,T_3) \arrow[d, dash, "\wr"] \\
H(S_1,T_1)\, H((S_2,T_2)(S_3,T_3)) \arrow[r, dash, "\simeq"] & H(S_1,T_1) H(S_2,T_2) H(S_3,T_3)
\end{tikzcd}$$

to commute. However this is easily seen to be the outside square of the diagram in Figure 1.

$$\begin{tikzcd}
F(S_1S_2S_3)G(T_1T_2T_3) \arrow[r, equal] \arrow[d, equal] & F(S_1S_2)FS_3\, G(T_1T_2)GT_3 \arrow[r, dash, "\simeq"] \arrow[d, equal] & F(S_1S_2)G(T_1T_2)FS_3GT_3 \arrow[dd, equal] \\
FS_1\, F(S_2S_3)GT_1\, G(T_2T_3) \arrow[r, equal] \arrow[dd, dash, "\wr"] & FS_1FS_2FS_3\, GT_1GT_2GT_3 \arrow[d, dash, "\wr"] & \\
 & FS_1FS_2GT_1FS_3\, GT_2GT_3 \arrow[r, dash, "\simeq"] \arrow[d, dash, "\wr"] & FS_1FS_2GT_1GT_2FS_3GT_3 \arrow[d, dash, "\wr"] \\
FS_1\, GT_1F(S_2S_3)G(T_2T_3) \arrow[r, equal] & FS_1GT_1FS_2FS_3\, GT_2GT_3 \arrow[r, dash, "\simeq"] & FS_1GT_1FS_2GT_2FS_3GT_3
\end{tikzcd}$$

Figure 1.

The axioms on a commutation isom imply the pentagons in this diagram commute. The lower right square commutes because the product is a bifunctor of its arguments. Hence the diagram commutes as desired.

~~In the strict case the unit isom for H is the identity map of $1_{\mathcal{U}}$ and the axiom that the comm. isom the~~

In the case where $\mathcal{S}, \mathcal{T}, \mathcal{U}, F, G$ are strict, the condition that c be compatible with the unit isom of F asserts that the comm. isom.

$$F(\mathbb{1})\,G(T) \simeq G(T)\,F(\mathbb{1})$$

is the identity map of $G(T)$. The unit isom for H is the identity map of $H(\mathbb{1},\mathbb{1}) = \mathbb{1}$. ~~Thus~~ Hence all the arrows in

$$\begin{tikzcd}
H((\mathbb{1},\mathbb{1})(S,T)) \arrow[r, phantom, "\simeq"] \arrow[d, phantom, "\wr"] & H(\mathbb{1},\mathbb{1})\,H(S,T) \arrow[d, phantom, "\wr"] \\
H(S,T) \arrow[r, phantom, "\simeq"] & \mathbb{1}\,H(S,T)
\end{tikzcd}$$

are equal to the identity map of $F(S)\,G(T)$. Thus H satisfies ~~[illegible]~~ the left unity condition for a monoidal functor, ~~and the~~ The right one is similar, so the lemma is proved.

By virtue of the lemma, it is clear that by associating to (F,G,c) the monoidal functor H, we obtain a functor

(**) $$\mathcal{H}(\mathcal{S},\mathcal{T};\mathcal{U}) \longrightarrow \underline{\mathrm{MHom}}(\mathcal{S}\times\mathcal{T};\mathcal{U}).$$

Lemma 2: The functors (**) and (*) are quasi-inverses of each other.

~~[illegible]~~

~~Given a mon. functor $H: \mathcal{S}\times\mathcal{T} \to \mathcal{U}$~~

Proof. Under the composition functor (**)(*) a mon. functor $H: \mathcal{S}\times\mathcal{T} \longrightarrow \mathcal{U}$ goes into the functor

$$H'(S,T) = H(S,\mathbb{1})\,H(\mathbb{1},T)$$

equipped with certain product and unity isos. The reader may verify ~~[illegible]~~ that the ~~[illegible]~~ isomorphism ~~of functors~~

$$H(S,\mathbb{1})\,H(\mathbb{1},T) \simeq H(S1,1T) \simeq H(S,T)$$

is an isomorphism of monoidal functors, ~~is compatible with product and unity isos~~, and further that in this way ~~[illegible]~~ one obtains an isomorphism of the composite (**)(*) with the identity.

<u>Lemma 2</u>: The functors (**) and (*) are quasi-inverses of each other.

<u>Proof</u>. Under the composite functor (**)(*) a monoidal functor $H: \mathcal{S}\times\mathcal{T}\to\mathcal{U}$ goes into the functor

$$(S,T) \longmapsto H(S,1)H(1,T),$$

equipped with certain product + unity isomorphisms. The reader may verify that the isomorphism

$$H(S,1)\,H(1,T) \simeq H(S1,1T) \simeq H(S,T)$$

is an isomorphism of monoidal functors (by considerations of naturality one reduces to the case where H is the identity functor of $\mathcal{S}\times\mathcal{T}$, in which case the verification is almost trivial). It is clear that in this way the ~~composite~~ functor (**)(*) is isom. to the identity.

Under the composite functor (*)(**) a triple (F,G,c) goes to the pair of functors

$$S \longmapsto F(S)\,G(1)$$
$$T \longmapsto F(1)\,G(T)$$

equipped with certain monoidal structures and a commutation isomorphism. The reader may verify that the isos.

$$F(S)\,G(1) \simeq F(S)\,1 \simeq F(S)$$
$$F(1)\,G(T) \simeq 1\,G(T) \simeq G(T)$$

constitute isomorphisms of monoidal functors compatible with the commutations isos. (One reduces to the case where $\mathcal{S}, \mathcal{T}, \mathcal{U}$, F, G are strict, in which case the verification is trivial.) It is thus clear that (*)(**) is isomorphic to the identity, and so the lemma follows.

Commuting actions

~~Let $\mathcal{S}, \mathcal{T}$ be monoidal categories and let~~

~~$\alpha : \mathcal{S} \longrightarrow \underline{\mathrm{Hom}}(\mathcal{X}, \mathcal{X})$~~

~~$\alpha' : \mathcal{T} \longrightarrow \underline{\mathrm{Hom}}(\mathcal{X}, \mathcal{X})$~~

~~be actions of $\mathcal{S}$ and $\mathcal{T}$ on a category $\mathcal{X}$~~

Let two monoidal categories $\mathcal{S}, \mathcal{T}$ act on the same category $\mathcal{X}$, the actions being defined by mon. functors

$$\alpha : \mathcal{S} \longrightarrow \underline{\mathrm{Hom}}(\mathcal{X}, \mathcal{X}) \quad , \quad S \mapsto (X \mapsto SX)$$

$$\beta : \mathcal{T} \longrightarrow \underline{\mathrm{Hom}}(\mathcal{X}, \mathcal{X}) \quad , \quad T \mapsto (X \mapsto TX).$$

A commutation isom. between α and β may be identified with an isomorphism

$$S(TX) \simeq T(SX)$$

of functors from $\mathcal{S} \times \mathcal{T} \times \mathcal{X}$ to $\mathcal{X}$, which is compatible in the evident sense with the associativity and unity isos. for the $\mathcal{S}$ and $\mathcal{T}$ action. It may also be identified with a lifting of α to a monoidal functor

$$\alpha' : \mathcal{S} \longrightarrow \underline{\mathrm{Hom}}^{\mathcal{T}}(\mathcal{X}, \mathcal{X})$$

or with a lifting of β to a monoidal functor

$$\beta' : \mathcal{T} \longrightarrow \underline{\mathrm{Hom}}^{\mathcal{S}}(\mathcal{X}, \mathcal{X})$$

By abuse of language, we will say that the $\mathcal{S}$ and $\mathcal{T}$ actions ~~composing α' with~~ commute if there is given a commutation isomorphism between α and β.

On composing α' with the canon functor

$$\underline{\mathrm{Hom}}^{\mathcal{T}}(\mathcal{X}, \mathcal{X}) \longrightarrow \underline{\mathrm{Hom}}(\langle \mathcal{T}, \mathcal{X} \rangle, \langle \mathcal{T}, \mathcal{X} \rangle)$$

ADD: $\underline{\mathrm{Hom}}^{\mathcal{T}}(\mathcal{X}, \mathcal{X}) \to \underline{\mathrm{Hom}}(\mathcal{T} \cdot \mathcal{X}, \mathcal{T} \cdot \mathcal{X})$.

we obtain an induced action of $\mathcal{S}$ on $\langle \mathcal{T}, \mathcal{X} \rangle$ and on $\mathcal{T}^{-1}\mathcal{X}$. Similarly there is an induced action of $\mathcal{T}$ on $\langle \mathcal{S}, \mathcal{X} \rangle$ and on $\mathcal{S}^{-1}\mathcal{X}$. ~~On the other hand from the preceding section (see Lemma 1) there is a monoidal functor~~

~~$$\mathcal{S}\times\mathcal{T} \longrightarrow \underline{\mathrm{Hom}}(\mathcal{X},\mathcal{X}), \quad (S,T) \mapsto (X \mapsto S(TX))$$~~

~~with product isomorphism~~

~~$S_1 T_1$~~

On the other hand the commutation isom between α and β gives rise to a monoidal functor

$$\mathcal{S}\times\mathcal{T} \longrightarrow \underline{\mathrm{Hom}}(\mathcal{X},\mathcal{X})$$

(see Lemma 1 ...), hence to an action of $\mathcal{S}\times\mathcal{T}$ on $\mathcal{X}$. This action is given by

$$(S,T)X = S(TX)$$

with the associativity isom

$$\begin{aligned}
\big((S_1,S_2)(T_1,T_2)\big)X &= (S_1S_2)\big[(T_1T_2)X\big] \\
&\simeq S_1(S_2(T_1(T_2X))) \\
&\simeq S_1(T_1(S_2(T_2X)) \\
&= (S_1,T_1)\big[(S_2,T_2)X\big]
\end{aligned}$$

and unity isom

$$(0,0)X = 0(0X) \simeq 0X \simeq X .$$

<u>Proposition</u>: (i) The categories $\langle \mathcal{S}, \langle \mathcal{T}, \mathcal{X}\rangle\rangle$, $\langle \mathcal{T}, \langle \mathcal{S}, \mathcal{X}\rangle\rangle$ and $\langle \mathcal{S}\times\mathcal{T}, \mathcal{X}\rangle$ are canonically isom.

(ii) The categories $\mathcal{S}^{-1}(\mathcal{T}^{-1}\mathcal{X})$, $\mathcal{T}^{-1}(\mathcal{S}^{-1}\mathcal{X})$, and $(\mathcal{S}\times\mathcal{T})^{-1}\mathcal{X}$ are canonically isomorphic.

Proof. (i): The categories in question have the same objects as $\mathcal{X}$. ~~On the other hand~~ As

$$
\begin{aligned}
\mathrm{Hom}_{\langle \mathcal{S}\times\mathcal{T},\, \mathcal{X}\rangle}(X, X') &= \varinjlim_{(S,T)} \mathrm{Hom}_{\mathcal{X}}((S,T)X, X') \\
&\simeq \varinjlim_{\mathcal{T}} \varinjlim_{\mathcal{S}} \mathrm{Hom}_{\mathcal{X}}(S(TX), X') \\
&= \varinjlim_{\mathcal{T}} \mathrm{Hom}_{\langle \mathcal{S},\mathcal{X}\rangle}(TX, X') \\
&= \mathrm{Hom}_{\langle \mathcal{T},\langle \mathcal{S},\mathcal{X}\rangle\rangle}(X, X')
\end{aligned}
$$

~~one obtains a 1-1 correspondence between arrows in the cats $\langle \mathcal{S}\times\mathcal{T}, \mathcal{X}\rangle$ and $\langle \mathcal{T}, \langle \mathcal{S}, \mathcal{X}\rangle\rangle$ which by associating to the $\langle \mathcal{S}\times\mathcal{T}, \mathcal{X}\rangle$-map represented by (S,T)X the $\mathcal{X}$-map $x: (S,T)$~~ ~~the following~~

one obtains a 1-1 correspondence between arrows in the cats. $\langle \mathcal{S}\times\mathcal{T}, \mathcal{X}\rangle$ and $\langle \mathcal{T}, \langle \mathcal{S}, \mathcal{X}\rangle\rangle$ which may be described as follows: To the $\langle \mathcal{S}\times\mathcal{T}, \mathcal{X}\rangle$-map $X \to X'$ represented by the $\mathcal{X}$-map $x: (S,T)X \to X'$ corresponds the $\langle \mathcal{T}, \langle \mathcal{S}, \mathcal{X}\rangle\rangle$-map represented by the $\langle \mathcal{S}, \mathcal{X}\rangle$-map $TX \to X'$ Rep. by the $\mathcal{X}$-map $x: S(TX) \longrightarrow X'$. Using this description, it is easy to see that the 1-1 correspondence is compatible with composition. Thus the cats $\langle \mathcal{S}\times\mathcal{T}, \mathcal{X}\rangle$ and $\langle \mathcal{T}, \langle \mathcal{S}, \mathcal{X}\rangle\rangle$ are isomorphic. ~~and by symmetry $\langle \mathcal{S}, \langle \mathcal{T}, \mathcal{X}\rangle\rangle$~~ Similarly $\langle \mathcal{S}, \langle \mathcal{T}, \mathcal{X}\rangle\rangle$ is isomorphic to $\langle \mathcal{S}\times\mathcal{T}, \mathcal{X}\rangle$.

(ii). Using (i) and (ref.——) we have

$$
\begin{aligned}
(\mathcal{S}\times\mathcal{T})^{-1}\mathcal{X} &= \langle \mathcal{S}\times\mathcal{T},\ \mathcal{S}\times\mathcal{T}\times\mathcal{X}\rangle \\
&\simeq \langle \mathcal{S}, \langle \mathcal{T},\ \mathcal{S}\times\mathcal{T}\times\mathcal{X}\rangle\rangle \\
&\simeq \langle \mathcal{S},\ \mathcal{S}\times\langle \mathcal{T}, \mathcal{T}\times\mathcal{X}\rangle\rangle = \mathcal{S}^{-1}\mathcal{T}^{-1}\mathcal{X}.
\end{aligned}
$$

Similarly for $\mathcal{T}^{-1}(\mathcal{S}^{-1}\mathcal{X})$.

QED

At the moment the writing process is stuck on ~~some aspects of~~ the $\mathcal{S}^{-P}$-section

Monoidal cat = cat $\mathcal{S}$ equipped with product functor, associativity isom, unit object, and left & right unit isos. $\ni$ various compatibilities hold

example: strict monoidal cats
$\underline{\mathrm{Hom}}(\mathcal{X},\mathcal{X})$

monoidal functor = functor equipped with a product isom + a unit isom subject to compat with the assoc + left unit & right-unit isos.

~~strict monoidal functors~~

composition of monoidal functors

~~morphism of m~~

morphism of monoidal functors = morphism of underlying functors comp. with prod. + unity iso.

$\underline{\mathrm{Hom}}^{+}(\mathcal{S},\mathcal{T})$ and 2-cat of modoid

~~[illegible]~~ equivalence of monoidal categories + Lemma

(left)-action of a monoidal cat. $\mathcal{S}$ on a category $\mathcal{X}$

= monoidal functor $\mathcal{S} \longrightarrow \underline{\mathrm{Hom}}(\mathcal{X},\mathcal{X})$

$$= \begin{cases} \text{product} & (S, X) \longmapsto SX \\ \text{assoc iso} & (S_1 S_2)X \simeq S_1(S_2X) \\ \text{unit iso} & 0X \simeq X \end{cases} \quad +$$

Example: ~~left~~ actions of $\mathcal{S}, \mathcal{S}^*$ on $\mathcal{S}$.

action-preserving functor $F : \mathcal{X} \longrightarrow \mathcal{Y}$ = functor equipped with an action-preserving isom

$$F(SX) \simeq S\,F(X)$$

$F(G(T_1)G(T_2))$

comp. with assoc. + unity isos.

composition

morphism of action-preserving functors

2-cat of cats ~~[illegible]~~ on which $\mathcal{S}$ ~~opera~~ acts.