$$
\begin{array}{ccccc}
L' & \simeq & (M_0 \times P_0) \times_{M \times P} L & \longrightarrow & L \\
\downarrow & & \downarrow & & \downarrow \\
M' \times P' & \longleftarrow & M_0 \times P_0 & \longrightarrow & M \times P .
\end{array}
$$

(bottom is product of a map in $Q(\mathcal{M})$ and one in $Q(\mathcal{P})$)
Equivalently a map from $(L' \to M' \times P')$ to $(L \to M \times P)$ may be identified with a map $L' \to L$ in $\mathcal{P}$ which induces a map from $M' \times P'$ to $M \times P$ in $Q(\mathcal{M})$ which is the product of maps $M' \to M$, $P' \to P$ in $Q(\mathcal{M})$ and $Q(\mathcal{P})$ respectively. In this last interpretation it is clear how to compose morphisms.

We have ~~two~~ functors

$$Q(\mathcal{M}) \xleftarrow{\ f\ } \mathcal{F} \xrightarrow{\ g\ } Q(\mathcal{P})$$

obtained in composing the functor $\mathcal{F} \to Q(\mathcal{P}) \times Q(\mathcal{M})$ forgetting L with the projections. Since the composition of two fibred functors is fibred, f and g are also fibred.

__Lemma 1:__ f is a heq.

~~Proof.~~ Since f is fibred, it suffices to show $f^{-1}(M)$ is contractible for a given M in $Q(\mathcal{M})$. Let R_M be the category whose objects are $\mathcal{M}$-admiss epis $L \to M$ with $L \in \mathcal{P}$, in which a map from $(L' \to M)$ to $(L \to M)$ is a $\mathcal{P}$-admissible mono $L' \to L$ over M. The category $f^{-1}(M)$ has objects $(L \to M \times P)$ for $\mathcal{M}$-admiss epis with $L, P \in \mathcal{P}$, and a map from $(L' \to M \times P')$ to $(L'' \to M'' \times P'')$ is a map $L' \to L$ in $\mathcal{P}$ which is over M and which ~~induces a~~ ~~the product~~ ~~map $M \times P' \to M \times P$ in $Q(\mathcal{M})$ which is~~ ~~of the identity of M and~~ a map $P' \to P$ in $Q(\mathcal{P})$.

~~(as above for $\overline{Q}$:) Thus we have an equivalence of cats~~

$$\begin{aligned} f^{-1}(M) &\longrightarrow \mathrm{Sub}(\mathcal{R}_M) \\ (L \to M \times P) &\longmapsto \left(\begin{tikzcd}[ampersand replacement=\&] \mathrm{Ker}(L \to P) \arrow[r] \arrow[dr] \& L \arrow[d] \\ \& M \end{tikzcd} \right). \end{aligned}$$

~~(Compare situation above for $\overline{Q}$.). Hence we are reduced~~

Proof. Since f is fibred, it suffices to show $f^{-1}(M)$ is contrac for any given M in $\mathcal{M}$. The cat $f^{-1}(M)$ has for its objects all $\mathcal{M}$-adm. epis $L \to M \times P$ with $L, P \in \mathcal{P}$, and a map from $(L' \to M \times P')$ to $(L \to M \times P)$ is a map $L' \to L$ in $\mathcal{P}$ which is over M and which induces a map $P' \to P$ in $Q(\mathcal{P})$. Let $\mathcal{R}_M$ denote the cat whose objects are $\mathcal{M}$-adm. epis $L \to M$, in which a map from $(L' \to M)$ to $(L \to M)$ is a triangle

$$\begin{tikzcd} L' \arrow[r] \arrow[dr] & L \arrow[d] \\ & M \end{tikzcd}$$

where $L' \rightarrowtail L$ is a $\mathcal{P}$-admissible mono. Then as above (see lemma in $\overline{Q}$) we have an equivalence of categories

$$\begin{aligned} f^{-1}(M) &\longrightarrow \mathrm{Sub}(\mathcal{R}_M) \\ (L \to M \times P) &\longmapsto \left(\begin{tikzcd}[ampersand replacement=\&] \mathrm{Ker}(L \to P) \arrow[r, hook] \arrow[dr] \& L \arrow[d] \\ \& M \end{tikzcd} \right). \end{aligned}$$

~~Because Sub(R_M) and R_M~~ Thus we are reduced to proving R_M is contractible.

But by hypothesis 2) R_M contains an object $(L_0 \to M)$, and for any other object $(L \to M)$ we have morphisms in R_M:

$$(L \to M) \longrightarrow (L \oplus L_0 \to M) \longleftarrow (L_0 \to M).$$

Thus R_M is conically contractible (ref.), so the lemma is done.

<u>Lemma 2.</u> g is a heq

The Proof is analogous to the preceding and will be omitted.

~~<u>Lemma 3.</u> The map in the homotopy category $g.f^{-1}: Q(P) \to Q(M)$~~

From the above two lemmas we see the categories $Q(P)$ and $Q(M)$ are heq. To finish the proof of the theorem it ~~suffices to~~ will be necessary to relate the homotopy equivalence obtained from f and g with the inclusion functor $i: Q(P) \longrightarrow Q(M)$. Recall that direct sum makes $Q(M)$ into a h-comm + assoc. connected H-space, so that homotopy classes of maps from a space X to $Q(M)$ form an abelian group.

<u>Lemma 3.</u> The functors f, ig: $\mathcal{F} \to Q(M)$ are negatives of each other for the H-space structure on $Q(M)$.

Proof: Follows immediately from the comm. diag

$$\begin{tikzcd}
\overline{\mathcal{F}} \arrow[rrr, "\lambda"] \arrow[d, "(f,g)"'] & & & \overline{Q}(M) \arrow[d, "\varepsilon"] \\
Q(M) \times Q(P) \arrow[r, "id \times i"'] & Q(M) \times Q(M) \arrow[rr, "\oplus"] & & Q(M)
\end{tikzcd}$$

where λ is the obvious inclusion, and the fact that $\overline{Q}(M)$ is contractible.

~~It follows from the preceding lemmas that~~
Thus we have that $(-1)f = ig$ where (-1) is the inverse for the H-space structure on $Q(M)$. From ~~the preceding~~ lemmas 1, 2 have f, g are heqs. $\Longrightarrow$ i is a heq $\Longrightarrow$ the theorem.