January 3, 1973

Gersten's conjecture: If A is a d.v.r. with residue field k, then the transfer map

$$K_*(k) \longrightarrow K_*(A)$$

is zero.

Remark first that any $A \otimes_{\mathbb{Z}} k$ module M which is finite over A induces an exact functor

$$\begin{tikzcd} \mathrm{Modf}(k) \arrow[r] & \mathrm{Modf}(A) \\ V \arrow[r, mapsto] & M \otimes_k V \end{tikzcd}$$

and that the transfer is the one given by $M = k$. Secondly because $K_*(k)$ is the limit of the K-groups ~~Gersten's conjecture is true in the equi-characteristic case: First remark that it suffices to show~~ of the subfields k' of k which are fin. gen., it suffices to show that

$$\begin{tikzcd} \mathrm{Modf}(k') \arrow[r] & \mathrm{Modf}(A) \\ V \arrow[r, mapsto] & k \otimes_{k'} V \end{tikzcd}$$

induces zero on K-groups.

The equi-characteristic case: This is where A is a P-algebra, $P =$ prime field of k. Let k' be a subfield of k finite type over P. Because P is perfect, one knows that k'/P has a separating transcendence basis, i.e. a transcendence basis S such that $k'/P(S)$ is finite and separable. (Zariski - Samuel p. 104-105). Lift S back to A. Then

$$\begin{tikzcd} P[S] \arrow[r] \arrow[d, phantom, "\cap"] & A \arrow[d] \\ P(S) \arrow[r, phantom, "\subset"] & k \end{tikzcd}$$

implies that $P(S)$ lifts back to A. Thus we can assume A is a k_0-algebra and $k_0 < k'$ is finite and separable.

Because the homo

$$K_i(k') \longrightarrow K_i(A)$$

induced by $\quad V \longmapsto k \otimes_{k'} V$

depends only on the class of k in the Grothendieck group of $A \otimes_{\mathbb{Z}} k'$ modules which are finite over A, it will clearly suffice to show that the class of k in the Grothendieck group of $A \otimes_{k_0} k'$ modules f.t. over A is zero. To show ∴ that

$$0 = [k] \in K_0'(A \otimes_{k_0} k').$$

But $A \otimes_{k_0} k'$ is etale and finite over A, hence $A \otimes_{k_0} k'$ is regular of dim. 1, hence a product of Dedekind domains. But it also has only fintely many max. ideals, hence $A \otimes_{k_0} k'$ is a product of PID's, hence

$$K_0'(A\otimes_{k_0} k') \xrightarrow{\sim} K_0'(F\otimes_{k_0} k')$$

$F\otimes_{k_0} k'$ being the total quotient ring of $A\otimes_{k_0} k'$. Conclude that any torsion f.t. $A\otimes_{k_0} k'$-module has [redacted] class zero in $K_0'(A\otimes_{k_0} k')$. done.

~~that $k'/P(S)$ is finite + separable. (Zariski-Samuel p. 104 + 105). Put $k_0 = P(S)$ so that k_0 lifts to A. Then we have seen already that k as an $A \otimes_{k_0} k'$ module has trivial image in $K_0(A \otimes_{k_0} k')$, whence the transfer is zero.~~

general. case k is of char. p.

$P = \mathbb{F}_p$

P k' again separable

$R = \mathbb{Z}_{(p)} \hookrightarrow P$

$$\begin{tikzcd}
R_0 \arrow[r] \arrow[d] & A \\
k_0 &
\end{tikzcd}$$

spectral sequence obtaining by filtering coh. sheaves by supports

~~$\mathcal{M}od_f(X)$~~

a

$\cup$

a_1 codim ≥ 1

$\cup$

$K a_1 \to K a \to \amalg$ a_2 codim ≥ 2

$\cup$

$K_i(a/a_1) \xrightarrow{\partial} K_{i-1}(a_1/a_2) \xrightarrow{\partial} K_{i-2}(a_2/a_3) \longrightarrow \dots$

$$\begin{tikzcd}
A \arrow[r] & A/tA \\
 & k \arrow[ul] \arrow[u, "\text{finite}"']
\end{tikzcd}$$

$$A \otimes_k A/tA$$

want to show $[A/tA] \in K_0'(A \otimes_k A/tA)$ is zero
so I should consider

$$\begin{tikzcd}
A \otimes_k A/tA \arrow[r, dashed, shift left] & A/tA \arrow[l, shift left] \\
A \arrow[u] \arrow[ur] & k \arrow[l] \arrow[u]
\end{tikzcd}$$

~~$A\otimes$~~

$$\longrightarrow A/tA \otimes_k A/tA \longrightarrow A/tA \longrightarrow 0$$

$$I \longrightarrow A \otimes_k A \longrightarrow A$$

$$\begin{tikzcd}
0 \arrow[r] & A \arrow[r, "t"] & A \arrow[r] & A/tA \\
 & & & k \arrow[ul] \arrow[u, "\text{finite flat}"']
\end{tikzcd}$$

t non-zero divisor.

typical residue situation

Then one expects that there
One knows in this situation that

res $\left[\begin{smallmatrix}\omega\\ t\end{smallmatrix}\right]$ defined. $\Omega^1_{A/k} \longrightarrow k$.

try
$$\begin{tikzcd}
0 \arrow[r] & A \arrow[r, "t"] & A \arrow[r] & A/tA \arrow[r] & 0 \\
 & & k \arrow[u] \arrow[ru, "\text{finite \& flat}"'] & &
\end{tikzcd}$$

Then A/tA is an $A \otimes_k (A/tA)$ - module
I want to compute its image in the Grothendieck group.

$$A/tA \otimes_{\underline{A}} \Omega^1_{A/k} \xrightarrow{\sim} \mathrm{Hom}_k(A/tA, k).$$

$$\omega \longmapsto \left(a \mapsto \mathrm{res}\begin{bmatrix} a\omega \\ t \end{bmatrix}\right)$$

~~Then suppose that A/tA is clear!!!!~~

~~$\Omega^1_{A/k}$~~

dt

$t = x^n$

$$k \to A \to A/tA$$

$$\to tA/t^2A \to \Omega^1_{A/k} \otimes_A A/tA \to \Omega^1_{A/tA,k} \to 0\ !$$

A/tA $\quad 0 \to I \otimes_A A/tA \to A \otimes_k A/tA \to A/tA \to 0$

$$0 \to I \to A \otimes_k A \to A \to 0$$

under good circumstance

January 6, 1973

Let $\mathcal{M}$ be an exact cat, and $\mathcal{P}$ a full exact subcat. ($\mathcal{P}$ is closed under extensions in $\mathcal{M}$) such that

a) $\forall M, \exists \quad 0 \to P' \to P \to M \to 0$

b) $0 \to M' \to P \to M \to 0 \implies M' \in \mathcal{P}$

(Note: granted a), b) is equivalent to

c) $0 \to M \to P \to P' \to 0 \implies M \in \mathcal{P}$

In effect, assuming a) + c), form

$$\begin{tikzcd}
 & & 0 & 0 \arrow[d] & \\
0 \arrow[r] & M' \arrow[r] & P \arrow[r] \arrow[u] & M \arrow[r] & 0 \\
0 \arrow[r] & M' \arrow[r] \arrow[u, equal] & Z \arrow[r] \arrow[u] \arrow[ur, phantom, "\text{cart}"] & P_1 \arrow[r] \arrow[u] & 0 \\
 & & P_0 \arrow[u] \arrow[r, equal] & P_0 \arrow[u] & \\
 & & & {} \arrow[u] &
\end{tikzcd}$$

then $Z \in \mathcal{P}$ as it is an extension of P by P_0; so $M' \in \mathcal{P}$ by c). Clearly b) $\Rightarrow$ c).)

Definition: $\mathcal{P}$ a full exact subcat of $\mathcal{M}$ will be called ~~[illegible]~~ (left) resolving if

1) $0 \to M \to P \to P' \to 0$ exact in $\mathcal{M} \implies M \in \mathcal{P}$.

2) $\forall M, \exists \quad 0 \to P_n \to \cdots \to P_0 \to M \to 0$.

To show $\mathcal{P}$ resolving $\implies Q(\mathcal{P}) \to Q(\mathcal{M})$ is a h.eq.

Let $\mathcal{M}_n \subset \mathcal{M}$ be the full subcat consisting of M admitting $\mathcal{P}$-resolutions of length $\leq n$. Standard lemma:

Assume $\mathcal{P}^{\subset \mathcal{M}}$ satisfies i) closed under extensions in $\mathcal{M}$
ii) $0 \to M \to P \to P' \to 0$ ex $\Rightarrow M \in \mathcal{P}$
iii) $\forall M, \exists$ $0 \to M' \to P \to M \to 0$ with $P \in \mathcal{P}$.

Let $\mathcal{M}_n \subset \mathcal{M}$ be those M which have $\mathcal{P}$ resolution length $\leq n$

Lemma: Given $0 \to M' \to M \to M'' \to 0$ exact in $\mathcal{M}$. Then we have

a)$_n$ $M'' \in \mathcal{M}_n, M \in \mathcal{M}_{n-1} \Longrightarrow M' \in \mathcal{M}_{n-1}$

b)$_n$ $M' \in \mathcal{M}_{n-1}, M \in \mathcal{M}_n \Longrightarrow M'' \in \mathcal{M}_n$ $(n \geq 1)$

c)$_n$ $M', M'' \in \mathcal{M}_n \Longrightarrow M \in \mathcal{M}_n$

Proof. Induction on n, starting from $n=0$, interpreting $\mathcal{M}_{-1} = \mathcal{M}_0 = \mathcal{P}$. a)$_0$, c)$_0$ are true, but not b)$_0$. Suppose now $(n \geq 1)$:

a)$_n$: Since $M'' \in \mathcal{M}_n$ can find $\exists$

$$\begin{tikzcd}
0 \arrow[r] & M' \arrow[r] & M \arrow[r] & M'' \arrow[r] & 0 \\
 & M' \arrow[u, equal] \arrow[r] & T \arrow[u] \arrow[r] & P^{\in \mathcal{M}_0} \arrow[u] & \\
 & & K \arrow[u] \arrow[r, equal] & K^{\in \mathcal{M}_{n-1}} \arrow[u] &
\end{tikzcd}$$

Apply c)$_{n-1}$ to $K \to T \to M$ to conclude $T \in \mathcal{M}_{n-1}$
Apply a)$_{n-1}$ to $M' \to T \to P$ to conclude $M' \in \mathcal{M}_{n-1}$ as desired.

b)$_n$: Since $M \in \mathcal{M}_n$ $\exists$ $\underset{\mathcal{M}_{n-1}}{\underset{\cap}{K}} \to \underset{\mathcal{M}_0}{\underset{\cap}{P}} \to M$

$$\begin{tikzcd}
0 \arrow[r] & M'^{\in \mathcal{M}_{n-1}} \arrow[r] & M \arrow[r] & M'' \arrow[r] & 0 \\
 & L \arrow[u] \arrow[r] & P^{\in \mathcal{M}_0} \arrow[u] \arrow[r] & M'' \arrow[u, equal] & \\
 & K \arrow[u] \arrow[r, equal] & K^{\in \mathcal{M}_{n-1}} \arrow[u] & &
\end{tikzcd}$$

Apply $c)_{n-1}$ to $K \to L \to M'$ to conclude $L \in \mathcal{M}_{n-1}$, whence $M'' \in \mathcal{M}_n$.

$c)_n$: We know $\exists\ P \twoheadrightarrow M$ for any $M \in \mathcal{M}$ by hyp. iii). Thus get

$$\begin{tikzcd}
0 \arrow[r] & M'^{\in \mathcal{M}_n} \arrow[r] & M \arrow[r] & M''^{\in \mathcal{M}_n} \arrow[r] & 0 \\
 & M' \arrow[u, equal] \arrow[r] & T \arrow[u] \arrow[r] & P^{\in \mathcal{M}_0} \arrow[u] \arrow[ul, dashed, "\exists"'] & \\
 & & K \arrow[u] \arrow[r, equal] & K \arrow[u] &
\end{tikzcd}$$

$T \simeq M' \oplus P$ so $T \in \mathcal{M}_n$.

Apply $a)_n$ to $K \to P \to M''$, get $K \in \mathcal{M}_{n-1}$

Apply $b)_n$ to $K \to T \to M$, get $M \in \mathcal{M}_n$ as desired.

QED.

~~[illegible]~~

Then one has from the resolution thm. keg's

$$Q(\mathcal{M}_{n-1}) \subset Q(\mathcal{M}_n) \subset \cdots$$

Problem: Suppose we have two resolving cats. $\mathcal{P}, \mathcal{Q} \subset \mathcal{M}$. Can they be compared?

Suppose to simplify we assume we are in the situation at the beginning: a) $\forall M.\ \exists\ 0 \to P' \to P \to M \to 0$

c) $0 \to M \to P \to P' \to 0 \Rightarrow M \in \mathcal{P}$.

and similarly for $\mathcal{Q}$. I was not able to show $\mathcal{P} \cap \mathcal{Q} \neq \phi$.

Why this is interesting: suppose I have $F : \mathcal{M} \to \mathcal{A}$ additive which is exact on $\mathcal{P}$ and on $\mathcal{Q}$. Then I would like to have commutativity in

$$\begin{tikzcd}
Q\mathcal{P} \arrow[dr, "\sim"'] \arrow[drr, "F/\mathcal{P}"] & & \\
& Q\mathcal{M} & Q\mathcal{A} \\
Q\mathcal{Q} \arrow[ur, "\sim"] \arrow[urr, "F/\mathcal{Q}"'] & &
\end{tikzcd}$$

Observe that F has a "relative" derived functor L_iF vanishing on $\mathcal{P}$ objects. Thus given M choose $0 \to P' \to P \to M \to 0$ and put

$$L_0F(M) = \text{Coker}\{F(P') \to F(P)\}, \quad L_1F(M) = \text{Ker}\{.\}$$

Note that if we take another choice

$$\begin{tikzcd}
0 \arrow[r] & P' \arrow[r] & P \arrow[r] & M \arrow[r] & 0 \\
0 \arrow[r] & P_1' \arrow[r] \arrow[u, two heads] \arrow[d, dash] & P_1 \arrow[r] \arrow[u, two heads] \arrow[d, dash] & M \arrow[r] \arrow[u, equal] & 0 \\
& K \arrow[r, equal] & K & &
\end{tikzcd}$$

we get that $K \in \mathcal{P}$, and

$$\begin{tikzcd}
0 \arrow[r] & F(K) \arrow[r] \arrow[d, equal] & F(P_1') \arrow[r] \arrow[d] & F(P') \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & F(K) \arrow[r] & F(P_1) \arrow[r] & F(P) \arrow[r] & 0
\end{tikzcd}$$

so get the same Ker & Coker. I will assume without checking that $L_i F$ gives usual long exact sequences.

Now because F is exact on $\mathcal{Q}$, we obtain from

$$\begin{tikzcd}
 & 0 & 0 & & \\
0 \arrow[r] & P' \arrow[r] \arrow[u] & P \arrow[r] \arrow[u] & Q \arrow[r] & 0 \\
0 \arrow[r] & Q_1 \arrow[r] \arrow[u] & Q_0 \arrow[r] \arrow[u] & Q \arrow[r] \arrow[u, "\|"'] & 0 \\
 & Q_2 \arrow[r, equal] \arrow[u] & Q_2 \arrow[u] & & \\
 & 0 \arrow[u] & 0 \arrow[u] & &
\end{tikzcd}$$

that first of all

$$\begin{tikzcd}
 & F(P) \arrow[dr] & \\
F(Q_0) \arrow[ur] \arrow[rr, two heads] & & F(Q)
\end{tikzcd}$$

so $P \twoheadrightarrow Q \implies F(P) \twoheadrightarrow F(Q)$. Dually $Q \twoheadrightarrow P \implies F(Q) \twoheadrightarrow F(P)$. Thus get

$$\begin{tikzcd}
 & F(P') \arrow[r] & F(P) \arrow[r] & F(Q) \arrow[r] & 0 \\
0 \arrow[r] & F(Q_1) \arrow[r] \arrow[u, two heads] & F(Q_0) \arrow[r] \arrow[u, two heads] & F(Q) \arrow[r] \arrow[u, "\|"'] & 0
\end{tikzcd}$$

which implies $L_0F(Q) = F(Q)$. Thus L_1F should be an exact functor from $\mathcal{Q}$ to $\mathcal{A}$ and what clearly ought to be the case is that

$$\boxed{(F|\mathcal{P})_* = (F|\mathcal{Q})_* \,\bar{\bullet}\, (L_1F|\mathcal{Q})_*}$$

Unfortunately, I have no examples yet, of two such $\mathcal{P}$, $\mathcal{Q}$ which have an empty intersection. To produce an example, one must avoid situations where there are projectives, e.g. Once

$$0 \longrightarrow P' \longrightarrow P \longrightarrow Q \longrightarrow 0$$

splits then $Q \in \mathcal{P}$, as it fits into

$$0 \longrightarrow Q \longrightarrow P \longrightarrow P' \longrightarrow 0.$$

<u>Problem</u>: Given $\mathcal{P} \subset \mathcal{M}$ such that

i) $0 \to M \to P \to P' \to 0$ exact in $\mathcal{M} \Rightarrow M \in \mathcal{P}$

ii) $\forall M \quad \exists \quad P \twoheadrightarrow M$.

so that the preceding lemma applies, can one find a ∂-functor T_i to an abelian category $\mathcal{A}$ such that

$$\mathcal{M}_n = \{M \mid T_i M = 0 \text{ for } i > n\}.?$$

This arises because, I propose to prove

<u>Corollary to res. thm</u>: Given a ∂-functor T_i $i \geq 1$ from $\mathcal{M}$ to an abelian category $\mathcal{A}$ which is effaceable [illegible] (if $\mathcal{P}$ = full subcat of $M \ni T_*(M) = 0$, then $\forall M$, $\exists P \twoheadrightarrow M$) and such that $\forall M$ $T_i(M) = 0$ $i >$ some n, then $K_i\mathcal{P} \xrightarrow{\sim} K_i\mathcal{M}$.

Observe first of all that if $T = \{T_i\}$ is given then

$$\mathcal{P} = \{M \mid T_i M = 0 \quad \forall i\}$$

satisfies i); it satisfies ii) when T is effaceable. In any case, it follows that

$$\{M \mid T_i M = 0 \quad i > n\} = \{M \mid M \text{ has a } \overset{\mathcal{P}}{\text{resolution}} \text{ of length} \leq n\}.$$

In effect, choose $0 \to M_n \to P_{n-1} \to \cdots \to P_0 \to M \to 0$ whence

$$T_i(M_n) = T_{i+n}(M) \qquad \forall i \geq 1.$$

so it's clear.

Summary: An effaceable ∂-functor $T = \{T_i\}_{i \geq 1}$ from $\mathcal{M}$ to $\mathcal{A}$ determines a $\mathcal{P} = \mathcal{M}_0$ and we have

$$K_i \mathcal{P} = K_i \mathcal{M}_1 = \cdots = K_i \mathcal{M}_\infty .$$

What I am interested in is the converse - whether any $\mathcal{P}$ occurs in this way. ~~This should reduce by induction~~ Consider the case where $\forall M \; \exists \quad 0 \to P' \to P \to M \to 0$ and try to determine an obstruction for M to be in $\mathcal{P}$. The point is that if $M \in \mathcal{P}$, then this sequence can be split by pushout in $\mathcal{P}$:

$$\begin{tikzcd}
0 \arrow[r] & P' \arrow[r] \arrow[d] & P \arrow[r] \arrow[d] & M \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & P \arrow[r] & Z \arrow[r] \arrow[l, dashed, bend right] & M \arrow[r] & 0
\end{tikzcd}$$

and conversely. Thus consider the ~~[illegible]~~ bifunctor

~~$(M,P) \mapsto$ iso [illegible] extensions~~

$$(M,P) \longmapsto \mathrm{Ext}^1(M,P)$$

the Ext^1 being computed in $\mathcal{M}$. Now keeping M fixed we can take the 0-th derived functor with resp. to P. This gives a functor $J(M,P) \ni$

$$0 \to K(P) \longrightarrow \mathrm{Ext}^1(M,P) \longrightarrow J(M,P) \to C(P) \to 0$$

where K, C are effaceable. ~~Recall R^0 is exact [illegible] and $R^0 F \leftarrow F \leftarrow F$ left exact. Thus if $0 \to M' \to M \to M'' \to 0$ is exact~~ ??

Recall R^0 is exact from $\mathrm{Hom}(\mathcal{P}, \mathcal{A}b)$ to $\mathrm{lex}(\mathcal{P}, \mathcal{A}b)$
This if $0 \to M' \to M \to M'' \to 0$ is exact

$$J(M'', ?) \longrightarrow J(M, ?) \longrightarrow J(M', ?)$$

is exact. To show that the last map is onto it suffices to show the cokernel of

$$\mathrm{Ext}^1(M, P) \longrightarrow \mathrm{Ext}^1(M', P)$$

is effaceable; so given

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & P \arrow[r] \arrow[d] & E \arrow[r] \arrow[d] & M' \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & P_2 \arrow[r] \arrow[d] & E \oplus P_0 \arrow[r] \arrow[d] & M \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & P_1 \arrow[r] \arrow[d] & P_0 \arrow[r] \arrow[d] \arrow[ur, dashed] & M'' \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

Choose $P_0 \twoheadrightarrow M$ and fill in as indicated. This shows that the image of $(0 \to P \to E \to M' \to 0)$ under

$$\mathrm{Ext}^1(M', P) \longrightarrow \mathrm{Ext}^1(M', P_2)$$

lifts back to $\mathrm{Ext}^1(M, P_2)$, as desired. So conclude that

$$J(M'', ?) \longrightarrow J(M, ?) \longrightarrow J(M', ?) \longrightarrow 0$$

is exact, whence I have a left exact functor

$$M \mapsto J(M), \qquad \mathcal{M} \longrightarrow \mathrm{lex}(\mathcal{P}, \mathcal{A}b)^\circ.$$

And I have seen already that $J(M) = 0 \Longleftrightarrow M \in \mathcal{P}$.

<u>General case:</u> Use the functors

$$T_i(M) = R^0\{P \mapsto \mathrm{Ext}^i_{\mathcal{M}}(M, P)\}.$$

The only thing you have to check is that for $i \geq 1$

$$P \longmapsto \mathrm{Ext}^i(P', P)$$

is effaceable. But any element of $\mathrm{Ext}^i_{\mathcal{M}}(P', P)$ is represented by

$$0 \to P \to M_{i-1} \to \cdots \to M_0 \to P' \to 0$$

and by the assumption that $\forall M \; \exists \; P \twoheadrightarrow M$, we can replace this by an equivalent extension such that the $M_j \in \mathcal{P}$. But then the condition that $\mathcal{P}$ is stable under kernels of epims. implies we can extend

$$\begin{tikzcd}
0 \arrow[r] & P \arrow[r] \arrow[d] & M_{i-1} \arrow[r] & \overset{\in \mathcal{P}}{Z_{i-2}} \arrow[r] & 0 \\
& M_{i-1} \arrow[r, dashed, leftrightarrow] & {} & &
\end{tikzcd}$$

and so efface the <u>given</u> element of $\mathrm{Ext}^i_{\mathcal{M}}(P', P)$.

January 10, 1973 transfer.

Let k be a field & consider quasi-proj. n.s. varieties over k. We then have the following operations on K_i:

f^* f arbitrary
f_* f proper
multiplication by $\xi \in K_0$

I want to understand the universal category generated by these operations.

First suppose we are given a family φ of supports on X, i.e. closed sets closed under subsets + finite union. Then we have a Serre subcategory

$$\mathrm{Modf}_\varphi(X) \subset \mathrm{Modf}(X) \quad .= \text{coh. } \mathcal{O}_X \text{ modules}$$

consisting of modules with support in φ. ~~Observe~~ Given $f: X \to Y$ such that each Z in φ is proper over Y, I want to define

$$f_* : K_i(\mathrm{Modf}_\varphi(X)) \longrightarrow K_i(Y)$$

The point is that $\exists$ very ample line bundle $\mathcal{O}(1)$ on X and a vector bundle exact sequence

$$0 \to \mathcal{O}_X \longrightarrow \mathcal{O}(N)^k \longrightarrow J \longrightarrow 0$$

for each N; k, J depend on N. Thus given F in $\mathrm{Modf}_\varphi(X)$, we have an exact sequence

$$0 \longrightarrow F \longrightarrow F(N)^k \longrightarrow J \otimes F \longrightarrow 0$$

such that $R^+ f_* \left(F(N)^k\right) = 0$, ~~[illegible]~~ (apply Serre's thm. to $Z \to X$ where F is an $\mathcal{O}_Z$-mod), so f_* is effaceable on the category $\mathrm{Modf}_\varphi(X)$.

Now given $(X, Y, \xi \in K_0 \,\mathrm{Modf}_{pr/X}(X\times Y))$ we can define a map from $K_i Y$ to $K_i X$ by

$$K_i Y \xrightarrow{pr_2^*} K_i (X\times Y) \xrightarrow{\xi\cdot} K_i \,\mathrm{Modf}_{pr/X}(X\times Y) \xrightarrow{pr_{1*}} K_i X$$

This map is induced by the functor

$$E \longmapsto pr_{1*}(C \otimes_Y E) = pr_{1*}\left(C \otimes_{X\times Y} pr_2^* E\right)$$

where C represents ξ, and E is restricted to suff. positive vector bundles on Y. (Observe can efface $E \mapsto R^+ pr_{1*}(C \otimes_Y E)$ on the cat $\mathcal{P}(Y)$, since we have that $pr_2^* \mathcal{O}_Y(1)$ is ample for $pr_1 : X\times Y \to X$.) Actually it is necessary that we use the difference of two C's to represent ξ.

Now for composition.

$$\begin{tikzcd}
X\times Y\times Z \arrow[rr, "pr_{23}"] \arrow[dr, "pr_{13}"] \arrow[dd, "pr_{12}"'] & & Y\times Z \arrow[r] \arrow[dd] & Z \\
& X\times Z \arrow[urr] \arrow[ddl] & & \\
X\times Y \arrow[rr] \arrow[d] & & Y & \\
X & & &
\end{tikzcd}$$

I want to argue as follows: Given $\xi \in K_0 \,\mathrm{Modf}_{pr/X}(X\times Y)$, $\eta \in K_0 \,\mathrm{Modf}_{pr/Y}(Y\times Z)$, and $z \in K_i Z$, then

$$pr_{1*}\left(\xi \cdot pr_2^* \, pr_{1*}\left(\eta \cdot pr_1^* z\right)\right)$$

$$\| \quad (1)$$

$$pr_{1*}\left(\xi \cdot pr_{12*}\, pr_{23}^*\left(\eta \cdot pr_1^* z\right)\right)$$

$$\| \quad (2)$$

$$pr_{1*}\, pr_{12*}\left(pr_{12}^*\xi \cdot pr_{23}^*\eta \cdot pr_{23}^*\, pr_1^* z\right)$$

$$\|$$

$$pr_{1*}\, pr_{13*}\left(pr_{12}^*\xi \cdot pr_{23}^*\eta \cdot pr_{13}^*\, pr_2^* z\right)$$

$$\|$$

$$pr_{1*}\left(\zeta \cdot pr_2^* z\right) \qquad \zeta = pr_{13*}\left(pr_{12}^*\xi \cdot pr_{23}^*\eta\right)$$

For (1) need commutativity

$$\begin{tikzcd}
K_0\mathrm{Modf}_{pr/X\times Y}(X\times Y\times Z) \arrow[d] & K_0\mathrm{Modf}_{pr/Y}(Y\times Z) \arrow[l] \arrow[d] \\
K_0\mathrm{Modf}_{pr/X}(X\times Y) & K_0\mathrm{Modf}(Y) \arrow[l]
\end{tikzcd}$$

which is OKAY because it is a flat base extension. In general given

$$\begin{tikzcd}
X' \arrow[r, "g'"] \arrow[d, "f'"'] & X \arrow[d, "f"] \\
S' \arrow[r, "g \text{ flat}"'] & S
\end{tikzcd}$$

we have

$$g^* \circ R^q f_* = R^q f'_* \circ g'^*$$

(Cech calculations).

For (2) we need f^* OKAY for products when f is flat, which is alright, and the projection formula, and an assoc. result.

$$\begin{tikzcd}
X \arrow[d, "f"'] & x \in K_i \mathrm{Modf}_{pr/Y}(X) \\
Y & \xi \in K_0 \mathrm{Modf}_{\varphi}(Y)
\end{tikzcd}$$

f_*

?

~~Related problem. Let φ, ψ be two families of supports on X.~~

Problem: The formula $i^* i_*(\alpha) = e(\nu_i)\alpha$ for an embedding.

Do first the case of codimension 1: $i : Y \to X$ where Y is a Cartier divisor, i.e.

$$0 \to I \to \mathcal{O}_X \to \mathcal{O}_Y \to 0$$

and I is a line bundle. Notation: $\mathcal{M} = Mod_f(\mathcal{O}_X)$, $\mathcal{P} \subset \mathcal{M}$ full subcat. of F such that $\underline{Tor}_1^{\mathcal{O}_X}(\mathcal{O}_Y, F) = 0$, $\mathcal{N} = Mod_f(\mathcal{O}_Y)$. The problem is to compute the composite

$$\begin{tikzcd} & Q\mathcal{P} \arrow[r, "P \mapsto \mathcal{O}_Y \otimes_X P"] \arrow[d, "heq"] & Q\mathcal{N} \\ Q\mathcal{N} \arrow[r, hook] & Q\mathcal{M} & \end{tikzcd}$$

and show it coincides with mult. by $1-[I]$.

Method: Start with the full subcat $\mathcal{E}'$ of $\mathcal{E}$ = exact sequences in $\mathcal{M}$ consisting of

$$0 \to M' \to M \to N \to 0$$

with $N \in \mathcal{N}$. Observe that we have a characteristic filtration

$$\begin{tikzcd} 0 \arrow[r] & M' \arrow[r] \arrow[d, no head] & M' \arrow[r] \arrow[d, no head] & 0 \arrow[r] \arrow[d, no head] & 0 \\ 0 \arrow[r] & M' \arrow[r] \arrow[d] & M \arrow[r] \arrow[d] & N \arrow[r] \arrow[d, no head] & 0 \\ 0 \arrow[r] & 0 \arrow[r] & N \arrow[r] & N \arrow[r] & 0 \end{tikzcd}$$

whence

$$Q\mathcal{E}' \xrightarrow{heq} Q\mathcal{M} \oplus Q\mathcal{N}.$$

Next let $\mathcal{E}'' \subset \mathcal{E}'$ consist of exact seq. of the

form
$$0 \longrightarrow P' \longrightarrow P \longrightarrow N \longrightarrow 0.$$
The resolution theorem implies $Q\mathcal{E}'' \xrightarrow{\text{heq}} Q\mathcal{E}'$.

$$\begin{tikzcd}
K_i\mathcal{E}'' \arrow[rrr] \arrow[ddd] & & & K_i\mathcal{P} \arrow[r] \arrow[ddd, "\wr"] & K_i\mathcal{N} \\
& {[P' \rightarrowtail P \twoheadrightarrow N]} \arrow[r, mapsto] \arrow[d, mapsto] \arrow[dr, mapsto] & -[P']+[P] \arrow[rr, mapsto] & & -[\mathcal{O}_Y \otimes_X P'] + [\mathcal{O}_Y \otimes_X P] \arrow[d, equal] \\
& {[N]} & -[P']+[P] & & -[I \otimes_Y N] + [N]. \\
K_i\mathcal{N} \arrow[rrr] & & & K_i\mathcal{M} &
\end{tikzcd}$$

This commutes and as $K_i\mathcal{E}'' \longrightarrow K_i\mathcal{N}$ is onto we win.

General case: Consider the exact category of exact sequences
$$0 \to M_n \to \cdots\cdots \to M_0 \longrightarrow N \longrightarrow 0$$
in $\mathcal{M}$ with $N \in \mathcal{N}$. By the characteristic filtration thm. this has K-groups $(K_i\mathcal{M})^n \oplus K_i\mathcal{N}$. On the other hand, it is clear that any sequence is a quotient of one with all $M_i \in \mathcal{P}$, provided $n \geq$ Tor dim of $\mathcal{O}_Y$ over $\mathcal{O}_X$. So done.

January 11, 1973

Let $\mathcal{P}$ be an additive category in which the Krull-Schmidt theorem holds, and let $\{I_\alpha\}$ be representatives for the isom. classes of indecomposables. Then there is a map

$$\bigoplus_\alpha K_i(\mathrm{End}(I_\alpha)^{op}) \longrightarrow K_i\mathcal{P}$$

($\mathcal{P}$ given the "split" exactness structure). Here is an example to show this map needn't be an isomorphism once $i \geq 0$. Take A to be a d.v.r. and let B be the ring

$$B = \begin{pmatrix} A & A \\ \pi A & A \end{pmatrix} \subset 2\times 2 \text{ matrices over } A$$

I claim that $K_iB = K_iA \oplus K_ik$. Here's why.

Start with the cat $\mathcal{E}$ of exact sequences

$$0 \longrightarrow M' \longrightarrow M \longrightarrow N \longrightarrow 0$$

with M', M in $Mod_f(A)$, $N \in Mod_f(k)$. Have a canonical filtration which shows that

$$K_i\mathcal{E} \xrightarrow{\sim} K_iA \oplus K_ik$$
$$[M' \rightarrowtail M \twoheadrightarrow N] \longmapsto [M'] + [N].$$

On the other hand any such sequence is a quotient of one where M is free. Call this subcategory $\mathcal{P}$. Every object is a direct sum of copies of the following

$$P: \quad 0 \longrightarrow A \xrightarrow{1} A \longrightarrow 0 \longrightarrow 0$$

$$Q: \quad 0 \longrightarrow A \xrightarrow{\pi} A \longrightarrow k \longrightarrow 0$$

and its clear that there can be no non-trivial extensions. Thus every exact sequence in $\mathcal{P}$ splits, and so $\mathcal{P}$ is equivalent to the category of projective f.g. mods over

$$\mathrm{End}(P\oplus Q)^{o}$$

Now consider the faithful functor $\mathcal{P} \to \mathcal{P}(A)$; it gives

$$\begin{array}{ccc} \mathrm{End}(P\oplus Q) & \subset & \mathrm{End}(A\oplus A) \\ \| & & \| \\ B = \begin{pmatrix} A & A \\ \underset{\overset{\|}{\mathrm{Hom}(P,Q)}}{\pi A} & A \end{pmatrix} & \subset & \begin{pmatrix} A & A \\ A & A \end{pmatrix} \end{array}$$

There are two homomorphisms

$$B \rightrightarrows \mathrm{End}(A^2)$$

$$\begin{pmatrix} a & b \\ c\pi & d \end{pmatrix} \longmapsto \begin{pmatrix} a & b \\ c\pi & d \end{pmatrix} \text{ and } \begin{pmatrix} a & b\pi \\ c & d \end{pmatrix}$$

which are conjugate by the matrix $\begin{pmatrix} \pi & \\ & 1 \end{pmatrix}$

$$\begin{pmatrix} \pi & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} a & b \\ c\pi & d \end{pmatrix}\begin{pmatrix} \pi^{-1} & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} \pi & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} a\pi^{-1} & b \\ c & d \end{pmatrix} = \begin{pmatrix} a & b\pi \\ c & d \end{pmatrix}$$

These two homos. represent the B action on the total module and submodule of an exact sequence in $\mathcal{P}$. To prove $K_i k \to K_i A$ is zero, must show the two maps $K_i B \rightrightarrows K_i A$ are the same.

Next suppose we have

$$0 \longrightarrow L_1 \xrightarrow[d]{} L_0 \longrightarrow V \longrightarrow 0$$

with L_i free and V a k-module. Since $pV=0$ one knows $\exists$ $h: L_0 \longrightarrow L_1$ such that

$$dh = p \cdot id_{L_0}$$
$$hd = p \cdot id_{L_1}$$

$p = \pi$

~~I claim that~~

$$\cancel{L_0 \xrightarrow{h} L_1}$$

~~has cokernel canonically isomorphic to V. In effect given $v \in V$ lift it to $x \in L_0$, apply h, and look at the image in Coker(h)~~

Now for

$P: \; 0 \longrightarrow A \xrightarrow{1} A \longrightarrow 0 \longrightarrow 0$ we have $\text{Coker}(h) = k$

$Q: \; 0 \longrightarrow A \xrightarrow{\pi} A \longrightarrow 0 \longrightarrow 0$ ———————— $= 0$.

Thus we have an exact functor

$$(0 \to L_1 \to L_0 \to V \to 0) \longmapsto \text{Coker}(h)$$

from $\mathcal{P}$ to k-modules. But

$$\text{Coker}(h) = \text{Im}\, \bar{d}$$

and

$$0 \to \text{Im}\, \bar{d} \to \bar{L}_0 \to V \to 0$$

so we don't get anything interesting

January 13, 1973. transfer

Given rings A, B let $\mathrm{Corr}(A,B)$ denote the additive category of $A \otimes_{\mathbb{Z}} B^{\circ}$ modules which are perfect over A with the usual notion of exact sequence for modules. Any $X \in \mathrm{Corr}(A,B)$ gives rise to an exact functor

$$P(B) \longrightarrow \mathrm{Modperf}(A) \qquad P \mapsto X \otimes_B P$$

hence to a homomorphism of K-groups

$$K_i B \longrightarrow K_i A.$$

Let $\mathrm{Corr}'(A,B)$ denote $A \otimes_{\mathbb{Z}} B^{\circ}$ modules which are of finite Tor dim$^+$ over B and f.t. over A. (Assume the rings are noetherian). Then $X \in \mathrm{Corr}'(A,B)$ induces

$$\begin{pmatrix} B\text{-modules of f.t.} \\ \text{flat wrt } M \end{pmatrix} \xrightarrow{X \otimes_B ?} \mathrm{Modf}(A).$$

+ hence a homomorphism of K-groups

$$K_i' B \longrightarrow K_i' A.$$

(+ enough to suppose $\forall$ B-mod M of f.t. that $\mathrm{Tor}_i^B(X, M) = 0$ for i large depending on M)

If A is regular, $\mathrm{Corr}(A,B)$ is abelian.

Note that the above construction provides products

$$K_0(\mathrm{Corr}(A,B)) \otimes K_i B \longrightarrow K_i A$$

in virtue of the exact sequence theorem. Now I want to show that composition does not produce new operations when things are regular. Thus suppose we have three rings A, B, C. [redacted] Then we have

$$(*) \qquad K_0(\mathrm{Corr}(A,B)) \otimes K_0(\mathrm{Corr}(B,C)) \longrightarrow K_0(\mathrm{Corr}(A,C)).$$

In effect given $X \in \mathrm{Corr}(A,B)$ and $Y \in \mathrm{Corr}(B,C)$, then

$$\mathrm{Tor}_i^B(X, Y) \in \mathrm{Corr}(A,C)$$

assuming A regular. Because if

$$(**) \qquad 0 \to P_n \to \cdots \to P_0 \to Y \to 0$$

is a B-resolution, then

$$\mathrm{Tor}_i^B(X,Y) = H_i(X \otimes_B P_\cdot)$$

and the complex $X \otimes_B P_\cdot$ is [redacted] f.g. over A. Thus we can define $(*)$ by

$$[X] \otimes [Y] \longmapsto \sum_{i \geq 0} (-1)^i \, \mathrm{Tor}_i^B(X,Y).$$

Now you want to prove the commutativity of

$$\begin{tikzcd}
K_0 \mathrm{Corr}(A,B) \otimes K_0 \mathrm{Cor}(B,C) \otimes K_i C \arrow[r] \arrow[d] & K_0 \mathrm{Corr}(A,B) \otimes K_i B \arrow[d] \\
K_0 \mathrm{Corr}(A,C) \otimes K_i C \arrow[r] & K_i A
\end{tikzcd}$$

To do this, would be easy if we could get C to act on the resolution $(**)$. For then

$$\mathcal{P}(C) \ni P \longmapsto [redacted] \; Y \otimes_C P$$

would be the alternating sum of

$$P \longmapsto P_i \otimes_C P \in \mathcal{P}(B)$$

so the composition with $Q \in \mathcal{P}(B)$, $Q \mapsto X \otimes_B Q$ would be the alternating sum of

$$P \longmapsto (X \otimes_B P_i) \otimes_C P$$

which would be the same as the alternating sum of

$$P \longmapsto \mathrm{Tor}_i^B(X, Y) \otimes_C P .$$

~~But we have to proceed as follows. Suppose we can find a differential graded ring resolution of C:~~

~~$0 \to \Lambda_n \to \quad \to \Lambda_1 \to \Lambda_0 \to C \to 0$~~

~~such that~~

~~$0 \to P_n \to \cdots \to P_1 \to P_0 \to Y \to 0$~~

~~is d.g.r. [illegible] $B \otimes \Lambda^{op}$ module resolving the $B \otimes C^{op}$ module N.~~

~~Then for any $Q \in \mathcal{P}(C)$~~

~~$P_\cdot \otimes_\Lambda Q \to N \otimes_C Q$~~

~~Consider the exact category $\mathcal{E}$ of d.g. Λ-module~~

~~$0 \to M_n \to \cdots\cdots \to M_0$~~

~~acyclic in degrees > 0. [illegible] By the char. filt. thm. one has~~

~~[illegible]~~

We can clearly suppose that $C = \mathrm{End}_B(Y)^{\circ}$. Let

$$0 \to P_n \to \cdots\cdots \to P_0 \to Y \to 0$$

be a fixed $\mathcal{P}(B)$-resolution of Y and put $\Lambda =$ the d.g. ring

$$0 \longrightarrow \mathrm{Hom}_n(P_\cdot, P_\cdot) \to \cdots \to \mathrm{Hom}_1(P_\cdot, P_\cdot) \longrightarrow \mathrm{Hom}_0(P_\cdot, P_\cdot)$$

which is a resolution of C. Now if $Q_\cdot$ is a proj f.g. Λ-module

$$P_\cdot \otimes_{\Lambda_\cdot} Q_\cdot$$

is a ~~[illegible]~~ perfect complex of B-modules and

$$H_i\left(X \otimes_B P_\cdot \otimes_{\Lambda_\cdot} Q_\cdot\right) = \mathrm{Tor}_i^B(X,Y) \otimes_C H_0(Q_\cdot)$$

so what I want to do is show that ~~[illegible]~~ the K-theory of d.g. Λ-modules $Q_\cdot$ ~~[illegible]~~ flat f.t. over $\Lambda_\cdot$ with $H_0(Q_\cdot)$ only non-zero group maps onto the K-theory of C.

?

January 19, 1973.

Let C be a category. I recall the basic cohomological object attached to C which is a homotopy-invariant is the full subcat $D_{lc}(C)$ of $D(C)$ (sheaves = functors $C \to$ sets). The way to think about it is to imagine there is a gadget C_h with a heq $i : C \to C_h$ such that

$$i^* : D(C_h) \longrightarrow D(C)$$

is fully faithful with image $D_{lc}(C)$. If this were the case, then we have adjoint functors

$$\begin{tikzcd} D(C_h) \arrow[r, shift left=2, "i_!"] & D(C) \arrow[l, "i^*" description] \arrow[l, phantom, shift right=2] \end{tikzcd}$$

$$\begin{tikzcd} D(C_h) \arrow[r, shift left=3, "i_!"] \arrow[r, shift right=3, "i_*"'] & D(C) \arrow[l, "i^*"'] \end{tikzcd}$$

(with suitable amplitude conditions perhaps). Thus we have for any F in $D(C)$ universal maps

$$i^{*}_{!}\, i_{*} F \longrightarrow F \longrightarrow i^{*}\, i_{!} F .$$

from and to an object of D_{lc}.

Here is another possibility. Recall we have the "subdivision" category $S(C)$ (= cof cat over $C \times C^o$ belonging to functor $(Y,X) \longmapsto \mathrm{Hom}(X,Y)$. Then have

$$\begin{tikzcd} & D(S(C)) & \\ D(C) \arrow[ur, "t^*"] & & D(C^o) \arrow[ul, "s^*"'] \\ & D_{lc}(C) = D_{lc}(C^o) \arrow[ul] \arrow[ur] & \end{tikzcd}$$

which is an intersection diagram. I review this:

t^* is fully-faithful because

$$\mathrm{Hom}(t^*K, t^*L) = \mathrm{Hom}(t_!\, t^*K, L) = \mathrm{Hom}(K, L);$$

indeed because t is cofibred

$$L_p t_!(F)(Y) = H_p(t/Y, F)$$

$$= H_p(t^{-1}(Y), F)$$

so

$$L_p t_!(t^*F)(Y) = H_p(t^{-1}(Y), t^*F) = \begin{cases} 0 & p>0 \\ F(tY) & p=0 \end{cases}$$

because $t^{-1}(Y)$ is contractible.

Once this known, it is formal that (at least for bounded complexes) the essential image of t^* is complexes whose homology sheaves come from $\mathcal{C}$. Observe that $F : S(\mathcal{C}) \longrightarrow \mathcal{C}'$ is of the form t^*G iff for every arrow $(X \to Y) \longrightarrow (X' \to Y')$ such that $Y \xrightarrow{\sim} Y'$ we have $F(X \to Y) \xrightarrow{\sim} F(X' \to Y')$. Similarly it is of the form s^*G when $X' \xrightarrow{\sim} X \Longrightarrow F(X \to Y) \xrightarrow{\sim} F(X' \to Y')$. Thus if a sheaf on $S(\mathcal{C})$ comes from $\mathcal{C}$ and from $\mathcal{C}^o$ it must be locally constant as the general map factors

$$(X \to Y) \longrightarrow (X \to Y') \longrightarrow (X' \longrightarrow Y')$$

Now the idea was to try the following process

$$F \longmapsto s_!\, t^*F \longmapsto t_!\, s^* s_!\, t^* F$$

and to iterate.

$$\begin{tikzcd}[column sep=small]
 & & S(\mathcal{C})\times_{\mathcal{C}^\circ}S(\mathcal{C}) \arrow[dl, "p_1"'] \arrow[dr, "p_2"] & & \\
 & S(\mathcal{C}) \arrow[dl, "t"'] \arrow[dr, "s"] & & S(\mathcal{C}) \arrow[dl, "s"'] \arrow[dr, "t"] & \\
\mathcal{C} & & \mathcal{C}^\circ & & \mathcal{C}
\end{tikzcd}$$

Since s cofibred, $s_!$ commutes with basechange so we have $s^*s_! = p_{1!}p_2^*$. Thus we are using the correspondence

$$\begin{tikzcd}
S(\mathcal{C})\times_{\mathcal{C}^\circ}S(\mathcal{C}) \arrow[r, "tp_2"] \arrow[d, "tp_1"'] & \mathcal{C} \\
\begin{matrix}\cancel{S(\mathcal{C})}\\ \mathcal{C}\end{matrix} &
\end{tikzcd}$$

Observe that $S\mathcal{C}\times_{\mathcal{C}^\circ}S\mathcal{C}$ consists $X \leftarrow Z \rightarrow Y$ with maps

$$\begin{tikzcd}
X \arrow[d] & Z \arrow[l] \arrow[r] & Y \arrow[d] \\
X' & Z' \arrow[l] \arrow[r] \arrow[u] & Y'
\end{tikzcd}$$

Thus we have the cat of triples $(Z, (X,Y), (Z,Z)\rightarrow(X,Y))$ in which the maps are

$$\begin{tikzcd}
Z & (X,Y) \arrow[d] \\
Z' \arrow[u] & (X',Y')
\end{tikzcd}$$

It is therefore cofibred over $(\mathcal{C}\times\mathcal{C})\times\mathcal{C}^\circ$ assoc. to $(X,Y,Z)\mapsto \mathrm{Hom}(Z,X\times Y)$

Unfortunately there does not seem to be a functor from F to $t_! s^* s_! t^* F$ or the other way. Thus we have

$$t_! s^* s_! t^* F \longleftarrow t_! t^* F \longrightarrow F.$$

Other possibility. Take standard factorization

$$\begin{tikzcd}
\mathcal{C} \arrow[rr, "\Delta"] \arrow[dr] & & \mathcal{C}\times\mathcal{C} \\
& M_\Delta \arrow[ur, "p"'] &
\end{tikzcd}$$

$$\begin{tikzcd}
(Z, \arrow[d] & (X,Y), \arrow[d] & (Z,Z) \arrow[r] \arrow[d] & (X,Y)) \arrow[d] \\
(Z', & (X',Y'), & (Z',Z') \arrow[r] & (X',Y'))
\end{tikzcd}$$

Anyway use M_Δ as your correspondence

$$\begin{tikzcd}
\mathcal{C} \arrow[dr, "i"] \arrow[drr, bend left, "id"] \arrow[ddr, bend right, "id"'] & & \\
& M_\Delta \arrow[r, "p_2"] \arrow[d, "p_1"] & \mathcal{C} \\
& \mathcal{C} &
\end{tikzcd}$$

and you have that there is a map

$$p_{1!} p_2^* \longleftarrow p_{1!} i_! i^* p_2^* = id$$

and
$$(p_{1!} p_2^* F)(Y) = \varinjlim_{Y \leftarrow Z \rightarrow X} F(X)$$

Conjectures: [redacted] $\varinjlim_{n} (p_{1!}\, p_2^*)^n F \in D_{lc}(\mathcal{C})$

[redacted]

$$\begin{tikzcd}
\mathcal{C} \arrow[dr, "i"] & & & \\
& \mathcal{M}_{\Delta} \arrow[dr, "p"] \arrow[ddr, bend right, "p_1"'] \arrow[drr, bend left, "p_2"] & & \\
& & \mathcal{C}\times\mathcal{C} \arrow[r, "pr_2"] \arrow[d, "pr_1"'] & \mathcal{C} \\
& & \mathcal{C} &
\end{tikzcd}$$

But it is much simpler, i.e.

$$p_{1!}p_2^* = pr_{1!}\, p_!\, p^*\, pr_2^*$$

and because p is cofibred

$$(p_! p^* F)(X,Y) = \varinjlim_{p^{-1}(X,Y)} F = \varinjlim_{\Delta Z \to (X,Y)} F(X,Y)$$

$$= \Big(\varinjlim_{\Delta Z \to (X,Y)} \mathbb{Z}\Big) \otimes F(X,Y)$$

$$= \text{[redacted]}\ (p_! \mathbb{Z} \otimes F)(X,Y)$$

and $p_! \mathbb{Z}^{(X,Y)} =$ [redacted] $\varinjlim_{\Delta Z \to (X,Y)} \mathbb{Z} = \Delta_! \mathbb{Z}^{(X,Y)}$.

Thus what we are doing is considering the operator

$$F \longmapsto pr_{1!}(\Delta_! \mathbb{Z} \otimes pr_2^* F)$$

~~[illegible]~~

$$\left(X \longmapsto \varinjlim_{Y} \left(\varinjlim_{\Delta Z \to (X,Y)} \mathbb{Z}, \right) \otimes F(Y) \right)$$

The conjecture on page 5 amounts to this:

If $F \xrightarrow{\sim} pr_{1!}(\Delta_! \mathbb{Z} \otimes pr_2^* F)$, then $F \in D_{lc}(\mathcal{C})$.

<u>Example</u>: Let $\mathcal{C}$ be the ordered set of simplices in a simplicial complex K. Then the category of Z included both in X and Y is either empty or it has a final object. Thus

$$L_p \varinjlim_{\Delta Z \to (X,Y)} \mathbb{Z} = \begin{cases} 0 & p > 0 \\ \mathbb{Z} & p = 0 \quad X \cap Y \neq \phi \\ 0 & p = 0 \quad X \cap Y = \phi . \end{cases}$$

Therefore it would seem that the operation takes F into

$$X \longrightarrow L \varinjlim_{Y \cap X \neq \phi} F(Y)$$

But observe: If $X \subset X'$ then we have commutative diagram

$$\begin{tikzcd}
F(X) \arrow[r, "\rho_X"] \arrow[d] & \varinjlim_{Y \cap X \neq \phi} F(Y) \arrow[d] \\
F(X') \arrow[r, "\rho_{X'}"] \arrow[ur] & \varinjlim_{Y \cap X' \neq \phi} F(Y)
\end{tikzcd}
\qquad
\begin{tikzcd}
\{X\} \arrow[r, phantom, "\subset"] \arrow[d] & \{Y \mid Y \cap X \neq \phi\} \arrow[d] \arrow[dl, phantom, "\Downarrow"] \\
\{X'\} \arrow[r, phantom, "\subset"] \arrow[ur] & \{Y \mid Y \cap X' \neq \phi\}
\end{tikzcd}$$

Thus if $\rho_X, \rho_{X'}$ are isomorphisms we conclude $F(X) \xrightarrow{\sim} F(X')$.

But this argument ought to work in general. Suppose F is a complex such that $F \xrightarrow{\sim} p_{1!}\, p_2^* F$.

?

~~i.e. for any X we have a quis~~

$$\begin{tikzcd}
F(X) \arrow[r] & L\varinjlim_{X \leftarrow Z \rightarrow Y} F(Y)
\end{tikzcd}$$

$$\begin{tikzcd}
pt = \{\overset{fix}{X}\} \arrow[r, "X \mapsto (X \Leftarrow X \Rightarrow X)"] \arrow[d] & \{\overset{fix}{X} \leftarrow Z \rightarrow Y\} \arrow[d] \\
pt = \{X'\} \arrow[r, "X' \mapsto (X' \Leftarrow X' \Rightarrow X')"'] \arrow[ur, "X' \mapsto (X \Leftarrow X \rightarrow X')"] & \{\overset{fix}{X'} \leftarrow Z \rightarrow Y\}
\end{tikzcd}$$

~~represents by the map~~

$$\begin{tikzcd}
X \arrow[r, equal] \arrow[d] & X \arrow[r, equal] \arrow[d] & X \arrow[d] \\
X' & X^{\#} \arrow[l, Leftarrow] \arrow[r, equal] & X^{\#}
\end{tikzcd}$$

January 24, 1973 (David is 9 today)

Suppose $\mathcal{S}$ is a groupoid with a $+$ and assume for each $X \in \mathcal{S}$ that $S \mapsto S+X$ is a faithful functor from $\mathcal{S}$ to itself. This means I can form the translation category $\mathcal{S}/\mathcal{S}$: a map $X \to X'$ is an isom. class of (S, α), $\alpha : S+X \xrightarrow{\sim} X'$. The point is that two isos. $(S,\alpha) \rightrightarrows (S',\alpha')$ θ_1, θ_2

$$\begin{tikzcd}
S+X \arrow[dr, "\alpha", "\sim"'] \arrow[dd, shift left] \arrow[dd, shift right] & \\
& X' \\
S'+X \arrow[ur, "\sim", "\alpha'"'] &
\end{tikzcd}$$

faithfulness of $S \mapsto S+X$ implies $\theta_1 = \theta_2$. In fact we can generalize this:

Suffices to have $\mathcal{S}$ acting on a cat $\mathcal{X}$, such that $S \mapsto S+X$ from $\mathcal{S}$ to $\mathcal{X}$ is faithful and such that every map in $\mathcal{X}$ is a monomorphism. Then I can consider the cat whose objects ~~are pairs~~ are the same as those of $\mathcal{X}$ but in which a map $X \to X'$ is an isom. class of pairs (S,α), where

$$\alpha : S+X \longrightarrow X'$$

~~Examples of interest. Take [illegible]~~
~~[illegible] the maximal groupoid of $\mathcal{M}$.~~

To form $Q(\mathcal{S})$ one wants to let $\mathcal{S} \times \mathcal{S}$ act on $\mathcal{S}$. Thus one needs to know that for any $T \in \mathcal{S}$

$$(S_1, S_2) \longmapsto S_1 + S_2 + T$$

is faithful.

Now suppose that we have a map of groupoids with +, $f: \mathcal{S} \to \mathcal{S}'$ which is such that for any $X \in \mathcal{S}'$, $S \mapsto fS + X$ from $\mathcal{S}$ to $\mathcal{S}'$ is faithful. Then I can form $\mathcal{S}'/\mathcal{S}$ with same objects as $\mathcal{S}'$ but in which the maps $X \to X'$ are ~~iso~~ classes of pairs (S, α) $\alpha : fS + X \xrightarrow{\sim} X'$. I want to show that when f is cofinal then we have a fibration

$$\mathcal{S}'/\mathcal{S} \longrightarrow Q\mathcal{S} \longrightarrow Q\mathcal{S}'.$$

So the idea will be to follow your proof of the localization theorem for $S^{-1}A$.

Over $Q\mathcal{S}'$ I have $\mathcal{E}$ the fibred cat which consists of $P \oplus V \twoheadrightarrow V$, i.e. pairs of objects of $\mathcal{S}'$. If I give a map in $Q\mathcal{S}'$:

$$V' \twoheadleftarrow R \oplus V' \hookrightarrow R \oplus V' \oplus S \simeq V.$$

i.e. $R \oplus V' \oplus S \xrightarrow{\sim} V$, then the base change sends $P \oplus V \twoheadrightarrow V$ to

$$P \oplus R \oplus V' \longrightarrow V'.$$

Thus we are adding the kernel R to P.

The pull-back over $Q\mathcal{S}$ of $\mathcal{E}$ consists of pairs (X,S), $X \in \mathcal{S}'$, $S \in \mathcal{S}$ with morphisms

$$\begin{tikzcd}
X+S_0+S' \arrow[d] & X+S \\
S_0 \oplus S' \arrow[d, dashed] & S \\
S' &
\end{tikzcd}$$

consisting of a map $S_0+S'+S_1 \xrightarrow{\sim} S$ in $Q\mathcal{S}$ from S' to S, and a map $X+S_0 \xrightarrow{\sim} X$. It is thus what I might think of as

$$\mathcal{S}'\times\mathcal{S} \,/\, \mathcal{S}\times\mathcal{S}$$

~~(x,s)~~

$$(S_0, S_1)(x, S) = (s_1+x,\ s_0+s+s_1)$$

Now we can project this gadget onto the first factor $(X,S) \longmapsto X \in \mathcal{S}'/\mathcal{S}$. Probably cofibred. The fibre is clearly the translation cat. So we get a heq.

So it remains to show the square

$$\begin{tikzcd}
\mathcal{S}'/\mathcal{S} & \mathcal{F} \arrow[l, "heq"'] \arrow[r] \arrow[d] \arrow[dr, phantom, "cart"] & \mathcal{E} \arrow[d] \\
& Q\mathcal{S} \arrow[r] & Q\mathcal{S}'
\end{tikzcd}$$

is homotopy cartesian. One considers translation by elements of $\mathcal{S}'$ on the fibres; suffices to show such a translation gives a heq of $\mathcal{F}$. The point is the action by an element of $\mathcal{S}$ is homotopic to identity on $\mathcal{S}'/\mathcal{S}$, so that because $\mathcal{S}$ is cofinal in $\mathcal{S}'$, at least the action of $\mathcal{S}'$ on $\mathcal{S}'/\mathcal{S}$ is invertible, so we are done.

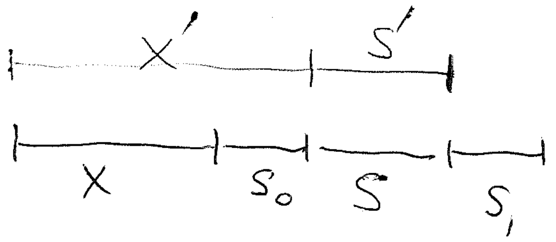
Can you directly show that

$$\mathcal{S}^{-1}\mathcal{S} \longrightarrow \mathcal{S}^{-1}\mathcal{S}' \longrightarrow \mathcal{S}'/\mathcal{S}$$

is a fibration, where $\mathcal{S}^{-1}\mathcal{S}' = \mathcal{S}\times\mathcal{S}'/\mathcal{S}$? Do this as follows. Over $\mathcal{S}'/\mathcal{S}$ we consider the previous category $\mathcal{F}$ which consists of pairs (X,S) and the maps $(X,S) \longrightarrow (X',S')$ are

$$S_0 + X \xrightarrow{\sim} X'$$
$$S_0 + S + S_1 \xrightarrow{\sim} S'$$

In fact this is the arrow category of $\mathcal{S}'/\mathcal{S}$, i.e. an arrow in $\mathcal{S}'/\mathcal{S}$ can be viewed as a pair (X,S) with source X, target $X * S$



??

(Probably the old sign problem).

In any case consider the arrow cat$^{\mathcal{A}}$ of $\mathcal{S}'/\mathcal{S}$ $X+S \xrightarrow{\sim} X'$. It is fibred over $Q\mathcal{S}$, the fibre being $\mathcal{S}$! So localize wrt $\mathcal{S}$; form $\mathcal{S}^{-1}\mathcal{A}$.

January 27, 1973

Lemma: Let $A \longrightarrow B$ be a flat local homomorphism of local noetherian rings. Then $\dim(A) \leq \dim(B)$.

Better: If $f: X \longrightarrow Y$ is a flat morphism, then codim $(x) \geq$ codim $f(x)$.

~~Have to prove this: If $y > f(x)$, that is $\overline{\{y\}} > \overline{\{f(x)\}}$, then there exists $x' > x$ such that [illegible] $y \geq f(x')$. Can assume by base extension that Y is [illegible] a be~~

What we have to show is that given a chain

$$y_p > \cdots > y_2 > y_1 = f(x)$$

that there exists $x_p > \cdots > x_1 = x$ such that $y_i \geq f(x_i)$. Can thus reduce to showing that given $y' > f(x)$, $\exists$ $x' > x$ with $y' \geq f(x')$. Can assume $X = \operatorname{Spec}(\mathcal{O}_x)$, $Y = \operatorname{Spec}(\mathcal{O}_{f(x)})$. Thus I have a local homo $A \longrightarrow B$ and a prime $\mathfrak{p} < m_A$ and I want to find $\mathfrak{q} < m_B$ such that $\mathfrak{p} \subset f^{-1}(\mathfrak{q})$. Can replace A by $A/\mathfrak{p}$ and B by $B/\mathfrak{p}B$. Then A is integral, and what must be shown is that m_B is not a minimal prime ideal in B, for then we can take $\mathfrak{q}$ to be a minimal prime ideal. But $0 < m_A \Rightarrow m_A$ contains a non-zero divisor t, hence by flatness t is a non-zero-divisor in B, so m_B contains a non-zero divisors, so it isn't minimal.

Example of a local domain A of dim 1 such that $F^* \xrightarrow{\varphi} \mathbb{Z}$ is not onto.

~~Example. $A \subset \mathbb{C}[[t]]$ subring over $\mathbb{R}$ generated by t, it. Thus $A = \mathbb{R}[$~~ [illegible]

~~Let A be the local ring at 0 of $\mathbb{R}[x,y]/(x^2+y^2)$. Then A is the subring of $\mathbb{C}$~~

$\mathbb{R}[x,y]/(x^2+y^2)$ is a domain. It may be veiwed as the subring of $\mathbb{C}[x]$ generated over $\mathbb{R}$ by x and $y = ix$. Thus $\mathbb{C}[x]$ is the integral closure of $\mathbb{R}[x,y]/(x^2+y^2)$ in its quotient field.

Let $A = \mathbb{R}[x,y]/(x^2+y^2)$ localized at the max ideal (x,y); [illegible] then $\bar{A}$ is $\mathbb{C}[x]$ localized at the max. ideal x. Thus we have a map of exact sequences

$$\begin{tikzcd}
{} \arrow[r] & K_1'A \arrow[r] & K_1F \arrow[r] & K_0\mathbb{R} \arrow[r] & K_0'A \arrow[r] & K_0F \arrow[r] & 0 \\
{} \arrow[r] & K_1\bar{A} \arrow[r] \arrow[u] & K_1F \arrow[r] \arrow[u, equal] & K_0\mathbb{C} \arrow[r] \arrow[u] & K_0\bar{A} \arrow[r] \arrow[u] & K_0F \arrow[r] \arrow[u, equal] & 0
\end{tikzcd}$$

Since $K_0\mathbb{C} \longrightarrow K_0\mathbb{R}$ is iso to $\mathbb{Z} \xrightarrow{2} \mathbb{Z}$ ~~it~~ follows that the transfer $K_0\mathbb{R} \longrightarrow K_0'A$ is not zero. ~~Similarly~~ $K_1\mathbb{C} \longrightarrow K_1\mathbb{R}$ is $\mathbb{C}^* \xrightarrow{\text{norm}} \mathbb{R}^*$ ~~so~~ also the transfer $K_1\mathbb{R} \longrightarrow K_1'A$ is not zero.

January 28, 1973 examples connected with Gersten's conjecture

Examples 1: Let A be the local ring on a (irred. reduced) curve over k alg. closed, F its quotient field, $\bar{A}$ the integral closure of A in F. Then we have

$$\begin{tikzcd} A \arrow[r, "\text{finite}"] \arrow[dr] & \bar{A} \arrow[d, "\exists"] \\ & k \end{tikzcd}$$

and $\bar{A}$ is a semi-local P.I.D. Since we know the transfer $K_i(k) \longrightarrow K_i(\bar{A})$ is zero it follows that the tranfer $K_i(k) \longrightarrow K_i'(A)$ is zero, so we have exact sequences

$$\begin{tikzcd} 0 \arrow[r] & K_i'A \arrow[r] & K_iF \arrow[r] & K_{i-1}k \arrow[r] & 0 \\ 0 \arrow[r] & K_i\bar{A} \arrow[r] \arrow[u] & K_iF \arrow[r] \arrow[u, phantom, "\parallel"] & \coprod_{m} K_{i-1}k \arrow[r] \arrow[u] & 0 \end{tikzcd}$$

2) Let A be the local ring of an isolated singular point on a surface. Then we get an exact sequence

$$\begin{tikzcd} K_1'(A) \arrow[r] & K_1(F) \arrow[r] \arrow[d, phantom, "\parallel"] & K_0(\text{torsion } A\text{-mods}) \arrow[r] & K_0'(A) \arrow[r] & K_0'(F) \arrow[r] \arrow[d, phantom, "\parallel"] & 0 \\ & F^* & & & \mathbb{Z} & \end{tikzcd}$$

and

$$\longrightarrow K_0(A/m) \longrightarrow K_0(\text{torsion } A\text{-mods}) \longrightarrow \coprod_{ht(p)=1} K_0'(k(p)) \longrightarrow 0$$

This is zero as it factors through

$K_i(k) \longrightarrow K_i(A/p)$ which is zero by example 1).

Thus we get

$$K_1'A \longrightarrow F^* \xrightarrow{\partial} \coprod_{ht(\mathfrak{p})=1} \mathbb{Z} \longrightarrow K_0'(A) \longrightarrow K_0'(F) \longrightarrow 0.$$

Since ∂ should assign to a rational function its divisor, it follows that when ∂ is not onto, i.e. A not a UFD that $K_0'(A) \longrightarrow K_0'(F) = \mathbb{Z}$ is not injective. Thus the Gersten conjecture doesn~~'t~~ hold in this case.

Example of $\mathbb{R}[[x,y]]/(x^2+y^2) \subset \mathbb{C}[[x]]$

more generally of $B \subset A$ d.v.r

$m(B) = m(A)$

$k(B) < k(A)$ finite ext.

shows that $\exists$ local domains B of dim 1 $\ni$ transfer $K_0(k) \longrightarrow K_0(B)$

not zero.

$$\begin{tikzcd}
 & \mathbb{Z} \arrow[d, equal] & & \mathbb{Z} \arrow[d, equal] & \\
K_1(F) \arrow[r, "\partial"] \arrow[d, equal] & K_0(k(A)) \arrow[r, "0"] \arrow[d, "{\text{mult by } [k(A):k(B)]}"'] & K_0(A) \arrow[r, hook] \arrow[d] & K_0(F) \arrow[d, equal] \\
K_1(F) \arrow[r] & K_0(k(B)) \arrow[r] & K_0'(B) \arrow[r] & K_0(F)
\end{tikzcd}$$

Example: K function field of a smooth variety X over $k = \bar{k}$, x a rational point, $\operatorname{Spec} k[[t]] \longrightarrow X$ a formal curve starting at x such that $K \hookrightarrow k((t))$. Let A be the induced valn. ring on K. Then A is a d.v.r. with quotient field K and residue field k, which is not essentially of finite type over a field. In effect if A is a local ring $\mathcal{O}_x$ of a variety with function field K one knows that

$$\text{tr. d.}(K) = \dim(\mathcal{O}_x) + \text{tr. d.}\left(\mathcal{O}_x/m_x\right)$$

This example shows that it is not always possible to write a d.v.r. as a limit of things e.f.t. over a field.

January 29, 1973

k_0 not perfect, A d.v.r. over k_0 with residue field $k = k_0[\alpha]$ $\quad \alpha = a^{1/p}$ $\quad a \in k_0 - k_0^p$. Let $\mathfrak{m} = Ax$. Then $y^p - a \in \mathfrak{m}$ so

$$y^p - a = f \cdot x \qquad f \in A.$$

So if $f \in \mathfrak{m}$, then if we make the base extension

$$k \otimes_{k_0} A = A'$$

we have the relation $(y-\alpha)^p = f \cdot x$. Thus A' has to be a d.v.r. with maximal ideal generated by $y - \alpha$.

Example: $Y^p - a - X^2$ is irreducible in $k_0[X,Y]$. because $a + X^2$ is not a p-th power in $k_0[X]$. $p \neq 2$.

$$(Y^p - a, X) = \operatorname{Ker}(k_0[X,Y] \twoheadrightarrow k_0[\alpha])$$

is a maximal ideal. Thus I will take the curve in the affine plane over k_0 defined by the equation

$$f(X,Y) = Y^p - a - X^2 = 0$$

Since $\frac{\partial f}{\partial X} = -2X$, $\frac{\partial f}{\partial Y} = 0$ the curve is smooth for $X \neq 0$. Let A be the local ring of the curve at the point $(0, \alpha)$, i.e. at the max. ideal $\mathfrak{m}$ whose inv. image in $k_0[X,Y]$ is $(Y^p - a, X)$. Then $\mathfrak{m}$ is principal since $y^p - a = x^2$ in A. Thus A is a d.v.r.

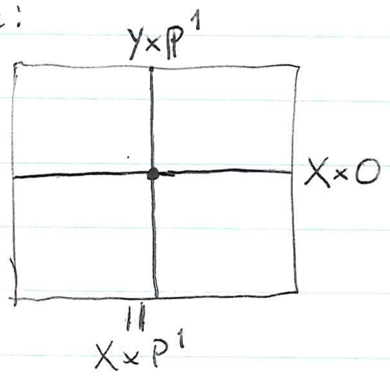
February 1, 1973. Mumford's proof of $\iota^*\iota_*(y) = c(\nu)\, y$.

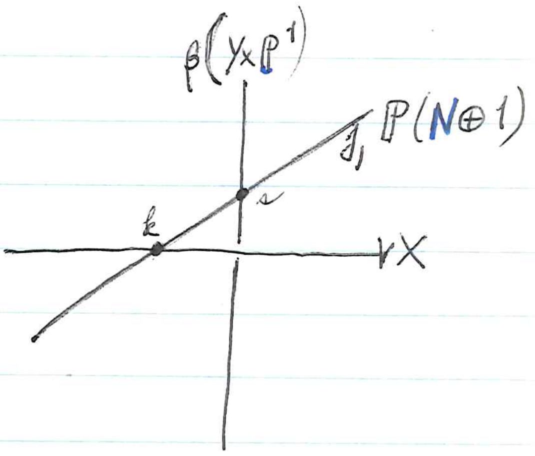
Suppose $\iota : Y \longrightarrow X$ is of codimension 1 and embed $X \longrightarrow X \times \mathbb{P}^1$ ($\times$ over $\mathbb{Z}$) and blow up $X \times \mathbb{P}^1$ along Y so that we have the following situation.

$$\begin{tikzcd}
\mathbb{P}(N \oplus 1) \arrow[rr, "j_1"] \arrow[d, shift right, "g_1"'] & & Z' \arrow[d, "f_1"] \\
Y \arrow[r, "\iota"'] \arrow[u, shift right, "k"'] & X \arrow[ur, "\nu"] \arrow[r] & X \times \mathbb{P}_1
\end{tikzcd}$$

$k : Y = \mathbb{P}(N) \hookrightarrow \mathbb{P}(N \oplus 1)$

Picture:





inside of Z'

Because ν and j_1 are transversal with intersection Y, we have

$$\iota_*(y) = \iota_* k^* g_1^* y$$
$$= \nu^* j_{1*}\, g_1^* y$$

Thus

$$\iota^*\iota_*(y) = \iota^*\nu^* j_{1*}\, g_1^* y = k^* j_1^* j_{1*}\, g_1^* y$$

Now consider $j_1^* j_{1*}\, g_1^* y \in K_i(\mathbb{P}(N \oplus 1))$

~~Any element $\alpha \in K_i(\mathbb{P}(N\oplus 1))$ can be uniquely~~

Have exact sequence

$$0 \longrightarrow K_i(\mathbb{P}N) \xrightarrow{k_*} K_i(\mathbb{P}(N+1)) \xrightarrow{s^*} K_i(\mathbb{P}1) \longrightarrow 0$$

and $\quad s^*\left(j_1^* j_{1*} g_1^* y\right) = 0$

because s can be moved down $Y \times \mathbb{P}^1$ off $\mathbb{P}(N\oplus 1)$. Thus we have a unique $\alpha(y) \in K_i Y$ such that

$$j_1^* j_{1*} g_1^* y = k_* \alpha(y)$$

so

$$\begin{aligned} \iota^* \iota_x y = k^* j_1^* j_{1*} g_1^* y &= k^* k_* \alpha(y) \\ &= k^* k_* k^* g_1^* \alpha(y) \\ &= k^* k_* 1 \cdot g_1^* \alpha(y) \\ &= k^* k_* 1 \cdot \alpha(y). \end{aligned}$$

which shows at least that this is zero if N is trivial.

Now however we have the diagram

$$\begin{tikzcd} Y \times \mathbb{P}^1 \amalg \mathbb{P}(N\oplus 1) \arrow[r, "\beta + j_1"] \arrow[d, "pr_1 + g_1"'] & Z' \arrow[d, "pr_1 f_1"] \\ Y \arrow[r, "i"'] & X \end{tikzcd} \qquad (*)$$

which is a proper intersection, so in the Chow ring gives rise to

$$pr_{1*}\beta^* + g_{1*} j_1^* = (pr_1 + g_1)_* (\beta + j_1)^* = \iota^* (pr_1 f_1)_*$$

Apply this to the element $j_{1*} g_1^* y$.

$$pr_{1*} \beta^* j_{1*} g_1^* y = y$$ as β, j_1 transversal with intersection Y

$$(pr_1, f)_* j_{1*} g_1^* y = i_* g_{1*} g_1^* y$$
$$= i_*(g_{1*} 1 \cdot y)$$
$$= 0$$ as $g_{1*} 1 = 0$ in the Chow theory.

Thus we get

$$y + g_{1*} j_1^* j_{1*} g_1^*(y) = 0$$

showing that $\alpha(y) = -y$. (Check:

$$k : \mathbb{P}N \longrightarrow \mathbb{P}(N+1)$$
$$\overset{\shortparallel}{Y}$$

N = normal bdle

$k(Y)$ is where $\mathcal{O}(-1) \subset g_1^*(N+1) \to g_1^* 1$ is zero

so $k_* 1 = e(\mathcal{O}(+1))$

and $k^* k_* 1 = e(k^* \mathcal{O}(+1)) = e(N^*) = -e(N)$ in Chow theory.

So it works.)

~~Here is how we can prove (3) [illegible]~~
~~for K theory~~

In K-theory $\quad e(N^\vee)\,\alpha(y) = e(N)\,y$

$$\therefore \quad \alpha(y) = \frac{e(N)}{e(N^\vee)}\,y = \frac{1-N^\vee}{1-N}\,y$$

$$= -N^\vee y$$

Thus $\quad y + \alpha(y) = (1-N^\vee)\,y = i^* i_* y$

which says amazingly that the square ( ) seems to be OKAY in K-theory too.

Here is how to prove ( ) commutes in K-theory:

$$0 \longrightarrow N^{-1} \longrightarrow \mathcal{O}_X \longrightarrow \mathcal{O}_Y \longrightarrow 0$$

We have diagram

$$\begin{tikzcd}
D_1 \amalg D_2 \arrow[r, "j_1+j_2"] \arrow[d, "h_1+h_2"'] & Z' \arrow[d, "h"] \\
Y \arrow[r, "i"'] & X
\end{tikzcd}$$

where Y is a divisor and $h^{-1}(Y) = D_1 \cup D_2$ is the sum of two effective divisors. Thus I have

$$h^*(N) = \mathcal{O}(D_1) \otimes \mathcal{O}(D_2)$$

~~so tensoring~~

[illegible]

~~with $\mathcal{O}(D_2)$~~

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{O}(D_1)^{-1} \otimes \mathcal{O}_{D_2} \arrow[r] & h^*\mathcal{O}_Y \arrow[r] & \mathcal{O}_{D_1} \arrow[r] & 0 \\
0 \arrow[r] & \mathcal{O}(+D_1)^{-1} \arrow[r] \arrow[u] & \mathcal{O}_{Z'} \arrow[r] \arrow[u] & \mathcal{O}_{D_1} \arrow[u] & \\
 & \mathcal{O}(D_1)^{-1} \otimes \mathcal{O}(D_2)^{-1} \arrow[r, equal] \arrow[u] & \mathcal{O}(D_1)^{-1} \otimes \mathcal{O}(D_2)^{-1} \arrow[u] & & \\
 & 0 \arrow[u] & 0 \arrow[u] & &
\end{tikzcd}$$

Let E be a vector bundle on Z' acyclic wrt h. Then we should have for E sufficiently untwisted that

$$\mathcal{O}_Y \otimes h_*(E) = h_*(h^*\mathcal{O}_Y \otimes E)$$

$$0 \to h_{2*}(\underline{\mathcal{O}_{D_2}(D_1)^{-1}} \otimes j_2^* E) \longrightarrow h_*(h^*\mathcal{O}_Y \otimes E) \longrightarrow h_{1*}\, j_1^* E \longrightarrow 0$$

and so in K-groups we should have

$$h_{1*}\, j_1^* + h_{2*}([\mathcal{O}_{D_2}(D_1)]^{-1} \cdot j_2^*) = \iota^* h_*$$

But since D_1, D_2 are transversal

$$0 \longrightarrow \mathcal{O}(D_1)^{-1} \longrightarrow \mathcal{O} \longrightarrow \mathcal{O}_{D_1} \longrightarrow 0$$

$$0 \longrightarrow \mathcal{O}_{D_2}(D_1)^{-1} \longrightarrow \mathcal{O}_{D_2} \longrightarrow \mathcal{O}_{D_1 \cap D_2} \longrightarrow 0$$

So we should have the formula

$$i^* h_* = h_{1*} j_1^* + h_{2*} j_2^* - h_{12*} j_{12}^*$$

So now return to the square

$$(*) \qquad \begin{tikzcd} Y \times \mathbb{P}^1 \amalg \mathbb{P}(N \oplus 1) \arrow[r, "\beta + j_1'"] \arrow[d] & Z' \arrow[d, "pr_1 f_1"] \\ Y \arrow[r, "i"'] & X \end{tikzcd}$$

and the intersection of $Y \times \mathbb{P}_1$ with $\mathbb{P}(N \oplus 1)$ is $s: Y \longrightarrow \mathbb{P}(N \oplus 1)$, and we know $s^*(j_{1*} g_1^*(y)) = 0$. Thus from the above formula we get

$$\begin{array}{ccccc} i^*(pr_1 f_1)_* j_{1*} g_1^* y & = & g_{1*} j_1^* j_{1*} g_1^*(y) & + & pr_{1*} \beta^* j_{1*} g_1^*(y) \\ \| & & \| & & \| \\ i^* i_* y & & \alpha(y) & & y \end{array}$$

So from before we have

$$i^* i_* y = k^* k_* 1 \cdot \alpha(y) = k^* k_* 1 \cdot (i^* i_* y - y)$$

or

$$(i^* i_* y)(1 - k^* k_* 1) = -k^* k_* 1 \cdot y$$

$$(i^* i_* y)(1 - (1 - N)) = -(1 - N)\, y$$

$$i^* i_* y = (1 - N^\vee)\, y$$

as was to be shown.

~~Useful observation: Suppose $i: Y \to X$ is a Cartier divisor such that $e(N) = 0$. Then for K′-groups we have $i^* i_* = 0$. Try to prove this directly. Start with a vector bundle~~

K_0-problem. Suppose $i: Y \to X$ is a regular closed embedding with X regular, but not Y. Then we have the composition

( )

$$\begin{tikzcd} K_0'(Y) \arrow[r, "i_*"] & K_0'(X) \arrow[d, equal] & \\ & K_0(X) \arrow[r, "i^*"] & K_0(Y) \end{tikzcd}$$

To understand, suppose to simplify that $i: Y \to X$ is a Cartier divisor. Then ~~we have the follo~~ given F over F, resolve it

$$0 \to P_d \to \cdots \to P_0 \to F \to 0$$

where the P_i are vector bundles over $\mathcal{O}_X$ and tensor

$$0 \to \mathcal{O}_Y \otimes P_d \to \cdots\cdots \to \mathcal{O}_Y \otimes P_0 \to 0$$

getting a complex whose homology groups are

$$\mathrm{Tor}_q^{\mathcal{O}_X}(\mathcal{O}_Y, F) = \begin{cases} F & q = 0 \\ I/I^2 \otimes F & q = 1 \\ 0 & q > 1 \end{cases}$$

$$0 \to I \to \mathcal{O}_X \to \mathcal{O}_Y \to 0$$

$$\begin{tikzcd} 0 \arrow[r] & \mathrm{Tor}_1^{\mathcal{O}_X}(\mathcal{O}_Y, F) \arrow[r] & I \otimes_{\mathcal{O}_X} F \arrow[r] \arrow[d, equal] & F \arrow[r, "\sim"] & F \arrow[r] & 0 \\ & & I/I^2 \otimes_{\mathcal{O}_Y} F & & & \end{tikzcd}$$

Assume now that I is generated by a function f. Then one expects that the complex $\mathcal{O}_Y \otimes P_\circ$ should represent the zero element of $K_0(Y)$ because its two homology groups are isomorphic. Actually we can suppose the resolution is of length one with P_1, P_0 flat with respect to $\mathcal{O}_Y$, i.e. such that mult by f is injective.

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & & \\
0 \arrow[r] & P_1 \arrow[r] \arrow[d, "f"'] & P_0 \arrow[r] \arrow[d, "f"] \arrow[dl, "h"'] & F \arrow[r] \arrow[d, "f=0"] & 0 \\
0 \arrow[r] & P_1 \arrow[r] \arrow[d] & P_0 \arrow[r] \arrow[d] & F \arrow[r] & 0 \\
 & \mathcal{O}_Y \otimes P_1 \arrow[r] \arrow[d] & \mathcal{O}_Y \otimes P_0 \arrow[d] & & \\
 & 0 & 0 & &
\end{tikzcd}$$

I feel it should be possible to prove that

$$0 = [\mathcal{O}_Y \otimes P_1] - [\mathcal{O}_Y \otimes P_0] \in K_0 Y.$$

But it certainly seems to be non-trivial.

Observe that one has ! $h : P_0 \longrightarrow P_1$ with $f = hd$, $f = dh$ and that h induces on the complex $\mathcal{O}_Y \otimes P_\circ$ an isom. of H_0 with H_1. Thus

$$\begin{tikzcd}
\mathcal{O}_Y \otimes P_1 \arrow[r, bend right, "d"'] & \mathcal{O}_Y \otimes P_0 \arrow[l, bend right, "h"']
\end{tikzcd}$$

is exact

Problem: Given $E, F \in \mathcal{P}(X)$ and maps $\psi: E \to F$, $\varphi: F \to E$ such that

$$\begin{tikzcd}F \arrow[r, bend right, "\varphi"'] & E \arrow[l, bend right, "\psi"']\end{tikzcd}$$

is exact, show that $[E] = [F]$ in $K_0 A$. NO see below p.10

If A is regular, there is no problem because then one has an exact sequence

$$0 \to \mathrm{Im}(\varphi) \to E \to F \to E \to F \to \cdots \to \mathrm{Im}(\varphi) \to 0$$

which shows that $\mathrm{Im}(\varphi)$ has projective dimension zero, which forces $\mathrm{Im}(\varphi)$ to be projective.

One should observe that $E \oplus F$, then has a nilpotent automorphism $(e, f) \longmapsto (\psi f, \varphi e)$

It seems that Mumford's argument sheds light on the original problem. NO

Suppose then $i: Y \to X$ is a Cartier divisor and form again the blow-up situation

$$\begin{tikzcd}
\mathbb{P}(N_S+1) \arrow[r] \arrow[d, shift left, "g_0"] \arrow[rrr, bend left, "j_0"] & \mathbb{P}(N+1) \arrow[rr, "j_1"] \arrow[d, shift left, "g_1"] & & Z \arrow[d, "f_1"] \\
S \arrow[r] \arrow[u, shift left, "k_0"] \arrow[rr, bend right, "i_0"'] & Y \arrow[r, "i"'] \arrow[u, shift left, "k_1"] & X \arrow[r, "u"'] \arrow[ur, "v"] & X \times P
\end{tikzcd}$$

Assume i_0 is of finite Tor dimension.

[redacted]

I will need j_0 of finite Tor dimension. Doesn't appear too likely.

Example

Let X be a scheme, and $X[\varepsilon]$ the dual nos. scheme over X. A vector bundle $E^{\#}$ over $X[\varepsilon]$ is the same thing as a vector bundle extension of a vector bundle $\bar{E}$ on X by itself.

$$0 \longrightarrow \bar{E} \longrightarrow E^{\#} \longrightarrow \bar{E} \longrightarrow 0$$

So take $\bar{E}$ to be the trivial line bundle, whence the iso classes of line bundles over $X[\varepsilon]$ extending $\bar{E}$ [redacted] are elements of $H^1(X, \mathcal{O}_X)$. The point is that because of the first Chern class, distinct line bundles represent different elements of K_0. Thus if $E^{\#}$ and F are two such extensions, we get

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{O}_X \arrow[r] \arrow[d, equal] & E^{\#} \arrow[r] \arrow[d, bend left, "\varphi"] & \mathcal{O}_X \arrow[r] \arrow[d, equal] & 0 \\
0 & \mathcal{O}_X \arrow[l] & F^{\#} \arrow[l] \arrow[u, "\psi"] & \mathcal{O}_X \arrow[l] & 0 \arrow[l]
\end{tikzcd}$$

as required on page 9, yet $[E]-[F] \neq$ in $K_0(X[\varepsilon])$.

February 8, 1973: Chow ring

To fix the ideas, suppose X is an irreducible non-sing variety over k of dimension r. I want to understand the subgroup $C^p_\ell(X) \subset C^p(X)$ of cycles of codim p lin. equivalent to zero. Generators for $C^p_\ell(X)$ are obtained as follows. Let W be a irred subvar of $X \times \mathbb{P}^1$ of codim p such that $W_0 = W \cap (X \times 0)$ and $W_\infty = W \cap (X \times \infty)$ are defined. Then $W_0 - W_\infty \in C^p_\ell(X)$ and these elements generate.

To say $W \cap (X \times a)$ is not ~~defined~~ a proper intersection means (since $X \times a$ is of codim 1) that $\dim(W \cap (X \times a)) = \dim W$, hence $W \cap (X \times a) = W$ as W is irred, $\Longrightarrow X \times a \supset W$. $\Longrightarrow t$ is constant on W. (t = basic rational fn. on $\mathbb{P}^1$). Thus we can assume t non-constant on $W \Longleftrightarrow$ the generic point of W maps to the generic point of $\mathbb{P}^1 \Longleftrightarrow k(t) \subset k(W)$.

Let Z be the image of the proper map $W \subset X \times \mathbb{P}^1 \longrightarrow X$, whence $W \subset Z \times \mathbb{P}^1$. If $W = Z \times \mathbb{P}^1$, then $W_0 - W_\infty = Z - Z = 0$, so we can forget this case. Thus $W < Z \times \mathbb{P}^1$, and $\dim(W) = \dim(Z)$, so $k(Z) \subset k(W)$ have the same transcendence degree, so $k(W)/k(Z)$ is finite, in fact $k(W) = k(Z)[t]$. Note Z has codim $p-1$ in X.

(Now conversely if I give an irred. subvar. Z in X of codim $p-1$ and an algebraic function t over $k(Z)$, then I get a W by taking the closure of the point of $Z \times \mathbb{P}^1$ lying over the generic point of Z determined by t. Thus the situation is satisfying.)

What I need to know now is how to compute $W_0 - W_\infty$

as an element of $C^p(X)$. Clearly it is the divisor of the rational function t on W, but this doesn't help very much.

Let V be a subvariety of Z of codim 1. I want to determine the multiplicity of V in $W_0 - W_\infty$. Let A be the local ring of Z at V. It is a local domain of dimension one with quotient field $k(Z)$, and residue field $k(V)$. Associated to V I define a homomorphism

$$\varphi_{V,Z} : k(Z)^* \longrightarrow \mathbb{Z}$$

by putting $\varphi_V(a) = \mathrm{length}_A \, A/aA$ and extending linearly. The formula to prove is

$$(*) \qquad \boxed{\text{multiplicity of } V \text{ in } W_0 - W_\infty = \varphi_{V,\mathbb{Z}}\left(\mathrm{Norm}_{k(W)\to k(\mathbb{Z})} t\right)}$$

Assuming this [redacted] we see that

$$C^p_e(X) = \mathrm{Im}\left(\coprod_{\mathrm{cod}(Z)=p-1} k(\mathbb{Z})^* \xrightarrow{\varphi} \coprod_{\mathrm{cod}(V)=p} \mathbb{Z} \right) \subset C^p(X)$$

[redacted] where φ restricted to $k(Z)^*$ is the map with

[redacted]

components $t \longmapsto \varphi_{VZ}(t)$ $\quad V$ codim 1 in Z.

Kleiman tells me that $(*)$ is in Chevalley's Chow ring seminar (?).

Here's how we might prove (*).

Let A be a local domain with fraction field F, $\overline{F}$ an algebraic closure of F, $t \in \overline{F}$. Then I consider in $A[T_0, T_1]$ the ideal of f such that $f(1,t) = 0$, (f should be homogeneous). This defines an irred subvar. W in $\mathbb{P}^1_A$ of codimension 1, and

$$W - W_\infty = \operatorname{Spec}\left(A[T] / \{f(T) \mid f(t) = 0\}\right)$$

$$= \operatorname{Spec} A[t]$$

$A[t] \subset \overline{F}$. Similarly

$$W - W_0 = \operatorname{Spec} A[\tfrac{1}{t}].$$

So

$$W_0 = \operatorname{Spec} A[t]/tA[t] = \operatorname{Spec} A/A \cap tA[t]$$

$$W_\infty = \operatorname{Spec} A[t^{-1}]/t^{-1}A[t^{-1}] = \operatorname{Spec} A/A \cap t^{-1}A[t^{-1}]$$

Consider the special case where A is a d.v.r. Then $A[T]$ is a U.F.D., so one knows that the ideal $\{f(T) \mid f(t) = 0\}$ is principal. In fact W is the subscheme defined by a single homogeneous polynomial

$$a_0 T_0^n + \cdots\cdots + a_n T_1^n \qquad \text{some } a_i \text{ unit}$$

where

$$t^n + \frac{a_1}{a_0} t^{n-1} + \cdots + \frac{a_n}{a_0} = 0$$

is a minimal equation for t over F. Thus

$$A[t]/tA[t] = A[T] / (T, a_0 T^n + \cdots + a_n) = A/a_n$$

$$A[t^{-1}]/t^{-1}A[t^{-1}] = A/a_0$$

so we do have

$$\text{mult}\,(W_0 - W_\infty) = \text{length } A/a_n - \text{length } A/a_0$$
$$= \varphi\left(\frac{a_n}{a_0}\right) = \varphi(\text{Norm}\,(t)).$$

So [illegible] in the general case I have to show

$$\varphi(\text{Norm}\,(t)) = \text{length } A/A \cap tA[t] - \text{length } A/A \cap t^{-1}A[t^{-1}].$$

Fact: W is finite over A. For W also has dimension 1 and [illegible] each fibre over A is finite, otherwise W would contain the fibre and coincide with it by irreducibility. Thus W is quasi-finite and proper over A, hence it is finite.

~~Method: Choose $k_0 \subset A$ so the residue field k of A is finite over k_0. Then [illegible] because I have assumed t to be non-constant on W, $k_0(t) \subset k(W)$. If $t \in k(W)$ is alg. over k_0, then $k(W) = k$~~

Method. Choose $k_0 \subset A$ so the residue field A/m of A is finite over k_0. If t is algebraic over k_0, then take a minimal equation for t over k_0, say

$$t^m + \lambda_1 t^{m-1} + \dots + \lambda_m = 0.$$

so $\lambda_m \neq 0$ in k_0, so t is a unit in $A[t]$, so $tA[t] = A[t]$ and $A \cap tA[t] = A \cap A[t] = A$. Ditto for t^{-1}. Thus the equation

~~[illegible]~~ we wish to prove is true. (Norm(t) will be a unit in A as it is alg. over k_0).

On the other hand if t is not algebraic over k_0, then $A[t]$ is torsion-free over $k_0[T]$, hence flat. Let $a_0T^n + \cdots + a_n$ be an equation $\neq 0$ satisfied by t with n least. Claim

$$A[T]/a_0T^n + \cdots + a_n$$

is flat over $k_0[T]$. In effect if $f(T) \in k_0[T]$ and $g(T), h(T) \in A[T]$ are such that

$$f(T) \cdot g(T) = h(T)(a_0T^n + \cdots + a_n),$$

Then in $F[T]$ we have

$$f(T) \frac{g(T)}{a_0T^n + \cdots + a_n} = h(T)$$

because $a_0T^n + \cdots + a_n$ is irreducible in $F[T]$, and we have assume t is not alg. over k_0, so $a_0T^n + \cdots + a_n$ cannot divide $f(T)$ in $F[T]$. Thus $f(T)$ divides $h(T)$ in $F[T]$, and since $f(T)$ can be assumed monic, $f(T)$ divides $h(T)$ in $A[T]$. (Point: $h(T) = g(T)f(T) + r(T)$ in $A[T]$ and $r(T) = 0$ in $F[T]$). Thus $g(T)$ is divisible by $(a_0T^n + \cdots + a_n)$ in $A[T]$, proving the claim.

But now define J by the ~~[illegible]~~ exact sequence

$$0 \longrightarrow J \longrightarrow A[T]/a_0T^n + \cdots + a_n \longrightarrow A[t] \longrightarrow 0$$

Then J is flat over $k_0[T]$. Also J is a $A/m^N[T]$-module. Therefore J is a free $k_0[T]$-modules so we get

$$\dim_{k_0} J/TJ + \dim_{k_0}\left(A[t]/tA[t]\right) = \dim_{k_0} A/a_n$$

Similarly define J' by

$$0 \longrightarrow J' \longrightarrow A[T^{-1}]/a_0+a_1T^{-1}+\cdots+a_nT^{-n} \longrightarrow A[t^{-1}] \to 0$$

and we have J' is flat finite type over $k_0[T^{-1}]$.

$$\dim_{k_0}\left(J'/T^{-1}J'\right) + \dim_{k_0}\left(A[t^{-1}]/t^{-1}A[t^{-1}]\right) = \dim_{k_0} A/a_0$$

But by flatness

$$\dim_{k_0} J/TJ = \operatorname{rank}_{k_0[T,T^{-1}]} J \otimes_{k_0[T]} k_0[T,T^{-1}]$$

$$\dim_{k_0} J'/T^{-1}J' = \operatorname{rank}_{k_0[T,T^{-1}]} J' \otimes_{k_0[T^{-1}]} k_0[T,T^{-1}]$$

and now we are done because the $\dim_{k_0}$ is a multiple of the length over A.

Preceding proof is a bit artificial as this ought to make sense for a general local domain A of dim. 1 (noetherian, of course).

Recall that the solution of Gersten's conjecture means that $H^p(X, \mathcal{K}_n)$ (X reg. var. fin. type over k) can be calculated as the homology of the complex

$$0 \longrightarrow \coprod_{x\in X_0} K_n(k(x)) \xrightarrow{d_1} \coprod_{x\in X_1} K_{n-1}(k(x)) \longrightarrow \cdots$$

where d_1 is the boundary map in ~~the~~ a suitable localization exact sequence, precisely for

$$\begin{tikzcd} M_{p+1}(X)/M_{p+2}(X) \arrow[r] & M_p(X)/M_{p+2}(X) \arrow[r] & M_p(X)/M_{p+1}(X) \\ \coprod_{x\in X_{p+1}} M(k(x)) \arrow[u, phantom, "\wr"] & & \coprod_{x\in X_p} M(k(x)) \arrow[u, phantom, "\wr"] \end{tikzcd}$$

and that hence d_1 will be known once one knows it for a local domain of dimension 1.

Now I want to check carefully that

$$\coprod_{x\in X_{p-1}} K_1(k(x)) \xrightarrow{d_1} \coprod_{x\in X_p} \mathbb{Z}$$

is what I think it is, namely the map which when restricted to $K_1(k(y)) = k(y)^*$ has the components $\varphi_{xy} : k(y)^* \longrightarrow \mathbb{Z}$ $\quad x \in X_p \cap \overline{\{y\}}$.

Use naturality: Given $f \in k(y)^*$. Clearly $d_1 f = 0$ and $\varphi(f)$ if f is algebraic over the ground field k. In the other case f defines a rational map to $\mathbb{P}^1_k$ which is dominant

Restrict attention to a local domain A of dim 1, with generic point y and special point x, and let t be a non-zero element of $k(y) = F$. Then form W as above whence W is finite over A and t defines a map $W \longrightarrow \mathbb{P}^1_k$. If A were normal we would have $W = \overset{Sp}{\ }A$. In any case we have that $Sp\,\bar{A} \longrightarrow W \longrightarrow Sp\,A$. So we get

$B = \Gamma(W, \mathcal{O}_W)$

$$\begin{tikzcd}
K_1(\mathbb{P}^1_k) \arrow[r] \arrow[d] & K_1(k(T)) \arrow[r, "\partial"] \arrow[d] & \coprod_{z \in \mathbb{P}^1_k,\ \text{closed}} K_0(k(z)) \arrow[r] \arrow[d] & K_0(\mathbb{P}^1_k) \arrow[r] \arrow[d] & \cdots \\
K_1(B) \arrow[r] \arrow[d] & K_1(k(y)) \arrow[r, "\partial"] \arrow[d] & \coprod_{\overset{+}{x} \in W_{cl}} K_0(k(\overset{+}{x})) \arrow[r] \arrow[d] & K_0(B) \arrow[d] & \\
K_1(A) \arrow[r] & K_1(k(y)) \arrow[r, "\partial"] & K_0(k(x)) \arrow[r] & K_0(A) &
\end{tikzcd}$$

Top map is OKAY because since t is not alg over k, W is flat over $\mathbb{P}^1$. Bottom map also OKAY. ~~It~~ should be clear that φ is OKAY for going between A and B. But now we can compute $K_1(k(T))$ and ∂ on it, by reduction to a local situation, where there is no choice for ∂ except for sign. So everything really is clear and we have $\therefore$ proved.

<u>Theorem</u>: X reg. scheme fin. type over a field. Then $\exists$ canonical isomorphism

$$A^p(X) \cong H^p(X, \mathcal{K}_p)$$

<u>Consequence</u>: We have a formula for the Chow ring which shows it is a contravariant functor of X without the "moving lemma". Thus we should have a Chow theory for non-quasi-projective non-singular varieties.

Moreover we have generalizations of the Chow ring

$$H^p(X, \mathcal{K}_q) \qquad p \leq q \quad (\text{zero for } p > q).$$

Using the resolution

$$0 \longrightarrow \mathcal{K}_q \longrightarrow \coprod_{x \in X_0} i_{x*} K_q(k(x)) \longrightarrow \cdots$$

or better the formula

$$H^p(X, \mathcal{K}_q) = H^p\Big(\nu \longmapsto \coprod_{x \in X_\nu} K_{q-\nu}(k(x))\Big)$$

try to derive basic formulas for the Chow ring.

First suppose $Y \xrightarrow{i} X$ is a regular closed subscheme with complement U, and suppose Y has codim d so that $X_\nu = U_\nu \amalg Y_{\nu-d}$, whence we have an exact sequence

$$0 \to \coprod_{x \in Y_{\nu-d}} K_{q-\nu}(k(x)) \longrightarrow \coprod_{x \in X_\nu} K_{q-\nu}(k(x)) \longrightarrow \coprod_{x \in U_\nu} K_{q-\nu}(k(x)) \to 0$$

certainly should be compatible with the differentials.

This is clear for the restriction to U and also for Y, since if $y \in Y$ then also does any $x \in \bar{y}$. Thus we get a long exact sequence

$$\to H^{p-d}(Y, \mathcal{K}_{q-d}) \longrightarrow H^p(X, \mathcal{K}_q) \longrightarrow H^p(U, \mathcal{K}_q) \longrightarrow \cdots .$$

~~[illegible]~~

~~[illegible] this we establish~~

Much more elementary is the Mayer-Vietoris sequence

$$\longrightarrow H^p(X, \mathcal{K}_q) \longrightarrow H^p(U, \mathcal{K}_q) \oplus H^p(V, \mathcal{K}_q) \longrightarrow H^p(U \cap V, \mathcal{K}_q) \to$$

if $X = U \cup V$, this being a general fact about sheaves. In fact this works for more complicated coverings.

Now if I want to establish the homotopy axiom:

$$H^p(X \times A^1, \mathcal{K}_q) = H^p(X, \mathcal{K}_q)$$

~~then by Mayer-Vietoris, I can restrict to the case where X is affine. In fact I have proved the Gersten conjecture for local rings essentially of finite type over a field. Thus [illegible] to establish the homotopy axiom we can restrict to X local whence the H^p [illegible] zero for~~

it suffices to prove this when X is local. Then I can remove the closed point and argue by induction on the dimension. Recall that Gersten's conjecture has been proved for semi-local regular rings ess. of finite type over a field. Thus I reduce to the case of a field, in which case one can compute everything as follows.

On A^1 we have just closed points and one gen. point, and we have the localization exact sequence which splits up into short exact sequences

$$0 \to K_q(k[T]) \longrightarrow K_q(k(T)) \longrightarrow \coprod_{m} K_{q-1}(k[T]/m) \longrightarrow 0$$

and since these are exact it's all clear.

Now with the homotopy axiom, and the exact sequence for a regular subscheme, the projective bundle theorem works. Wait you need c_1. OKAY.

Next point is that with the projective bundle theorems, we can, at least in the affine case, define Chern classes for representations with coefficients in the $H^p(X, \mathcal{K}_q)$ cohomology. So therefore I find Chern class maps

$$c_i : K_q(X) \longrightarrow H^{i-q}(X, \mathcal{K}_i)$$

and a Chern character map provided I tensor with $\mathbb{Q}$. (For X affine.) The fact that this should be a section of the spectral sequence should be clear on the representation level, so things really ought to work.

?

February 10, 1973

In the following I work with regular schemes whose local rings are essentially of finite type over a field, so that we have a resolution

$$0 \longrightarrow \mathcal{K}_n \longrightarrow \coprod_{x\in X_0} i_{x*} K_n(k(x)) \longrightarrow \coprod_{x \in X_1} i_{x*}(K_{n-1}(k(x))) \longrightarrow \cdots$$

and so we can identify

$$H^p(\mathcal{K}_n) = E_2^{p,-n}$$

Set $H^{pq}(X) = H^q(X, \mathcal{K}_p)$; the Bloch cohomology of X. I want to develop its properties.

Ring structure - anti-commutative with respect to the degree $p+q$.

Exact sequence: $Y \xrightarrow{i} X$ closed immersion with X, Y regular. Then one gets

$$\begin{tikzcd}
H^{pq}(X,U) \arrow[r] & H^{pq}(X) \arrow[r] & H^{pq}(U) \arrow[r] & H^{p,q+1}(X,U) \\
H^{p-d,q-d}(Y) \arrow[u, "\wr"'] & & & H^{p-d,q-d+1}(Y) \arrow[u, "\wr"']
\end{tikzcd}$$

where $d = \operatorname{codim}(Y \text{ in } X)$.

Question: If $x \in X$, is $\varinjlim_{U \ni x} \underset{\overset{\parallel}{K_p(\mathcal{O}_x)}}{H^{pq}(U)} = \underset{\overset{\parallel}{K_p(\mathcal{O}_x)}}{H^{pq}(\operatorname{Spec}\mathcal{O}_x)}$?

YES.

Consider now the map $X \times \mathbb{A}^1 \xrightarrow{f} X$. I want to show $f^*: H^{pq}(X) \longrightarrow H^{pq}(X \times \mathbb{A}^1)$ is an isom.

$$\begin{array}{ccc} H^{pq}(X) & \longrightarrow & H^{pq}(X\times\mathbb{A}^1) \\ \| & & \| \\ H^q(X, \mathcal{K}_p) & & H^q(X\times\mathbb{A}^1, \mathcal{K}_p) \Leftarrow E_2^{st} = H^s(X, R^t f_*(\mathcal{K}_p)) \end{array}$$

It is enough to show $\mathcal{K}_p \xrightarrow{\sim} f_* \mathcal{K}_p$, $R^t f_*(\mathcal{K}_p) = 0$.

Fix $x \in X$. Then

$$R^q f_*(\mathcal{K}_p)_x = \varinjlim_{U \ni x} H^q(U \times \mathbb{A}^1, \mathcal{K}_p).$$

~~It would seem to be better to consider the proper map $f: X \times \mathbb{P}^1 \longrightarrow X$. Then~~

~~$$\varinjlim_{U \ni x} H^q(U \times \mathbb{P}^1, \mathcal{K}_p) = \varinjlim_{U \ni x} H^q\Big(\cdots \to \coprod_{y \in (U \times \mathbb{P}^1)_q} K_{p-q}(k(y)) \to \cdots\Big)$$~~

~~But [illegible] $y \in U \times \mathbb{P}^1 = f^{-1}(U)$ for all $U \ni x \Leftrightarrow f(y) \in \overline{\{x\}}$~~

$$= \varinjlim_{U \ni x} H^q\Big(\longrightarrow \coprod_{y \in (U \times \mathbb{A}^1)_q} K_{p-q}(k(y)) \longrightarrow \cdots\Big)$$

$$= H^q\Big(\longrightarrow \varinjlim_{U \ni x} \coprod_{y \in (U \times \mathbb{A}^1)_q} K_{p-q} k(y) \longrightarrow \cdots\Big)$$

But $y \in U \times \mathbb{A}^1 = f^{-1}(U)$ for all $U \ni x \Leftrightarrow f(y) \in \overline{\{x\}} = \operatorname{Spec}(\mathcal{O}_x)$. And the codimension of y, being the dim of $\mathcal{O}_y$, doesn't change in restricting from X to $\mathcal{O}_x$. Thus it ~~[illegible]~~ is clear that $$\boxed{R^q f_*(\mathcal{K}_p)_x = H^q(f^{-1}(\mathrm{Sp}\,\mathcal{O}_x), \mathcal{K}_p)}$$

for any map $f: Y \to X$.

So to prove the homotopy axiom I can now argue by induction on the dimension of X. First $\dim(X)=0$ whence $X=$ field k, and here one computes, using the fact that the localization theorem sequence breaks up into short exact sequences

$$0 \longrightarrow K_p(k[T]) \longrightarrow K_p(k(T)) \longrightarrow \coprod_m K_{p-1}(k[T]/m) \longrightarrow 0$$

so $H^+(k[T], \mathcal{K}_p) = 0$, $H^0(k[T], \mathcal{K}_p) = K_p(k[T]) = K_p(k)$.

Then one argues that for a general X we have only to prove it for $X = \mathrm{Spec}(\mathcal{O}_x)$, in which case we have $Y = \mathrm{Spec}(\mathcal{O}_x/m_x) \longrightarrow X$ and a map of exact sequences

$$\begin{tikzcd}
{} \arrow[r] & H^{q-d}(Y, \mathcal{K}_{p-d}) \arrow[r] \arrow[d, "(?)"] & H^q(X, \mathcal{K}_p) \arrow[r] \arrow[d] & H^q(U, \mathcal{K}_p) \arrow[r] \arrow[d] & {} \\
{} \arrow[r] & H^{q-d}(Y \times \mathbb{A}^1, \mathcal{K}_{p-d}) \arrow[r] & H^q(X \times \mathbb{A}^1, \mathcal{K}_p) \arrow[r] & H^q(U \times \mathbb{A}^1, \mathcal{K}_p) \arrow[r] & {}
\end{tikzcd}$$

It is necessary to check that (?) is what you think it is, i.e. that the Thom isom.

$$H^q(Y, \mathcal{K}_p) \xrightarrow{\sim} H^{q+d}(X, U; \mathcal{K}_{p+d})$$

is compatible with transversal base change.

~~[illegible]~~ In the case of a flat map it is pretty simple because in general if $f: X \to Y$ is flat then f induces a map of E_1 terms for the support filtration spectral sequence. Thus I can compute $f^*: H^q(Y, \mathcal{K}_p) \longrightarrow H^q(X, \mathcal{K}_p)$ using the standard complexes.

$$\begin{tikzcd}
\cdots \arrow[r] & \coprod_{y \in Y_q} K_{p-q}(k(y)) \arrow[r] \arrow[d] & \cdots \\
\cdots \arrow[r] & \coprod_{x \in X_q} K_{p-q}(k(x)) \arrow[r] & \cdots
\end{tikzcd}$$

e.g. if y has codim q, then an~~d~~y x over y has $\operatorname{codim}(x) \geq q$. and the map does what it should for just those x of codim q.

Ultimately you will want to deal with transversal intersections, but not for now.

Anyway it is now clear that we can prove the <u>homotopy property</u>:

$$H^{pq}(X) \xrightarrow{\sim} H^{pq}(X \times \mathbb{A}^1).$$

Next we want the projective bundle theorem. First note $H^1(X, \mathcal{K}_1) = H^1(X, \mathcal{O}_X^*) = \operatorname{Pic}(X)$, so for any line bundle L on X we have $c_1(L) \in H^{11}(X)$. Now to prove the projective bundle theorem, the argument used above for the homotopy property reduces one to the case of a field.

In this case we can proceed by induction on the dimension of the projective space, using the exact sequence

$$\mathbb{P}1 \xrightarrow{i} \mathbb{P}(E\oplus 1) \xleftarrow{j} \mathcal{O}(1) \sim \mathbb{P}E$$

which thanks to the induction hypothesis will give rise to short exact sequences

$$0 \to h(X) \xrightarrow{i_*} h(\mathbb{P}E\oplus 1)) \xrightarrow{j^*} h(\mathbb{P}E) \longrightarrow 0$$

$h(X) = \oplus H^{pq}(X)$. (In general, one has to prove the projection formula, i.e. that i_* is a module homomorphism for i^*. Assume this for now.) I know that $h(\mathbb{P}E)$ is a free module over $h(X)$ with basis $1, \xi, \ldots, \xi^{n-1}$, $\xi = c_1(\mathcal{O}_{\mathbb{P}E}(1))$, and that $\xi^n = 0$ by induction. To prove

$$i_* 1 = \xi^n \quad \text{in } h(\mathbb{P}E\oplus 1)$$

But if $E = L_1 + \cdots + L_n$, then have $H_i = L_1 + \cdot + \hat{L}_i + \cdot + L_n$ and

$$\mathbb{P}1 = \bigcap_{i=1}^{n} H_i$$

This is a transversal intersection. Thus I have several things to establish about the Thom isomorphism:

A) <u>projection formula</u>:

$$i_* : H^{pq}(Y) \xrightarrow{\sim} H^{p+d,q+d}(X, U)$$

module homo for i^*.

B). <u>transversal intersection formula</u>:

Given
$$\begin{tikzcd} Y' \arrow[r, "f'"] \arrow[d, "i'"'] & Y \arrow[d, "i"] \\ X' \arrow[r, "f"'] & X \end{tikzcd}$$

transversal, then $f^* i_* = i'_* f'^* : h(Y) \longrightarrow h(X', X'-Y')$.

To solve these it seems we must come to grips with the following problem

c). Given $f : X' \longrightarrow X$, X' and X regular, show that f induces a map f^* from the spectral sequence of X to that of X':

$$\begin{tikzcd} E_2^{pq}(X) = H^p(X, \mathcal{K}_{-q}) \arrow[r, Rightarrow] \arrow[d] & K_{-p-q}(X) \arrow[d] \\ H^p(X', \mathcal{K}_{-q}) \arrow[r, Rightarrow] & K_{-p-q}(X') \end{tikzcd}$$

This would follow if I were sure that Brown-Gersten spectral sequence <u>was</u> the same as mine from E_2 on.

<u>Special case</u>: Suppose X' is a divisor in X.

Then

?

February 11, 1973

Summary: Using support filtration I can construct a spectral sequence

$$E_2^{pq} = H^p(X, \mathcal{K}_{-q}) \Longrightarrow K_{-p-q}(X)$$

covariant for proper morphisms. (This uses the Gersten conjecture, so one assumes X regular and its local rings should be essentially of finite type over a field). On the other hand Brown-Gersten construct a similar spectral sequence which is evidently contravariant. Perhaps Gersten can prove these are the same spectral sequence. Assume this to be true.

Now we would like to show the differentials in the above spectral sequence are torsion. To do this we construct Chern classes with values in the Bloch cohomology $\oplus H^p(X, \mathcal{K}_q)$.

Let E be a vector bundle over X on which a group operates. Then G operates on the space $\mathbb{P}E$, and $\mathcal{K}_q$ is a G-sheaf over $\mathbb{P}E$, so we can speak of the equivariant cohomology

$$H^*(\mathbb{P}E, G; \mathcal{K}_*)$$

$\mathcal{O}(1)$ is a G-sheaf on $\mathbb{P}E$, so it has a class in $H^1(\mathbb{P}E, G; \mathcal{O}^*_{\mathbb{P}E}) = H^1(\mathbb{P}E, G; \mathcal{K}_1)$. Since there are spectral sequences

$$H^p(G, H^q(\mathbb{P}E, \mathcal{K}_*)) \Longrightarrow H^{p+q}(\mathbb{P}E, G; \mathcal{K}_*)$$

$$H^p(G, H^q(X, \mathcal{K}_*)) \Longrightarrow H^{p+q}(X, G; \mathcal{K}_*)$$

and since we have (more or less) seen the projective bundle theorem is true, we have that the projective bundle theorem holds in the equivariant situation. Thus I get Chern classes

$$c_i(E) \in H^i(X, \mathcal{K}_i)$$

for any vector bundle E on X. This I knew. But more, I get

$$\begin{tikzcd} c_i : K(X,G) \arrow[r] & H^i(X,G; \mathcal{K}_i) \arrow[d, equal] \\ & \bigoplus_a H^a(G, H^{i-a}(X, \mathcal{K}_i)) \end{tikzcd}$$

and so therefore, when X is affine, I should get maps

$$c_i : K_a(X) \longrightarrow H^{i-a}(X, \mathcal{K}_i).$$

For X local we get only maps

$$c_a : K_a(X) \longrightarrow K_a(X)$$

What is surprising about this is that in general we expect the Chern character to be an isomorphism

$$ch : K_a \otimes \mathbb{Q} \xrightarrow{\sim} \bigoplus_{i \geq 0} H^{i-a}(X, \mathcal{K}_i)$$

We know this is true for $a = 0$.

<u>Paradox:</u> We have operations Ψ^k on $K(X, G)$ which induce Ψ^k on $K_a(X)$. One knows for any Chern class theory that

$$c_i(\Psi^k E) = k^i c_i(E).$$

(This follows from the splitting principle and $c_1(\Psi^k L) = c_1(L^{\otimes k}) = k\, c_1(L)$.) Thus we have

$$\begin{tikzcd}
K(X,G) \arrow[r, "c_i"] \arrow[d, "\Psi^k"] & H^i(X,G;\mathcal{K}_i) & \left(= H^i(G, K_i(X))\right) \\
K(X,G) \arrow[r, "c_i"] & H^i(X,G;\mathcal{K}_i) \arrow[from=u, "k^i"] & \left(= H^i(G, K_i(X))\right)
\end{tikzcd}$$

(in local case)

so it follows that

$$\begin{tikzcd}
K_i X \arrow[r, "c_i"] \arrow[d, "\Psi^k"] & K_i X \arrow[d, "k^i"] \\
K_i X \arrow[r, "c_i"] & K_i X
\end{tikzcd}$$

should commute in the local case. But for $X =$ algebraic closure of $\mathbb{F}_p$ this seems to be nonsense for $i=3$, since I know Ψ^p acts on K_3 by multiplying by p^2.

History of paper – On the cohomology and K-theory of the general linear groups over a finite field.

finished Feb 13, 1972.

the final writing spree started not too long after Jan. 4, 1972, except we started with the paper on K-theory then decided to switch.

research done at IAS spring 1970.
~~[illegible]~~ upper bound for $H^*(GL_n \mathbb{F}_q, \mathbb{F}_\ell)$ understood from cohomology of the symmetric groups, Dec. 1969
Sullivan showed me $F\Psi q$ in Jan. 1970
Basic idea for K-groups obtained May 28, 1970

First version written June & May 1970, finished June 19, 1970 at that time did not know about triviality of mod p cohomology. That was discovered sometime in the summer in July, along with the stable splitting theorem.

Elementary proof for $p = 2$ discovered Oct 21, 1971

Topics omitted in the paper

1. Wu formulas
2. symmetric functions in de Rham cx.
3. dependence on the choice of the embedding $\rho: \bar{k}^* \hookrightarrow \mathbb{C}^*$
4. The Hurewicz map $K_{2i-1}(\mathbb{F}_q) \longrightarrow \mu_{q^i-1}^{\otimes i}$ defined by c_i

5. $(F\Psi^q)_\ell$ depends only on the subgroup of $\mathbb{Z}_\ell^*$ generated by q.

6. Identification of λ-ring ~~str~~ structure $[X, BGL(k)^+]$ with that on $[X, F\Psi^q]$.

7. Identification of Groth. classes with ones constructed by the Brauer lifting

8. Sullivan's fast argument for getting $K_*(\overline{\mathbb{F}}_p)$ from A.C. paper.

$BGL(\overline{\mathbb{F}}_p)$ is $\bigoplus_{\ell \neq p} \mathbb{Q}_\ell/\mathbb{Z}_\ell$ version of BU

---

problems avoided for lack of understanding

5. above

7. why Groth's classes $H^{2i-1}(GL_n\mathbb{F}_q, \mu_{q^i-1}^{\otimes i})$ coincide with mine

Also why $SL_2(k)$ 1-connected (Steinberg, Moore, Matsumoto theory)?

stability of mod p coh. in n. GL_∞

Bockstein spec. seq for $H^*(GL_n k, \mathbb{Z}_\ell)$

February 1973

Exact sequence of a closed subscheme.

1.) Z closed in X, $U = Z - X$, $i : Z \to X$, $j : U \to X$ canonical immersions. $j^* : M(X) \to M(U)$ = localization wrt Serre subcat $\mathcal{B}$ of coh. sheaves on X with support on Z; deivssage $\Rightarrow$ $K_q M(Z) \simeq K_q \mathcal{B}$, so get from loc. thm. an exact seq.

$$\cdots\cdots \longrightarrow K'_{q+1} U \xrightarrow{\partial} K'_q Z \xrightarrow{i_*} K'_q X \xrightarrow{j^*} K'_q U \xrightarrow{\partial} \cdots\cdots$$

Mention the fibration $QM(Z) \xrightarrow{i_*} QM(X) \xrightarrow{j^*} QM(U)$.

2.) Natural with respect to inclusion $Z_1 \subset Z_2$.

3.) Mayer-Vietoris. If $X = U_1 \cup U_2$, then

$$\longrightarrow K'_q(X) \longrightarrow K'_q U_1 \oplus K'_q U_2 \longrightarrow K'_q(U_1 \cap U_2) \xrightarrow{\partial} \cdots\cdots$$

Mention Gersten and Brown-Gersten.

4.) Covariant character. Suppose X, Z, U, i, j as above; $f : X' \to X$, $Z' = f^{-1}(U)$ etc. If f proper, then claim have a map of long exact sequences

$$\begin{tikzcd}
{} \arrow[r] & K'_{q+1} U' \arrow[r] \arrow[d, "(f_U)_*"] & K'_q Z' \arrow[r] \arrow[d, "(f_Z)_*"] & K'_q X' \arrow[r] \arrow[d, "f_*"] & K'_q U' \arrow[r] \arrow[d, "(f_U)_*"] & \cdots \\
{} \arrow[r] & K'_{q+1} U \arrow[r] & K'_q Z \arrow[r] & K'_q X \arrow[r] & K'_q U \arrow[r] & \cdots
\end{tikzcd}$$

Proof.

$$\begin{tikzcd}
& M(Z') \arrow[r] & M(X') \arrow[r] & M(U') \\
\text{heq's} \arrow[r] & \mathcal{F}(Z', f_Z) \arrow[r] \arrow[u] \arrow[d, "(f_Z)_*"] & \mathcal{F}(X', f) \arrow[r] \arrow[u] \arrow[d, "f_*"] & \mathcal{F}(U', f_U) \arrow[u] \arrow[d, "(f_U)_*"] \\
& M(Z) \arrow[r] & M(X) \arrow[r] & M(U).
\end{tikzcd}$$

5.) Contravariant character. $X, Z, U, i, j, X', Z', U', i', j'$, etc. as above. Suppose now that f is of fin. Tor dim. Then have

$$\begin{tikzcd}
\mathcal{M}(X') \arrow[r] & \mathcal{M}(U') \\
\mathcal{P}(X,f) \arrow[r] \arrow[u, "f^*"] & \mathcal{P}(U,f_U) \arrow[u, "f_U^*"'] \\
\mathcal{M}(X) \arrow[r] \arrow[u, phantom, "\cup"] & \mathcal{M}(U) \arrow[u, phantom, "\cup"]
\end{tikzcd}$$

where inclusion ~~bottom vertical arrows~~ give isos. on K-theory. Thus get a map of long exact seq.

$$\begin{tikzcd}
(*) & \arrow[r] & K'_q Z' \arrow[r] & {} \arrow[r] & {} \\
 & \arrow[r] & K'_q Z \arrow[r] \arrow[u, "(?)"] & {} \arrow[r] \arrow[u, "f^*"] & {} \arrow[u, "f_U^*"]
\end{tikzcd}$$

where (?) is the composition

$$\begin{tikzcd}
K'_q Z' \arrow[r, "\sim"] & K_q(\mathcal{M}(X') \to \mathcal{M}(U')) \\
 & K_q(\mathcal{P}(X,f) \to \mathcal{P}(U,f_U)) \arrow[u] \arrow[d, "\wr"] \\
K_q Z \arrow[r, "\sim"] & K_q(\mathcal{M}(X) \to \mathcal{M}(U))
\end{tikzcd}$$

Following prop. identifies $(*)$ in some special cases.

Prop. a) $\mathcal{O}_{X'}, \mathcal{O}_Z$ tor ind. $\Rightarrow$ (?) $=$ f_Z^*.

b) If $T_p = \mathrm{Tor}_p^{\mathcal{O}_X}(\mathcal{O}_{X'}, \mathcal{O}_Z)$ fin. Tor dim over $\mathcal{O}_Z$ $\forall p$, ~~the (?) is the map~~ let $\xi_p : K'_q Z \to K'_q Z'$ be the map induced by the exact functor

$$F \longmapsto T_p \otimes_Z F$$

on the subcat of $\mathcal{M}(Z)$ con. of $F \ni \mathrm{Tor}^{\mathcal{O}_Z}(T_p, F) = 0$

Then $(?) = \sum(-1)^p \xi_p$.

c) If $\mathcal{O}_{Z'}$ is of finite Tor dim $/\mathcal{O}_Z$ and T_p is of finite tor dimension over $\mathcal{O}_{Z'} = T_0$ for all p, ~~then (?) = ... f_Z^*~~ whence have

$$e = \sum(-1)^p T_p \in K_0 Z'$$

then (?) = composition

$$K_q' Z \xrightarrow{f_Z^*} K_q' Z' \xrightarrow{e \cdot} K_q' Z'.$$

a) easy because ~~we have for $M \in \mathcal{M}(Z)$~~

$$i'_* \mathrm{Tor}_p^{\mathcal{O}_Z}(\mathcal{O}_{Z'}, M) = \mathrm{Tor}_p^{\mathcal{O}_X}(\mathcal{O}_{X'}, i_* M)$$

so f_Z is of finite Tor dimension and have $i_* \mathcal{P}(Z, f_Z) \subset \mathcal{P}(X, f)$, $f^* i_* = i'_*(f_Z^*)^*$. so clear.

We will outline proof of b) below.

exact sequence of a closed subscheme.

X noeth, with an ample line bundle; Z closed subscheme, $U = X - Z$

$i : Z \to X$ inclusions

$j : U \to X$

$X' - Z'$

$\cup$

$f^{-1}(X - fZ') \supset f^{-1}U$

$\Uparrow$

$(fZ' \subset Z)$

Then have an exact sequence

$$\xrightarrow{\partial} K'_q(Z) \xrightarrow{i_*} K'_q(X) \xrightarrow{j^*} K'_q(U) \longrightarrow \cdots$$

Covariant character. $f : X' \to X$, $f(Z') \subset Z$, Claim then that we get a ~~[illegible]~~ map of long exact sequences

$$\begin{tikzcd}
\cdots \arrow[r, "\partial"] & K'_q Z' \arrow[r, "i'_*"] \arrow[d, "(f_Z)_*"] & K'_q X' \arrow[r, "j'^*"] \arrow[d, "f_*"] & K'_q U' \arrow[r, "\partial"] \arrow[d, "(f_U)_* \ \mathrm{res}^{U'}_{f^{-1}(U)}"] & \cdots \\
\cdots \arrow[r, "\partial"] & K'_q Z \arrow[r, "i_*"'] & K'_q X \arrow[r, "j^*"'] & K'_q U \arrow[r, "\partial"] & \cdots
\end{tikzcd}$$

Proof: first reduce to case $Z' = f^{-1}Z$, $U' = f^{-1}(U)$.

$$\begin{tikzcd}
\mathcal{M}(Z') \arrow[r, "i'_*"] & \mathcal{M}(X') \arrow[r, "j'^*"] & \mathcal{M}(U') \\
\mathcal{F}\ell(Z', f) \arrow[r, "i'_*"] \arrow[u, hook] \arrow[d, "f_{Z*}"] & \mathcal{F}\ell(X', f) \arrow[r, "j'^*"] \arrow[u, hook] \arrow[d, "f_*"] & \mathcal{F}\ell(U', f) \arrow[u, hook] \arrow[d, "f_{U*}"] \\
\mathcal{M}(Z) \arrow[r, "i_*"] & \mathcal{M}(X) \arrow[r, "j^*"] & \mathcal{M}(U)
\end{tikzcd}$$

so its clear.

Contravariant character: $f: X' \to X$, $f^{-1}Z \subset Z'$

again reduce easily to case of $f^{-1}Z = Z'$. Assume f finite Tor dimension. ~~and that~~ Then have diagram of exact functors

$$\begin{tikzcd}
 & \mathcal{M}(Z') \arrow[r] & \mathcal{M}(X') \arrow[r] & \mathcal{M}(U') \\
(*)\ \boxtimes & & \mathcal{P}(X,f) \arrow[r] \arrow[u, "f"] \arrow[d] & \mathcal{P}(U,f) \arrow[u, "f_U^*"] \arrow[d] \\
 & \mathcal{M}(Z) \arrow[r] & \mathcal{M}(X) \arrow[r] & \mathcal{M}(U)
\end{tikzcd}$$

and so one gets a map of exact sequences

$$\begin{tikzcd}
{} \arrow[r, "\partial"] & K'_q(Z') \arrow[r, "i'_*"] & K'_q X' \arrow[r, "j'^*"] & K'_q U' \arrow[r] & {} \\
{} \arrow[u] \arrow[r, "\partial"] & K'_q(Z) \arrow[r, "i_*"] \arrow[u, "(*)"] & K'_q X \arrow[r, "j^*"] \arrow[u, "f^*"] & K_q U \arrow[r] \arrow[u, "f_U^*"] & {}
\end{tikzcd}$$

and the problem is to identify the map$^{(*)}$ on the fibre.

Case 1: Z, X' are Tor independent over X, i.e.

$$\mathrm{Tor}_+^{\mathcal{O}_X}(\mathcal{O}_{X'}, \mathcal{O}_Z) = 0. \qquad \text{Then } (f_{Z*})^{=(*)}$$

Then we have for any $M \in \mathcal{M}(Z)$

$$\mathrm{Tor}_q^{\mathcal{O}_X}(\mathcal{O}_{X'}, i_* M) = i'_* \mathrm{Tor}_q^{\mathcal{O}_Z}(\mathcal{O}_{Z'}, M)$$

so that $M \in \mathcal{P}(Z, f_Z) \Leftrightarrow i_* M \in \mathcal{P}(X, f)$. Thus can add to $\boxtimes$ the row

$$\mathcal{P}(Z, f_Z)$$

and so the assertion is clear.

Case 2: $\mathrm{Tor}_q^{\mathcal{O}_X}(\mathcal{O}_{X'}, \mathcal{O}_Z)$ is of fin. Tor dim over $\mathcal{O}_Z$ for each q. Then have maps

$$K_q'(Z) \longrightarrow K_q'(Z')$$

$$F \longmapsto \left(\mathrm{Tor}_q^{\mathcal{O}_X}(\mathcal{O}_{X'}, \mathcal{O}_{Z'}) \otimes_{f^!\mathcal{O}_{Z'}} F\right) \; *$$

and the claim is that

$$(*) = \sum (-1)^q \left[\mathrm{Tor}_q^{\mathcal{O}_X}(\mathcal{O}_{X'}, \mathcal{O}_{Z'}) \otimes_{\mathcal{O}_Z} ?\right].$$

Proof. $\mathcal{R}$ cat cons. of $\mathcal{P}(X, f)$-resol. of an $\mathcal{O}_Z$ module $F \in \mathcal{F}(Z, \mathrm{Tor}.(X', Z))$.

$$0 \to P_n \to \cdots \to P_0 \to F \to 0$$

I first want to show that

(1)
$$K_q(\mathcal{P}) \xrightarrow{\sim} K_q'(X)^{n-1} \oplus K_q'(Z).$$

$$(P_\cdot) \longmapsto \left((Z_i(P_\cdot))_{i=0}^{n-1}, H_0(P_\cdot)\right)$$

but assume this.

Then I have maps of K-theories

$$\begin{tikzcd}
 & \mathcal{M}(Z') \arrow[r] & \mathcal{M}(X') \\
\mathcal{R} \arrow[ur, "\text{alt. sum of } \mathrm{Tor}_i(X',Z) \otimes H_0P."] \arrow[rr, "\text{alt sum of } P_i"] \arrow[dr, "P. \mapsto H_0P."'] & & \mathcal{P}(X,f) \arrow[u, "f^*"'] \arrow[d, "h_0"] \\
 & \mathcal{M}(Z) \arrow[r] & \mathcal{M}(X)
\end{tikzcd}$$

which because of the exact sequence thm. is commutative.

__Blowing-up.__ Suppose $i : Z \to X$ is a reg. im. i.e. a closed immersion whose defining ideal is gen. loc. by a reg. sequence, $f : X' \to X$ the blowing-up of Z, N the normal bundle, $Z' = \mathbb{P}N$, and F the canonical bundle on Z' defined by

$$0 \to F \to f_Z^* N \to \mathcal{O}_{Z'}(1) \to 0$$

$$\begin{tikzcd}
Z' \arrow[r, "i'"] \arrow[d, "f_Z"] & X' \arrow[d, "f"] & U' \arrow[l] \arrow[d, equal, "f_U"] \\
Z \arrow[r, "i"'] & X & U \arrow[l]
\end{tikzcd}$$

Then in $(*)$ get f_U

so get bicart. squares

$$\begin{tikzcd}
K'_q(Y') \arrow[r, "i'_*"] & K'_q(X') \\
K'_q(Y) \arrow[u, hook, "\lambda_{-1}(F) g^*"] \arrow[r, "i_*"] & K'_q(X) \arrow[u, hook, "f^*"']
\end{tikzcd}$$

Using projective bundle thm. get therefore the gen. of K_0(blown-up scheme to K' situation.

__Prop:__ $i : Y \to X$ regular immersion

then $i^* i_* y = \lambda_{-1}(N^\vee) \cdot y$ $\qquad y \in K_q Y$.

Difference construction: Let $f: \mathcal{M} \to \mathcal{M}'$ be an exact functor between exact categories. ~~Denote by $C^b(\mathcal{M})$ the exact cat of bdd complexes and by $C^{b\phi}(\mathcal{M})$ the sub cat. of bdd. acyclic complexes.~~

$$Q\mathcal{M} \xrightarrow{Qf} Q\mathcal{M}'$$

fibre of induced map of classifying spaces $\mathrm{Fib}(Qf)$

$$K_q(\mathcal{M} \xrightarrow{f} \mathcal{M}') = \pi_{q+1}(\mathrm{Fib}(Qf)).$$

Now let $C^b(\mathcal{M})$ be bdd complexes in $\mathcal{M}$, and $C^{b\phi}(\mathcal{M})$ the subcat. of acyclic complexes. I propose to define a homo.

$$K_q\left[C^b(\mathcal{M}) \times_{C^b(\mathcal{M}')} C^{b,\phi}(\mathcal{M}')\right] \longrightarrow K_q(\mathcal{M} \xrightarrow{f} \mathcal{M}').$$

In other words to give an exact functor from $\mathcal{N}$ to bdd complexes in $\mathcal{M}$ which become acyclic in $\mathcal{M}'$ defines a map from $K_q\mathcal{N}$ to $K_q(\mathcal{M} \to \mathcal{M}')$.

The construction: Will limit myself to $C^{[-n,0]}(\mathcal{M})$ complexes with amplitude in $[-n,0]$.

$$\begin{tikzcd}
 & C^{[-n,0]}(\mathcal{M}) & \\
C^{[-n,0],\phi}(\mathcal{M}) \arrow[r, hook] & C^{[-n,1],\phi}(\mathcal{M}) \arrow[u, hook] \arrow[r] & \mathcal{M}
\end{tikzcd}$$

$$\begin{tikzcd}
Q\mathcal{N} \arrow[rr, "\alpha"] \arrow[dddr, "\beta"'] & & QC^{[-n,0]}(\mathcal{M}) \arrow[d] & & \\
& & QC^{[-n,0]}(\mathcal{M}') & & \\
& QC^{[-n,0]^{\phi}}(\mathcal{M}) \arrow[rr, "\varepsilon"] \arrow[d] & & QC^{[-n,1],\phi}(\mathcal{M}) \arrow[r, "\pi"] \arrow[d] \arrow[uuul, "\gamma"'] & Q\mathcal{M} \arrow[d] \\
& QC^{[-n,0],\phi}(\mathcal{M}') \arrow[rr, "\varepsilon'"] & & QC^{[-n,1],\phi}(\mathcal{M}') \arrow[r, "\pi'"] \arrow[uul, "\gamma'"] & Q\mathcal{M}'
\end{tikzcd}$$

The idea we have the maps α, β ~~[illegible]~~ with the same image in $C^{[-n,0]}(\mathcal{M}')$. Now γ is a heq, so we have a map $\pi\gamma^{-1}\alpha$ which represents the alternating sum $\sum(-1)^q F_q$. Now

$f\pi\gamma^{-1}\alpha = \pi' f\gamma^{-1}\alpha$. ~~But γ^{-1} is a homotopy-inverse for γ.~~

$$\gamma' f\gamma^{-1} = f\gamma\gamma^{-1} \sim f$$

$$\gamma' f\gamma^{-1}\alpha \sim f\alpha = \gamma'\varepsilon'\beta$$

so $\quad f\gamma^{-1}\alpha \sim \varepsilon'\beta$

so $f\pi\gamma^{-1}\alpha = \pi' f\gamma^{-1}\alpha \sim \pi'\varepsilon'\beta = 0.$

Thus the map $\pi\gamma^{-1}\alpha$, with this contracting homotopy, defines a map $Q\mathcal{N} \to \mathrm{Fib}(Q\mathcal{M} - Q\mathcal{M}')$.

$\boxed{\text{Not clear!}}$

diagram is this maybe

$$\begin{tikzcd}
C^{[-n,0]}(\mathcal{M}) \arrow[d] & & C^{[-n,1],\phi}(\mathcal{M}) \arrow[ll, "heq"', "\gamma"] \arrow[d] \arrow[rr, "\pi"] & & \mathcal{M} \arrow[d] \\
C^{[n,0]}(\mathcal{M}') & & C^{[-n,1],\phi}(\mathcal{M}') \arrow[ll, "heq"'] \arrow[rr, "\pi'"] & & \mathcal{M}' \\
 & \mathcal{N} \arrow[uul, "\alpha"] \arrow[ur, "\beta"] \arrow[urrr, "0"'] & & &
\end{tikzcd}$$

Because the indicated arrows are heqs, there is ~~a way of lifting α to~~ a unique triple $\alpha' : BQ\mathcal{N} \to BQC^{[-n,1],\phi}(\mathcal{M})$ with homotopies joining the images $f\alpha' \sim \beta$, $\gamma\alpha' \sim \alpha$. Then one projects into $\mathcal{M}$ getting $f\pi\alpha' = \pi' f\alpha' \sim \pi'\beta = 0$, hence one gets a well-defined map $Q\mathcal{N} \to \text{Fibre}$.

Additivity. __Notation:__ If $F : \mathcal{N} \to C^b(\mathcal{M})$ is basic functor how about $\Delta_F : K_0\mathcal{N} \to K_0(\mathcal{M} \to \mathcal{M}')$ for the induced map. Then

$$\boxed{\Delta_F = \Delta_{F'} + \Delta_{F''}}$$

for $0 \to F' \to F \to F'' \to 0$. __Proof.__ $Ex(\mathcal{M}) \to Ex(\mathcal{M}')$.

Proof of ~~coboundary of~~ variance formula:

$\mathcal{R} \subset C^{b\cdot}_{\geq 0}(\mathcal{M}(X))$ ch. complexes of modules in $\mathcal{P}(X,f)$ which resolve an $F \in \mathcal{P}(\mathrm{Tor}_\cdot(X';Z))$

Now you have a commutative diagr

$$\begin{tikzcd}
K_q(\mathcal{M}(Z')) \arrow[r, "\sim"] & K_q(\mathcal{M}(X') \to \mathcal{M}(U')) \\
K_q(\mathcal{R}) \arrow[u] \arrow[r, "\alpha"] \arrow[d, "\beta"] & K_q(\mathcal{P}(X', f) \to \mathcal{P}(U, f_U)) \arrow[u] \arrow[d, "\gamma"] \\
K_q(\mathcal{M}(Z)) \arrow[r, "\sim"'] & K_q(\mathcal{M}(X) \to \mathcal{M}(U))
\end{tikzcd}$$

α defined by assoc. to $P_\cdot \to M$ in $\mathcal{R}$, the complex $P_\cdot$ in $\mathcal{P}(X, f)$ which becomes acyclic in $\mathcal{U}$.

$$F(P_\cdot \to M) = F_\cdot \qquad \alpha = \Delta_F$$

β defined by $(P_\cdot \to M) \mapsto M.$

γ by alt. sum of $(P_\cdot \to M) \longmapsto \mathrm{Tor}_\beta(\mathcal{O}_{X'}, \mathcal{O}_Z) \otimes_{\mathcal{O}_Z} M)$

Proof. Let n be the Tor dimension of $\mathcal{O}_{X'}$ over $\mathcal{O}_X$. Let $\mathcal{R}$ be the exact category consisting of exact sequences in $\mathcal{M}(X)$ of the form

$$(*) \qquad 0 \longrightarrow M_n \longrightarrow \cdots\cdots \longrightarrow M_0 \longrightarrow M_{-1} \longrightarrow 0$$

with M_{-1} in $\mathcal{M}(Z)$. Let $\mathcal{R}'$ be the full subcat of $\mathcal{R}$ consisting of those exact sequences such that $M_i \in \mathcal{P}(X,f)$ for $i \geq 0$ and such that $\mathrm{Tor}_q^{\mathcal{O}_Z}(T_p, M) = 0$ for $q > 0$, and all p where

$$T_p = \mathrm{Tor}_p^{\mathcal{O}_X}(\mathcal{O}_{X'}, \mathcal{O}_Z) \in \mathcal{M}(Z').$$

Using the resolution thm one can show that $K_q \mathcal{R}' \xrightarrow{\sim} K_q \mathcal{R}$. However any object $(*)$ in $\mathcal{R}$ has an exact char. filtration whose quotients are elementary complexes:

$$0 \longrightarrow M_i \xrightarrow{\sim} M_{i-1} \longrightarrow 0 \cdot \quad -$$

Thus by the exact sequence thm. get isom.

$$K_q \mathcal{R} \xrightarrow{\sim} K_q \mathcal{M}(Z) \times K_q \mathcal{M}(X)^{n-1}$$

$$(M_\cdot) \longmapsto \left(M_{-1} ; \mathrm{Ker}(M_{i+1} \to M_i),\ 0 \leq i < n\right).$$

—

~~So anyway the situation is this~~

$$0 \to P_n \to \cdots \cdots \to P_1 \to P_0 \to P_{-1} \to 0$$

exact in $\mathcal{M}(X)$, i.e. $P_\bullet \in C^{[n,1],\phi}(\mathcal{M}(X))$.

want $P_i \in \mathcal{P}(X,f)$, $i \geq 0$, and $P_{-1} \in \mathcal{P}(\mathrm{Tor}(X', Z))$

Start off with $C^{[n,1],\phi}(\mathcal{M}(X))$ with M_{-1} an $\mathcal{O}_Z$-mod. and you let $\mathcal{R}$ be this subcat. Claim

$$K_q(\mathcal{R}) \xrightarrow{\sim} K_q(\mathcal{M}(X))^{n-1} \times K_q(\mathcal{M}(Z))$$

$$(P_\bullet) \longmapsto (dP_{i+2}, P_{-1}).$$

You have to apply the resolution theorem, so you have to be able to resolve. Classical.

$$\begin{tikzcd}
0 \arrow[r] & P_n \arrow[r] \arrow[d] & \cdots \arrow[r] & P_0 \arrow[r] \arrow[d] & P_{-1} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & M_n \arrow[r] & \cdots \arrow[r] & M_0 \arrow[r] & M_{-1} \arrow[r] & 0
\end{tikzcd}$$

and $P_n \in \mathcal{P}(X,f)$ provided $n \geq$ Tor dim $\mathcal{O}_{X'}$ over $\mathcal{O}_X$.

$\longrightarrow$