February 4, 1972

Let $\mathcal{M}$ be an exact category, and let $\mathcal{C}(\mathcal{M})$ be the category of chain complexes in $\mathcal{M}$ which are bounded. Then if $\mathcal{M} \subset \mathcal{A}$ we have the homology functor $M_\cdot \mapsto H_i(M_\cdot)$ which is effaceable, hence

$$K_q(\mathcal{P}) \xrightarrow{\sim} K_q(\mathcal{C}(\mathcal{M}))$$

where $\mathcal{P}$ is the full subcategory consisting of complexes such that $H_i(M_\cdot) = 0$ for $i > 0$.

It would be better to proceed as follows. Let $\mathcal{P} \subset \mathcal{C}(\mathcal{M})$ consist of complexes $M_\cdot$ such that $H_0(M_\cdot) \in \mathcal{M}$ and $H_+(M_\cdot) = 0$. Then any complex is a quotient of a member of $\mathcal{P}$.

$$0 \to M_n \to \cdots\cdots \to M_0 \to 0$$
$$\uparrow$$
$$M_n \to \begin{matrix} M_n \\ \oplus \\ M_{n-1} \end{matrix} \to \begin{matrix} M_{n-1} \\ \oplus \\ M_{n-2} \end{matrix} \cdot \to \begin{matrix} M_1 \\ \oplus \\ M_0 \end{matrix}$$

$$M_n \to \cdots \to M_1$$

And it is clear that every object has a finite resolution by members of $\mathcal{P}$. Check that $0 \to M \to P \to P' \to 0$ $\Rightarrow M \in \mathcal{P}$. But

$$\begin{tikzcd}
 & 0 \arrow[d] & & & 0 \arrow[d] & 0 \arrow[d] & & \\
0 \arrow[r] & M_n \arrow[r] \arrow[d] & \cdots \arrow[r] & & M_1 \arrow[r] \arrow[d] & M_0 \arrow[r] \arrow[d] & ? \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & P_n \arrow[r] \arrow[d] & \cdots \arrow[r] & & P_1 \arrow[r] \arrow[d] & P_0 \arrow[r] \arrow[d] & H_0 \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & P'_n \arrow[r] \arrow[d] & \cdots \arrow[r] & & P'_1 \arrow[r] \arrow[d] & P'_0 \arrow[r] \arrow[d] & H'_0 \arrow[r] \arrow[d] & 0 \\
 & 0 & & & 0 & 0 & 0 &
\end{tikzcd}$$

<u>NO</u>: Example of free modules. Let M be $M \oplus A^p \simeq A^q$ and let M be free modules.

Then

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] \\
0 \arrow[r] & A^p \arrow[r] \arrow[d] & A^q \arrow[r] \arrow[d, "r"] & M \arrow[d] \\
0 \arrow[r] & A^q \arrow[r] \arrow[d] & A^q \times A^q \arrow[r] \arrow[d] & A^q \arrow[d] \\
0 \arrow[r] & P_1 \arrow[r] \arrow[d] & P_0 \arrow[r] \arrow[d] & A^p \arrow[d] \\
 & 0 & 0 & 0
\end{tikzcd}$$

but P_0, P_1 are stably-free. Now we can add some big free module to $A^q \times A^q$, A^q, P_0, P_1 to make P_1, P_0 free.

How to do it. Start with a complex of length n and consider the functorial exact sequence

$$\begin{array}{ll} M_\cdot : & M_n \to \cdots \to M_0 \\ \uparrow & \\ F(M_\cdot) : & M_n \to M_{n-1} \oplus M_n \to \cdots \to M_0 \oplus M_1 \\ \uparrow & \\ \sigma_1(M_\cdot) : & M_n \to \cdots \to M_1 \end{array}$$

and so one has that (length $\leq n$) can be replaced by P and length $\leq (n-1)$. Thus one is done. So can prove

Prop. $\mathcal{C}(\mathcal{M})$ bounded chain complexes $\tilde{\mathcal{C}}(\mathcal{M}) \subset \mathcal{C}(\mathcal{M})$ full subcat. containing $M_\cdot$ such that $H_+ M_\cdot = 0$, $H_0 M_\cdot \in \mathcal{M}$. Then $\tilde{\mathcal{C}}(\mathcal{M}) \to \mathcal{C}(\mathcal{M})$ induces isos. on K-groups.

Now you have the exact functor: $\tilde{\mathcal{C}}(\mathcal{M}) \to \mathcal{C}(\mathcal{M})$, $M_\cdot \mapsto H_0(M_\cdot)$. Thus you get a map of K-groups

$$K_i(\mathcal{C}(\mathcal{M})) \longrightarrow K_i \mathcal{M}$$

which is clearly the alternating sum of the exact functors

$$M_\cdot \longmapsto M_q .$$

We have a basic fibration (immediate, because you can compute everything)

$$\begin{tikzcd} Q\begin{pmatrix} \tilde{\tilde{\mathcal{C}}}(\mathcal{M}) \\ \text{bdd "acyclic} \\ \text{complexes} \end{pmatrix} \arrow[r] \arrow[dr] & Q\begin{pmatrix} \mathcal{C}(\mathcal{M}) \\ \text{bdd chain} \\ \text{complexes} \end{pmatrix} \arrow[r, squiggly, "\text{alt. sum}"] & Q(\mathcal{M}) . \\ & Q(\tilde{\mathcal{C}}(\mathcal{M})) \arrow[u, "\text{heq.}"'] \arrow[ur, "H_0"'] & \end{tikzcd}$$

Now suppose I am given $j : \mathcal{M} \to \mathcal{M}'$ exact. I now want to consider chain complexes (bdd) in $\mathcal{M}$ which become acyclic when j is applied.

mostly wrong
mistake p. 4.

February 16, 1972:

Let k be a field to simplify. I propose to define a category having the homotopy type of $BGL(k)^+$ and hopefully permitting a good proof of the stable splitting theorem.

Begin with the following category $\mathcal{B}'$:

objects: fin. diml. vector space $/k$.

morphisms: an element of $\mathrm{Hom}_{\mathcal{B}'}(V',V)$ consists of an injection $f: V' \hookrightarrow V$ together with a complement $Q \subset V$ for the image of f: notation: $(Q,f): V' \hookrightarrow V$

composition:

$$\begin{array}{ccccc} V'' & \xrightarrow{f'} & V' & \xrightarrow{f} & V \\ & & \cup & & \cup \\ & & Q' & & Q \end{array}$$

$$(Q,f)(Q',f') = (Q + fQ', ff').$$

Next define a functor $\mathcal{B}' \longrightarrow (\text{groups})$:

$$V \longmapsto \mathrm{Aut}(V)$$

given $(Q,f): V' \longrightarrow V$ define

$$(Q,f)_* : \mathrm{Aut}(V') \longrightarrow \mathrm{Aut}(V)$$

by

$$\begin{cases} (Q,f)_*(\theta')(fv') = f(\theta' v') \\ (Q,f)_*(\theta')(q) = q \quad \text{if } q \in Q. \end{cases}$$

Thus $(Q,f)_*\theta'$ is the identity on Q and the f-transform of θ' on fV. Then this is a functor

$$(Q,f)_*(Q',f')_*\,\theta'' = \begin{cases} ff' \text{ transf. of } \theta'' & \text{on } ff'V'' \\ f \text{ transf of id} & \text{on } f^*Q' \\ \text{id} \quad \text{on} & Q \end{cases}$$

$$= \begin{cases} ff' \text{ transf of } \theta'' & \text{on } ff'V'' \\ \text{id} \quad \text{on} & Q+fQ' \end{cases}$$

$$= (Q+fQ', ff')_*\,\theta''.$$

Now define the category $\mathcal{E}'$ to be the cofcinded category over $\mathcal{B}$ whose fibres over V is the ~~category~~ group $\mathrm{Aut}(V)$ and whose cobase change functors are $(Q,f)_*$ as above. Thus $\mathcal{E}'$ has

objects: V

morph: $\mathrm{Hom}_{\mathcal{E}'}(V',V) = \left\{ (\theta, Q, f) \;\middle|\; \begin{array}{l} f: V' \hookrightarrow V \\ Q \text{ comp. to } fV' \text{ in } V \\ \theta \in \mathrm{Aut}(V) \end{array} \right\}$

and composition

$$(\theta, Q, f)(\theta', Q', f') = (\theta\cdot(Q,f)_*\theta', Q+fQ', ff')$$

Next let $\mathcal{B}$ be the category

objects: V f.d. v.s. $/k$

morph: $\mathrm{Hom}_{\mathcal{B}}(V',V) = \{\text{injections } f: V' \hookrightarrow V\}$

so there is an evident functor

$$(Q, f) \longmapsto f$$
$$\mathcal{B}' \longrightarrow \mathcal{B}.$$

We want to define a corresponding quotient $\mathcal{E}$ of $\mathcal{E}'$. Thus given $f : V' \hookrightarrow V$ let

$$\Gamma_f = \left\{ \theta \in \operatorname{Aut} V \;\middle|\; \begin{array}{l} \theta f = f \\ \operatorname{Im}(\theta - 1) \subset \operatorname{Im} f \end{array} \right\}.$$

$$\Gamma_f \simeq \operatorname{Hom}(V/fV', V')$$

Introduce the equivalence relation on the arrows of $\mathcal{E}'$ lying over f:

$$(\theta, Q, f) \sim (\theta\gamma^{-1}, \gamma Q, f) \qquad \gamma \in \Gamma_f$$

I claim this equivalence relation is compatible with composition in $\mathcal{E}'$: Thus suppose we are given

$$\begin{array}{ccccc} & & \theta' & & \theta \\ V'' & \xrightarrow{f'} & V' & \xrightarrow{f} & V \\ & & Q' & & Q \end{array}$$

and $\gamma \in \Gamma_f$

$$(\theta\gamma^{-1}, \gamma Q, f)(\theta', Q', f') = (\theta\gamma^{-1} \cdot (\gamma Q, f)_* \theta', \gamma Q + fQ', ff')$$
$$(\theta, Q, f)(\theta', Q', f') = (\theta \cdot (Q, f)_* \theta', Q + fQ', ff')$$

Since $\quad \gamma(Q+fQ') = \gamma Q + fQ' \quad (\gamma f = f)$

we must prove that

$$\gamma^{-1}(\gamma Q, f)_* \theta' \cdot \gamma = (Q, f)_* \theta'$$

But

$$(\gamma^{-1}(\gamma Q, f)_* \theta' \cdot \gamma)(fv') = (\gamma^{-1}(\gamma Q, f)_* \theta')(fv')$$

$$= \gamma^{-1} f \theta' v' = \theta' v' = ((Q, f)_* \theta')(fv')$$

$$(\gamma^{-1}(\gamma Q, f)_* \theta' \cdot \gamma)(g) = \gamma^{-1}(\gamma g) = g.$$

so OKAY here. NO $\Gamma_f \not\subset \Gamma_{ff'}$ so this is no good.

If on the other hand we have $\gamma' \in \Gamma_{f'}$

$$(\theta, Q, f)(\theta' \gamma'^{-1}, \gamma' Q', f') = (\theta \cdot (Q,f)_* (\theta' \gamma'^{-1}), Q + f\gamma' Q', ff')$$

$$= (\theta \cdot (Q,f)_* \theta' \cdot [(Q,f)_* \gamma']^{-1}, Q + f\gamma' Q', ff')$$

so all we need show is that

$$((Q,f)_* \gamma')(Q + fQ') \overset{?}{=} Q + f\gamma' Q'$$

which is clear. Thus we define $\mathcal{E}$ by

objects: $\quad V$

morph: $\quad \mathrm{Hom}_{\mathcal{E}}(V', V) = \mathrm{Hom}_{\mathcal{E}'}(V', V)/\Gamma$

i.e. $$\mathrm{Hom}_{\mathcal{E}}(V',V) = \coprod_{f\in \mathrm{Hom}_{\mathcal{B}}(V',V)} \mathrm{Hom}_{\mathcal{E}'}(V',V)_f \big/ \Gamma_f$$

The next point is that the set of maps of $\mathcal{E}$ lying over $f: V' \hookrightarrow V$ in $\mathcal{B}$ is

$$\mathrm{Aut}(V) \times \{Q \text{ comp. to } fV'\} \big/ \Gamma_f$$

which is a simply-transitive $\mathrm{Aut}(V)$-space. Thus the category $\mathcal{E} \longrightarrow \mathcal{B}$ is cofibred with fibre over V equivalent to $\mathrm{Aut}(V)$.

I want now to show that $\mathcal{E}$ has the homotopy type of $BGL(k)^+$. Because the functor $p: \mathcal{E} \longrightarrow \mathcal{B}$ is cofibrant one has the Leray spectral sequence for $p: \mathcal{E}^\wedge \longrightarrow \mathcal{B}^\wedge$, the induced morphism of topoi

$$E_2^{pq} = H^p(\mathcal{B}, R^q p_*(F)) \Longrightarrow H^{p+q}(\mathcal{E}, F)$$

$$R^q p_*(F)(V) = H^q(\mathrm{Aut}(V), F)$$

Thus what I want to show is

$$H^p(\mathcal{B}, V \longmapsto H^*(\mathrm{Aut}(V), \mathbb{Z})) = 0 \qquad p > 0.$$

It will be better to use covariant functors, ~~thus~~ and homology. Thus have for F in $\mathcal{E}^\vee$

$$E^2_{pq} = H_p(\mathcal{B}, L_q p_*(F)) \Longrightarrow H_{p+q}(\mathcal{B}, F)$$

$$L_q p_*(F)(V) = H_q(\mathrm{Aut}(V), F)$$

and what I want to show is the vanishing of

$$H_p(\mathcal{B}, V \longmapsto H_q(\mathrm{Aut}(V), \mathbb{Z}))$$

for $p > 0$.

Let W be an infinite dimensional vector space over k. It is thus an ind-object of $\mathcal{B}$, its endos being the injective ~~maps of~~ maps, which form a monoid M. Now consider the morphism of topoi

$$M^\vee \xrightarrow{\quad f \quad} B^\vee$$

constituted by the **adjoint** functors

$$(f_* F')(V) = \varprojlim_{V \to W} F'(W) \qquad F'(W) \text{ is an } M\text{-set}$$

$$(f^* F)(W) = \varinjlim_{V \subset W} F(V)$$

(since W is an ind-object, this is indeed a morphism of topoi!)

I want to show that

$$H_i(M, f^*F) = H_i(\mathcal{B}, F)$$

for all i. Now I think this ought to result from

$$\mathbb{R}f_*(\mathbb{Z}) = \mathbb{Z}.$$

Both sides are homological functors of F, and for $i=0$ they coincide since

$$\varinjlim_{V \in \mathcal{B}} F(V) \longleftarrow \varinjlim_{M} \varinjlim_{V \subset W} F(V)$$

(here we need to know that given any injection

$$\begin{tikzcd} V_1 \arrow[r, hook] \arrow[d, phantom, "\cap"] & V_2 \arrow[d, phantom, "\cap"] \\ W \arrow[r, dotted] & W \end{tikzcd} \qquad \begin{tikzcd} V_1 \arrow[r] \arrow[d] & V_2 \arrow[r, hook] \arrow[d] & W \\ W \arrow[r] & W \coprod_{V_1} V_2 \arrow[ur, dotted] & \end{tikzcd}$$

a dotted arrow exists.)

So it is necessary to check effaceability of the right side. Try F of the form

$$F(V) = \coprod_{\mathrm{Hom}_{\mathcal{B}}(V_0, V)} \mathbb{Z}$$

then

$$H_i(M, f^*F) = H_i\Big(M, \coprod_{\mathrm{Hom}_{\mathcal{B}}(V_0, W)} \mathbb{Z}\Big) = H_i(V_0 \to W, \mathbb{Z})$$
$$= \Big(L_i \varinjlim_{V_0 \to W}\Big)(\mathbb{Z}).$$

Thus what we need to show is the contractibility (for each V) of the category of arrows $V \longrightarrow W$, i.e. injections $V \hookrightarrow W$ under the action of the monoid of injections. This is clear by the existence of cofibred products, i.e. we have the following ~~The next point to understand is the existence of the~~ functor and morphisms of functors from the category $B(\mathrm{Inj}(W,W), \mathrm{Inj}(V,W))$ ($B(M,S)$ = cat with objects s and arrows $ms \xleftarrow{m} s$) which is equivalent to the cat. of countable dimensional v.s. under V:

$$\begin{tikzcd} W_0 \arrow[r, Rightarrow, "in_1"] & W_0 \amalg_V W & W \arrow[l, Rightarrow, "in_2"'] \\ & W \arrow[ul, mapsto, "\text{const. functor}"] \arrow[u, mapsto] \arrow[ur, mapsto, "\text{identity}"'] & \end{tikzcd}$$

Define

$$\mathcal{E} \longrightarrow \mathcal{B}'$$

$$V \longmapsto V$$

$$(\theta, Q, f) \longmapsto (\theta Q, \theta f)$$

this is well-defined on $\mathcal{E}$

$$(\theta\gamma, \gamma^{-1}Q, f) \longmapsto (\theta Q, \theta\gamma f) = (\theta Q, \theta f)$$

and it's a functor since

$$(\theta\cdot(Q,f)_*\theta',\ Q+fQ',\ ff') \mapsto$$

$$\left([\theta\cdot(Q,f)_*\theta'](Q+fQ'),\ [\theta\cdot(Q,f)_*\theta']ff'\right)$$

$$= (\theta Q + \theta f\theta' Q',\ \theta f\theta' f') = (\theta Q, \theta f)(\theta' Q', \theta' f').$$

Consider the change of coordinates

$$\langle \theta, Q, f \rangle = (\theta, \theta^{-1}Q, \theta^{-1}f).$$

$$\langle \theta, Q, f\rangle\langle \theta', Q', f'\rangle = (\theta, \theta^{-1}Q, \theta^{-1}f)(\theta', \theta'^{-1}Q', \theta'^{-1}f')$$

$$= (\theta\cdot(\theta^{-1}Q, \theta^{-1}f)_*\theta',\ \theta^{-1}Q + \theta^{-1}f\theta'^{-1}Q',\ \theta^{-1}f\theta'^{-1}f')$$

$$= ((Q,f)_*\theta'\cdot\theta,\ \quad '' \quad,\ \quad '' \quad)$$

$$= \langle (Q,f)_*\theta'\cdot\theta,\ (Q,f)_*\theta'(Q+f\theta'^{-1}Q'),\ (Q,f)_*\theta'(f\theta'^{-1}f')\rangle$$

$$= \langle (Q,f)_*\theta'\cdot\theta,\ Q+fQ',\ ff'\rangle.$$

(Have used formula

$$\theta\cdot(\theta^{-1}Q, \theta^{-1}f)_*\theta' = (Q,f)_*\theta'\cdot\theta \qquad \text{or}$$

$$\theta\cdot(\theta^{-1}Q, \theta^{-1}f)_*\theta'\cdot\theta^{-1} = (Q,f)_*\theta'$$

To prove must check for $q \in Q$, $fv' \in V$:

$$\theta\cdot(\theta^{-1}Q, \theta^{-1}f)_* \theta' \cdot \theta^{-1} \begin{cases} q \\ fv' \end{cases} = \begin{cases} \theta\theta^{-1}q \\ \theta\theta^{-1}f\theta' v' \end{cases} = \begin{cases} q \\ f\theta' v' \end{cases}.$$

Thus we have the composition formula

$$\boxed{\langle \theta, Q, f \rangle \langle \theta', Q', f' \rangle = \langle (Q,f)_* \theta' \cdot \theta, Q + fQ', ff' \rangle}$$

On the other hand observe that

$$\theta^{-1} \Gamma_f \theta = \Gamma_{\theta^{-1}f}$$

so if $\gamma \in \Gamma_f$

$$\langle \gamma\theta, Q, f \rangle = (\gamma\theta, \theta^{-1}\gamma^{-1}Q, \theta^{-1}\gamma^{-1}f)$$

$$= (\theta\theta^{-1}\gamma\theta, \theta^{-1}\gamma^{-1}Q, \theta^{-1}f) \qquad \Gamma_f f = f$$

$$\sim (\theta, \theta^{-1}\cancel{\gamma\theta\theta^{-1}\gamma^{-1}}Q, \theta^{-1}f)$$

$$= \langle \theta, Q, f \rangle.$$

$$\langle\!\langle \theta, Q, f \rangle\!\rangle = \langle \theta^{-1}, Q, f \rangle = (\theta^{-1}, \theta Q, \theta f)$$

$$\langle\!\langle \theta, Q, f \rangle\!\rangle \langle\!\langle \theta', Q', f' \rangle\!\rangle = (\theta^{-1}, \theta Q, \theta f)(\theta'^{-1}, \theta' Q', \theta' f')$$

$$= (\theta^{-1} \cdot (\theta Q, \theta f)_* \theta'^{-1}, \ \theta Q + \theta f \theta' Q', \ \theta f \theta' f')$$

$$= ((Q,f)_* \theta'^{-1} \cdot \theta^{-1}, \qquad)$$

$$= \left([\theta \cdot (Q,f)_* \theta']^{-1}, \ [\theta \cdot (Q,f)_* \theta'](Q + fQ'), \ [\theta \cdot (Q,f)_* \theta'] ff' \right)$$

$$= \langle\!\langle \theta \cdot (Q,f)_* \theta', \ Q + fQ', \ ff' \rangle\!\rangle$$

Also if $\gamma \in \Gamma_f$, then

$$\langle\!\langle \theta\gamma, Q, f \rangle\!\rangle = (\gamma^{-1}\theta^{-1}, \ \theta\gamma Q, \ \theta\gamma f)$$

$$= (\theta^{-1}\theta\gamma^{-1}\theta^{-1}, \ \theta\gamma Q, \ \theta\cancel{\gamma} f) \qquad \theta\gamma^{-1}\theta^{-1} \in \Gamma_{\theta f}$$

$$\simeq (\theta^{-1}, \theta Q, \theta f)$$

$$= \langle\!\langle \theta, Q, f \rangle\!\rangle.$$

Thus the category $\mathcal{E}$ I have constructed ~~[illegible] discretely fibred over the [illegible]~~ is equivalent to category with

objects V

morphisms: a complemented injection $(Q,f): V' \longrightarrow V$

+ a coset in Aut V / Γ_f.

February 18, 1972.

I want now to understand passage to the limit a bit better.

Let W be an infinite dimensional vector space (countable dimension, to simplify). Then we can ~~define~~ restrict the category $\mathcal{E}$ to $\mathcal{B}_{/W}$, i.e. consider the category $\tilde{\mathcal{E}}_W$ with objects $V \subset W$ and morphisms equivalence classes of (θ, Q, i), where $i: V' \to V$ is the inclusion, and Q is a complement of V', and $\theta \in \mathrm{Aut}(V)$. Now following SGA4 we form the inductive limit category

$$\mathcal{E}_W = \varinjlim_{V \subset W} (e, \mathrm{Aut}(V))$$

by inverting all the arrows in $\tilde{\mathcal{E}}_W$. Thus the objects of $\mathcal{E}_W$ are the f.d. subspaces $V \subset W$ and

$$\mathrm{Hom}_{\mathcal{E}_W}(V'', V') = \varinjlim_{V''+V' \subset V \subset W} \mathrm{Aut}(V).$$

This has to be made clear! The first thing to note is that given $i: V' \subset V$ when we say that $i_*(V') \simeq V$ we must given as cartesian arrow over i, i.e. a triple (θ, Q, i). Now of course we will pick the arrow belonging to some splitting Q, i.e.

$$i_*(V') \simeq V \quad \text{means} \quad \begin{cases} V' \xrightarrow{(id_V, Q, i)} V \\ V' \xrightarrow{i} V. \end{cases}$$

Thus it would seem that the category $\mathcal{E}_W$ is a groupoid with group isomorphic to

$$\varinjlim_{T\subset S} \mathrm{Aut}(k[T])$$

if $W = k[S]$.

What is the relation between this thing and the ~~the~~ subgroup of $\mathrm{Aut}(W)$ consisting of automorphisms of the form $I+K$, $K: W \longrightarrow W$ of finite rank?

First observe that if $\Theta = I + K$ is of this form then we can choose $Q \subset \mathrm{Ker}(K)$ of finite codimension such that $Q \cap \mathrm{Im}(K) = 0$, and we can choose $V' \subset W$ such that $\mathrm{Im}(K) \subset V'$, $V' \oplus Q = W$. Then

$$K : Q \longrightarrow Q \quad \text{is zero}$$

$$K : V' \longrightarrow V'$$

and so Θ comes from an auto. of V' i.e

$$\theta = (Q, f)_* \theta'$$

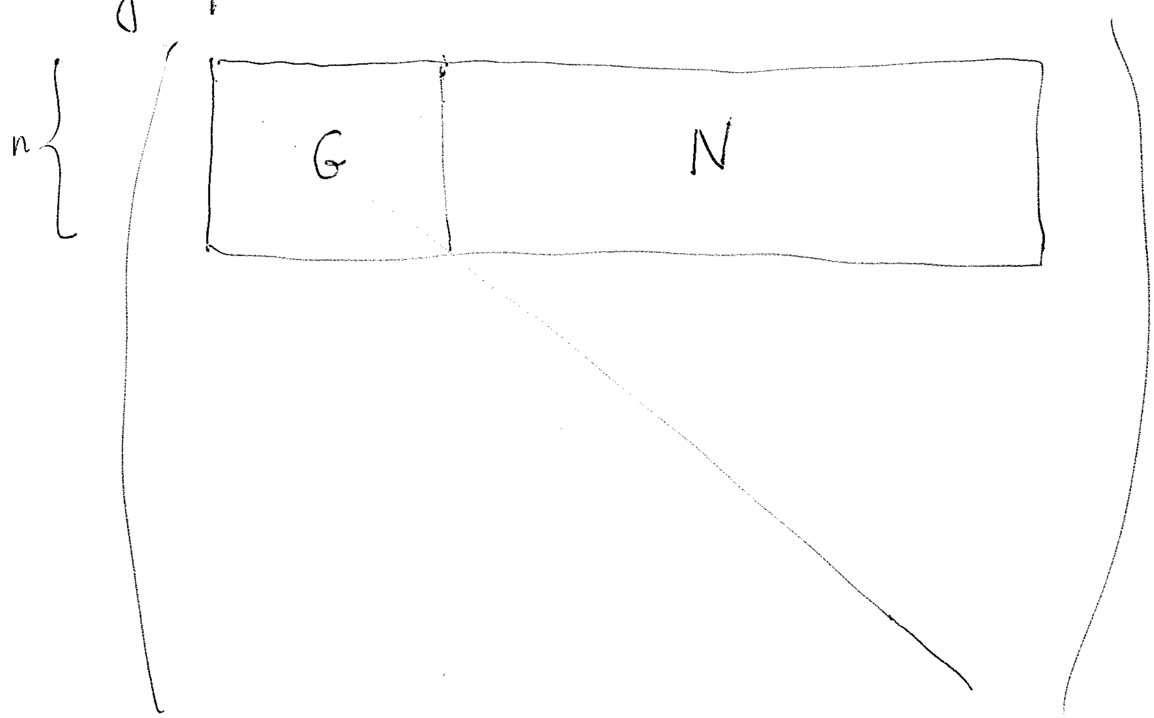
where $\Theta' : V' \longrightarrow V'$ is $I + K|V'$.

<u>Conjecture:</u> $GL_\infty(R) \subset \mathrm{Aut}^f(R^{(\infty)})$ induces an isomorphism on cohomology.

$\mathrm{Aut}^f(R^{(\infty)})$ is the subgroup of automorphisms of the form $1+K$, where K is of finite rank. This means that K lies in the submodule

$$E \otimes E^* \subset \mathrm{End}(E).$$

In terms of matrices K has only finitely many non-zero rows. Thus it is union of the subgroups



$G \tilde{\times} N$. Again we consider the extension

$$1 \longrightarrow N \longrightarrow G \tilde{\times} N \longrightarrow G \longrightarrow 1$$

and the unstable argument goes through. So I need only check the norm argument.

Actually the following argument should be simpler. Observe that if H is a fin. gen. subgroup of $\mathrm{Aut}^f(\mathbf{k}^{(\infty)})$, then it should be possible to find a decomposition

$$k^{(\infty)} = V' \oplus Q$$

so that H comes from a subgroup of V'. Then ~~[illegible]~~ all you have to do is show that $\mathrm{Aut}(k^{(\infty)})$ acts trivially on the cohomology of $\mathrm{Aut}^f(k^{(\infty)})$.

February 19, 1972

Consider a ring R and an automorphism of $R^{(S)}$ of the form $1+K$ where $K: R^{(S)} \to R^{(S)}$ is an endomorphism of finite rank, i.e contained in R^T for some finite subset T of S. (In general if E is an R-module we have a map

$$\mathrm{Hom}_R(E,R) \otimes_R E \longrightarrow \mathrm{Hom}_R(E,E)$$

and elements in the image are what I mean by endomorphisms of finite rank. More generally, I can consider

$$\mathrm{Hom}_R(E,R) \otimes_R F \longrightarrow \mathrm{Hom}_R(E,F)$$

which will be an isomorphism for F f.g. projective, and injective for F infinitely generated and free.)

I want now to understand when given $1+K$, does there exist a splitting $R^{(S)} = V \oplus Q$ with V finitely generated such that $K(V) \subset V$, $K(Q) = 0$. Consider $\mathrm{Im}\, K \subset R^T$ for some finite $T \subset S$. If $\mathrm{Im}\, K$ is finitely generated let Ke_i $i \in T_1$ generate it and write

$$Ke_j = \sum_{i \in T_1} r_{ji} Ke_i \qquad j \in S - T_1$$

and let Q' be spanned by

$$e_j - \sum_i r_{ji} e_i \qquad j \in S - T_1 .$$

Then Q' is a direct summand of V and by making the evident change of coordinates we can suppose

$$e_j \in \mathrm{Ker}(K) \qquad\qquad j \in S - T_1$$

Now take Q to be spanned by

$$e_j \qquad\qquad j \in S - T_1 - T$$

and V to be spanned by

$$e_j \qquad\qquad j \in T_1 \cup T.$$

Then $V \supset R^T \supset \mathrm{Im}\, K$, $Q \subset \mathrm{Ker}(K)$ and $Q \oplus V = R^{(S)}$, so ~~the~~ we have what we want.

Thus if $\mathrm{Im}\, K$ is finitely generated $\exists$ splitting $R^{(S)} = V \oplus Q$ with $K(Q) = 0$ $K(V) \subset V$. and V finitely generated.

Conversely if such a splitting exists ~~B~~ $\mathrm{Im}\, K = K(V)$ is finitely generated.

So if R is not noetherian, then there are finite rank maps

$$R^{(S)} \xrightarrow{\quad K \quad} R$$

whose kernels are so small they contain no ~~projective~~ summand Q of finite corank.

Now suppose R noetherian, or better, a field k, (since I know already there are limitations on the generality possible). Then for each decomposition

$$R^{(S)} = V \oplus Q$$

I get a subgroup

$$(V,Q)_* \operatorname{Aut}(V) \subset \operatorname{Autf}(R^{(S)})$$

and the union of these subgroups in $\operatorname{Autf}(R^{(S)})$. Suppose I partially order the decompositions:

$$(V,Q) \leq (V',Q') \quad \text{if} \quad \begin{matrix} V \subset V' \\ Q \supset Q' \end{matrix}$$

I then want to show that the partially-ordered set is directed. So given (V_1,Q_1) and (V_2,Q_2), I consider the map

$$R^{(S)} \xrightarrow{\varphi} R^{(S)}/Q_1 \times R^{(S)}/Q_2 \simeq V \oplus V'$$

The image is finitely generated, as R is noetherian, hence I know there exists a summand Q of finite corank in the kernel, i.e. in $Q_1 \cap Q_2$. In fact I consider this by choosing $T \subset S$ so that $\varphi(e_i)$, $i \in T$, spans $\operatorname{Im}\varphi$ and letting Q be a complement to R^T spanned by vectors of the form $e_j - \sum_{i \in T} r_{ij} e_i$, $j \notin T$. Choosing T so large that V_1 and V_2 are contained in $V = R^T$, we are done.

Therefore when R is noetherian

$$\mathrm{Autf}(R^{(S)}) = \varinjlim_{(V,Q)} \mathrm{Aut}(V)$$

where (V,Q) runs over the directed set of decompositions

$$R^{(S)} = V \oplus Q$$

with V finitely generated.

It follows that

$$H_*(\mathrm{Autf}(R^{(S)})) = \varinjlim_{(V,Q)} H_*(\mathrm{Aut}(V))$$

but we have already seen that if $V' \subset V^*$ is a direct injection, the arrow

$$H_*(\mathrm{Aut}(V')) \longrightarrow H_*(\mathrm{Aut}(V)),$$

induced by choosing a complement, is in fact independent of the choices. Thus

$$H_*(\mathrm{Autf}(R^{(S)})) = \varinjlim_{V} H_*(\mathrm{Aut}(V))$$

where V runs over the finitely generated direct summands of $R^{(S)}$. By cofinality this is the same as

$$H_*(GL(R)).$$

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Suppose k a field to simplify. Fix a vector space W and consider the category whose objects are exact sequences

$$0 \longrightarrow V_0 \longrightarrow V \longrightarrow W \longrightarrow 0$$

with V_0 finite-dimensional, and whose morphisms are isomorphisms of exact sequences inducing the identity on W. Now this category has a direct sum operation: Given $V \twoheadrightarrow W$ and $V' \to W$ we get

$$V \times_W V' \twoheadrightarrow W$$

with kernel $\simeq V_0 \oplus V_0'$. The operation is commutative, associative, and unitary (e.g. permutative category). In addition it is cofibred over the category of finite diml k-vector spaces meaning that given $V_0 \longrightarrow Z$ we have a functor

$$\begin{tikzcd}
0 \arrow[r] & V_0 \arrow[r] \arrow[d] & V \arrow[r] \arrow[d] & W \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & Z \arrow[r] & Z \amalg_{V_0} V \arrow[r] & W \arrow[r] & 0.
\end{tikzcd}$$

And finally Grothendieck has signalled that it is left exact in the sense that given

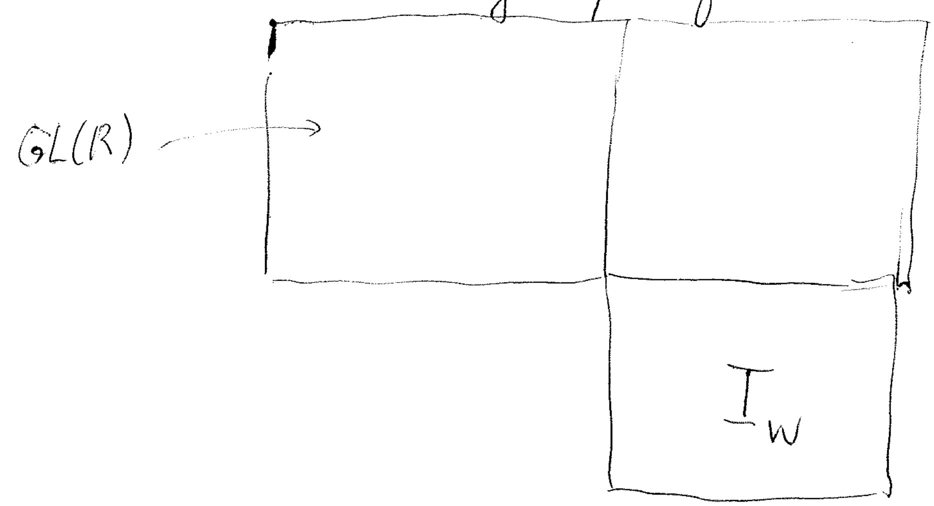
$$0 \longrightarrow V_0' \longrightarrow V_0 \longrightarrow V_0'' \longrightarrow 0$$

then $\underline{\mathrm{Ext}}(W, V_0')$ is the fibre of $\underline{\mathrm{Ext}}(W, V_0) \longrightarrow \underline{\mathrm{Ext}}(W, V_0'')$.

Now I believe the old norm argument works. Thus let me suppose given a stable exponential class Θ for representations of groups in $\underline{\mathrm{Ext}}(?, W)$. Observe that the monoid of isomorphism classes of such extensions is $\mathbb{N}$, so Θ should be the same as a ring homomorphism

$$H_*(G) \longrightarrow S$$

where G is the group of autos of $R^{(\mathbb{N})} \oplus W$



consisting of matrices $I + K$, where $K(W) = 0$ and $\mathrm{Im}\, K \subset R^T$ T finite. Better

$$G = \bigcup G_n$$

where
$$G_n = GL_n R \,\tilde{\times}\, \mathrm{Hom}_R(W, R^n) \subset \mathrm{Aut}(R^n \oplus W)$$

$$(\alpha \;,\; \beta)(v, w) = \alpha(v) + \alpha\beta w + w$$

Thus G_n is a semi-direct product where the second factor is an abelian group, an R-module. Thus it is now clear to me that the norm argument should work.

Conclusions:

I). The group of matrices

$$G_n = GL_n R \tilde{\times} \mathrm{Hom}_R(W, R^n)$$

for any R-module W, and any ring R has the same cohomology as $GL_n R$ in the limit as $n \to \infty$.

II) If W is an infinitely-generated free R-module, then the group of autom. of W of the form $1+K$ where K is an endomorphism of W of "trace class" (comes from $W \otimes W^*$) has the same cohomology as $GL(R)$.

~~Note: three subgroups~~

~~$GL(R) \subset \{1+K \mid \mathrm{Im}\, K \text{ fin. gen}\} \subset \{1+K \mid$~~

D. Quillen

February 24, 1972.

three diml. real class belonging to a codim 1 foliation. Attempt at finding the corresponding 2-diml class for group of diffeos of $\mathbb{R}$ with comp. support.

Let L be a real line bundle over a manifold X stratified with respect to a foliation of codim 1. To define ~~an~~ element of $H^3(X,\mathbb{R})$. Observe $p_1(L) = \pm c_1(L)^2$ vanishes for two reasons - by Bott's theorem, and because $c_1(L)$ is torsion.

Choose a metric $|\ |$ on L. If D' is a connection preserving the metric and s is a local section

$$D's = s\theta'$$

then

$$\begin{aligned} d\,|s|^2 &= \langle s\theta, s\rangle + \langle s, s\theta\rangle \\ &= 2|s|^2\theta' \end{aligned}$$

or

$$\theta' = \frac{1}{2}\, d\log|s|^2 = d\log|s|$$

Observe the curvature $d\theta' = 0$ in this connection.

On the other hand, let D be connection adapted to the S-connection given on L, and let s be an S-flat local section (S = subbundle of T of tangents to the leaves). Then

$$Ds = s\,\theta$$

where θ is a local section of Q^*, ($Q = T/S$). Then the curvature in this connection is $d\theta$, which is a local section of $Q^* T^*$, so $(d\theta)^2 \in \Lambda^2 Q^* \cdot \Lambda^2 T^* = 0$, since $\dim(Q) = 1$.

Now $\Theta - \Theta'$ is a global 1-form (independent of the choice of s), and so is $d\Theta$, so also is

$$(\Theta - \Theta')\, d\Theta$$

But this form is closed, since

$$d\left[(\Theta-\Theta')d\Theta\right] = (d\Theta)^2 = 0$$

so we get an element of $H^3(X,\mathbb{R})$. I omit the verification that this class is independent of the choices. (see page 3)

Next take $L = Q^*$. Here the S-connection $\tilde{D}$ may be identified with d:

$$\begin{tikzcd}
Q^* \arrow[r, "d"] \arrow[d, equal] & Q^* \cdot T^* \arrow[r, "d"] & Q^* \cdot \Lambda^2 T^* \\
Q^* \arrow[r, "\tilde{D}"] & Q^* \otimes S^* \arrow[r, "\tilde{D}"] \arrow[u, "s"'] & Q^* \otimes \Lambda^2 S^* \arrow[u]
\end{tikzcd}$$

Choose a metric on Q^* and let ω denote a local unit section. Then to lift $\tilde{D}$ to a connection D means we choose a form $\bar{\Theta}$ with

$$d\omega = \omega \wedge \bar{\Theta}$$

and put

$$D\omega = \omega \cdot \tilde{\Theta}$$

Let $f\omega$ be an S-flat local section. Then

$$\Theta' = d \log |f\omega| = d \log f$$

$$D(f\omega) = \omega\, df + f\omega\tilde{\Theta} = f\omega\left(\frac{df}{f} + \tilde{\Theta}\right)$$

$$\theta = \frac{df}{f} + \tilde{\theta}$$

Thus our three form is

$$(\theta - \theta')\,d\theta = \tilde{\theta}\cdot d\tilde{\theta}.$$

<u>Procedure</u>: To find the three form, choose a non-vanishing section ω of Q^* (possible over double covering) and choose $\tilde{\theta}$ so that

$$d\omega = \omega\tilde{\theta}$$

(possible by integrability). $\tilde{\theta}$ exists globally over X. Then the three form is $\tilde{\theta}\,d\tilde{\theta}$.

Why well-defined: (write θ for $\tilde{\theta}$)

$d(\theta\cdot d\theta) = (d\theta)^2$. $0 = d^2\omega = d\omega\cdot\theta - \omega\, d\theta = -\omega\, d\theta$

so as ω non-vanishing $\exists\eta$ $d\theta = \eta\omega \Rightarrow (d\theta)^2 = 0$. Thus $\theta\cdot d\theta$ is a closed form.

Change θ to $\theta + f\omega$. Then

$$(\theta + f\omega)(d\theta + df\omega + fd\omega) = \theta\cdot d\theta + \theta\cdot df\,\omega + \cancel{\theta f d\omega}$$
$$\cancel{f\omega d\theta} + f\omega df\omega + \underbrace{\cancel{f\omega f d\omega}}_{\omega\theta}$$
$$= \theta\cdot d\theta \pm d(f\,d\omega)$$

Change ω to $f\omega$ where f non-vanishing.

$$d(f\omega) = df\,\omega + f\,\omega\theta = f\omega\left(-\frac{df}{f} + \theta\right)$$

$$\left(-\frac{df}{f} + \theta\right)(0 + d\theta) = \theta\cdot d\theta + d\left(\frac{df}{f}\theta\right)$$

so the cohomology class is well-defined.

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Let $Y = X \times \mathbb{R}$ be endowed with a codim 1 foliation $S \subset T_Y$ transversal to the fibres. Thus if $\pi : Y \longrightarrow X$ is the projection, the composite

$$S \longrightarrow T_Y \longrightarrow \pi^* T_X$$

is an isomorphism, and so the two exact sequences

$$0 \longrightarrow S \longrightarrow T_Y \longrightarrow Q \longrightarrow 0$$

$$0 \longrightarrow T_\pi \longrightarrow T_Y \longrightarrow \pi^* T_X \longrightarrow 0$$

split each other, i.e. have Cartan-Eilenberg picture

$$\begin{tikzcd}
0 \arrow[dr] & & & & 0 \\
& S \arrow[rr, "\sim"] \arrow[dr] & & \pi^* T_X \arrow[ur] & \\
& & T_Y \arrow[ur] \arrow[dr] & & \\
& T_\pi \arrow[ur] \arrow[rr, "\sim"'] & & Q \arrow[dr] & \\
0 \arrow[ur] & & & & 0
\end{tikzcd}$$

Let $t : Y \longrightarrow \mathbb{R}$ be the second projection. Then we can project $dt \in \Gamma(Y, T_Y^*)$ onto $\Gamma(Y, Q^*)$ to obtain a section ω of the latter which locally is of the form

$$\omega = dt + \sum_{i=1}^{m} a_i \, dx_i \qquad \begin{array}{l} m = \dim X \\ a_i = a_i(x, t) \end{array}$$

A vector field

$$v = u_0 \frac{\partial}{\partial t} + \sum u_i \frac{\partial}{\partial x_i}$$

is tangent to the foliation iff

$$\langle v, \omega \rangle = u_0 + \sum a_i u_i = 0$$

Thus the ~~[illegible]~~ bundle S has the frame

$$v_i = -a_i \frac{\partial}{\partial t} + \frac{\partial}{\partial x_i} \qquad 1 \le i \le m$$

Since

$$[v_i, v_j] = \left[-a_i \frac{\partial}{\partial t} + \frac{\partial}{\partial x_i} , -a_j \frac{\partial}{\partial t} + \frac{\partial}{\partial x_j} \right]$$

$$= \left[\begin{array}{l} a_i \dfrac{\partial a_j}{\partial t} - a_j \dfrac{\partial a_i}{\partial t} \\ -\dfrac{\partial a_j}{\partial x_i} + \dfrac{\partial a_i}{\partial x_j} \end{array} \right] \frac{\partial}{\partial t}$$

and this must lie in S by integrability. Thus the equations

$$a_i \frac{\partial a_j}{\partial t} - a_j \frac{\partial a_i}{\partial t} + \frac{\partial a_i}{\partial x_j} - \frac{\partial a_j}{\partial x_i} = 0 \qquad \forall i, j$$

express the integrability of S.

Suppose now $\tilde{\theta}$ such that $d\omega = \omega \tilde{\theta}$. By removing $f\omega$ from $\tilde{\theta}$ we can suppose that $\tilde{\theta} \in \Gamma(Y, \pi^* T_X^*)$, which locally means $\tilde{\theta}$ is of the form

$$\tilde{\theta} = \sum b_i \, dx_i .$$

Thus

$$\omega \tilde{\theta} = \sum b_i \, dt \, dx_i + \sum a_i b_j \, dx_i \, dx_j$$

$$= \sum_i b_i \, dt \, dx_i + \sum_{i<j} (a_i b_j - a_j b_i) \, dx_i \, dx_j$$

$$d\omega = \sum_i \frac{\partial a_i}{\partial t}\, dt\, dx_i + \sum_{i,j} \frac{\partial a_i}{\partial x_j}\, dx_j\, dx_i$$

$$= \sum_i \frac{\partial a_i}{\partial t}\, dt\, dx_i + \sum_{i<j} \left(\frac{\partial a_j}{\partial x_i} - \frac{\partial a_i}{\partial x_j}\right) dx_i\, dx_j$$

so we conclude that

$$b_i = \frac{\partial a_i}{\partial t}\ .$$

Thus

$$\boxed{\tilde{\Theta} = \sum_{i=1}^{m} \frac{\partial a_i}{\partial t}\, dx_i}$$

$$d\tilde{\Theta} = \sum \frac{\partial^2 a_j}{\partial t^2}\, dt\, dx_j + \sum \frac{\partial^2 a_k}{\partial t\, \partial x_j}\, dx_j\, dx_k$$

$$\tilde{\Theta}\cdot d\tilde{\Theta} = \sum_{i,j} - \frac{\partial a_i}{\partial t} \frac{\partial^2 a_j}{\partial t^2}\, dt\, dx_i\, dx_j + \sum_{i,j,k} \frac{\partial a_i}{\partial t} \frac{\partial^2 a_k}{\partial t\, \partial x_j}\, dx_i\, dx_j\, dx_k$$

Recall the foliation for $|t|$ large is flat, i.e. $a_i \equiv 0$ for $|t|$ large, so $\tilde{\Theta} d\tilde{\Theta}$ is a form with support proper over X. We want then the 2-form

$$\pi_*(\tilde{\Theta}\cdot d\tilde{\Theta}) = \int \text{wrt to } t$$

Thus last term disappears and we get

$$\pi_*(\tilde{\Theta}\cdot d\tilde{\Theta}) = \sum_{i<j} \int_{t=-\infty}^{t=\infty} \left[\frac{\partial^2 a_i}{\partial t^2}\cdot\frac{\partial a_j}{\partial t} - \frac{\partial a_i}{\partial t}\frac{\partial^2 a_j}{\partial t^2}\right] dt \cdot dx_i dx_j$$

I want to review Bott's formulas for Chern classes in terms of transition functions.

Suppose $E \longrightarrow X$ is a complex vector bundle over a manifold X. Let $\{U\}$ be a covering of X over which the bundle is trivial, and choose trivializations

$$s_U : U \times \mathbb{C}^d \xrightarrow{\sim} E_U .$$

I will think of s_U as a map from sections of the trivial bundle to sections of E_U. Thus

$$s_U = s_V \, g_{VU}$$

where $g_{VU} : V \cap U \longrightarrow GL_d(\mathbb{C})$ is a smooth function. Similarly if D is a connection, and $v : U \to \mathbb{C}^d$ is a section of the trivial bundle

$$D \, s_U(v) = s_U(\Theta_U + d)(v)$$

where Θ_U is a $d\times d$-matrix of 1-forms on X, and d denotes the canonical connection on the trivial bundle. Abbreviation:

$$D \, s_U = s_U(\Theta_U + d)$$

Next compute change in forms going from

U to V:

$$s_V = s_U g_{UV}$$

$$D(s_V) = s_V(\theta_V + d) = s_U g_{UV}(\theta_V + d)$$

$$\| $$

$$D(s_U g_{UV}) = s_U(\theta_U + d) g_{UV}$$

$$= s_U(\theta_U g_{UV} + d g_{UV} + g_{UV} d)$$

so

$$\boxed{g_{UV}\theta_V = \theta_U g_{UV} + d g_{UV}}$$

or

$$\boxed{\theta_V = g_{UV}^{-1}\theta_U g_{UV} + g_{UV}^{-1} d g_{UV}}$$

Formula for curvature

$$D^2 s_U = D s_U(\theta_U + d)$$

$$= s_U(\theta_U + d)(\theta_U + d)$$

$$= s_U(\theta_U\theta_U + \cancel{\theta_U d} + d\theta_U - \cancel{\theta_U d})$$

so

$$\boxed{K_U = \theta_U\theta_U + d\theta_U}$$

Now suppose $\{\rho_U\}$ is a partition of unity. Then we start with the local connections

$$D_U s_U = s_U d$$

$$D_V s_U = D_V(s_V g_{VU})$$

$$= s_V d \cdot g_{VU}$$

$$= s_V (dg_{VU} + g_{VU} d)$$

$$= s_U (g_{VU}^{-1} dg_{VU} + d)$$

and average: $D = \sum \rho_V D_V$.

$$D(s_U) = \left(\sum \rho_V D_V\right) s_U$$

$$= s_U \left(\sum \rho_V g_{VU}^{-1} dg_{VU} + d\right)$$

hence

$$\boxed{\theta_U = \sum_V \rho_V g_{VU}^{-1} dg_{VU}}$$

Back to $\pi : Y \longrightarrow X$, let $\{U\}$ be a covering of X over which the bundle is "trivial" and let $f_U : \pi^{-1} U \longrightarrow \mathbb{R}$ be flat coordinate functions. Then we can suppose

$$t = \sum_V{}' \rho_V f_V \qquad (\text{strictly } \rho_V \circ \pi)$$

where $\{\rho_U\}$ is a partition of unity. Also we can take

$$\omega = \sum_V \rho_V \, df_V$$

for basis of Q^*.
Since

$$dt = \sum d\rho_V \cdot f_V + \omega$$

$$\omega \doteq dt - \sum f_V \, d\rho_V$$

so ω is locally of the form

$$\omega = dt + \sum a_i \, dx_i$$

$$-a_i = \sum_V f_V \frac{\partial \rho_V}{\partial x_i}$$

Thus

$$\tilde{\theta} = -\sum_{i,V} \frac{\partial f_V}{\partial t} \frac{\partial \rho_V}{\partial x_i}$$

$$\pi_*(\tilde{\theta} \cdot d\tilde{\theta}) = \sum_{i,j} \left(\int_{-\infty}^{\infty} \frac{\partial^2 a_i}{\partial t^2} \frac{\partial a_j}{\partial t} \, dt \right) dx_i \, dx_j$$

$$= \sum_{i,j} \left(\int_{-\infty}^{\infty} dt \sum_{v} \frac{\partial^2 f_v}{\partial t^2} \frac{\partial \rho_v}{\partial x_i} \cdot \sum_{w} \frac{\partial f_w}{\partial t} \frac{\partial \rho_w}{\partial x_j} \right) dx_i \, dx_j$$

$$= \sum_{v,w} \left(\int_{-\infty}^{\infty} \frac{\partial^2 f_v}{\partial t^2} \frac{\partial f_w}{\partial t} \, dt \right) d\rho_v \, d\rho_w$$

The trouble with this formula is that we don't know what A is.

obsolete

February 25, 1972. {stable splitting of exact sequences of representations.

Fix a f.d. v.s. V_0 and consider the category of arrows $V_0 \hookrightarrow V$ and isomorphisms inducing the identity on V_0. Then this category, call it $\mathcal{A}$, has a permutative operation. Moreover we have the identity

$$\left(V \overset{V_0}{\amalg} V\right) = V \overset{V_0}{\amalg} (V_0 \oplus V/V_0)$$

The point is that the map

$$\begin{array}{ccc} V & \longrightarrow & V \overset{V_0}{\amalg} V \\ w & & (w, -w) \end{array}$$

kills V_0 and hence induces a map

$$V/V_0 \longrightarrow V \overset{V_0}{\amalg} V$$

so that

$$\begin{array}{ccc} V \oplus V/V_0 & \xrightarrow{\sim} & V \overset{V_0}{\amalg} V \\ v \quad\; w & & (v-w, w) \end{array}$$

Now the isomorphism classes of $\mathcal{A}$, $Is(\mathcal{A})$, is an abelian monoid, $\mathbb{N}$ in the case of modules over a field. Characteristic classes are thus classified by points of the Hopf algebra

$$\mathrm{Aut}(P_s \oplus V_0\,;\,V_0)$$
$$\|$$
$$\bigoplus_{s \in S} H_*\left(\mathrm{Aut}(P_s) \tilde{\times} \mathrm{Hom}(P_s, V_0)\right)$$

where P_α are representatives for the different isom. classes of projective f.g. R-modules.

If θ is an invertible exponential char class, then we have for any representation E

$$\theta(E \overset{V_0}{\amalg} E) = \theta(E)\,\theta(E)$$

$$\parallel$$

$$\theta\big(E \overset{V_0}{\amalg} (V_0 \oplus E/V_0)\big) = \theta(E)\,\theta(V_0 \oplus E/V_0)$$

so by invertibility, we have

$$\theta(E) = \theta(V_0 \oplus E/V_0) .$$

Now I recall we can split invertible exponential classes into inessential and stable components, the stable ones being classified by points of

$$\varinjlim_n H_*\big(GL_n R \tilde{\times} \mathrm{Hom}(R^n, V_0)\big)$$

$$\left(\begin{array}{c|c} \mathrm{Id}_{V_0} & \mathrm{Hom}(R^n, V_0) \\ \hline 0 & GL_n R \end{array}\right)$$

since we have the identity $\theta(E) = \theta(V_0 \oplus E/V_0)$ we conclude that

$$\varinjlim_n H_*(GL_n R \tilde{\times} \mathrm{Hom}(R^n, V_0)) \simeq \varinjlim_n H_*(GL_n R).$$

Similarly in the other direction we can consider the category of surjections

$$V \longrightarrow V_0$$

with V_0 fixed. This has permutative structure

$$V \times_{V_0} V'$$

and again we have

$$V \underset{V_0}{\times} V \simeq V \underset{V_0}{\times} (V_0 \oplus \mathrm{Ker})$$

so again we can conclude

$$\varinjlim_n H_*(GL_n R \tilde{\times} \mathrm{Hom}(V_0, R^n)) = \varinjlim_n H_*(GL_n R).$$

In a general small abelian category consider representations on extensions

$$V_0 \hookrightarrow V$$

inducing the identity on V_0. Claim that any stable characteristic class ($\Theta(E \amalg^{V_0} V) = \Theta(E)$ if G acts trivially on V) ~~[illegible]~~ satisfies

$$\Theta(E) = \Theta(E/V_0 \oplus V_0)$$

at least with field coefficients. In effect, let S be the monoid of iso. classes of such extensions, and $\{V_s\}$ a system of representatives. Then char. classes for all repns. are linear functionals on

$$\bigoplus_s H_*(\mathrm{Aut}\, V_s)$$

and the stable classes are linear functions on

$$\varinjlim_s H_*(\mathrm{Aut}\, V_s).$$

Now let $T \subset S$ be the class of split extensions, so that $T \cong \mathrm{Is}(\mathcal{a})$. Then we are claiming that

$$(*) \qquad \varinjlim_t H_*(\mathrm{Aut}\, V_t) \xrightarrow{\sim} \varinjlim_s H_*(\mathrm{Aut}\, V_s)$$

(A stable Θ is a linear fn. on latter; its composition with the map is the restriction to split representations. Since $\varphi(E) = \Theta(E/V_0 \oplus V_0)$ is a stable class, ~~[illegible]~~ the map is has a retraction, so the only point is its surjectivity, i.e. whether $\varphi(E) = \Theta(E)$.) Now the point is that

$$\varinjlim_{s} H_*(\mathrm{Aut}\, V_s)$$

is a connected Hopf algebra whose points are the stable exponential classes. Previous arg:

$$\Theta(E)\cdot\Theta(E) = \Theta(E \amalg^{V_0} E) = \Theta(E)\cdot\Theta(E/V_0 \oplus V_0)$$

$$\Theta(E) = \Theta(E/V_0 \oplus V_0)$$

Θ invertible by connectedness of Hopf algebra. Thus the map (*) in question induces an isom. on points, and so is an isomorphism.

Question: Let $\mathcal{A}$ be a small abelian category, and Θ an exponential class for representations in $\mathcal{A}$, i.e.

$$\Theta(E_1 \oplus E_2) = \Theta(E_1)\Theta(E_2).$$

Assume also that for any exact sequence of repns.

$$0 \longrightarrow V \longrightarrow E \longrightarrow E'' \longrightarrow 0$$

with V a trivial representation, we have

$$\Theta(E) = \Theta(E'')$$

Then is Θ exponential for all exact sequences of representations?

Proposition: Let Θ be a characteristic class for representations in a small abelian category $\mathcal{A}$. Assume that for any exact sequence of repns.

$$0 \longrightarrow E' \longrightarrow E \longrightarrow E'' \longrightarrow 0$$

with E'' trivial we have

$$\Theta(E') = \Theta(E).$$

Then we have $\Theta(E) = \Theta(E'')$ whenever E' is trivial.

Proof: Fix V_0 and consider repns. in the category of injections $V_0 \hookrightarrow V$. Then if E is such a representation ($E' = V_0$) and if V is a trivial repn. in the category, we have an exact sequences of repns

$$0 \longrightarrow E \longrightarrow E \overset{V_0}{\amalg} V \longrightarrow V/V_0 \longrightarrow 0$$

hence

$$\Theta(E) = \Theta(E \overset{V_0}{\amalg} V)$$

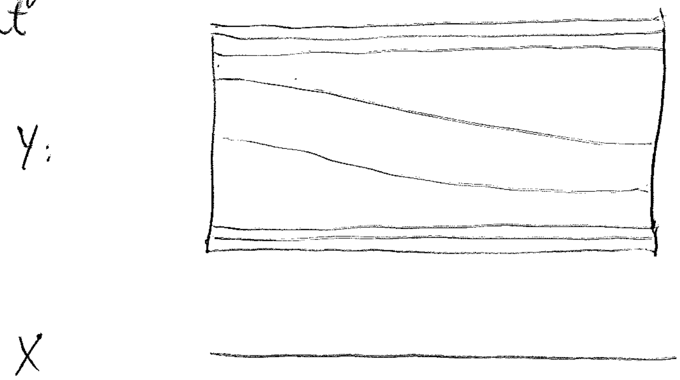
by hypothesis. By previous arguments we know then that

$$\Theta(E) = \Theta(E/V_0)$$

q.e.d.

February 26, 1972:

Another approach: suppose given a bundle $Y \to X$ with foliation transversal to the fibres and flat "way out"



and suppose also given $\{U\}$ and $f_U : Y_U \longrightarrow \mathbb{R}$ flat such that

$$f_V = g_{VU} \circ f_U$$

where g_{VU} is a diffeom. with compact support of $\mathbb{R}$. Following the Bott-Segal idea, for each simplex $\sigma = \{U_0, \ldots, U_q\}$ of the nerve of the covering consider $U_0 \cap \cdots \cap U_q \times \Delta(q)$. Over this we have

$$\omega_{U_0 \cdots U_q} = \sum_{i=0}^{q} t_i \, df_{U_i}$$

where t_i are the barycentric coordinates. Observe

$$f_{U_i} = g_{U_i U_0} \circ f_{U_0}$$

$$df_{U_i} = (g'_{U_i U_0} \circ f_{U_0}) \, df_{U_0}$$

so

$$\omega_{U_0 \cdots U_q} = \sum t_i (g'_{U_i U_0} \circ f_{U_0}) \cdot df_{U_0}$$

$$d\omega_{U_0 \cdots U_q} = \sum_i dt_i \, (g'_{U_i U_0} \circ f_{U_0}) \, df_{U_0}$$

so

$$\theta_{u_0 \cdots u_q} = - \frac{\sum dt_i (g'_{u_i u_0} \circ f_{u_0})}{\sum t_i (g'_{u_i u_0} \circ f_{u_0})}$$

because this lies in $\pi^* T_{X'}$ $\quad X' = (U_0 \cdots U_q) \times \Delta(q)$.

Now if

$$\theta = \frac{\alpha}{f} \qquad \alpha \text{ odd degree form}$$

$$\theta \cdot d\theta = \cancel{\frac{\alpha}{f}} \cdot \left(\frac{d\alpha}{f} - \frac{1}{f^2} df \, \alpha \right) = \frac{\alpha . d\alpha}{f^2}$$

so

$$\theta_{u_0 \cdots u_q} \cdot d\theta_{u_0 \cdots u_q} = \frac{\sum dt_i (g'_{u_i u_0} \circ f_{u_0}) \cdot \sum dt_i (g''_{u_i u_0} \circ f_{u_0}) \cdot df_{u_0}}{\left[\sum t_i (g'_{u_i u_0} \circ f_{u_0}) \right]^2}$$

Now we have to integrate this over fibres of π, and we can use the coordinate furnished by f_{u_0}.

So

$$\pi_* \left(\theta_{u_0 \cdots u_q} . d\theta_{u_0 \cdots u_q} \right) = \int_{-\infty}^{\infty} \frac{\sum dt_i \, g'_{u_i u_0}(z) \cdot \sum dt_i \, g''_{u_i u_0}(z)}{\left[\sum t_i \, g'_{u_i u_0}(z) \right]^2} dz$$

Note that this is independent of x and hence is a closed 2-form on $\Delta(q)$ pulled up to $U_0 \cdots U_q \times \Delta(q)$.

$$\iint\limits_{\substack{0\le x+y\le 1\\ 0\le x,y}} \frac{dx\,dy}{(1+ax+by)^2} = \int_{x=0}^{1} dx \int_{y=0}^{1-x} dy \, \frac{1}{(1+ax+by)^2}$$

$$= \int_0^1 dx \, \frac{1}{b}\left[\frac{-1}{1+ax+by}\right]_0^{1-x} = \frac{1}{b}\int_0^1 dx \left[\frac{1}{1+ax} - \frac{1}{1+b+(a-b)x}\right]$$

$$= \frac{1}{ab}\Big[\log(1+ax)\Big]_0^1 - \frac{1}{b(a-b)}\Big[\log(1+b+(a-b)x)\Big]_0^1$$

$$= \frac{1}{ab}\log(1+a) - \frac{1}{b(a-b)}\Big[\log(1+a) - \log(1+b)\Big]$$

$$= \frac{\log(1+a)}{a(b-a)} + \frac{\log(1+b)}{b(a-b)}$$

since

$$\frac{1}{ab} - \frac{1}{b(a-b)} = \frac{1}{b}\left[\frac{\cancel{a}-b-\cancel{a}}{a(a-b)}\right] = \frac{1}{a(b-a)}$$

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Conjecture: Let X be a manifold (smooth) and $\mathfrak{U} = \{U\}$ an open covering. Let

$$\tilde{X} = \bigcup_{\sigma \in \mathrm{Nerv}(\mathfrak{U})} U_\sigma \times \Delta(\dim \sigma)$$

be Segal's space and suppose we are given a closed n-form ω over $\tilde{X}$, that is, a compatible family of closed n-forms: ω_σ on $U_\sigma \times \Delta(\dim \sigma)$. If $\{\rho_U\}$ is a partition of unity subordinate to $\{U\}$, it defines a section $\rho: X \longrightarrow \tilde{X}$ and we can pull back ω to obtain a closed n-form $\rho^*(\omega)$ on X. On the other hand, given σ of dim. q we can integrate over the fibres of the projection

$$pr_1^\sigma : U_\sigma \times \Delta(\dim \sigma) \longrightarrow U_\sigma$$

and obtain a form

$$\tau_\sigma = pr^\sigma_{1*}\, \omega_\sigma \in \Gamma(U_\sigma, \Omega^{n-q}).$$

Thus from ω we obtain elements

$$\tau^q = \{\tau_\sigma\} \in C^q(\mathfrak{U}, \Omega^{n-q}).$$

The first claim is that $\{\tau\}$ is a n-cocycle in the double complex $C^\cdot(\mathfrak{U}, \Omega^\cdot)$, i.e.

$$d\tau^q = \delta\tau^{q-1} \qquad \text{(modulo signs)}.$$

If this is so τ gives rise to an element of

$$\check{H}^n(\mathfrak{U}, \Omega^\bullet) \longrightarrow H^n(X, \Omega^\bullet) = H^n_{DR}(X)$$

and the second claim is that τ coincides with $\rho^*\omega$ when pushed into $H^n_{DR}(X)$.

The point should be that for the map

$$\pi: \quad U \times \Delta(q) \longrightarrow U$$

we have a Stokes formula:

$$d\,\pi_* = (-1)^q \pi_* d \; + \; \dot{\pi}_*$$

Stokes formula should then give a map of complexes

$$\Gamma(\tilde{X}, \Omega^\bullet) \longrightarrow C^\bullet(\mathfrak{U}, \Omega^\bullet)$$

probably compatible with the two maps from $\Gamma(X, \Omega^\bullet)$, so what we want will then follow from the fact that the latter two maps are quasi-isomorphism.

Special case: If ω form on $X \times I$, then

$$\iota_1^*\omega - \iota_0^*\omega = d\pi_*\omega + \pi_* d\omega$$

where if $\omega = dt \wedge \alpha + \beta$ with $\alpha, \beta \in \Gamma(\pi^* T_X^*)$, then

$$\pi_*\omega = \int_{t=0}^{t=1} dt\,(\iota_t^*\alpha) \;\stackrel{\text{defn}}{=}\; \lim_{\|\Delta t_i\| \to 0} \sum (t_i - t_{i-1})\, \iota_{t_i}^*\alpha$$

Now back to Γ: Assuming the above conjecture is true, the Čech 2-cocycle we are after is obtained by integrating the two form on

$$U \cap V \cap W \times \Delta(2)$$

with respect to the first projection. So here we have

$$\sum t_i\, g'_{u_i u_0}(z) = t_0 + t_1 g'_{vu} + t_2 g'_{wu}$$

$$= 1 + t_1(g'_{vu} - 1) + t_2(g'_{wu} - 1)$$

$$\sum dt_i\, g'_{u_i u_0} = dt_1(g'_{vu} - 1) + dt_2(g'_{wu} - 1)$$

$$\sum dt_i\, g''_{u_i u_0} = dt_1 \cdot g''_{vu} + dt_2 \cdot g''_{wu}$$

so we want

$$\int_{-\infty}^{\infty} dz \int_{\substack{t_1 + t_2 \leq 1 \\ 0 \leq t_1, t_2}} \frac{\begin{vmatrix} g'_{vu} - 1 & g'_{wu} - 1 \\ g''_{vu} & g''_{wu} \end{vmatrix} dt_1\, dt_2}{\left[1 + t_1(g'_{vu} - 1) + t_2(g'_{wu} - 1)\right]^2}$$

$$\boxed{= \int_{-\infty}^{\infty} dz \begin{vmatrix} g'_{vu} - 1, & g'_{wu} - 1 \\ g''_{vu} & g''_{wu} \end{vmatrix} \cdot \left[\frac{\log(g'_{vu})}{(g'_{vu} - 1)(g'_{wu} - g'_{vu})} + \frac{\log(g'_{wu})}{(g'_{wu} - 1)(g'_{vu} - g'_{wu})} \right]}$$

If we put $\varphi = g'vu - 1$, $\psi = g'wu - 1$ then this takes the more pleasant form

$$\int_{-\infty}^{\infty} (\varphi\psi' - \psi\varphi') \frac{f(\varphi) - f(\psi)}{\varphi - \psi} \, dz$$

where $$f(x) = \frac{\log(1+x)}{x}.$$

This is still unworkable but perhaps there should be some infinitesimal version of this cocycle which could be recognized. (This is probably where Gelfand-Fuchs enters.)

Motivation for preceding work

Find the central extension of G (group of diffeos. of $\mathbb{R}$ with compact support) by $\mathbb{R}$ which one knows exists by Mather theorem. This extension should be an analogue of the Steinberg group in alg. K-theory.