Second part

1. **Definition** $GL(A)$, $E(A)$, $BGL(A)^+$, $\tilde{R}(X,A)$, $K(X,A)$

   Bass–Milnor + Dror

   Top. K-theory

2. H-space:

   simplicity

   H-space ~~structure~~

   Rational K-groups

3. Canonical arrow:

$$R(X,A) \longrightarrow K(X,A)$$

define this for any space X

Introduce

$$R'(X,A),\ R(X,A),\ \tilde{R}',\ \tilde{R}$$

$Vect(X,A)$ $\quad St(X,A) = \varinjlim_n [X, BGL_n(A)]$.

Vect

Stable splitting thm.

4. $K(X,A)$ as a universal functor:

Now restrict to finite complex and prove successively the universal props of

$$St(X,A) \longrightarrow \tilde{K}(X,A)$$

$$\tilde{R}'(X,A) \longrightarrow \tilde{R}(X,A)$$

$$R'(X,A) \longrightarrow K(X,A)$$

as maps to $[X, \Gamma]$ where $\Gamma \ni$ the fund. gp each comp. contains no [redacted] non-identity perfect subgrp.

5.

~~Properties of $\tilde{K}(X;A)$.~~

$\boxed{\text{Defn:}}$ $$\tilde{K}(X;A) = [X, BGL(A)^+]$$

$$K(X;A) = [X, K_0A \times BGL(A)^+]$$

$$\cong H^0(X, K_0A) \times \tilde{K}(X;A).$$

Properties of $\tilde{K}(X;A)$ on category of finite complexes

(0) map $St(X;A) \longrightarrow \tilde{K}(X;A)$

(1) abelian group structure: $\exists !$ map

$$\tilde{K} \times \tilde{K} \longrightarrow \tilde{K}$$

natural + compatible with given map on St. Moreover, makes $\tilde{K}$ into an abelian group.

(2) These map (0) extends uniquely to a map of abelian groups

$$R(X;A) \longrightarrow \tilde{K}(X;A)$$

(3) splitting theorem implies $\exists$ canonical map

$$R(X,A) \longrightarrow K(X;A)$$

§3. Products and λ-operations.

For any finite complex X we set

$$K(X;A) = [X, K_0A \times BGL(A)^+].$$

Thus if X is a pointed connected finite complex, then

$$K(X;A) = K_0A \oplus \tilde{K}(X;A)$$

as $BGL(A)^+$ is simple. If E is a representation of $\pi_1 X$, set $[E] = \varepsilon[E] \oplus \eta[E]$, where $\varepsilon[E]$ denotes the class in K_0A represented by the underlying A-module of the representation E, and where $\eta(E)$ has been defined above. Since $[E \oplus E'] = [E] + [E']$ and $\tilde{K}(X;A)$ is an abelian group, the map $E \longmapsto [E]$ from representations to virtual vector bundles induces a homomorphism

$$\begin{array}{ccc} R'_A(\pi_1 X) & \longrightarrow & K(X;A) \\ d(E) & \longmapsto & [E] \end{array}$$

where $R'_A(G)$ is the abelian group associated to the monoid of isomorphism classes of representations of G over A.

disjoint union of representable functors on $\mathcal{C}$

<u>Proposition:</u> The above arrow is a universal natural transformation from $R_A'(\pi_1 ?)$ to a representable functor on C (= homotopy category of pointed connected finite complexes.)

false

<u>Proof:</u> We have splittings

$$R_A'(\pi_1 X) = K_0 A \oplus \bar{R}_A'(\pi_1 X)$$

$$K(X; A) = K_0 A \oplus \tilde{K}(X; A)$$

so it evidently suffices to prove that the reduced map $\bar{R}_A'(\pi_1 X) \longrightarrow \tilde{K}(X; A)$ is a universal arrow to a representable functor. But in the category of sets we have a diagram

$$\begin{array}{ccccc}
St^{\#}(\pi_1 X; A)^3 & \rightrightarrows & St^{\#}(\pi_1 X; A)^2 & \longrightarrow & \bar{R}_A'(\pi_1 X) \\
\downarrow & & \downarrow & & \downarrow \\
\tilde{K}(X; A)^3 & \rightrightarrows & \tilde{K}(X; A)^2 & \longrightarrow & \tilde{K}(X; A)
\end{array}$$

where the rows are exact. (In general if M is an abelian monoid and $M \longrightarrow M^{\#}$ is a universal arrow to an abelian group the diagram

$$M^3 \rightrightarrows M \longrightarrow M^{\#}$$

$$(m_1, m_2, m_3) \begin{array}{l} \longmapsto (m_1 + m_2, m_3) \\ \longmapsto (m_1, m_2 + m_3) \end{array}$$

is exact. We've seen that the two left-most vertical arrows ~~[illegible]~~ are universal, hence we can argue that given $\bar{R}'_A(\pi_1 ?) \longrightarrow F(?)$ with F representable, the composition ~~[illegible]~~ $St(?;A)^2 \longrightarrow F(?)$ extends uniquely to ~~define~~ a map $\tilde{R}(?,A)^2 \longrightarrow F(?)$ which by uniqueness equalizes ~~the~~ two maps $\tilde{K}^3 \rightrightarrows \tilde{K}^2$, hence ~~[illegible]~~ induces a ~~[illegible]~~ natural transformation $\tilde{K}(?,A) \longrightarrow F$. q.e.d.

~~Applications~~

The same proof works for the arrows $(R'_A)^n \longrightarrow (K)^n$. (n-fold product). We use this to ~~[illegible]~~ extend properties of (fin) ~~virtual~~ representations to virtual bundles.

Products: If A and A' are two rings tensor product ~~[illegible]~~ gives rise to a pairing

$$R'_A(G) \times R'_{A'}(G) \longrightarrow R'_{A\otimes_{\mathbb{Z}} A'}(G)$$

which is unitary, associative, + commutative in standard sense ~~[illegible]~~ By the proposition ~~[illegible]~~ this induces a product

$$K(X,A) \times K(X,A') \longrightarrow K(X, A\otimes_{\mathbb{Z}} A'),$$

which is bilinear associative and commutative. ~~[illegible]~~

If X, Y are $\in \mathcal{C}$, then

$$0 \longrightarrow \tilde{K}(X \wedge Y; A'') \longrightarrow K(X \times Y; A'') \longrightarrow K(X \vee Y; A'') \to 0$$

is exact so we have pairings

$$\tilde{K}(X; A) \otimes \tilde{K}(X, A') \longrightarrow \tilde{K}(X \wedge Y, A \otimes_{\mathbb{Z}} A')$$

which permit us to define products

$$K_i A \otimes K_j A' \longrightarrow K_{i+j}(A \otimes_{\mathbb{Z}} A').$$

in well-known fashion. In particular $\underbrace{}_{}$ (when **A** is commutative) $K(X, A)$ is a commutative ring augmented over $K_0 A$ and $K_* A$ is a graded anti-commutative ring.

<u>λ-operations:</u> Suppose A is commutative. Then the map $\lambda^i : R'_A(G) \longrightarrow R'_A(G)$ induced by sending a representation E to $\Lambda^i E$ induces a map $\lambda^i : K(X; A) \longrightarrow K(X; A)$.

<u>Proposition:</u> There are unique natural maps $\lambda^i : K(X; A) \to K(X; A)$ for $i > 0$ such that

$$\lambda^i(E) = [\Lambda^i E]$$

Moreover these operators make $K(X; A)$ into a λ-ring, i.e.

the identities

$$\lambda^i(xy) = P_i(\lambda^1 x, \ldots, \lambda^i x, \lambda^1 y, \ldots \lambda^n y)$$

$$\lambda^j \lambda^i(x) = P_{i,j}(\qquad\qquad)$$

hold.

<u>Proof:</u> It is necessary to know that the canonical map $R'_A(\pi_1 X) \longrightarrow K(X;A)$ factors to define a map $R_A(\pi_1 X) \longrightarrow K(X;A)$. One knows that $R_A(\pi_1 X)$ is a λ-ring (Grothendieck seminar [ ], or Swan [ ]), hence the identities hold for elements x and y of the form $[E]$, hence for all elements by the universal property.

<u>Local nilpotence of the γ-filtration:</u> Since $K(X;A)$ is a λ-ring it has γ-operations and Adams operations satisfying habitual identities $\Psi^k \Psi^m = \Psi^{km}$

<u>Proposition:</u> If $x \in \tilde{K}(X;A)$, then $\exists\, N \ni$

$$\gamma^{i_1} x \cdots \gamma^{i_k} x = 0$$

if $i_1 + \cdots + i_k > N$.

Proof: By the same argument as in topological K-theory. If $n \geq 3$ we can form $(BGL_n A)^+$ by killing $E_n A$. Then

$$BGL(A)^+ = \bigcup_n BGL_n(A)^+$$

so

$$[X, BGL(A)^+]_0 = \varinjlim_n [X, BGL_n(A)^+]_0 .$$

Let $\dim_\gamma(x)$ = least n such that x is in the image of the map $[X, BGL_n(A)^+] \to [X, BGL(A)^+]$. Then

$$\gamma^i x = 0 \quad \text{for} \quad i > \dim_\gamma x$$

since to prove this it suffices (by the universal property) to show that $\gamma^i([E]-n) = 0$ in $R_A(G)$ for $i > n$ if E is a representation of G on A^n. As in Atiyah [ ] this follows since $\Lambda^i E = 0$ for $i > n$. Again as in Atiyah the proposition follows from the identity

$$\gamma_t(x)\, \gamma_t(-x) = 1$$

and the finiteness of $\dim_\gamma(x)$ and $\dim_\gamma(-x)$.

~~Corollary: $K(X;A) = \bigoplus_{i \geq 0} N_i$ decomposes as a direct sum of eigenspaces under the Adams operations where $\Psi^k x = k^i x$ for all~~

Cor: Let $K(X;A)^{(i)} = \left\{ x \in K(X;A)\otimes\mathbb{Q} \;\middle|\; \begin{array}{l}\Psi^k x = k^i x \\ \text{all } k \geq 1\end{array} \right\}$

Then $$K(X;A)\otimes\mathbb{Q} = \bigoplus_{i\geq 0} K(X;A)^{(i)}.$$

Proof: One knows that any $x \in K(X;A)$ which is locally nilpotent for the γ-filtration defines a λ-ring homomorphism

$$\begin{array}{ccc} K(G_{nn}) & \longrightarrow & K(X;A) \\ \| & \nearrow & \\ \mathbb{Z}[\lambda_1',\ldots,\lambda_n',\lambda_1'',\ldots,\lambda_m''] \big/ (\lambda_t'\cdot\lambda_t'' = 1) & & \end{array}$$

$$\lambda_i' \longmapsto \lambda_i(x)$$

$$\lambda_j'' \longmapsto \lambda_j(-x)$$

for n sufficiently large. But $K(G_{nn})\otimes\mathbb{Q} \cong H^{ev}(G_{nn},\mathbb{Q})$ with $H^{2i}(G_{nn},\mathbb{Q})$ being the eigenspace with Ψ^k having ~~eigenvalues~~ k^i. This shows that if $x \in \tilde{K}(X;A)\otimes\mathbb{Q}$ we can write

$$x = \sum_{i\geq 0} x_i$$

with $x_i \in K(X;A)^{(i)}$.

Finally one knows that $K_0(A)\otimes\mathbb{Q}$ can be similarly decomposed into eigenspaces for Ψ^k, these being the components of the rational Chow ring of Spec A.

paper on K-theory :

Acyclic maps and algebraic K-theory.

Part I. acyclic maps.

definition characterization acyclic maps.
universal property
classification with fixed target.

Dror theory. introduction to

Part II. K-theory

$BGL(A) \longrightarrow BGL(A)^+$ $\quad \operatorname{Ker} \pi_1 = E(A)$.

Defn: $K_i A = \pi_i BGL(A)^+$

to do

(1) agreement with Bass–Milnor. $K_3 A =$
Dror yoga.

(2) structure on the K-groups

$\tilde{K}(X;A) = [X, BGL(A)^+]$ $\quad$ X finite cx.

i) universal property (?) will show that structure descends uniquely for the map $\tilde{R}_A(X) \longrightarrow \tilde{K}(X,A)$.

then should all work
nilpotence of γ-filtration by Atiyah argument
application to K of a perfect ring

computations.

~~The first point:~~

~~[illegible]~~

It seems that the first ~~[illegible]~~ problem to examine

~~is~~ $\boxed{\text{the map} \quad R(X;A) \longrightarrow K(X;A)}$

~~and its universal property.~~

~~Defn. of map~~

~~[illegible] to start~~

~~X space. E A-bundle over X.~~

~~$E^{\#} \in [X, BGL(A)]$.~~ ~~well-defined.~~

~~how.~~

$Vect(X;A)$ iso classes, abel. monoid

If X connected, then

~~$Vect(X;A) = \coprod_{P} Vect_P(X;A) =$~~

~~where P runs over iso. classes of f.g. proj~~ ~~reps~~

~~A-modules. Then have~~

~~$Vect_P(X;A) \to Vect_{P+Q}(X;A)$.~~

$$Vect(X;A) = \coprod_{s \in S} [X, B\,Aut(P_s)].$$

where ~~[illegible]~~ S = iso classes of f.g. proj. A-modules

$s \in S$, P_s rep.

Now one ~~sees~~ has X connected

$$St\,Vect(X;A) = \varinjlim_{s\in S} [X, BAut(P_s)].$$

S being viewed as a category. Since $\mathbb{N}$ cofinal

$$St\,Vect(X;A) = \varinjlim_{n} [X, BGL_n A].$$

This maps ~~to~~ in an evident way to

$$[X, BGL(A)].$$

So we have maps

$$Vect(X;A) \twoheadrightarrow St\,Vect(X,A) \longrightarrow [X, BGL(A)].$$

Can continue with

$$[X, BGL(A)] \longrightarrow [X, BGL(A)^+].$$

In this way we have associated to a vector bundle E over X an element

$$cl(E) \in [X, BGL(A)^+].$$

~~Next~~ ~~to~~

<u>Theorem:</u> $0 \to E' \longrightarrow E \longrightarrow E'' \longrightarrow 0$

leads to $\underline{cl(E) = cl(E') + cl(E'')}$

$+$ being defined rel. to H-space structure introduced already in $BGL(A)^+$.

Set

might not be good if $\pi_0 X$ infinite

$R(X;A)$ = Groth group of A-bundles / X.

Because of thm. and fact that $BGL(A)^+$ is a comm. H-space, hence invertible, we get

$$R(X;A) \longrightarrow [X, BGL(A)^+].$$

<u>Logic</u>: 1 $[X, BGL(A)^+]$ is an abelian group

2 $\forall E \in Vect(X;A)$ have $d(E) \in [X, BGL(A)^+]$

$\Rightarrow$ ~~$d(E) = d(E') + d(E'')$~~ additive function

so extends by universal prop. to a map

$$R(X;A) \longrightarrow [X, BGL(A)^+] = \tilde{K}(X;A)$$

---

on the other have have evident map

$$R(X;A) \longrightarrow [X, K_0 A]$$

so putting these together get

$$R(X;A) \longrightarrow [X, K_0 A \times BGL(A)^+] = K(X;A)$$

K-theory part.

1. $BGL(A)^+$, $K_i A$, comparison Bass-Milnor $K_3 A$.

2. H-space structure (elementary properties of $BGL(A)^+$).

3. map $R(X,A) \longrightarrow K(X,A)$
   stable splitting theorem

4. universal property of the above map

5. structure on $K(X,A)$.

   products
   λ operations
   γ filtration
   application to perfect rings. Ψ^p = Frobenius in char. p

   naturality
   ext. scalars
   rest. "
   proj. fmlas. Galois case

Computation for a finite field and its alg. closure.

Have now to understand paper on K-theory.

Defn. $\pi_1 BGL(A) = GL(A) \supset E(A)$ which is perf. + so $\exists$ acyclic map, unique á homotopy pres,

$$f : BGL(A) \longrightarrow BGL(A)^+$$

$\ni Ker(\pi_1 f) = E(A)$.

$$K_i(A) = \cancel{[S^i, BGL(A)^+]}. \quad \pi_i BGL(A)^+$$

Then from theory of acyclic maps I can prove

$K_i A$ == Bass, Milnor $i=1,2$

$K_3 A$ == $H_3(St(A))$.

relation to ~~topological~~ K-theory

H-space structure of $BGL(A)^+$: ~~...~~

Results:

1) $BGL(A)^+$ homotopy-commutative + assoc. H-space. (simple space)

2) $PH_*(BGL(A)) = K_*(A) \otimes \mathbb{Q}$.

3) lemma: $Inj(\mathbb{N}, \mathbb{N})$ trivial.

[$u : \mathbb{N} \to \mathbb{N}$ injective. Then $u^+ : BGL(A)^+$ is a heq. and so can apply the lemma.

4) remark explicit formula for $BGL(A)^+$.

5) $K_*(A * B) =$

$$K_*(\varinjlim A_i) = \varinjlim_i K_*(A_i).$$

important to mention $[\ , BGL(A)^+]$ is an abelian group

$\tilde{K}(X;A) = [X, BGL(A)^+]$ as a functor on the category of finite complexes.

idea is the universality of the map from repns.?

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the real point is the λ-ring structure on

$K(X;A)$

nilpotence of γ-filtration

consequences for perfect rings.

one problem appears to be to define the map

$$R(X;A) \longrightarrow K(X;A).$$

seems to require

stable splitting theorem

invertibility of H-space structure on $BGL(A)^+$

The γ-filtration on a λ-ring R.
problem with definition of $\tilde{R}$

<u>Thm</u>: $\tilde{K}(X;A)$ (λ-ring without 1)
assertion: the γ filtration is nilpotent

~~Obs~~ $\tilde{K}(X;A) = F_1 \supset F_2 \supset \cdots$

$F_p\tilde{K}(X;A) =$ ~~[illegible]~~ subgroup gener. by monomials
$\gamma^{i_1}(x_1)\cdots\gamma^{i_p}(x_p)$
with $i_1+\cdots+i_p \geq p$

Assertion $\Psi^k \equiv k^p$ on F_p/F_{p+1} ?

OKAY ATIYAH-TALL

The next point to be understood is the structure on K.

~~[scribble]~~ products: $K(X;A)\otimes K(Y;B)\to K(X\times Y, A\otimes B)$

λ-operations for A-commutative.

extension of scalars
restriction of scalars } projection formulas
trace formulas.
for Galois covering
$f^*f_*x=\sum_{\sigma\in gal} x^\sigma$.

It was my hope to be able to deduce these things from a universal property for the map

$$R(X;A)\longrightarrow K(X;A)$$

on the homotopy category of finite CW complexes.

$$K(X;A)=[X,\underbrace{K_0A\times BGL(A)^+}]$$

space $\ni$ each component has no perfect subgroup ~~[scribble]~~ in π_1

<u>Claim:</u> Γ space $\ni$ no component has a non-trivial perfect subgroup of its π_1 (equivalently $\Gamma\xrightarrow{u}\Gamma'$ acyclic $\Longrightarrow$ u heq.). Then

$$\mathrm{Hom}\Big(\prod_{i=1}^{n}R(\,;A_i)^*,[\ ,\Gamma]\Big)=\mathrm{Hom}\Big(\prod_{i=1}^{n}K(X;A_i)^*,[\ ,\Gamma]\Big)$$

as functors on the homotopy category of finite cxs.

outline. acyclic maps.

space of homotopy type CW cx., maps cont. maps.
reference to Milnor, sheaf + sing. coh. coincide here.

Def 1. acyclic space

acyclic space with $\pi_1 = 0$ is contractible (Whitehead)

homotopy-fibers of $f \overset{\text{defn}}{=}$ fibres of $X \times_Y Y^I \to Y$.

Def. 2: f **acyclic** if all homot.-fibres are acyclic spaces

If f fibration, this is same as saying all fibres acyclic.

observe if f acyclic, then $\pi_0 X \simeq \pi_0 Y$. Thus in the following to simplify we supp. X, Y conn. + endowed with basepoint.

~~In the following, suppose Y is~~
~~To simplify, suppose spaces are connected to simplify~~

to simplify (the map $f: X \to Y$ fixed)

In the following, we suppose Y connected and provided with basepoint, $F \overset{(hom)}{=}$ fibre of f over y, $\pi_1 Y$ = fdl group at basepoint.

Proposition 1: TFAE

(i) f acyclic

(ii) $\forall$ local coeff. sys. on Y, $H^*(Y, L) \xrightarrow{\sim} H^*(X, f^*L)$

(iii) suppose in the square

$$\begin{tikzcd} \tilde{X} \arrow[r, "\tilde{f}"] \arrow[d, "u'"'] & \tilde{Y} \arrow[d, "u"] \\ X \arrow[r, "f"'] & Y \end{tikzcd}$$

$\tilde{Y}$ is the universal covering of Y, and $\tilde{X}$ is the induced covering of X, ~~s.t. the square is cos~~ Then $\tilde{f}$ induces isom on integral homology.

Cor. 1: f acyclic + $\pi_1(f) \cong \implies f$ h.eq.

Remark: That (ii) and + $\pi_1(f) \cong$ implies f h.eq. is the Whitehead thm. as formulated by A-M.

~~Cor. 2: If~~

$$\cancel{\begin{array}{ccc} X & - & X' \\ | & & \\ Y & - & Y' \end{array}}$$

~~has M-V, then f acyclic $\to f'$ acyclic~~

We say that the commutative square

$$\begin{tikzcd} X \arrow[r, "g"] \arrow[d, "f"'] & X' \arrow[d] \\ Y \arrow[r] & Y' \end{tikzcd}$$

has the MV property if for every local coeff. system L on Y' have MV sequence

|

This will be case if the square is cocartesian and if either f or g is a cofibration.

Cor 2: If square $(*)$ has M-V, then f acyclic $\implies g$ acyclic.

Universal property

Prop. 2:

~~$\tilde{Y}$ 1-connected $\Rightarrow \pi_0 \tilde{F} = e$~~

~~considering Leray spectral sequence~~

$$\cancel{E_2^{pq} = H^p(\tilde{Y}, H^q(F, A)) \Longrightarrow H^{p+q}(\tilde{X}, A)}$$

~~local coeff. system trivial $\Rightarrow H^q(F, A) = \begin{cases} A & q = 0 \\ 0 & q > 0 \end{cases}$~~

~~for any A abelian group $\Longrightarrow$ F acyclic~~

~~[illegible]~~

~~Thus $\tilde{f}$ induces isoms on integral homology $\Longleftrightarrow$ F acyclic~~

~~[illegible]~~ assume Y connected
F = fibre of $X \longrightarrow Y$ over basepoint
First consider the case ~~whence~~ Y is 1-connected.
~~[illegible]~~ then

Proposition 1:

(i) f acyclic

(ii) $\forall L, \quad H^*(Y, L) \xrightarrow{\sim} H^*(X, f^*L)$

(iii)
$$\begin{tikzcd} \tilde{X} \arrow[r, "\tilde{f}"] \arrow[d, "u'"'] \arrow[dr, phantom, "\text{cart.}"] & \tilde{Y} \arrow[d, "u"] \\ X \arrow[r, "f"'] & Y \end{tikzcd}$$

$\tilde{Y}$ 1-conn. covering of Y
$\tilde{X}$ induced covering

Then $\tilde{f}$ induces isomorphisms on integral homology.

may assume f a fibration. F = fibre over basepoint also = fibre of $\tilde{f}$. Spectral sequence

$$E_2^{pq} = H^p(Y, H^q(F, E')) \Longrightarrow H^{p+q}(X, f^*L)$$

where E' is the pull-back of f^*L to F. E' is trivial, so

s.s. deg.

(i) $\Longrightarrow$ (ii) by spectral sequence for $f: X \to Y$.

(ii) $\Longrightarrow$ (iii) suffices to show for all abelian groups A that $\tilde{f}^* : H^*(\tilde{Y}, A) \xrightarrow{\sim} H^*(\tilde{X}, A)$.

But if $u_* A = L$ then $f^*(L) = u'_* A$ and have

$$\begin{tikzcd} H^n(Y, L) \arrow[r, "f^*"] \arrow[d, "\wr"] & H^n(X, f^*L) \arrow[d, "\wr"] \\ H^n(\tilde{Y}, A) \arrow[r, "\tilde{f}^*"] & H^n(\tilde{X}, A) \end{tikzcd}$$

(iii) $\Longrightarrow$ (i). F also fibre of $\tilde{f}$, so spec. seq

$$E_2^{pq} = H^p(\tilde{Y}, H^q(F, A)) \Longrightarrow H^{p+q}(\tilde{X}, A)$$

where local coeff. system on $\tilde{Y}$ is trivial because $\tilde{Y}$ is 1-connected. Since $\tilde{f}$ induces isom. on coh. with coeff. in A, one deduces that $H^q(F, A) = \begin{cases} A \\ 0 \end{cases}$ hence F acyclic.

Cor 1:

f acyclic + $\pi_1(f)$ isom. $\Longrightarrow$ f heq.

Proof: In this case, $\tilde{X}$ is also 1-connected, so by Hurewicz $\pi_2 \tilde{X} \simeq H_2 \tilde{X}$ will be mapped isomorphically to $\pi_2 \tilde{Y} \simeq H_2 \tilde{Y}$. Thus by homot long exact sequence $\pi_1 F = 0$. As F is acyclic it is contractible, so f is a homotopy equivalence (by obstruction theory).

Remark: This corollary + (ii) is the Whitehead theorem as formulated by A-M.

Cor 2: If

$$\begin{tikzcd} X \arrow[r, "g"] \arrow[d, "f"'] & X' \arrow[d, "f'"] \\ Y \arrow[r, "g'"'] & Y' \end{tikzcd}$$

has M-V property, then f acyclic $\Rightarrow$ f' acyclic.

Prefaced by a discussion of when this is the case

(α) g cofibration, meaning what? HEP.

(β) f cofibration ________.

Prop. 2: (Universal Property). $X_1 \times I \cup Y \times \dot{I} \to Y \times I$ heq gives injectivity

follows using van Kampen + cobase change.

Cor: Given $N \subset \pi_1 X$ perf. normal, $\exists$ at most one acyclic map $f: X \to Y$ up to homotopy equivalence with $\operatorname{Ker} \pi_1(f) = N$.

Given $N \subset \pi_1 X$ perf. + normal

Prop. 3: Existence of acyclic $f: X \to Y$ with $N = \operatorname{Ker} \pi_1(f)$.

first, take case $N = \pi_1 X$, then attach 2 + 3 cells to get Y. Then form cocartesian square

$$(*) \quad \begin{array}{ccc} \tilde{X} & \xrightarrow{f'} & \tilde{X}^+ \\ \downarrow & & \downarrow \\ X & \xrightarrow{f} & Y \end{array}$$

Remark: $\tilde{X}^+ \to \tilde{Y}$ heq because both space 1-conn. ~~both spaces~~ both $\tilde{X} \to \tilde{X}^+$ and $\tilde{X} \to \tilde{Y}$ are homology isom.
Thus square is both cartesian + cocartesian up ~~...~~ to homotopy equivalence.

$\boxed{\text{The theory of Emmanuel Dror.}}$

DRORI   First give Dror's construction.

Given X conn. ptd. + $N \subset \pi_1 X$ perf. normal

$X_0 = X$, $X_1 =$ covering space with $\pi_1 X_1 \xrightarrow{\sim} N$.

for $n \geq 2$ assume X_{n-1} found so that

$$H_i(X_{n-1}) = 0 \qquad 0 < i \leq n-1$$

~~Then have canonical isom~~

$$\cancel{[X, K(A,n)] \xrightarrow{\sim} \mathrm{Hom}(H_n X, A)}$$

~~for any abelian group A, defined by sending~~ ~~$f: X \to K(A,n)$ to~~

$$\cancel{H_n X \xrightarrow{f_*} H_n K(A,n) \overset{\text{canon. isom}}{\simeq} A.}$$

~~(origins of this isom~~

$$\cancel{[Z, K(A,n)] \xrightarrow{\sim} H^n(Z, A)}$$

~~For any space Z and abels we have a canonical map~~

$$\cancel{[Z, K(A,n)] \longrightarrow \mathrm{Hom}(}$$

Let A be an abelian group and $K(A,n)$ an E-M space type (A,n). Then for any space Z, we have a canonical map

$$(*) \qquad [Z, K(A,n)] \longrightarrow \mathrm{Hom}(H_n Z, A)$$

sending a map f into the composition

$$H_n X \xrightarrow{f_*} H_n K(A,n) \overset{h}{\simeq} \cancel{A}\ \pi_n K(A,n) = A$$

where ~~this last map~~ h is the ~~canonical~~ Hurewicz isomorphism. ~~Then~~

~~One knows (*) is surjective with kernel $Ext^1(H_{n-1}Z, A)$~~ ~~by the universal~~ By the universal coeff. thm and le fait que EM spaces rep. coh., $(*)$ is surj. with kernel $\mathrm{Ext}^1(H_{n-1}Z, A)$. ~~Then~~ As $H_{n-1}X_{n-1} = 0$, we obtain a canon. isom.

$$[X_{n-1}, K(A,n)] \xrightarrow{\sim} \mathrm{Hom}(H_n X_{n-1}, A).$$

for all abelian groups A. ~~It follows, on taking $A = H_n X_{n-1}$, that~~ Let $f_n : X_{n-1} \to K(H_{n-1}X_n, n)$ be a basepoint-preserving map whose homotopy class corresponds to the identity of $H_n X_{n-1}$, ~~Then by general nonsense on representable functors, the~~ i.e. such that

$$H_n X_{n-1} \xrightarrow{f_{n*}} H_n K(H_{n-1}X_n, n) \simeq H_{n-1}X_{n-1}$$

is the identity. We let X_n be the homotopy-fibre of f_n.

From the spectral sequence of f_n we ~~obtain~~ ~~a long exact sequence~~ obtain isoms.

$$\cancel{H_i X_n \to H_i X_{n-1} \to H_i K_n \to H_{i-1} X_n \to \cdots}$$

$$H_i X_n \xrightarrow{\sim} H_i X_{n-1} \qquad i < n-1$$

and an exact sequence

$$H_{n+1} K \xrightarrow{d_n} H_n X_n \to H_n X_{n-1} \xrightarrow{(f_n)_*} H_n K,$$

where we have put $K = K(H_n X_{n-1}, n)$. By the Hurewicz theorem ~~$0 = \pi_{n+1}K$ maps onto~~ ~~$H_{n+1}K$~~ $\pi_{n+1}K$ maps onto $H_{n+1}K$, so $H_{n+1}K$ is zero. As $(f_n)_*$ is an isom. by the above, we ~~have~~ see that $H_i X_n = 0$ for $1 \le i \le n$. ~~[illegible]~~ This completes the inductive const. of X_n and provides a tower of fibrations

$$
\begin{tikzcd}
X_2 \arrow[r] \arrow[d] & K(H_3 X_2, 3) \\
X_1 \arrow[r] \arrow[d] & K(H_2 X_1, 2) \\
X_0 \arrow[r] & K(\pi_1 X/N, 1)
\end{tikzcd}
$$

~~Let X_∞ ...~~

Let

$$X_\infty = \text{lim.inv. } X_n .$$

Then X_∞ will have same homot. type as X_n in degrees $\le n$

~~Then~~ $\cancel{H_i X_\infty = H_i X_n \qquad i < n}$

so X_∞ is an acyclic space. Moreover

$$\pi_1 X_\infty \xrightarrow{\sim} \pi_1 X_2$$

maps onto $\pi_1 X_1$ which maps isomorphically onto $N \subset \pi_1 X_0$. If

$$Y = \text{Cone}\,(X_\infty \longrightarrow X)$$

~~that the canonical ... embedding~~ and $f: X \longrightarrow Y$ is the canonical embedding, then f is an acyclic map (special case of Cor 2 to prop 1) with $\operatorname{Ker} \pi_1(f) = N$ (van Kampen).

~~Let F = homotopy fibre of f.~~

Proposition: ~~The canonical map $X_\infty \to F$ is a homotopy equivalence.~~ (X_∞ is homotopy equiv. to the homotopy-fibre F of $f: X \to Y$.)

~~Proof: F also homotopy fibre of $\tilde{X} \to \tilde{Y}$, so we~~

~~For any acyclic space P with basepoint~~

~~$[P,F]_\bullet \xrightarrow{\ ?\ } \{$~~

Let P be an acyclic space with basepoint. Then

$$^\bullet[P,X_n]_m \xrightarrow{\sim} {}^\bullet[P,X_{n-1}]_m \cdots\cdots \xrightarrow{\sim} {}^\bullet[P,X_1]_m$$

~~and $D_0 \to [P,X_1] \to [P,X_0] \to$~~

and $[P,X_1]$ injects into $[P,X_0]$

with image the set of maps ~~α~~ Im ~~α~~ $\pi_1(P) \subset N$.
Thus

$$^\bullet[P,X_\infty]_m \xrightarrow{\sim} \{\alpha \in {}^\bullet[P,X] \mid \operatorname{Im}\pi_1(\alpha) \subset N\}.$$

On the other hand, ~~we have a Puppe sequence~~ ~~know that two elements of $[P,F]_\bullet$ became equal in $[P,X]_\bullet$ iff they are~~ ~~from the Puppe-style sequence associated to the fibration f we know that the group $[P,\Omega Y]_\bullet$ acts on $[P,F]_\bullet$ and the set of orbits is~~ from the Puppe-style sequence

$$[P,\Omega Y]_\bullet \longrightarrow {}^\bullet[P,F]_\bullet \xrightarrow{i_*} {}^\bullet[P,X]_\bullet \xrightarrow{f_*} {}^\bullet[P,Y]_\bullet$$

we know that two elements of $[P,F]$ become $=$ in $[P,X]$ iff they are conjugate under the action

~~$\pi_1(\alpha)$~~ ~~$f_*(\alpha) = 0$~~

~~may assume that $N = \pi_1 X$ and Y simply-connected.~~
~~We know already that E is acyclic~~

~~We have~~

~~On the other hand we can compute $[P, F]$. by the Puppe style sequence associated to the fibration f. Thus~~

On the other hand, one knows by the Puppe-style sequence assoc. to the fibration f, that two elements of ${}^{\bullet}[P, F]$. which become equal are in the same orbit of the natural action of the gp.

$$ {}^{\bullet}[P, \Omega Y]_{\bullet} \simeq {}^{\bullet}[\Sigma P, Y]_{\bullet} $$

Since P is acyclic, ΣP is both acyclic and 1-connected, hence contractible. Thus ~~${}^{\bullet}[P, F]$ injects and is isomorphic to~~

$$ {}^{\bullet}[P, F] \xrightarrow{\sim} \{ \alpha \in {}^{\bullet}[P, X] \mid f_*(\alpha) = 0 \}. $$

Applying the universal property to the acyclic map $P \to c$, one sees that the condition $f_*(\alpha) = 0$ is equivalent to ~~$\pi_1(\alpha)$~~ $\operatorname{Im} \pi_1(\alpha) \subset N$. Therefore ~~we see that F and X_∞ both represent the same functor on the homotopy category of acyclic~~ the functors represented by F and X_∞ are isomorphic, and the proposition follows.

~~Corollary:~~

~~$$\pi_n(Y) \simeq H_n(X_{n-1}) \qquad \text{for } n \geq 2.$$~~

~~Proof: The inverse image of the tower (*) over the basepoint of x, is the Postnikov system of the fibre of $X_\infty \to X$ which is ΩX. Hence $H_n(X_{n-1}) = \pi_{n-1}$~~

Corollary: $\pi_n(Y) \simeq H_n(X_{n-1})$ for $n \geq 2$.

In effect, the inverse image of the tower (*) over the basepoint of x, is ~~[illegible]~~ the Postnikov system of the fibre of $X_\infty \to X$, which is ΩX. Thus $H_n(X_{n-1}) = \pi_{n-1}(\Omega X)$ for $n \geq 2$, proving the corollary. (In fact the tower (*) is the Postnikov system of the map $X_\infty \to X$.)